A Two-Stage Model of Assignment and Market

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Introduction

We consider a two-stage economy with non-monetary assignment in the first stage and market trades in the second.

- College students foreseeing the future job prospects
- Office allocation with subsequent exchange

Introduction

The second stage market makes the assignment stage a totally different ball game from the one without it, e.g.,

- An agent may go for a less preferable good, expecting to sell it later, and therefore, both the first and second stage outcome may be neither efficient nor stable.
- This is true even with or without money.
- We present equivalent conditions under which we recover efficiency in the economy with money and stability in the economy with no money.

Literature

- Non-market assignment of indivisible goods Gale=Shapley (1962), Roth=Sotomayor (1989), Ergin (2002), Kojima=Manea (2010), ...
- Market for indivisible goods: comparison with assignment Shapley=Scarf (1974), Kaneko (1982), Gale (1984), Quinzii (1984), Piccione=Rubinstein (2007), ...
- Property right assignment with resale Coase (1960), Demsetz (1964), Jehiel=Moldovanu (1999), Pagnozzi (2007), Hafalir=Krishna (2008), ...
- Mechanism with renegotiation Maskin=Moore (1999), Segal=Whinston (2002), [Maskin=Tirole (1999)]

Plan of the talk

- Introduction
- Model
- Market with Money
- Market with no Money
- Conclusion

Preliminaries A two stage economy

Model: Players and Objects

$$\begin{split} N: \text{ a finite set of players, } & |N| \geq 2 \\ O: \text{ a finite set of indivisible (tangible) objects} \\ \phi: \text{ the null object} \\ \bar{O} &= O \cup \{\phi\} \\ q^a: \text{ quota for } a \in \bar{O} \\ q^a &< |N| \ (a \in O), \ q^\phi = |N|, \ q = (q^a)_{a \in O} \end{split}$$

Each player in N consumes one unit in \overline{O} .

Preliminaries A two stage economy

Preferences

Preferences are represented by quasi-linear utility functions, i.e., for i~ with $(a_i,m_i)\in \bar{O}\times \mathbb{R},$

$$u_i(a_i, m_i) = v_i(a_i) + m_i$$

$$v_i(\phi) = 0$$
, $v = (v_i)_{i \in N}$, $m_i = 0$ if no money

Payoffs are generic (unless otherwise mentioned).

Preliminaries A two stage economy

A two stage economy

First stage: Assignment via M

 $P \subset N$: Participants in MObjects are assigned to P via M based on priority \succ . Each agent i obtains one object in \overline{O} ($i \in N \setminus P$ obtains ϕ). ω : object allocation of the first stage (not consumed yet) M: either Boston or DA \blacktriangleright Formal Definition \blacktriangleright Boston \blacktriangleright DA Model Market with Money Market with no Money Conclusion

Preliminaries A two stage economy

A two stage economy

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Second stage: Market with Money

Market opens with ω as endowments. N: market participants $(p, (\mu, m))$: the eventual outcome, p: price, (μ, m) : allocation μ : object allocation, m: money allocation

Agents are price-takers.

Model Market with Money Market with no Money Conclusion

Preliminaries A two stage economy

A two stage economy

First stage: Assignment via M

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Second stage: Market with no Money

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Agents are price-takers.

Preliminaries A two stage economy

${\sf Priority} \, \operatorname{in} \, M$

- \succ_a : strict total order over $P \subset N$ at $a \in O$
- $i \succ_a j$ means that *i* has higher priority than *j* at *a*. $\succ = (\succ_a)_{a \in O}$: a priority profile

Preliminaries A two stage economy

Equilibrium concept

Perfect Market Equilibrium (PME)

- The second stage outcome is a market equilibrium both on-path and off-path.
- The first stage outcome is a Nash equilibrium in the game induced by the second stage outcomes.

Market equilibrium (ME)

Perfect Market equilibrium (PME)

Preliminaries A two stage economy

Pareto Optimality and Social Welfare

Definition

 $\begin{array}{l} (\mu,m)=(\mu_i,m_i)_{i\in N} \mbox{ Pareto dominates } (\mu',m')=(\mu'_i,m'_i)_{i\in N} \mbox{ if } \\ u_i(\mu_i,m_i)\geq u_i(\mu'_i,m'_i) \mbox{ for all } i\in N, \\ u_j(\mu_j,m_j)>u_j(\mu'_j,m'_j) \mbox{ for some } j\in N. \\ (\mu,m) \mbox{ is Pareto optimal if no allocation Pareto dominates } (\mu,m). \\ \mbox{Replace } (\mu,m) \mbox{ with } \mu \mbox{ for the no money case.} \end{array}$

Preliminaries A two stage economy

Pareto Optimality and Social Welfare

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Definition

 (μ,m) (or μ) is *efficient* (a social welfare maximizer) if

$$\mu \in \arg\max_{\mu'} W(\mu') = \sum_{i \in N} v_i(\mu'_i).$$

Existence and efficiency Example 1 Results

Market with Money: Existence

P = Nm: money profile (μ, m) : allocation

Claim (Quinzii, 1984)

For all ω , there exists at least one ME under ω .

Proposition

There exists at least one PME.

Existence and efficiency Example 1 Results

Example 1: Market with Money

Values and Priority

$v_i(a)$	A	B
x	10	50
y	20	85

Values

i = A, B: agents a = x, y: tangible objects $A \succ_a B, a = x, y$: priority

Existence and efficiency Example 1 Results

Example 1: Market with Money

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Outcome when no second stage market

 $\mu = (y, x)$ u = (20, 50)

Existence and efficiency Example 1 Results

Example 1: Market with Money

Values and Priority

$v_i(a)$	A	B
x	10	50
y	20	85

Values

i = A, B: agents a = x, y: tangible objects $A \succ_a B, a = x, y$: priority

Outcome when they anticipate the future trade

$$\begin{split} & \omega = (x, y) \\ & p = (p_x, p_y) = (30, 10), \ \mu = (y, x), \ m = (20, -20) \\ & u = (40, 30) = (20, 50) + (20, -20) \end{split}$$

Existence and efficiency Example 1 Results

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Existence and efficiency Example 1 Results

