A Two-Stage Model of Assignment and Market

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Introduction

We consider a two-stage economy with non-monetary assignment in the first stage and market trades in the second.

- College students foreseeing the future job prospects
- Office allocation with subsequent exchange

Introduction

The second stage market makes the assignment stage a totally different ball game from the one without it, e.g.,

- An agent may go for a less preferable good, expecting to sell it later, and therefore, both the first and second stage outcome may be neither efficient nor stable.
- This is true even with or without money.
- We present equivalent conditions under which we recover efficiency in the economy with money and stability in the economy with no money.

Literature

- Non-market assignment of indivisible goods
  Gale=Shapley (1962), Roth=Sotomayor (1989), Ergin (2002), Kojima=Manea (2010), ...
- Market for indivisible goods: comparison with assignment
  Shapley=Scarf (1974), Kaneko (1982), Gale (1984), Quinzii (1984), Piccione=Rubinstein (2007), ...
- Property right assignment with resale
  Coase (1960), Demsetz (1964), Jehiel=Moldovanu (1999), Pagnozzi (2007), Hafalir=Krishna (2008), ...
- Mechanism with renegotiation
  Maskin=Moore (1999), Segal=Whinston (2002), [Maskin=Tirole (1999)]

Plan of the talk

- Introduction
- Model
- Market with Money
- Market with no Money
- Conclusion

Model: Players and Objects

N: a finite set of players, $|N| \geq 2$
O: a finite set of indivisible (tangible) objects
ϕ: the null object
$\bar{O} = O \cup \{\phi\}$
q^a: quota for $a \in \bar{O}$
$q^a < |N| \ (a \in O), \ q^\phi = |N|, \ q = (q^a)_{a \in O}$

Each player in N consumes one unit in $\bar{O}$.

Preferences

Preferences are represented by quasi-linear utility functions, i.e., for i with $(a_i, m_i) \in \bar{O} \times \mathbb{R}$,

$$u_i(a_i, m_i) = v_i(a_i) + m_i$$

$v_i(\phi) = 0$, $v = (v_i)_{i \in N}$, $m_i = 0$ if no money

Payoffs are generic (unless otherwise mentioned).

A two stage economy

First stage: Assignment via M

$P \subset N$: Participants in M

Objects are assigned to P via M based on priority $\succ$.

Each agent i obtains one object in $\bar{O}$ ($i \in N \setminus P$ obtains ϕ).

ω: object allocation of the first stage (not consumed yet)

M: either Boston or DA ▸ Formal Definition ▸ Boston ▸ DA

A two stage economy

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ω: object allocation of the first stage (not consumed yet)

M: either Boston or DA Formal Definition Boston DA

Second stage: Market with Money

Market opens with ω as endowments. N: market participants

$(p, (\mu, m))$: the eventual outcome, p: price, (μ, m): allocation

μ: object allocation, m: money allocation

Agents are price-takers.

A two stage economy

First stage: Assignment via M

$P \subset N$: Participants in M
Objects are assigned to P via M based on priority $\succ$.
Each agent i obtains one object in $\bar{O}$ ($i \in N \setminus P$ obtains ϕ).
ω: object allocation of the first stage (not consumed yet)
M: either Boston or DA ▸ Formal Definition ▸ Boston ▸ DA

Second stage: Market with no Money

Market opens with ω as endowments. N: market participants
(p, μ): the eventual outcome, p: price,
μ: (object) allocation
Agents are price-takers.

Priority in M

$\succ_a$: strict total order over $P \subset N$ at $a \in O$

$i \succ_a j$ means that i has higher priority than j at a.

$\succ = (\succ_a)_{a \in O}$: a priority profile

Equilibrium concept

Perfect Market Equilibrium (PME)

- The second stage outcome is a market equilibrium both on-path and off-path.
- The first stage outcome is a Nash equilibrium in the game induced by the second stage outcomes.

▸ Market equilibrium (ME)

▸ Perfect Market equilibrium (PME)

Pareto Optimality and Social Welfare

Definition

$(\mu, m) = (\mu_i, m_i)_{i \in N}$ *Pareto dominates* $(\mu', m') = (\mu'_i, m'_i)_{i \in N}$ if

$u_i(\mu_i, m_i) \geq u_i(\mu'_i, m'_i)$ for all $i \in N$,

$u_j(\mu_j, m_j) > u_j(\mu'_j, m'_j)$ for some $j \in N$.

(μ, m) is *Pareto optimal* if no allocation Pareto dominates (μ, m).

Replace (μ, m) with μ for the no money case.

Pareto Optimality and Social Welfare

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(μ, m) is *Pareto optimal* if no allocation Pareto dominates (μ, m).
Replace (μ, m) with μ for the no money case.

Definition

(μ, m) (or μ) is *efficient* (a social welfare maximizer) if

$$\mu \in \arg\max_{\mu'} W(\mu') = \sum_{i \in N} v_i(\mu'_i).$$

Market with Money: Existence

$P = N$
m: money profile
(μ, m): allocation

Claim (Quinzii, 1984)

For all ω, there exists at least one ME under ω.

Proposition

There exists at least one PME.

