# Continuous Time Random Matching

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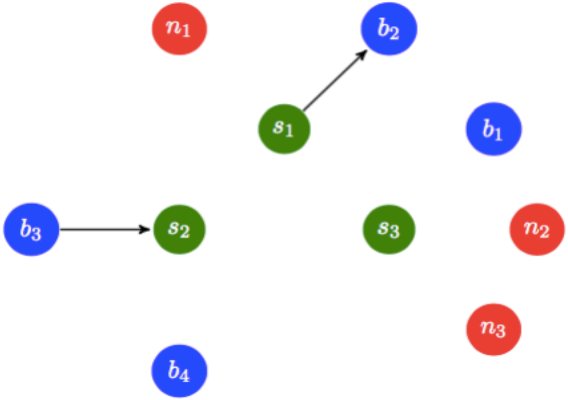
# Random Matching Markets



# Reliance on Continuous-Time Random Matching

- Many researchers have worked with continuous-time models that assume a large number of agents who meet their partners randomly according to a Poisson process with a given arrival rate.
- The intuition is that when a large number of agents conduct searches without explicit coordination, random searches by different agents can be considered to be independent.
- By the law of large numbers, there should be an almost-sure constant cross-sectional distribution of types, which will simplify the analysis dramatically.

# Infinite Agent Space

- However, the matching processes cannot be mathematically independent, as long as there are only finitely many agents in the economy.

- This naturally leads to the consideration of infinitely many agents in random matching models.

# Why Continuum?

- If the agent space is countable, the measure can not be countably additive.
- Failure of convergence theorems.
- Non-existence of equilibrium of simple models in games and economies: Khan, Qiao, Rath and Sun (working paper)

# Reliance on Continuous-Time Random Matching

- **Monetary theory.** Hellwig (1976), Diamond-Yellin (1990), Diamond (1993), Trejos-Wright (1995), Shi (1997), Zhou (1997), Postel-Vinay-Robin (2002), Moscarini (2005).
- **Labor markets.** Pissarides (1985), Hosios (1990), Mortensen-Pissarides (1994), Acemoglu-Shimer (1999), Shimer (2005), Flinn (2006), Kiyotaki-Lagos (2007).
- **Over-the-counter financial markets.** Duffie-Garleanu-Pedersen (2005), Weill (2008), Vayanos-Wang (2007), Vayanos-Weill (2008), Weill (2008), Lagos-Rocheteau (2009), Hugonnier-Lester-Weill (2014), Lester, Rocheteau, Weill (2015).
- **Biology (genetics and epidemiology).** Hardy-Weinberg (1908), Crow-Kimura (1970), Eigen (1971), Shashahani (1978), Schuster-Sigmund (1983), Bomze (1983).
- **Game theory.** Mortensen (1982), Foster-Young (1990), Binmore-Samuelson (1999), Battalio-Samuelson-Van Huycjk (2001), Burdzy-Frankel-Pauzner (2001), Bena m-Weibull (2003), Currarini-Jackson-Pin (2009), Hofbauer-Sandholm (2007).

# Static Random Matching

- Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space of agents.
- $S = \{1, \ldots, K\}$ a set of finite types.
- $\alpha : I \to S$ a type function with type distribution $p$ on $S$.
- $(\Omega, \mathcal{F}, P)$, another probability space modeling randomness in matching.

# Static Random Matching II

- A mapping $\pi$ from $I$ to $I$ is called a (deterministic) matching if for any $i \in I$, $\pi(\pi(i)) = i$. If $\pi(i) = i$, then $i$ is not matched.
- A mapping $\pi$ from $I \times \Omega$ to $I$ is called a random matching if for any $\omega \in \Omega$, $\pi_\omega$ is a (deterministic) matching.
- For $k, l \in S$, let $q_{kl} \in \mathbb{R}$ be the matching probability for a type-$k$ agent to meet a type-$l$ agent. Assume that $p_k q_{kl} = p_l q_{lk}$ for any $k, l \in S$ and $\sum_{l=1}^{K} q_{kl} \leq 1$ for each $k \in S$.
- Measurability problem of a continuum of independent random variables.