Example 1: Efficient Equilibrium

Efficient equilibrium $v_i(a)$ ABx1050y205 $A \succ_a B$, a = x, y

	(ω_A,ω_B)	(p_x, p_y)	(μ_A,μ_B)	(u_A, u_B)	W	
Eqm on-path	(x,y)	(30, 10)	(y,x)	(40, 30)	70	
off-path	(x,ϕ)	(30, -)	(ϕ, x)	(30, 20)	50	

Existence and efficiency Example 1 Results

Example 1: Inefficient Equilibrium

Inefficient equilibrium $v_i(a)$ Ax1050yy205 $A \succ_a B$, a = x, y

	(ω_A,ω_B)	(p_x, p_y)	(μ_A,μ_B)	(u_A, u_B)	W	
Eqm on-path	(x,ϕ)	(20, -)	(ϕ, x)	(20, 30)	50	
off-path	(x,y)	(40, 10)	(y,x)	(50, 20)	70	

Existence and efficiency Example 1 Results

Example 1: Inefficient Equilibrium

Inefficient equilibrium $v_i(a)$ Ax10y205 $A \succ_a B$, a = x, y

	(ω_A,ω_B)	(p_x, p_y)	(μ_A,μ_B)	(u_A, u_B)	W	
Eqm on-path	(x,ϕ)	(20, -)	(ϕ, x)	(20, 30)	50	
off-path	(x,y)	(40, 10)	(y,x)	(50, 20)	70	

Existence and efficiency Example 1 Results

Example 1: Inefficient Equilibrium

Inefficient e	equilibri	um		
	$v_i(a)$	A	В	
	x	10	50	
	y	20	5	$A \succ_a B, \ a = x, y$

	(ω_A,ω_B)	(p_x, p_y)	(μ_A,μ_B)	(u_A, u_B)	W	
Eqm on-path	(x,ϕ)	(20, -)	(ϕ, x)	(20, 30)	50	
off-path	(x, \mathbf{y})	(40, 10)	(y, x)	(50, 20)	70	

Existence and efficiency Example 1 Results

Example 1': Inefficient PME disappears

Values and Priority

$v_i(a)$	A	B	C
x	10	50	4
y	20	5	4

 $A \succ_a B \succ_a C, \ a = x, y$

Existence and efficiency Example 1 Results

Example 1': Inefficient PME disappears

Values a	nd Prior	rity			
	$v_i(a)$	A	В	C	
	x	10	50	4	
	y	20	5	4	$A \succ_a B \succ_a C, \ a = x, y$

	$(\omega_A, \omega_B, \omega_C)$	(p_x, p_y)	(μ_A,μ_B,μ_C)	$\left(u_{A},u_{B},u_{C} ight)$	W
on-path	(x, ϕ, ϕ)	(20, -)	(ϕ, x, ϕ)	(20, 30, 0)	50
deviation	(x,ϕ,y)	(40, 10)	(y, x, ϕ)	(50, 10, 10)	70

Existence and efficiency Example 1 Results

Example 1': Inefficient PME disappears

Values and Priority									
Γ	$v_i(a)$	A	В	C					
Γ	x	10	50	4					
	y	20	5	4	$A \succ_a B \succ_a C, \ a = x, y$				

	$(\omega_A, \omega_B, \omega_C)$	(p_x, p_y)	(μ_A,μ_B,μ_C)	$\left(u_{A},u_{B},u_{C} ight)$	$W \mid$
on-path	(x,ϕ,ϕ)	(20, -)	(ϕ, x, ϕ)	(20, 30, 0)	50
deviation	(x, ϕ, \mathbf{y})	(40, 10)	(y,x,ϕ)	(50, 10, 10)	70

Existence and efficiency Example 1 Results

Scarcity

Definition

Given $k = 1, 2, \ldots$, let

$$V_k = \left\{ v \in \mathbb{R}^{N \times \bar{O}} \mid \min_{a \in O} |\{i \in P | v_i(a) > 0\}| = k \right\},\$$

i.e., for each a, there are at least k players who value a.

DEF. Objects are scarce w.r.t. k if

$$2Q - \min_{a \in O} q^a \le k$$

where $Q = \sum_{a \in O} q^a$.

Existence and efficiency Example 1 Results

Efficiency of PME

Theorem

The following two statements are equivalent for each $k \ge 3$:

- If or all v ∈ V_k, a pure PME exists, and every pure PME allocation is efficient;
- **2** objects are scarce w.r.t. k.



Existence and optimality Example 2 Results

Market with no Money

No money is available for transaction.

Conditions

(Value) All tangible objects have positive intrinsic values for all:

$$V_{+} = \{ v \in \mathbb{R}^{\bar{O} \times N} | \forall i \in N \forall a \in O \ v_{i}(a) > 0 \}$$

(Quota1) Quota is one for all tangible objects.

Existence and optimality Example 2 Results

Existence

Lemma

[Shapley=Scarf] Assume (Value) and (Quota1). For all ω , ME exists.

Proposition

Assume (Value) and (Quota1). There exists at least one PME.

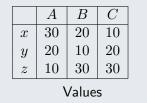
Counterexample if (Value) is violated

Counterexample if (Quota1) is violated

Existence and optimality Example 2 Results

Example 2: Market with no Money

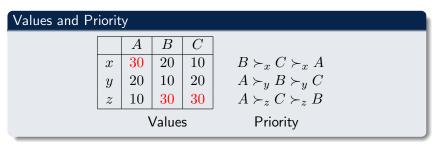
Values and Priority



$$B \succ_x C \succ_x A$$
$$A \succ_y B \succ_y C$$
$$A \succ_z C \succ_z B$$
Priority

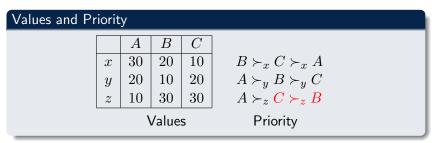
The first stage mechanism: DA

Existence and optimality Example 2 Results



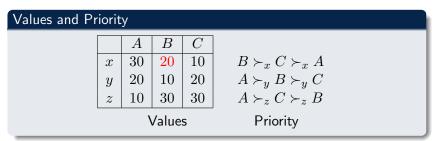
DA: Truth-telling strategies			
x A	y	$\overset{z}{BC}$	