Example 1: Market with Money

Values and Priority

$v_i(a)$	A	B
x	10	50
y	20	35

Values

$i = A, B$: agents

$a = x, y$: tangible objects

$A \succ_a B$, $a = x, y$: priority

Example 1: Market with Money

Values and Priority

$v_i(a)$	A	B
x	10	50
y	20	5

Values

$i = A, B$: agents
$a = x, y$: tangible objects
$A \succ_a B$, $a = x, y$: priority

Outcome when no second stage market

$\mu = (y, x)$
$u = (20, 50)$

Example 1: Market with Money

Values and Priority

$v_i(a)$	A	B
x	10	50
y	20	5

Values

$i = A, B$: agents
$a = x, y$: tangible objects
$A \succ_a B$, $a = x, y$: priority

Outcome when they anticipate the future trade

$\omega = (x, y)$
$p = (p_x, p_y) = (30, 10)$, $\mu = (y, x)$, $m = (20, -20)$
$u = (40, 30) = (20, 50) + (20, -20)$

Example 1: Market with Money

Values and Priority

$v_i(a)$	A	B
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$u = (40, 30) = (20, 50) + (20, -20)$

Example 1: Efficient Equilibrium

Efficient equilibrium

$v_i(a)$	A	B
x	10	50
y	20	5

$$A \succ_a B, \quad a = x, y$$

	(ω_A, ω_B)	(p_x, p_y)	(μ_A, μ_B)	(u_A, u_B)	W
Eqm on-path	(x, y)	$(30, 10)$	(y, x)	$(40,\ 30)$	70
off-path	(x, ϕ)	$(30, -)$	(ϕ, x)	$(30,\ 20)$	50

Example 1: Inefficient Equilibrium

Inefficient equilibrium

$v_i(a)$	A	B
x	10	50
y	20	5

$$A \succ_a B, \quad a = x, y$$

	(ω_A, ω_B)	(p_x, p_y)	(μ_A, μ_B)	(u_A, u_B)	W
Eqm on-path	(x, ϕ)	$(20, -)$	(ϕ, x)	$(20,\ 30)$	50
off-path	(x, y)	$(40, 10)$	(y, x)	$(50,\ 20)$	70

Example 1: Inefficient Equilibrium

Inefficient equilibrium

$v_i(a)$	A	B
x	10	50
y	20	5

$A \succ_a B, \quad a = x, y$

	(ω_A, ω_B)	(p_x, p_y)	(μ_A, μ_B)	(u_A, u_B)	W
Eqm on-path	(x, ϕ)	$(20, -)$	(ϕ, x)	$(20,\ 30)$	50
off-path	(x, y)	$(40, 10)$	(y, x)	$(50,\ 20)$	70

Example 1: Inefficient Equilibrium

Inefficient equilibrium

$v_i(a)$	A	B
x	10	50
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$A \succ_a B, \ \ a = x, y$

	(ω_A, ω_B)	(p_x, p_y)	(μ_A, μ_B)	(u_A, u_B)	W
Eqm on-path	(x, ϕ)	$(20, -)$	(ϕ, x)	$(20, \ 30)$	50
off-path	(x, y)	$(40, 10)$	(y, x)	$(50, \ 20)$	70

Example 1': Inefficient PME disappears

Values and Priority

$v_i(a)$	A	B	C
x	10	50	4
y	20	5	4

$$A \succ_a B \succ_a C, \ a = x, y$$

Example 1': Inefficient PME disappears

Values and Priority

$v_i(a)$	A	B	C
x	10	50	4
y	20	5	4

$A \succ_a B \succ_a C, \ a = x, y$

	$(\omega_A, \omega_B, \omega_C)$	(p_x, p_y)	(μ_A, μ_B, μ_C)	(u_A, u_B, u_C)	W
on-path	(x, ϕ, ϕ)	$(20, -)$	(ϕ, x, ϕ)	$(20, 30, 0)$	50
deviation	(x, ϕ, y)	$(40, 10)$	(y, x, ϕ)	$(50, 10, 10)$	70

Example 1': Inefficient PME disappears

Values and Priority

$v_i(a)$	A	B	C
x	10	50	4
y	20	5	4

$A \succ_a B \succ_a C,\ a = x, y$

	$(\omega_A, \omega_B, \omega_C)$	(p_x, p_y)	(μ_A, μ_B, μ_C)	(u_A, u_B, u_C)	W
on-path	(x, ϕ, ϕ)	$(20, -)$	(ϕ, x, ϕ)	$(20, 30, 0)$	50
deviation	(x, ϕ, y)	$(40, 10)$	(y, x, ϕ)	$(50, 10, 10)$	70

Scarcity

Definition

Given $k = 1, 2, \ldots$, let

$$V_k = \left\{ v \in \mathbb{R}^{N \times \bar{O}} \;\middle|\; \min_{a \in O} |\{i \in P | v_i(a) > 0\}| = k \right\},$$

i.e., for each a, there are at least k players who value a.

DEF. Objects are scarce w.r.t. k if

$$2Q - \min_{a \in O} q^a \leq k$$

where $Q = \sum_{a \in O} q^a$.

Efficiency of PME

Theorem

The following two statements are equivalent for each $k \geq 3$:

1. for all $v \in V_k$, a pure PME exists, and every pure PME allocation is efficient;
2. objects are scarce w.r.t. k.

▸ Proof of (⇒) ▸ Illustration of the Proof of (⇒)

▸ Proof of existence (⇐)

▸ Proof of efficiency (⇐)

Market with no Money

No money is available for transaction.

Conditions

(Value) All tangible objects have positive intrinsic values for all:

$$V_+ = \{v \in \mathbb{R}^{\bar{O} \times N} | \forall i \in N \forall a \in O \ v_i(a) > 0\}$$

(Quota1) Quota is one for all tangible objects.

Existence

Lemma

[Shapley=Scarf] Assume (Value) and (Quota1).
For all ω, ME exists.

Proposition

Assume (Value) and (Quota1).
There exists at least one PME.