# Fubini Extension

Let $(I \times \Omega, \mathcal{W}, Q)$ be a probability space extending the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{A}, \lambda \times P)$. The extension is said to be a **Fubini extension** of the usual product probability space if it retains the Fubini property, i.e., for any real-valued $\mathcal{W}$-integrable function $f$ on $T \times \Omega$,

$$\begin{aligned}\int_{I \times \Omega} f dQ &= \int_I \left( \int_\Omega f_t dP \right) d\lambda \\ &= \int_\Omega \left( \int_I f_\omega d\lambda \right) dP.\end{aligned}$$

The extension is denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{A}, \lambda \boxtimes P)$.

**Exact Law of Large Numbers** [Sun 1998, 2006]: Let $f$ be a process defined on a Fubini extension. If $f$ is essentially pairwise independent, then for $P$-almost all $\omega \in \Omega$,

$$P f_\omega^{-1} = (\lambda \boxtimes P) f^{-1}.$$

# Static Random Matching III

Let $\pi$ be a mapping from a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $I$. $\pi$ is said to be a independent random matching if:

- For any $\omega \in \Omega$, $\pi_\omega$ is a (deterministic) matching.
- Let

$$g(i,\omega) = \begin{cases} \alpha(\pi(i,\omega)) & \pi(i,\omega) \neq i \\ J & \pi(i,\omega) = i, \end{cases}$$

  and $g$ is $\mathcal{I} \boxtimes \mathcal{F}$ measurable.
- For any type-$k$ agent $i \in I$, $P(g_i = l) = q_{kl}$.
- The process $g$ is essentially pairwise independent in the sense that for $\lambda$-almost all $i, j \in I$, $g_i$ and $g_j$ are independent.

# Static Random Matching IV

**Proposition (Duffie, Qiao and Sun, 2018)**

**(a)** *Let $\pi$ be an independent random matching with parameters $(p, q)$. Then, for $P$-almost every $\omega \in \Omega$, we have $\lambda(\{i : \alpha(i) = k, g_\omega(i) = l\}) = p_k q_{kl}$ for any $k, l \in S$.*

**(b)** *For any given $(p, q)$, there exists an independent random matching .*

# Continuous-Time Random Matching

In our most basic model:

- $S = \{1, \ldots, K\}$ is a set of finite types.
- $\alpha^0 : (I, \mathcal{I}, \lambda) \to S$ is the initial a type function.
- $p^0$ is the distribution of $\alpha^0$.

# Continuous-Time Random Matching

- $\alpha(i,t)$ is the random type of agent $i$ at time $t$.
- $\varphi(i,t)$ is the last partner of agent $i$ up to time $t$.
- $h(i,t)$ is the type of the last partner of agent $i$ up to time $t$.