Existence and optimality Example 2 Results



DA: Truth-telling strategies			
x A	y	z <mark>B</mark> C	

Existence and optimality Example 2 Results



DA: Truth-telling strategies			
x A	y	$\overset{z}{}_{C}$	
В			

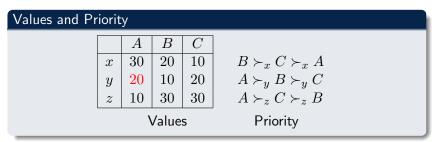
Existence and optimality Example 2 Results

Values and Priority								
		A	B	C				
	x	30	20	10	$B \succ_x C \succ_x A$			
	y	20	10	20	$\begin{array}{c} A \succ_{y} B \succ_{y} C \\ A \succ_{z} C \succ_{z} B \end{array}$			
	z	10	30	30	$A \succ_z C \succ_z B$			
	Values				Priority			

DA: Truth-telling strategies			
x X B	y	${z \atop C}$	

Existence and optimality Example 2 Results

Example 2: Market with no Money

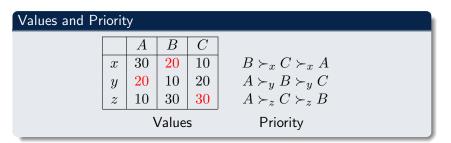




Matsui and Murakami Assignment and Market

Existence and optimality Example 2 Results

Example 2: Market with no Money

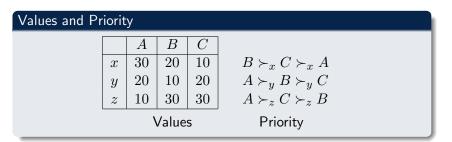


DA: Outcome

$$\label{eq:second} \begin{split} &\omega = (y,x,z) \\ \text{Pareto optimal. Also stable, i.e.,} \\ &\text{no player wants an object held by another with lower priority;} \\ &\text{no player wants a left-over (=unassigned tangible object).} \end{split}$$

Existence and optimality Example 2 Results

Example 2: Market with no Money



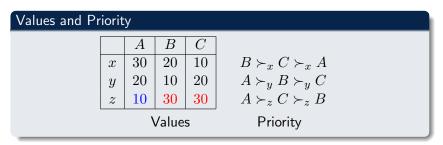
Two-stage economy

But, if there is the second stage,

A has an incentive to obtain z in the first stage.

Existence and optimality Example 2 Results

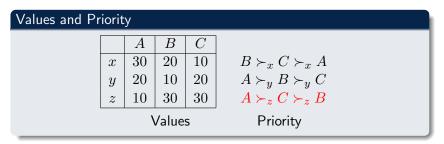
Example 2: Market with no Money

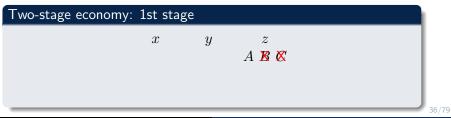


Two-stage economy	: 1st stag	ge		
	x	y	z	
			A B C	

Existence and optimality Example 2 Results

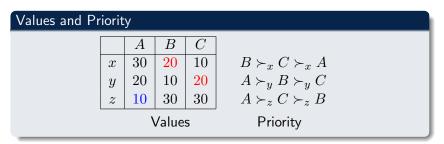
Example 2: Market with no Money





Existence and optimality Example 2 Results

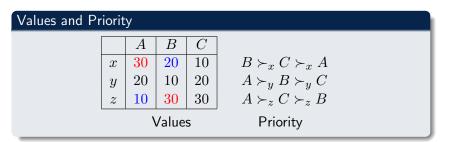
Example 2: Market with no Money



Two-stage economy:	1st stage			
	x	y	z	
	B	C	A	
	D	U		

Existence and optimality Example 2 Results

Example 2: Market with no Money



Two-stage economy: 2nd stage

In the second stage, given $\omega = (z, x, y)$, the (essentially) unique market eqm is $\mu = (x, z, y)$ with $p_x = p_z > p_y$ Pareto optimal but NOT Stable • Example 3

Existence and optimality Example 2 Results

Pareto optimality

Lemma

For all ω , an ME allocation is Pareto optimal under ω .

Proposition 4.1

A pure PME allocation is Pareto optimal.

Existence and optimality Example 2 Results

Stability

Definition

An object allocation μ is stable if

- no player wants an object held by another player with lower priority;
- no player wants a leftover.

Formal Definition

Existence and optimality Example 2 Results

Stable market equilibrium (SME)

Definition

Given u and \succ , (p, μ) is a stable market equilibrium (SME) if

- (p,μ) is a market equilibrium under μ itself,
- μ is stable.

Existence and optimality Example 2 Results

Priority Cycles

Definition

A priority cycle consists of distinct $i,j,k\in N$ and $a,b\in O$ such that:

Cycle condition: $i \succ_a j \succ_a k \succ_b i$.

 \succ is acyclical if there is no cycle.

Ergin (2002)

Existence and optimality Example 2 Results

Main result for no money

Theorem

Assume $|O| \ge 3$, $|N| \ge 3$, and (Quota1). The following two are equivalent:

- For any P with |P| ≥ 3 and any v ∈ V₊, an SME exists, and its allocation is always sustained by a pure PME;
- \succ is acyclical.



Conclusion Appendices

Conclusion

- We have considered a two-stage economy with non-monetary assignment in the first stage and market trades in the second.
- The second stage market makes the assignment stage a different ball game from the one without it.
- We have analyzed the economy with money and without.
- We have identified necessary and sufficient conditions for some properties of PME like efficiency and stability:
 - With money, "efficiency" and "scarcity" are equivalent;
 - With no money, "stability" and "acyclicity" are equivalent.

Conclusion Appendices

Thank you!