▸ Counterexample if (Value) is violated ▸ Counterexample if (Quota1) is violated

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$$B \succ_x C \succ_x A$$
$$A \succ_y B \succ_y C$$
$$A \succ_z C \succ_z B$$

Priority

The first stage mechanism: DA

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$
$A \succ_y B \succ_y C$
$A \succ_z C \succ_z B$

Priority

DA: Truth-telling strategies

x	y	z
A		B C

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$
$A \succ_y B \succ_y C$
$A \succ_z C \succ_z B$

Priority

DA: Truth-telling strategies

x	y	z
A		~~B~~ C

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$

$A \succ_y B \succ_y C$

$A \succ_z C \succ_z B$

Priority

DA: Truth-telling strategies

x	y	z
A		C
B		

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$$B \succ_x C \succ_x A$$
$$A \succ_y B \succ_y C$$
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Priority

DA: Truth-telling strategies

x	y	z
~~A~~		C
B		

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$
$A \succ_y B \succ_y C$
$A \succ_z C \succ_z B$

Priority

DA: Truth-telling strategies

x	y	z
		C
B		
	A	

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$$B \succ_x C \succ_x A$$
$$A \succ_y B \succ_y C$$
$$A \succ_z C \succ_z B$$

Priority

DA: Outcome

$\omega = (y, x, z)$

Pareto optimal. Also stable, i.e.,

no player wants an object held by another with lower priority;

no player wants a left-over (=unassigned tangible object).

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$
$A \succ_y B \succ_y C$
$A \succ_z C \succ_z B$

Priority

Two-stage economy

But, if there is the second stage,
A has an incentive to obtain z in the first stage.

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$$B \succ_x C \succ_x A$$
$$A \succ_y B \succ_y C$$
$$A \succ_z C \succ_z B$$

Priority

Two-stage economy: 1st stage

x y z

A B C

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$
$A \succ_y B \succ_y C$
$A \succ_z C \succ_z B$

Priority

Two-stage economy: 1st stage

x y z

A ~~B~~ ~~C~~

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$
$A \succ_y B \succ_y C$
$A \succ_z C \succ_z B$

Priority

Two-stage economy: 1st stage

x	y	z
		A
B	C	

Example 2: Market with no Money

Values and Priority

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

$B \succ_x C \succ_x A$
$A \succ_y B \succ_y C$
$A \succ_z C \succ_z B$

Priority

Two-stage economy: 2nd stage

In the second stage, given $\omega = (z, x, y)$, the (essentially) unique market eqm is $\mu = (x, z, y)$ with $p_x = p_z > p_y$
Pareto optimal but NOT Stable Example 3

Pareto optimality

Lemma

For all ω, an ME allocation is Pareto optimal under ω.

Proposition 4.1

A pure PME allocation is Pareto optimal.

Stability

Definition

An object allocation μ is stable if

- no player wants an object held by another player with lower priority;
- no player wants a leftover.

▸ Formal Definition

Stable market equilibrium (SME)

Definition

Given u and $\succ$, (p, μ) is a $\mathrm{stable\ market\ equilibrium}$ (SME) if

- (p, μ) is a market equilibrium under μ itself,
- μ is stable.

Priority Cycles

Definition

A priority cycle consists of distinct $i, j, k \in N$ and $a, b \in O$ such that:
Cycle condition: $i \succ_a j \succ_a k \succ_b i$.

$\succ$ is acyclical if there is no cycle.

Ergin (2002)

Main result for no money

Theorem

Assume $|O| \geq 3$, $|N| \geq 3$, and (Quota1). The following two are equivalent:

- For any P with $|P| \geq 3$ and any $v \in V_+$, an SME exists, and its allocation is always sustained by a pure PME;
- $\succ$ is acyclical.

▸ Sketch of Proof

Conclusion

- We have considered a two-stage economy with non-monetary assignment in the first stage and market trades in the second.
- The second stage market makes the assignment stage a different ball game from the one without it.
- We have analyzed the economy with money and without.
- We have identified necessary and sufficient conditions for some properties of PME like efficiency and stability:
  - With money, "efficiency" and "scarcity" are equivalent;
  - With no money, "stability" and "acyclicity" are equivalent.

Thank you!

Appendix: Feasibility in the 2nd stage

Definition 5.1

Given ω, an allocation $x = (\mu, m)$ is ω-feasible if
for all $a \in O$, $|\mu^a| \leq |\omega^a|$ holds.
$\mathcal{A}^\omega$: the set of ω-feasible allocations.
$O^\omega = \{a \in O | |\omega^a| > 0\}$: the set of available objects
$\bar{O}^\omega = O^\omega \cup \{\phi\}$.

return

ω-Pareto optimality and ω-efficiency

Definition 5.2

Given ω,

- an allocation x is ω-Pareto optimal (ω-optimal) if $\nexists\ x' \in \mathcal{A}^\omega$ that Pareto dominates x.
- an allocation (μ, m) is ω-efficient if $\nexists\ (\mu', m') \in \mathcal{A}^\omega$ s.t. $W(\mu') > W(\mu)$.

▸ return

Market equilibrium (ME)

Definition 5.3

Given ω, $(p,(\mu,m))$ is a $\mathrm{market\ equilibrium\ (ME)}$ under ω if $p_\phi = 0$, and

1. budget constraint
2. individual optimization
3. no excess demand, and excess supply implies zero price for each object

formal definition
return

Market Equilibrium (ME) with Money

Definition 5.4

Given $\omega \in \mathcal{A}$, $(p, \mu, m) \in \mathbb{R}_+^{\bar{O}^\omega} \times \mathcal{A}^\omega \times \mathbb{R}^N$ is a market $\mathrm{equilibrium}$ $\mathrm{(ME)}$ under ω if $p_\phi = 0$, and

1. $\forall i \in N \; p_{\mu_i} + m_i = p_{\omega_i}$
2. $\forall i \in N \; \mu_i \in \arg\max_{a \in \bar{O}} v_i(a) - p_a$
3. $\forall a \in O^\omega [|\mu^a| \leq |\omega^a|] \wedge [|\mu^a| < |\omega^a| \Rightarrow p_a = 0]$

return

Perfect Market Equilibrium (PME)

Definition

$(\rho, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ is a perfect market equilibrium (PME) if

1. for all $\omega \in \mathcal{A}$, $(p(\omega), \mu(\omega), m(\omega))$ is an ME under ω;
2. ρ is a Nash equilibrium of the game of which payoffs are induced by the second stage ME outcomes.