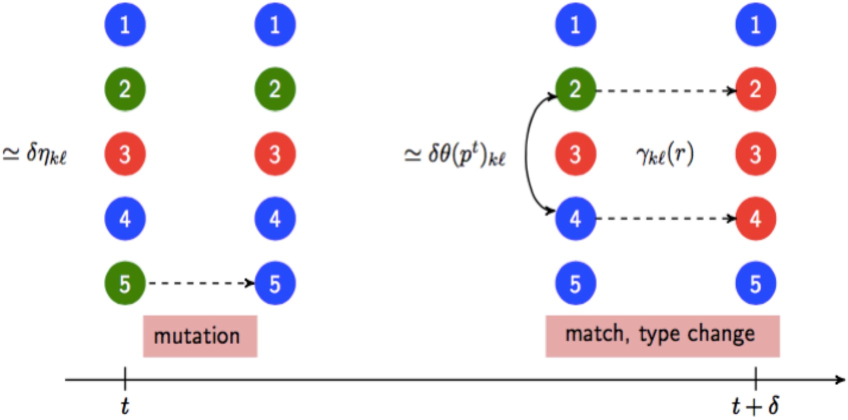
# Continuous-Time Random Matching

- $d_i^n$ is the $n$-th matching time for agent $i$.
- For any agent $i$ and any matching time $d_i^n$ for agent $i$, if $\varphi(i, d_i^n) = j$, then $\varphi(j, d_i^n) = i$ $P$-almost surely, or equivalently
$$\varphi(\varphi(i, d_i^n), d_i^n) = i$$
$P$-almost surely.

# Continuous-Time Random Matching

- Mutation intensity $\eta_{kl}$.
- Matching intensity $\theta_{kl} : \Delta(S) \to \mathbb{R}_+$, continuous, satisfying the balance identity $p_k\theta_{kl}(p) = p_l\theta_{lk}(p)$.
- Match-induced type changing probability distribution $\gamma_{kl} \in \Delta(S)$.

# Random Matching with Enduring Partnership



# Continuous-Time Random Matching: Theorem

Fixing any parameters $(p^0, \eta, \theta, \gamma)$, there exists a continuous-time dynamical system with random mutation, matching and type changing such that

- The expected cross-sectional type distribution $\bar{p}^t$satisfies

$$\frac{d\bar{p}^t}{dt} = \bar{p}^t R^t, \qquad \bar{p}^0 = p^0,$$

where

$$R^t_{kl} = \eta_{kl} + \sum_{l=1}^{K} \theta_{kr}(\bar{p}^t)\gamma_{kr}(l)$$

for any $k \neq l$, and $R^t_{kk} = -\sum_{l \in S, l \neq k} R^t_{kl}$.

# Continuous-Time Random Matching: Theorem

- With probability one, the realized cross-sectional type $p^t_\omega$ is equal to the expected cross-sectional type distribution $\bar{p}^t$.
- $\{\alpha_i\}_{i \in I}$ forms a continuum of independent continuous-time Markov chains with transition intensity matrix $R^t$ at time $t$, where

$$R^t_{kl} = \eta_{kl} + \sum_{l=1}^{K} \theta_{kr}(\bar{p}^t)\gamma_{kr}(l)$$

for any $k \neq r$, and $R^t_{kk} = -\sum_{l \in S, l \neq k} R^t_{kl}$.

# Continuous-Time Random Matching: Theorem

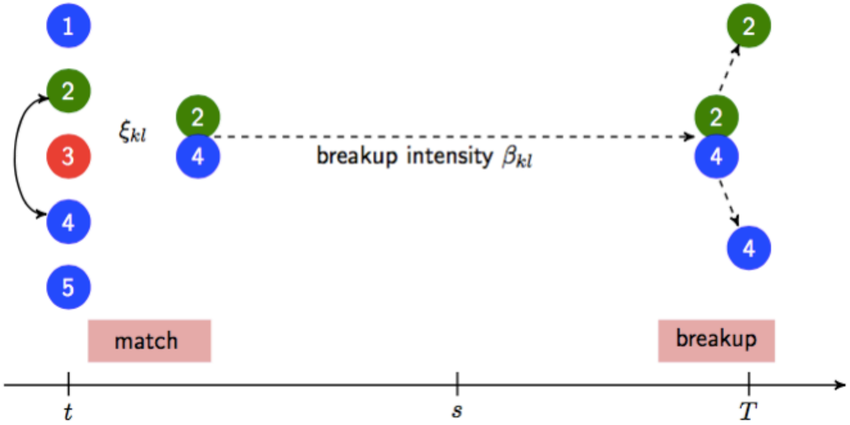
- $\{(\alpha_i, h_i)\}_{i\in I}$ forms a continuum of independent continuous-time Markov chains.

- For any $(\eta, \theta, \gamma)$, there exists $p^0$ such that with probability one, the realized cross-sectional type distribution $p^t_\omega = p^0$ for all $t \geq 0$.