Conclusion Appendices

Appendix: Feasibility in the 2nd stage

Definition 5.1

Given ω , an allocation $x = (\mu, m)$ is ω -feasible if for all $a \in O$, $|\mu^a| \le |\omega^a|$ holds. \mathcal{A}^{ω} : the set of ω -feasible allocations. $O^{\omega} = \{a \in O | |\omega^a| > 0\}$: the set of available objects $\bar{O}^{\omega} = O^{\omega} \cup \{\phi\}.$

Conclusion Appendices

$\omega\textsc{-Pareto}$ optimality and $\omega\textsc{-efficiency}$

Definition 5.2

Given ω ,

- an allocation x is ω-Pareto optimal (ω-optimal) if
 A x' ∈ A^ω that Pareto dominates x.
- an allocation (μ, m) is ω -efficient if $\nexists (\mu', m') \in \mathcal{A}^{\omega}$ s.t. $W(\mu') > W(\mu)$.

return

Conclusion Appendices

Market equilibrium (ME)

Definition 5.3

Given $\omega,\,(p,(\mu,m))$ is a market equilibrium (ME) under ω if $p_{\phi}=0,$ and

- budget constraint
- individual optimization
- on excess demand, and excess supply implies zero price for each object

formal definition

Conclusion Appendices

Market Equilibrium (ME) with Money

Definition 5.4

Given $\omega \in \mathcal{A}$, $(p, \mu, m) \in \mathbb{R}^{\overline{O}^{\omega}}_{+} \times \mathcal{A}^{\omega} \times \mathbb{R}^{N}$ is a market equilibrium (ME) under ω if $p_{\phi} = 0$, and $\forall i \in N \ p_{\mu_{i}} + m_{i} = p_{\omega_{i}}$

2
$$\forall i \in N \ \mu_i \in \arg\max_{a \in \bar{O}} v_i(a) - p_a$$

 $\begin{tabular}{ll} \bullet &\forall a \in O^{\omega}[|\mu^a| \leq |\omega^a|] \wedge [|\mu^a| < |\omega^a| \Rightarrow p_a = 0] \end{tabular} \end{tabular} \end{tabular}$

Conclusion Appendices

Perfect Market Equilibrium (PME)

Definition

 $(\rho,(p(\omega),\mu(\omega),m(\omega))_{\omega\in\mathcal{A}})$ is a perfect market equilibrium (PME) if

- $\label{eq:constraint} \textbf{0} \mbox{ for all } \omega \in \mathcal{A}, \ (p(\omega), \mu(\omega), m(\omega)) \mbox{ is an ME under } \omega;$

PIPME return

Conclusion Appendices

Market Equilibrium (ME) with no Money

Definition 5.5

Given $\omega \in \mathcal{A}$, $(p, \mu) \in \mathbb{R}^{\bar{O}^{\omega}}_{+} \times \mathcal{A}^{\omega}$ is a market equilibrium (ME) under ω if $p_{\phi} = 0$, and

$$\forall i \in N \ \mu_i \in \operatorname{arg\,max}_{a \in \bar{O}} v_i(a) - p_a$$

Conclusion Appendices

Properties under scarcity

Lemma 5.1

Assume (Scarcity).
∀a ∈ O |μ^a| = q^a if (μ, m) is Pareto optimal;
given ω ∈ A, ∀a ∈ O |μ^a| = |ω^a| if (μ, m) is ω-optimal;
given ω ∈ A, ∀a ∈ O p_a > 0, |μ^a| = |ω^a| if (p, μ, m) is ME under ω.

Conclusion Appendices

Permutation Independent PME (PIPME)

Definition 5.6

 $(\rho,(p(\omega),x(\omega))_{\omega\in\mathcal{A}})$ is a permutation independent PME (PIPME) if

it is a PME;

2
$$p(\omega) = p(\omega')$$
 whenever $|\omega| = |\omega'|$.

The price is unchanged unless the total endowment changes.

Conclusion Appendices

The first stage: Assignment

Assignment Mechanism

$$M = \langle \Sigma, \lambda \rangle$$

 $\Sigma \equiv \times_{i \in N} \Sigma_i$: the finite set of strategy profiles $\sigma_i \in \Sigma_i$: *i*'s strategy, $\sigma = (\sigma_i)_{i \in N}$

$$\begin{split} \lambda: \Sigma \to A: \text{ an outcome function.} \\ \lambda(\sigma) \in A: \text{ object outcome in the first stage} \\ \lambda_i(\cdot) \in A_i: \text{ the set of available objects for } i \in N \\ A_i = \{\phi\} \text{ or } \bar{O}, \ A = \times_{i \in N} A_i \end{split}$$

Conclusion Appendices

Boston Mechanism

Each player submits a list of objects ordered from the best to the worst. The rest is determined by the algorithm:

Step 1 The players go to the first object in their respective lists.

- * If # of the players choosing a does not exceed q^a , they are assigned to a (and it's final).
- * If # exceeds q^a , then players with higher priority are assigned to a (final), and the rest go to the next in their resp list.
- Step k Repeat \star 's in Step 1 with leftovers and remaining players. If the chosen object is already taken, the player goes to the next step with the (k + 1)th object in her list.

Terminate the process when all are assigned to an object in \bar{O} .



Conclusion Appendices

Deferred Acceptance Algorithm (DA)

Each player submits a list of objects ordered from the best to the worst. The rest is determined by the algorithm.

Step 1 The players go to the first object in their respective lists.

- $\star~$ If # of the players choosing a does not exceed $q^a,$ they are temporarily assigned to a.
- \star If # exceeds q^a , then players with higher priority are assigned to a, and the rest go to the next in the list.
- Step k Those assigned to a before and those who choose a in this step go to a, and repeat \star 's in Step 1.

Terminate the process when all are assigned to an object in \overline{O} .