PIPME return

Market Equilibrium (ME) with no Money

Definition 5.5

Given $\omega \in \mathcal{A}$, $(p, \mu) \in \mathbb{R}_+^{\bar{O}^\omega} \times \mathcal{A}^\omega$ is a $\text{market equilibrium (ME)}$ under ω if $p_\phi = 0$, and

1. $\forall i \in N \ p_{\mu_i} \leq p_{\omega_i}$
2. $\forall i \in N \ \mu_i \in \arg\max_{a \in \bar{O}} v_i(a) - p_a$
3. $\forall a \in O^\omega [|\mu^a| \leq |\omega^a|] \wedge [|\mu^a| < |\omega^a| \Rightarrow p_a = 0]$

return

Properties under scarcity

Lemma 5.1

Assume (Scarcity).

1. $\forall a \in O \ |\mu^a| = q^a$ if (μ, m) is Pareto optimal;
2. given $\omega \in \mathcal{A}$, $\forall a \in O \ |\mu^a| = |\omega^a|$ if (μ, m) is ω-optimal;
3. given $\omega \in \mathcal{A}$, $\forall a \in O \ p_a > 0, \ |\mu^a| = |\omega^a|$ if (p, μ, m) is ME under ω.

return

Permutation Independent PME (PIPME)

Definition 5.6

$(\rho, (p(\omega), x(\omega))_{\omega \in \mathcal{A}})$ is a $\text{permutation independent PME (PIPME)}$ if

1. it is a PME;
2. $p(\omega) = p(\omega')$ whenever $|\omega| = |\omega'|$.

The price is unchanged unless the total endowment changes.

return

The first stage: Assignment

Assignment Mechanism

$$M = \langle \Sigma, \lambda \rangle$$

$\Sigma \equiv \times_{i \in N} \Sigma_i$: the finite set of strategy profiles

$\sigma_i \in \Sigma_i$: i's strategy, $\sigma = (\sigma_i)_{i \in N}$

$\lambda : \Sigma \to A$: an outcome function.

$\lambda(\sigma) \in A$: object outcome in the first stage

$\lambda_i(\cdot) \in A_i$: the set of available objects for $i \in N$

$A_i = \{\phi\}$ or $\bar{O}$, $A = \times_{i \in N} A_i$

Return

Boston Mechanism

Each player submits a list of objects ordered from the best to the worst. The rest is determined by the algorithm:

Step 1 The players go to the first object in their respective lists.

- ⋆ If # of the players choosing a does not exceed q^a, they are assigned to a (and it's final).
- ⋆ If # exceeds q^a, then players with higher priority are assigned to a (final), and the rest go to the next in their resp list.

Step k Repeat ⋆'s in Step 1 with leftovers and remaining players. If the chosen object is already taken, the player goes to the next step with the $(k+1)$th object in her list.

Terminate the process when all are assigned to an object in $\bar{O}$.

▸ Return

Deferred Acceptance Algorithm (DA)

Each player submits a list of objects ordered from the best to the worst. The rest is determined by the algorithm.

Step 1 The players go to the first object in their respective lists.

- $\star$ If # of the players choosing a does not exceed q^a, they are temporarily assigned to a.
- $\star$ If # exceeds q^a, then players with higher priority are assigned to a, and the rest go to the next in the list.

Step k Those assigned to a before and those who choose a in this step go to a, and repeat $\star$'s in Step 1.

Terminate the process when all are assigned to an object in $\bar{O}$.

Return

Illustration of the proof of $(1 \Rightarrow 2)$

Construction of PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'} - 1$

$O = \{a, b\}, q^a = 5, q^b = 3, N = \{1, \ldots, 12\}$

	1,...,5	6,...,10	11,12
a	10	20	1
b	1	1	1

$i \succ_a j$ if $i \leq 5, j > 5$

Illustration of the proof of $(1 \Rightarrow 2)$

Construction of PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'} - 1$

$O = \{a, b\}, q^a = 5, q^b = 3, N = \{1, \ldots, 12\}$

	1,...,5	6,...,10	11,12
a	10	20	1
b	1	1	1

$i \succ_a j$ if $i \leq 5, j > 5$

ω^a	1	2	3	4	5	$\omega^b = \mu^b$	11	12	$\circ$ ↑ leftover
μ^a	6	7	8	9	10				

$p = (p_a, p_b) = (15, 1)$ on path
$p = (p_a, p_b) = (16, 1)$ off path if someone $(6, \ldots, 10)$ takes the leftover
$\Rightarrow$ Nobody has an incentive to deviate

Illustration of the proof of $(2 \Rightarrow 1)$

No PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'}$

$O = \{a, b\}, q^a = 5, q^b = 3, N = \{1, \dots, 12, 13\}$
$i \succ_a j$ if $i \leq 5, j > 5$

Illustration of the proof of $(2 \Rightarrow 1)$

No PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'}$

$O = \{a, b\}, q^a = 5, q^b = 3, N = \{1, \ldots, 12, 13\}$
$i \succ_a j$ if $i \leq 5, j > 5$

ω^a	1	2	3	4	5	$\omega^b = \mu^b$	11	12	$\circ$
									$\uparrow$
μ^a	6	7	8	9	10				leftover

If there is a leftover like the above, 13 has an incentive to take it.

return

Proof: College Theorem ($\Rightarrow$)