# Continuous-Time Random Matching: Theorem

- Let $N_{ikl}(t)$ be the number of matches by agent $i$ up to time $t$, when of type $k$, to an agent of type $l$.
- Then the cumulative total quantity $\Theta_{kl}(\omega, t)$ of matches can be defined as $\int_I N_{ikl}(\omega, t)\, d\lambda(i)$.
- For $P$-almost all $\omega \in \Omega$, for any types $k$ and $l$, the cumulative total quantity $\Theta_{kl}(\omega, t)$ equals to its expectation $\mathbb{E}(\Theta_{kl}(t))$ and grows at the rate $\dot{\Theta}_{kl}(\omega, t) = \bar{p}_k^t \theta_{kl}(\bar{p}^t)$.

# Random Matching with Enduring Partnership



# Transition Intensity Matrix for Extended Types

$\{(\alpha_i, g_i)\}_{i\in I}$ forms a continuum of independent continuous-time Markov chains with transition intensity matrix $Q^t$ at time $t$, where

$$
\begin{aligned}
Q^t_{(k_1 l_1)(k_2 l_2)} &= \eta_{k_1 k_2}\delta_{l_1}(l_2) + \eta_{l_1 l_2}\delta_{k_1}(k_2),\\
Q^t_{(k_1 l_1)(k_2 J)} &= \beta_{k_1 l_1}\gamma_{k_1 l_1}(k_2),\\
Q^t_{(k_1 J)(k_2 l_2)} &= \sum_{l_1=1}^{K} \theta_{k_1 l_1}(\check{p}(t))\xi_{k_1 l_1}\sigma_{k_1 l_1}(k_2, l_2),\\
Q^t_{(k_1 J)(k_2 J)} &= \eta_{k_1 k_2} + \sum_{l_1=1}^{K} \theta_{k_1 l_1}(\check{p}(t))(1-\xi_{k_1 l_1})\gamma_{k_1 l_1}(k_2),\\
Q^t_{(kl)(kl)} &= -\sum_{(k',l')\neq(k,l)} Q^t_{(kl)(k'l')},
\end{aligned}
$$

where $\check{p}(t) = \mathbb{E}\left(\hat{p}(t)\right)$

# Compact Type Space

- $S$ is a compact metric space.
- $$\begin{aligned} Q^t_{(kl)}(\hat{A}) &= \eta_k(\hat{A}_l) + \eta_l(\hat{A}^T_k) + \vartheta^S_{kl}(\hat{A}_J) \\ Q^t_{(kJ)}(\hat{B}) &= \eta_k(\hat{B}_J) + \int_{l' \in S} \xi_{kl'} \sigma_{kl'}(\hat{B} \cap (S \times S)) d\theta(k, \check{p}(t)) \\ &+ \int_{l' \in S} (1 - \xi_{kl'}) \varsigma^S_{kl'}(\hat{B}_J) d\theta(k, \check{p}(t)) \end{aligned}$$
- $\frac{d\check{p}^t}{dt} = \int_{\hat{S}} Q^t_{kl} d\check{p}^t, \qquad \check{p}^0 = \hat{p}^0.$

# $\sigma$-Compact Type Space

- $S$ is a $\sigma$-compact metric space.
- **Boundedness:** $\eta(\cdot)(S)$, $\theta(\cdot,\cdot)(S)$, $\vartheta(\cdot,\cdot)$ are bounded.
- **Compact Tightness:** For any $t \in \mathbb{R}_+$, for any $\epsilon > 0$ and any compact $K \in \mathcal{S}$, there exists a compact $K'$ in $\mathcal{S}$ , such that

$$\eta_k(K') > 1 - \epsilon,$$
$$\sigma_{kl}^S(K') > 1 - \epsilon,$$
$$\varsigma_{kl}(K') > 1 - \epsilon$$

for any $k \in K$, $l \in S$.

Thanks!