▶ Return

Conclusion Appendices

Illustration of the proof of $(1 \Rightarrow 2)$

Construction of PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'} - 1$

$$\begin{array}{c} O = \{a,b\}, q^a = 5, q^b = 3, N = \{1, \dots, 12\} \\ \hline 1, \dots, 5 & 6, \dots, 10 & 11, 12 \\ \hline a & 10 & 20 & 1 \\ \hline b & 1 & 1 & 1 \\ \hline i \succ_a j \text{ if } i \leq 5, j > 5 \end{array}$$

Conclusion Appendices

Illustration of the proof of $(1 \Rightarrow 2)$

Construction of PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'} - 1$

$$\begin{array}{c} O = \{a,b\}, q^a = 5, q^b = 3, N = \{1, \dots, 12\} \\ \hline 1, \dots, 5 & 6, \dots, 10 & 11, 12 \\ \hline a & 10 & 20 & 1 \\ \hline b & 1 & 1 & 1 \\ i \succ_a j \text{ if } i \leq 5, j > 5 \end{array}$$

 $p = (p_a, p_b) = (15, 1)$ on path $p = (p_a, p_b) = (16, 1)$ off path if someone $(6, \ldots, 10)$ takes the leftover \Rightarrow Nobody has an incentive to deviate

Conclusion Appendices

Illustration of the proof of $(2 \Rightarrow 1)$

No PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'}$

$$O = \{a, b\}, q^a = 5, q^b = 3, N = \{1, \dots, 12, 13\}$$

$$i \succ_a j \text{ if } i \le 5, j > 5$$

Conclusion Appendices

Illustration of the proof of $(2 \Rightarrow 1)$

No PME with leftovers when $k = 2Q - \min_{a' \in O} q^{a'}$

$$O = \{a, b\}, q^a = 5, q^b = 3, N = \{1, \dots, 12, 13\}$$

$$i \succ_a j \text{ if } i \le 5, j > 5$$

If there is a leftover like the above, 13 has an incentive to take it. • return

Proof: College Theorem (\Rightarrow)

Suppose that objects are not scarce', i.e., either $|N_s| < Q$ or $v \in V_k^f$ with $k \leq Q$ (or both).

Case I. $|N_{\rm s}| < k$: efficiency is trivially violated as the economy cannot deliver all the objects to the firms who need them.

Case II. $k \leq |N_s|$: construct v as follows. Align the objects in an arbitrary manner, $\{a_1, \ldots, a_{\bar{L}}\}$. There is $L = 1, \ldots, \bar{L}$ such that $q_{a_1} + \cdots + q_{a_{L-1}} < k \leq q_{a_1} + \cdots + q_{a_L}$. Fix L. Let $\hat{N}_f \subset N_f$ satisfy $|\hat{N}_f| = k$ and $\forall i \notin \hat{N}_f \forall a \in O[v_i(a) < 0]$.

Assign $v_i(a)$ $(i \in \hat{N}_f, a \in O)$ in such a way that for each $\ell = 1, \ldots, \bar{L} - 1$, and for all $i, j \in \hat{N}_f$, $v_i(a_\ell) > v_j(a_{\ell+1}) > 0$.

Let μ^* be the efficient object allocation given v. It must be the case that $|\mu^{*a}| = q^a$ for $a = a_1, \ldots, a_{L-1}$ and that $0 < |\mu^{*a}L| \leq q^aL$. Consider ω with $|\omega| = |\mu^*|$. Then (p,μ^*,m) becomes an ME under ω for some p and m. It is verified, due to the way we construct v, that $p_{a_1} \geq p_{a_2} \geq \ldots \geq p_{a_L}$. Then there is another ME (p^*,μ^*,m^*) such that $p_{a_\ell}^* = p_{a_\ell} - p_{a_L}$ holds for all $\ell = 1,\ldots,L$. Note $p_{a_L}^* = 0$. Assign objects to the players in N_s in the first stage from a_1 to a_{L-1} to fill their respective quotas, using \succ . As

For a_L to the remaining students so that the total number of the students assigned to some tangible objects becomes k. Assign the other students to ϕ . Denote this assignment profile ω^* .

Remove one player, say, *i* from ω^{*a_L} to obtain ω^{**} . We would like to have this ω^{**} as the PME allocation of the first stage. On the equilibrium path, we have the second stage outcome.

Let us check if there is no incentive to deviate. Under ω^{**} , there is one firm that cannot buy a tangible object in the second stage, and there is at least one student who does not obtain a leftover in the first stage. If such a student obtains the object, then the first stage object allocation becomes ω^* (or some ω' with $|\omega'| = |\omega^*|$ to be precise), and therefore, the price of the object this student obtains is zero. Thus, the student has no incentive to deviate in the first stage. An inefficienct outcome arises as a PME.

return

Conclusion Appendices

Proof: College Theorem (\Leftarrow)

Suppose that objects are scarce, i.e., $|N_s| \geq Q$ and $v \in V_k^f$ with k > Q. Take v as given along with other parameters, \succ and q. Existence: Take some ω with $|\omega| = q$. Let (p^*, μ^*, m) be an ME under ω . Align $O = \{a_1, \ldots, a_L\}$ in such a way that $p_{a_1}^* \geq p_{a_2}^* \geq \ldots \geq p_{a_L}^*$ holds. Since k > Q holds, there exists $j \in N_f$ such that $\mu_j^* = \phi$ and $v_j(a_L) > 0$ hold. Therefore, $p_{a_L}^* \geq v_j(a_L) > 0$. Assign objects to the players in N_s in the first stage from a_1 to a_{L-1} to fill their respective quotas, using \succ . We can do it as $|N_s| \geq Q$. Assign the other students to ϕ . Denote this assignment profile ω^* .

Under ω^* , (p^*, μ^*, m^*) becomes an ME for some m^* .

Let ω^* be the outcome of the first stage. Then together with appropriate off-path ME's, we have a PME as nobody has an incentive to deviate.

Efficiency: Suppose $(\sigma, (p(\omega), \mu(\omega), m(\omega))$ is a PME. Let $\omega^* = \lambda(\sigma)$. Take any ω . Since k > Q holds, for all $a \in O$, there exists $j \in N_f$ such that $\mu_j(\omega) = \phi$ and $v_j(a) > 0$ hold. Therefore, $p_a(\omega) \ge v_j(a) > 0$ for all $a \in O$; otherwise, j would buy a in ME. Suppose that $a \in O$ has some leftover, i.e., $|\omega^{*a}| < q^a$.

Since $|N_s| \ge Q$, there exists at least one student who does not obtain any tangible object. This player has an incentive to obtain the leftover a since under any ω , $p_a(\omega) > 0$ as we have shown.