Suppose that objects are *not* scarce', i.e., either $|N_s| < Q$ or $v \in V_k^f$ with $k \leq Q$ (or both).
Case I. $|N_s| < k$: efficiency is trivially violated as the economy cannot deliver all the objects to the firms who need them.
Case II. $k \leq |N_s|$: construct v as follows. Align the objects in an arbitrary manner, $\{a_1, \ldots, a_{\bar{L}}\}$. There is $L = 1, \ldots, \bar{L}$ such that $q_{a_1} + \cdots + q_{a_{L-1}} < k \leq q_{a_1} + \cdots + q_{a_L}$. Fix L.
Let $\hat{N}_f \subset N_f$ satisfy $|\hat{N}_f| = k$ and $\forall i \notin \hat{N}_f \forall a \in O[v_i(a) < 0]$.
Assign $v_i(a)$ ($i \in \hat{N}_f, a \in O$) in such a way that for each $\ell = 1, \ldots, \bar{L} - 1$, and for all $i, j \in \hat{N}_f$, $v_i(a_\ell) > v_j(a_{\ell+1}) > 0$.
Let μ^* be the efficient object allocation given v. It must be the case that $|\mu^{*a}| = q^a$ for $a = a_1, \ldots, a_{L-1}$ and that $0 < |\mu^{*a_L}| \leq q^{a_L}$. Consider ω with $|\omega| = |\mu^*|$. Then (p, μ^*, m) becomes an ME under ω for some p and m. It is verified, due to the way we construct v, that $p_{a_1} \geq p_{a_2} \geq \ldots \geq p_{a_L}$. Then there is another ME (p^*, μ^*, m^*) such that $p^*_{a_\ell} = p_{a_\ell} - p_{a_L}$ holds for all $\ell = 1, \ldots, L$. Note $p^*_{a_L} = 0$.
Assign objects to the players in N_s in the first stage from a_1 to a_{L-1} to fill their respective quotas, using $\succ$. As for a_L to the remaining students so that the total number of the students assigned to some tangible objects becomes k. Assign the other students to ϕ. Denote this assignment profile ω^*.
Remove one player, say, i from ω^{*a_L} to obtain ω^{**}. We would like to have this ω^{**} as the PME allocation of the first stage. On the equilibrium path, we have the second stage outcome.
Let us check if there is no incentive to deviate. Under ω^{**}, there is one firm that cannot buy a tangible object in the second stage, and there is at least one student who does not obtain a leftover in the first stage. If such a student obtains the object, then the first stage object allocation becomes ω^* (or some ω' with $|\omega'| = |\omega^*|$ to be precise), and therefore, the price of the object this student obtains is zero. Thus, the student has no incentive to deviate in the first stage. An inefficienct outcome arises as a PME. $\square$

return

Proof: College Theorem ($\Leftarrow$)

Suppose that objects are scarce, i.e., $|N_s| \geq Q$ and $v \in V_k^f$ with $k > Q$.
Take v as given along with other parameters, $\succ$ and q.
Existence: Take some ω with $|\omega| = q$. Let (p^*, μ^*, m) be an ME under ω.
Align $O = \{a_1, \ldots, a_L\}$ in such a way that $p^*_{a_1} \geq p^*_{a_2} \geq \ldots \geq p^*_{a_L}$ holds.
Since $k > Q$ holds, there exists $j \in N_f$ such that $\mu^*_j = \phi$ and $v_j(a_L) > 0$ hold. Therefore, $p^*_{a_L} \geq v_j(a_L) > 0$.
Assign objects to the players in N_s in the first stage from a_1 to a_{L-1} to fill their respective quotas, using $\succ$.
We can do it as $|N_s| \geq Q$. Assign the other students to ϕ. Denote this assignment profile ω^*.
Under ω^*, (p^*, μ^*, m^*) becomes an ME for some m^*.
Let ω^* be the outcome of the first stage. Then together with appropriate off-path ME's, we have a PME as nobody has an incentive to deviate.

Efficiency: Suppose $(\sigma, (p(\omega), \mu(\omega), m(\omega))$ is a PME. Let $\omega^* = \lambda(\sigma)$.
Take any ω. Since $k > Q$ holds, for all $a \in O$, there exists $j \in N_f$ such that $\mu_j(\omega) = \phi$ and $v_j(a) > 0$ hold. Therefore, $p_a(\omega) \geq v_j(a) > 0$ for all $a \in O$; otherwise, j would buy a in ME.
Suppose that $a \in O$ has some leftover, i.e., $|\omega^{*a}| < q^a$.
Since $|N_s| \geq Q$, there exists at least one student who does not obtain any tangible object. This player has an incentive to obtain the leftover a since under any ω, $p_a(\omega) > 0$ as we have shown. □

return

Formal Definition of Stability

Definition

μ is stable if

- $\forall i, j \in N \; [\mu_j \in O \wedge i \succ_{\mu_j} j \Rightarrow u_i(\mu_i, 0) \geq u_i(\mu_j, 0)]$
- $\forall a \in \bar{O} \; \forall i \in N \; [|\mu^a| < q^a \Rightarrow u_i(\mu_i, 0) \geq u_i(a, 0)]$

return

Construction of PME with leftovers

Construction of PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'} - 1$

$q, \succ$: given. Let $a \in \arg\min_{a' \in O} q^{a'}$. Proof for k less than this is similar or (easier).
Let an auxiliary value profile $\hat{v}$ be given by
$\hat{v}_i(a) = 1$ for all $i \in N$,
$\hat{v}_i(b) = 10$ for all $i \in N$, $b \neq a, \phi$.
Find a NE of the first stage (not necessarily PME). Let ω be its outcome.
Let $S = \{i \in N | \ \omega_i = b \text{ for some } b \neq a, \phi\}$.
Let $W = N \setminus S$. Pick $J \subset W$ where $|J| = q^a - 1$. Note $|S| = |W \setminus J| = Q - q^a$.
Construct v:

$$v_i(b) \begin{cases} \leq 10 & \text{if } b \neq a, \phi, \ i \in S \\ \in [24, 25] & \text{if } b \neq a, \phi, \ i \in W \setminus J \\ \in [1, 2] & \text{if } i \in J \vee b = a \end{cases}$$

Let $p_b = 20$ $(b \neq a, \phi)$ and $p_a = 1$ on path, or off path when $i \in S$ deviates.
Let $p_b = 21$ $(b \neq a, \phi)$ and $p_a = 1$ off path when $j \in W$ deviates.
- On path, $i \in S$ gets 20.
- Off path when $i \in S$ deviates, it gets either 20 or at most 10.
- On path, $j \in J$ gets at most 2, while $k \in W \setminus J$ gets between 4 and 5.
- Off path when $j \in W$ deviates, it gets at most the same.

▸ return

Proof of Existence

Proof of existence when $k = 2Q - \min_{a \in O} q^a$

Proof for k greater than this is similar.
Let ω be an allocation with no leftover. Sps (p^*, μ^*, m^*) is an ME under ω (such an ME exists). We may assume $p^*_a > 0$ for $a \in O$ since there is a sufficient amount of demand for each $a \in O$.
For any ω' with no leftover, let $p(\omega') = p^*$. Adjusting m' appropriately, we obtain an ME (p^*, μ^*, m') under ω'.
Consider an auxiliary $\hat{v}$ as follows:
$\hat{v}_i(a) = p^*_a$ $(i \in N,\ a \in \bar{O})$.
Use this $\hat{v}$ and run DA with the truth-telling strategies σ^* to obtain ω^*.
Note ω^* is stable w.r.t. $\hat{v}$.
Also, no leftover under σ^*.
Moreover, even if one, say, player i, makes a unilateral deviaiton to, say, σ_i, no leftover under $(\sigma_i, \sigma^*_{-i})$.
This σ^* constitutes a pure PME along with ME's mentioned above (and appropriately chosen ME's for other ω's).
This completes the proof for DA.

return

Proof of Efficiency

Proof of efficiency when $k = 2Q - \min_{a \in O} q^a$

Proof for k greater than this is similar.
Suppose $a \in O$ has some left-over, i.e., $|\omega^a| < q^a$.
Observe at least q^a agents who cannot obtain $b \neq a, \phi$ in neither stage and have a positive value for a.
Let L be the set of such agents. Note $|L| \geq q^a > |\omega^a|$. Then
$p_a \geq \min_{i \in L} v_i(a) > 0$, (for if not, there would be excess demand).
Then $\exists \ell \in L[\omega_\ell = \phi]$.
This agent ℓ has an incentive to obtain the left-over to obtain $v_\ell(a)$ instead of $v_\ell(a) - p_a$.

return

Proof of PME $\Leftarrow$ ACY under DA

Assume (No Money), (ACY), (Quota), and (DA).
Sps $\exists A, v \in V_+ \ \varphi(v|A)$ is not PME.
WTS $\exists$ a cycle.
Remove j' with $A_{j'} = \{\phi\}$ from the economy. Hereafter, N means those players j with $A_j = \bar{O}$. $\succ$ is reduced to N as well.
$(p(\omega), \mu(\omega), 0)_{\omega \in \mathcal{A}}$: ME profile
$\zeta^* = (\zeta_j^*)_{j \in N}$: truth-telling strategy.
Player i has an incentive to deviate by submitting ζ_i. Fix i.
Let $\omega^* = \lambda(\zeta^*)$ and $\hat{\omega} = \lambda(\zeta_i, \zeta_{-i}^*)$.
DA implies $v_i(\omega_i^*) \geq v_i(\hat{\omega}_i)$, and i will trade through a TC:
$(k_0, k_1, \ldots, k_{\bar{n}})$ with $k_0 = k_{\bar{n}} = i$ s.t. $v_{k_n}(\hat{\omega}_{k_{n+1}}) > v_{k_n}(\hat{\omega}_{k_n})$. Note
(*) $\quad k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} k_n$

Proof of PME $\Leftarrow$ ACY under DA

Auxiliary DA:
We run DA without i and then add i. No change in result.
At t^*, i is put in DA.
i follows ζ_i'. Sps i obtains ω_i in step $\bar{t}$

Lemma. i is never accepted at $a_1 \neq \omega_i$ before step $\bar{t}$.
Pf of Lemma. Sps not, i.e., $\exists t_1 \exists a_1 \neq \omega_i$, i obtains a_1 in $t_1 < \bar{t}$.
In t_1, either a_1 is a leftover, which will end the process $\rightarrow\leftarrow$, or
j_1 is rejected at a_1 by i. j_1 is the only loser.
Rejection chain is needed to push out i from a_1 as i must obtain $\omega_i \neq a_1$:
$(a_1, j_1, t_1), (a_2, j_2, t_2), \ldots, (a_{\bar{\kappa}}, j_{\bar{\kappa}}, t_{\bar{\kappa}}) = (a_1, i, t')$
where j_κ is rejected at a_κ by $j_{\kappa-1}$ at t_κ. Then $\exists$ a generalized cycle with
$a_\kappa \neq a_1$ for $\kappa \neq 1$:
$j_{\bar{\kappa}-1} \succ_{a_1} i \succ_{a_1} j_1 \succ_{a_2} j_2 \cdots \succ_{a_{\bar{\kappa}-1}} j_{\bar{\kappa}-1}$
Then $\exists$ a cycle. $\rightarrow\leftarrow$ $\diamondsuit$