Conclusion Appendices

Formal Definition of Stability

Definition

 $\boldsymbol{\mu}$ is stable if

- $\forall i, j \in N \ [\mu_j \in O \land i \succ_{\mu_j} j \Rightarrow u_i(\mu_i, 0) \ge u_i(\mu_j, 0)]$
- $\forall a \in \overline{O} \ \forall i \in N \ [|\mu^a| < q^a \Rightarrow u_i(\mu_i, 0) \ge u_i(a, 0)]$

Conclusion Appendices

Construction of PME with leftovers

Construction of PME with leftovers when $k = 2Q - \min_{a' \in Q} q^{a'} - 1$

 $\begin{array}{l} q,\succ: \text{given. Let } a\in\arg\min_{a'\in O}q^{a'}. \text{ Proof for }k\text{ less than this is similar or (easier).}\\ \text{Let an auxilirary value profile }\hat{v}\text{ be given by}\\ \hat{v}_i(a)=1 \text{ for all } i\in N, \\ \hat{v}_i(b)=10 \text{ for all } i\in N, \\ \hat{v}_i(b) \begin{cases} \leq 10 & \text{ if } b\neq a,\phi, \\ \in [24,25] & \text{ if } b\neq a,\phi, \\ \in [1,2] & \text{ if } i\in J \lor b=a \end{cases}\\ \hat{v}_i(b) \begin{cases} \leq 10 & \text{ if } b\neq a,\phi, \\ \in [1,2] & \text{ if } i\in J \lor b=a \end{cases} \\ \text{Let } p_b=20 \ (b\neq a,\phi) \text{ and } p_a=1 \text{ on path, or off path when } i\in S \text{ deviates.} \\ \text{Let } p_b=21 \ (b\neq a,\phi) \text{ and } p_a=1 \text{ off path when } i\in S \text{ deviates.} \\ \text{On path, } i\in S \text{ gets } 20. \\ \text{Off path when } i\in S \text{ deviates, it gets either } 20 \text{ or at most } 10. \\ \text{On path, } j\in J \text{ gets at most } 2, \text{ while } k\in W \setminus J \text{ gets between 4 and 5.} \\ \text{Off path when } j\in W \text{ deviates, it gets at most the same.} \end{cases}$

return

Conclusion Appendices

Proof of Existence

Proof of existence when $k = 2Q - \min_{a \in O} q^a$

Proof for k greater than this is similar.

Let ω be an allocation with no leftover. Sps (p^*, μ^*, m^*) is an ME under ω (such an ME exists). We may assume $p_a^* > 0$ for $a \in O$ since there is a sufficient amount of demand for each $a \in O$. For any ω' with no leftover, let $p(\omega') = p^*$. Adjusting m' appropriately, we obtain an ME (p^*, μ^*, m') under ω' . Consider an auxiliary \hat{v} as follows:

 $\hat{v}_i(a) = p_a^*$ $(i \in N, a \in \overline{O}).$ Use this \hat{v} and run DA with the truth-telling strategies σ^* to obtain ω^* .

Note ω^* is stable w.r.t. \hat{v} .

Also, no leftover under σ^* .

Moreover, even if one, say, player *i*, makes a unilateral deviation to, say, σ_i , no leftover under (σ_i, σ_i^{*}) .

This σ^* constitutes a pure PME along with ME's mentioned above (and appropriately chosen ME's for other ω 's). This completes the proof for DA.

Conclusion Appendices

Proof of Efficiency

Proof of efficiency when $k = 2Q - \min_{a \in O} q^a$

Proof for k greater than this is similar. Suppose $a \in O$ has some left-over, i.e., $|\omega^a| < q^a$. Observe at least q^a agents who cannot obtain $b \neq a, \phi$ in neither stage and have a positive value for a. Let L be the set of such agents. Note $|L| \ge q^a > |\omega^a|$. Then $p_a \ge \min_{i \in L} v_i(a) > 0$, (for if not, there would be excess demand). Then $\exists \ell \in L[\omega_\ell = \phi]$. This agent ℓ has an incentive to obtain the left-over to obtain $v_\ell(a)$ instead of $v_\ell(a) - p_a$.

Conclusion Appendices

$\mathsf{Proof of PME} \Leftarrow \mathsf{ACY} \mathsf{ under DA}$

Assume (No Money), (ACY), (Quota), and (DA). Sps $\exists A, v \in V_+ \varphi(v|A)$ is not PME. WTS \exists a cvcle. Remove j' with $A_{i'} = \{\phi\}$ from the economy. Hereafter, N means those players j with $A_i = O_i \succ is$ reduced to N as well. $(p(\omega), \mu(\omega), 0)_{\omega \in \mathcal{A}}$: ME profile $\zeta^* = (\zeta^*_i)_{i \in N}$: truth-telling strategy. Player *i* has an incentive to deviate by submitting ζ_i . Fix *i*. Let $\omega^* = \lambda(\zeta^*)$ and $\hat{\omega} = \lambda(\zeta_i, \zeta^*_i)$. DA implies $v_i(\omega_i^*) \ge v_i(\hat{\omega}_i)$, and *i* will trade through a TC: $(k_0, k_1, \ldots, k_{\bar{n}})$ with $k_0 = k_{\bar{n}} = i$ s.t. $v_{k_n}(\hat{\omega}_{k_{n+1}}) > v_{k_n}(\hat{\omega}_{k_n})$. Note (*) $k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} k_n$

Conclusion Appendices

$\mathsf{Proof of PME} \Leftarrow \mathsf{ACY} \mathsf{ under DA}$

Auxiliary DA: We run DA without *i* and then add *i*. No change in result. At t^* , *i* is put in DA. *i* follows ζ'_i . Sps *i* obtains ω_i in step \bar{t}