Proof of PME $\Leftarrow$ ACY under DA

Lemma implies $\omega_i^* \neq \hat{\omega}_i$. Thus, we have the following argument.
(1) If $\omega_{k_{\bar{n}-1}}^* = \hat{\omega}_{k_{\bar{n}-1}}$, then $\hat{\omega}_i$ is not a leftover in ω^*; for if not, $k_{\bar{n}-1}$ would have obtained it. i pushed out, say, $\ell \neq k_{\bar{n}-1}$ from $\hat{\omega}_i$. Since $k_{\bar{n}-1}$ could not obtain $\hat{\omega}_i$ from ℓ, stability of DA implies $i \succ_{\hat{\omega}_i} \ell \succ_{\hat{\omega}_i} k_{\bar{n}-1}$. Together with (*), $\exists$ a generalized cycle.
(2) In general, sps $\omega_{k_{n'}}^* \neq \hat{\omega}_{k_{n'}}$ for $n' = n+1, \ldots, \bar{n}-1$ and $\omega_{k_n}^* = \hat{\omega}_{k_n}$. Then k_{n+1} must have pushed out, say, ℓ' ($\because \hat{\omega}_{k_{n+1}}$ is not a leftover similar to (1)). k_n wanted $\omega_{\ell'}^*$ but could not. Thus,
$k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} \ell' \succ_{\hat{\omega}_{k_{n+1}}} k_n$. Together with (*), a generalized cycle exists.
(3) Sps $\omega_{k_{n'}}^* \neq \hat{\omega}_{k_{n'}}$ for $n' = 1, \ldots, \bar{n}-1$. Then
k_1 must have pushed out, say, ℓ'' from $\hat{\omega}_{k_1}$ ($\because \hat{\omega}_{k_1}$ is not a leftover similar to (1)). Then
$k_1 \succ_{\hat{\omega}_{k_1}} \ell'' \succ_{\hat{\omega}_{k_1}} i$. Together with (*), a generalized cycle exists.

Proof of PME $\Leftarrow$ ACY under DA

The remaining task is to show that the generalized cycle found above uses distinct players. Note that $k_0, k_1, \ldots, k_{\bar{n}-1}$ are distinct as they form a TC.

(1) $i \succ_{\hat{\omega}_i} \ell \succ_{\hat{\omega}_i} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} \ldots \succ_{\hat{\omega}_{k_2}} k_1 \succ_{\hat{\omega}_{k_1}} i$
Note i, ℓ, $k_{\bar{n}-1}$ are distinct.
If $k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} i$, then we have a cycle with distinct players:
$i \succ_{\hat{\omega}_i} \ell \succ_{\hat{\omega}_i} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} i$
If $i \succ_{\hat{\omega}_{k_{\bar{n}-1}}} k_{\bar{n}-1}$, then we can shorten the cycle:
$i \succ_{\hat{\omega}_{k_{\bar{n}-1}}} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} k_{\bar{n}-2} \ldots \succ_{\hat{\omega}_{k_2}} k_1 \succ_{\hat{\omega}_{k_1}} i$,
which is a generalized cycle with distinct players.
(2) $k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} \ell' \succ_{\hat{\omega}_{k_{n+1}}} k_n \succ_{\hat{\omega}_{k_n}} k_{n-1} \ldots \succ_{\hat{\omega}_{k_{n+2}}} k_{n+1}$
Note k_{n+1}, ℓ', k_n are distinct. The rest is similar to (1).
(3) $k_1 \succ_{\hat{\omega}_{k_1}} \ell'' \succ_{\hat{\omega}_{k_1}} i \succ_{\hat{\omega}_i} k_{\bar{n}-1} \ldots k_2 \succ_{\hat{\omega}_{k_2}} k_1$
Note k_1, ℓ'', i are distinct. The rest is similar to (1). □

return

Proof of PME $\Leftarrow$ ACY under Boston

Assume (No Money), (ACY), (Quota), and (Boston).
Sps $\exists A, v \in V_+$ $\varphi(v|A)$ is not PME.
WTS $\exists$ a cycle.
Given $\omega^* = \varphi(v|A)$, there exists a NE σ^* such that the players obtain their final objects ω^* in the first step.
$\exists i$ who gains by deviation. Fix i.
Let σ_i be the deviating strategy, and let $\omega^* = \lambda(\sigma^*)$ and $\hat{\omega} = \lambda(\sigma_i, \sigma^*_{-i})$.
In σ^* under Boston, there exists at most one player who is affected by i's deviation.
i has an incentive to deviate only when there is a TC after i's deviation.
TC: $i = k_0, k_1, \ldots, k_{\bar{n}} = i$ where k_n wants k_{n+1}'s object.

Proof of PME $\Leftarrow$ ACY under Boston

Claim. $\exists j$ directly affected by i's deviation, taking over ω_j^*, i.e., $\hat{\omega}_i = \omega_j^*$, $\hat{\omega}_j \neq \omega_j^*$.
Pf. Sps not, i.e., i does not affect any player in the first stage.
i must have taken a leftover or ϕ. $\exists$ no new TC since nobody wants a leftover or ϕ under SME. $\diamondsuit$

Claim. j is not in TC, i.e., $j \neq k_0, k_1, \ldots, k_{\bar{n}}$.
Pf. j, after i's deviation, can go for either one of ω_i^*, a leftover, and ϕ.
If j goes for a leftover or ϕ, j is not in TC as nobody is interested in the leftover under SME.
So, sps $\hat{\omega}_j = \omega_i^*$. Sps also j is in TC. Then $k_n = j$ for some $n = 1, \ldots, \bar{n} - 1$.
Then $i = k_0, k_1, \ldots, k_{n-1}, k_n = i$ form a nontrivial TC under ω^* since $k_{n'-1}$ wants $k_{n'}$'s object $(n' = 1, 2, \ldots, n)$.
This contradicts with the premise that ω^* is a SME allocation. $\diamondsuit$

Proof of PME $\Leftarrow$ ACY under Boston

The previous claim implies agents in TC and j are all distinct.
Since $i \succ_{\omega_j^*} j \succ_{\omega_j^*} k_{\bar{n}-1}$ ($\omega_j^* = \hat{\omega}_i$), we have a generalized cycle of priority:
$i \succ_{\hat{\omega}_i} j \succ_{\hat{\omega}_i} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} \ldots \succ_{\hat{\omega}_{k_2}} k_1 \succ_{\hat{\omega}_{k_1}} i$.