Lemma. *i* is never accepted at $a_1 \neq \omega_i$ before step \bar{t} . Pf of Lemma. Sps not, i.e., $\exists t_1 \exists a_1 \neq \omega_i$, *i* obtains a_1 in $t_1 < \bar{t}$. In t_1 , either a_1 is a leftover, which will end the process $\rightarrow \leftarrow$, or j_1 is rejected at a_1 by *i*. j_1 is the only loser. Rejection chain is needed to push out *i* from a_1 as *i* must obtain $\omega_i \neq a_1$: $(a_1, j_1, t_1), (a_2, j_2, t_2), \dots, (a_{\bar{\kappa}}, j_{\bar{\kappa}}, t_{\bar{\kappa}}) = (a_1, i, t')$ where j_{κ} is rejected at a_{κ} by $j_{\kappa-1}$ at t_{κ} . Then \exists a generalized cycle with $a_{\kappa} \neq a_1$ for $\kappa \neq 1$: $j_{\bar{\kappa}-1} \succ_{a_1} i \succ_{a_2} j_2 \cdots \succ_{a_{\bar{\kappa}-1}} j_{\bar{\kappa}-1}$ Then \exists a cycle. $\rightarrow \leftarrow$ \diamondsuit

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Conclusion Appendices

$\mathsf{Proof} \text{ of } \mathsf{PME} \Leftarrow \mathsf{ACY} \text{ under } \mathsf{DA}$

Lemma implies $\omega_i^* \neq \hat{\omega}_i$. Thus, we have the following argument. (1) If $\omega_{k_{\bar{n}-1}}^* = \hat{\omega}_{k_{\bar{n}-1}}$, then $\hat{\omega}_i$ is not a leftover in ω^* ; for if not, $k_{\bar{n}-1}$ would have obtained it. *i* pushed out, say, $\ell \neq k_{\bar{n}-1}$ from $\hat{\omega}_i$. Since $k_{\bar{n}-1}$ could not obtain $\hat{\omega}_i$ from ℓ , stability of DA implies $i \succ_{\hat{\omega}_i} \ell \succ_{\hat{\omega}_i} k_{\bar{n}-1}$. Together with (*), \exists a generalized cycle. (2) In general, sps ω_k^* , $\neq \hat{\omega}_{k_n}$ for $n' = n + 1, \dots, \bar{n} - 1$ and $\omega_{k_n}^* = \hat{\omega}_{k_n}$. Then k_{n+1} must have pushed out, say, ℓ' (:: $\hat{\omega}_{k_{n+1}}$ is not a leftover similar to (1)). k_n wanted $\omega_{\ell'}^*$ but could not. Thus, $k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} \ell' \succ_{\hat{\omega}_{k_{n+1}}} k_n$. Together with (*), a generalized cycle exists. (3) Sps ω_k^* , $\neq \hat{\omega}_{k_n}$, for $n' = 1, \ldots, \bar{n} - 1$. Then k_1 must have pushed out, say, ℓ'' from $\hat{\omega}_{k_1}$ ($\therefore \hat{\omega}_{k_1}$ is not a leftover similar to (1)). Then $k_1 \succ_{\hat{\omega}_{k_1}} \ell'' \succ_{\hat{\omega}_{k_1}} i$. Together with (*), a generalized cycle exists.

Conclusion Appendices

$\mathsf{Proof} \text{ of } \mathsf{PME} \Leftarrow \mathsf{ACY} \text{ under } \mathsf{DA}$

The remaining task is to show that the generalized cycle found above uses distinct players. Note that $k_0, k_1, \ldots, k_{\bar{n}-1}$ are distinct as they form a TC.

(1) $i \succ_{\hat{\omega}_i} \ell \succ_{\hat{\omega}_i} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} \dots \succ_{\hat{\omega}_{k_2}} k_1 \succ_{\hat{\omega}_{k_1}} i$ Note $i, \ell, k_{\bar{n}-1}$ are distinct. If $k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} i$, then we have a cycle with distinct players: $i \succ_{\hat{\omega}_i} \ell \succ_{\hat{\omega}_i} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} i$ If $i \succ_{\hat{\omega}_{k_{\bar{n}-1}}} k_{\bar{n}-1}$, then we can shorten the cycle: $i \succ_{\hat{\omega}_{k_{\bar{n}-1}}} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} k_{\bar{n}-2} \dots \succ_{\hat{\omega}_{k_2}} k_1 \succ_{\hat{\omega}_{k_1}} i$, which is a generalized cycle with distinct players. (2) $k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} \ell' \succ_{\hat{\omega}_{k_{n+1}}} k_n \succ_{\hat{\omega}_{k_n}} k_{n-1} \dots \succ_{\hat{\omega}_{k_{n+2}}} k_{n+1}$ Note k_{n+1}, ℓ', k_n are distinct. The rest is similar to (1). (3) $k_1 \succ_{\hat{\omega}_{k_1}} \ell'' \succ_{\hat{\omega}_{k_1}} i \succ_{\hat{\omega}_i} k_{\bar{n}-1} \dots k_2 \succ_{\hat{\omega}_{k_2}} k_1$ Note k_1, ℓ'', i are distinct. The rest is similar to (1).

▶ return

Conclusion Appendices

$\mathsf{Proof} \text{ of } \mathsf{PME} \Leftarrow \mathsf{ACY} \text{ under Boston}$

Assume (No Money), (ACY), (Quota), and (Boston). Sps $\exists A, v \in V_+ \varphi(v|A)$ is not PME. WTS \exists a cycle. Given $\omega^* = \varphi(v|A)$, there exists a NE σ^* such that the players obtain their final objects ω^* in the first step. $\exists i$ who gains by deviation. Fix *i*.

Let σ_i be the deviating strategy, and let $\omega^* = \lambda(\sigma^*)$ and $\hat{\omega} = \lambda(\sigma_i, \sigma_{-i}^*)$. In σ^* under Boston, there exists at most one player who is affected by *i*'s deviation.

i has an incentive to deviate only when there is a TC after i's deviation.

TC: $i = k_0, k_1, \ldots, k_{\bar{n}} = i$ where k_n wants k_{n+1} 's object.