$\square$

No ME if (Quota1) is violated

Example 5.1

	A	B	C
x	10	20	20
y	20	10	10

$$\omega = (x, y, y)$$

no ME under ω, and therefore, no PME

$\because$ (i) $p_x \leq p_y$: B and C demand x. $\Rightarrow$ Excess demand
(ii) $p_x > p_y$: No demand for x (A demands y)

return

No ME if (Value) is violated

Example 5.2

	A	B	C
x	20	-10	20
y	10	-20	10
ϕ	0	0	0

$$\omega = (\phi, x, y)$$

no ME under ω, and therefore, no PME

$\because$ (i) $p_x > p_y$: Excess supply of x
(ii) $p_x \leq p_y$, $p_y > 0$: Excess supply of y (C demands x)
(iii) $p_x = 0$: Excess demand for x

return

Proof of Necessity using Example 2

Construction of v under $A \succ_z C \succ_z B \succ_x A$

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Values

DA and Boston:
outcome is $(y, x, z) \Rightarrow A$ has an incentive to obtain z.

Technical detail in the proof: $A_i = \{\phi\}$ for all $i \neq A, B, C$.

Proof of Sufficiency: Demo for a particular case

Consider (DA). Sps $\succ$ is acyclical.
Sps $\exists v \in V_+$ A has an incentive to deviate from SME allocation μ^v.
WTS $\rightarrow\leftarrow$

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y. Stability	$\Rightarrow B \succ_x A$

Proof of Sufficiency: Demo for a particular case

Consider (DA). Sps $\succ$ is acyclical.
Sps $\exists v \in V_+$ A has an incentive to deviate from SME allocation μ^v.
WTS $\rightarrow\leftarrow$

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y. Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	x	y	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$

Proof of Sufficiency: Demo for a particular case

Consider (DA). Sps $\succ$ is acyclical.
Sps $\exists v \in V_+$ A has an incentive to deviate from SME allocation μ^v.
WTS $\rightarrow\leftarrow$

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y. Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	x	y	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$
$\hat{\mu}$	x	z	y	B must prefer z to x. Stability	$\Rightarrow C \succ_z B$

Proof of Sufficiency: Demo for a particular case

Consider (DA). Sps $\succ$ is acyclical.
Sps $\exists v \in V_+$ A has an incentive to deviate from SME allocation μ^v.
WTS $\rightarrow\leftarrow$

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y. Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	x	y	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$
$\hat{\mu}$	x	z	y	B must prefer z to x. Stability	$\Rightarrow C \succ_z B$

$$\Rightarrow \ A \succ_z C \succ_z B \succ_x A$$

$$\rightarrow\leftarrow$$

Proof of Sufficiency: Demo for a particular case

Consider (DA). Sps $\succ$ is acyclical.
Sps $\exists v \in V_+$ A has an incentive to deviate from SME allocation μ^v.
WTS $\rightarrow\leftarrow$

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y. Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	x	y	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$
$\hat{\mu}$	x	y	z	$\hat{\mu}$ must be the DA outcome	$\rightarrow\leftarrow$

Proof of Sufficiency: Demo for a particular case

Consider (DA). Sps $\succ$ is acyclical.
Sps $\exists v \in V_+$ A has an incentive to deviate from SME allocation μ^v.
WTS $\rightarrow\leftarrow$

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y. Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	y	x	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$
				C must push out B from x	$\Rightarrow C \succ_x B$

Proof of Sufficiency: Demo for a particular case

Consider (DA). Sps $\succ$ is acyclical.
Sps $\exists v \in V_+$ A has an incentive to deviate from SME allocation μ^v.
WTS $\rightarrow\leftarrow$

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y. Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	y	x	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$
				C must push out B from x	$\Rightarrow C \succ_x B$

$$\Rightarrow \; C \succ_x B \succ_x A \succ_z C$$

$$\rightarrow\leftarrow$$

Return

Example 3: TTC $\neq$ PME

Values and Priority

	A	B	C	D
x	40	20	40	10
y	20	40	30	20
z	30	30	20	30
w	10	10	10	40

Values

$D \succ_x B \succ_x C \succ_x A$
$A \succ_y C \succ_y B \succ_y D$
$D \succ_z C \succ_z B \succ_z A$
$A \succ_z D \succ_z B \succ_z C$

Priority

TTC (single stage): (x, z, y, w)
PME with DA: (x, y, z, w), DA only: (z, y, x, w)

return

Example 3: TTC $\neq$ PME

Values and Priority

	A	B	C	D
x	40	20	40	10
y	20	40	30	20
z	30	30	20	30
w	10	10	10	40

Values

$D \succ_x B \succ_x C \succ_x A$
$A \succ_y C \succ_y B \succ_y D$
$D \succ_z C \succ_z B \succ_z A$
$A \succ_z D \succ_z B \succ_z C$

Priority

TTC (single stage): $A \to x \to D \to w \to A$
PME with DA: $\omega = (y, x, z, w)$ $\mu = (x, y, z, w)$

return