Conclusion Appendices

$\mathsf{Proof} \text{ of } \mathsf{PME} \Leftarrow \mathsf{ACY} \text{ under Boston}$

Claim. $\exists j$ directly affected by *i*'s deviation, taking over ω_j^* , i.e., $\hat{\omega}_i = \omega_j^*$, $\hat{\omega}_j \neq \omega_j^*$. Pf. Sps not, i.e., *i* does not affect any player in the first stage. *i* must have taken a leftover or ϕ . \exists no new TC since nobody wants a leftover or ϕ under SME. \diamondsuit

Claim. j is not in TC, i.e., $j \neq k_0, k_1, \ldots, k_{\bar{n}}$. Pf. j, after i's deviation, can go for either one of ω_i^* , a leftover, and ϕ . If j goes for a leftover or ϕ , j is not in TC as nobody is interested in the leftover under SME. So, sps $\hat{\omega}_j = \omega_i^*$. Sps also j is in TC. Then $k_n = j$ for some $n = 1, \ldots, \bar{n} - 1$. Then $i = k_0, k_1, \ldots, k_{n-1}, k_n = i$ form a nontrivial TC under ω^* since $k_{n'-1}$ wants $k_{n'}$'s object $(n' = 1, 2, \ldots, n)$. This contradicts with the premise that ω^* is a SME allocation. \diamondsuit

Conclusion Appendices

$\mathsf{Proof} \text{ of } \mathsf{PME} \Leftarrow \mathsf{ACY} \text{ under Boston}$

The previous claim implies agents in TC and j are all distinct. Since $i \succ_{\omega_j^*} j \succ_{\omega_j^*} k_{\bar{n}-1}$ ($\omega_j^* = \hat{\omega}_i$), we have a generalized cycle of priority: $i \succ_{\hat{\omega}_i} j \succ_{\hat{\omega}_i} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} \ldots \succ_{\hat{\omega}_{k_2}} k_1 \succ_{\hat{\omega}_{k_1}} i$.

Conclusion Appendices

No ME if (Quota1) is violated

Example 5.1

	A	B	C
x	10	20	20
y	20	10	10

$$\omega = (x,y,y)$$

no ME under $\omega,$ and therefore, no PME

: (i) $p_x \leq p_y$: B and C demand x. \Rightarrow Excess demand (ii) $p_x > p_y$: No demand for x (A demands y) return

Conclusion Appendices

No ME if (Value) is violated

Example 5.2

$$\omega = (\phi, x, y)$$

no ME under $\omega,$ and therefore, no PME

: (i) $p_x > p_y$: Excess supply of x(ii) $p_x \le p_y$, $p_y > 0$: Excess supply of y (C demands x) (iii) $p_x = 0$: Excess demand for x

Conclusion Appendices

Proof of Necessity using Example 2

Construction of v under $A \succ_z C \succ_z B \succ_x A$

	A	B	C	
x	30	20	10	
y	20	10	20	
z	10	30	30	
Values				

DA and Boston: outcome is $(y, x, z) \Rightarrow A$ has an incentive to obtain z.

Technical detail in the proof: $A_i = \{\phi\}$ for all $i \neq A, B, C$.

Conclusion Appendices

Proof of Sufficiency: Demo for a particular case

 $\begin{array}{l} \mbox{Consider (DA). Sps} \succ \mbox{ is acyclical.} \\ \mbox{Sps } \exists v \in V_+ \ A \ \mbox{has an incentive to deviate from SME allocation } \mu^v. \\ \mbox{WTS} \rightarrow \leftarrow \end{array}$

 $\begin{array}{cccc} A & B & C \\ \mu^v & y & x & z \end{array} \text{ Sps } A \text{ prefers } x \text{ to } y. \text{ Stability } \qquad \Rightarrow B \succ_x A \end{array}$

Conclusion Appendices

Proof of Sufficiency: Demo for a particular case

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y . Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	x	y	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$

Conclusion Appendices

Proof of Sufficiency: Demo for a particular case

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y . Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	x	y	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$
$\hat{\mu}$	x	z	y	B must prefer z to x . Stability	$\Rightarrow C \succ_z B$

Conclusion Appendices

Proof of Sufficiency: Demo for a particular case

 $\begin{array}{l} \mbox{Consider (DA). Sps} \succ \mbox{ is acyclical.} \\ \mbox{Sps } \exists v \in V_+ \ A \ \mbox{has an incentive to deviate from SME allocation } \mu^v. \\ \mbox{WTS} \rightarrow \leftarrow \end{array}$

Conclusion Appendices

Proof of Sufficiency: Demo for a particular case

	A	B	C		
μ^v	y	x	z	Sps A prefers x to y . Stability	$\Rightarrow B \succ_x A$
$\hat{\omega}$	z	x	y	A must go to z and get x thru TC.	$\Rightarrow A \succ_z C$
$\hat{\mu}$	x	y	z	$\hat{\mu}$ must be the DA outcome	$\rightarrow \leftarrow$

Conclusion Appendices

Proof of Sufficiency: Demo for a particular case

 $\begin{array}{l} \mbox{Consider (DA). Sps} \succ \mbox{ is acyclical.} \\ \mbox{Sps } \exists v \in V_+ \ A \ \mbox{has an incentive to deviate from SME allocation } \mu^v. \\ \mbox{WTS} \rightarrow \leftarrow \end{array}$

Conclusion Appendices

Proof of Sufficiency: Demo for a particular case



Conclusion Appendices

Example 3: TTC \neq PME

Values and Priority

	A	B	C	D
x	40	20	40	10
y	20	40	30	20
z	30	30	20	30
w	10	10	10	40
	Values			es

$$D \succ_x B \succ_x C \succ_x A$$
$$A \succ_y C \succ_y B \succ_y D$$
$$D \succ_z C \succ_z B \succ_z A$$
$$A \succ_z D \succ_z B \succ_z C$$
Priority

TTC (single stage): (x, z, y, w)

PME with DA: (x, y, z, w), DA only: (z, y, x, w)

▶ return

Conclusion Appendices

Example 3: TTC \neq PME

Values and Priority

	A	B	C	D
x	40	20	40	10
y	20	40	30	20
z	30	30	20	30
w	10	10	10	40
	Values			

$$D \succ_x B \succ_x C \succ_x A$$
$$A \succ_y C \succ_y B \succ_y D$$
$$D \succ_z C \succ_z B \succ_z A$$
$$A \succ_z D \succ_z B \succ_z C$$
Priority

TTC (single stage): $A \rightarrow x \rightarrow D \rightarrow w \rightarrow A$ PME with DA: $\omega = (y, x, z, w) \ \mu = (x, y, z, w)$

▶ return