Mislaid Pieces in Finitely Additive Population Games

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1. Large Population Games

2. Finitely Additive Probabilities

3. PFAs in Population Games

4. PFAs in Economic Models

Basics

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- Or $\mathbb{U} \subset C(A \times M)$, $M = \{q \in \Delta(T \times A) : q(E \times A) = \mu(E)\}$.
- $\mathcal{G} : T \to \mathbb{U}$, $P = \mathcal{G}(\mu) \in \Delta(\mathbb{U})$.

Population-Wide Maximizing Behavior

If $a : T \to \Delta(A)$ is the population strategy, the distribution is $\nu_a(E) = \int a(t)(E)\, d\mu(t)$, and agent t receives utility $\mathcal{G}(t)(a(t), \nu_a)$.

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A strategy $a(\cdot)$ is an ϵ-**equilibrium** if

$$\mu(\{t : \mathcal{G}(t)(a(t), \nu_a) \geq \max_{b \in A} \mathcal{G}(t)(b, \nu_a) - \epsilon\}) \geq 1 - \epsilon, \qquad (1)$$

and is an **equilibrium** if it is a 0-equilibrium.

Definitions

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Countable additivity is **not** "just a technical assumption."

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If the deficiency is 1, then μ is **purely finitely additive**. A probability is pfa iff there exists a strictly positive g with $\int g \, d\mu = 0$.

Weak* Compactness

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Alaoglu's Theorem: the set of finitely additive probabilities is weak*-compact.

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- FIDI's — define $\mu' : \mathcal{P}^\circ \to [0, 1]$ by $\mathcal{L}(\{\text{proj}_{t_m}(\mu') - \text{proj}_{t_{m-1}}(\mu') : m = 1, \ldots n\})$ to be independent Poissons with parameters $(\lambda \cdot (t_m - t_{m-1}))$.

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Compactness implies non-emptiness of the intersection over all finite $0 =: t_0 \leq t_1 < \cdots < t_n$. Any μ in the intersection is purely finitely additive.

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Fix arbitrary $\epsilon > 0$ and interval $[0, 1/\epsilon]$. There exists K such that $\tau_K \leq (1/\epsilon) < \tau_{K+1}$. There exists an $f \in \mathbb{P}$ with slope at least $1/\epsilon$ such that for $1 \leq k \leq K$,

$$[k \leq h(t) < (k+1)] \Rightarrow [k \leq f(t) < (k+1)]$$
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The finitely additive μ is "trying to" put mass 1 on polynomials having slopes at least $1/\epsilon$ for every $\epsilon > 0$.

Representing Infinitely Steep Functions

Let ${}^*\mathbb{P}$ be the nonstandard version of the polynomials. By overspill, there exists a strictly positive $\epsilon \simeq 0$ such that for every Poisson realization h, there is an $f \in {}^*\mathbb{P}$ such that for $1 \leq k \leq K$,

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${}^*\mu$ or $L({}^*\mu)$ is a probability on ${}^*\mathbb{P}$ having the FIDIs of a Poisson process.

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- Let η be a pfa probability on $\mathbb{N}$ with $\eta(E) = 0$ or $\eta(E) = 1$ for all $E \subset \mathbb{N}$.

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- If $\epsilon = \langle \epsilon_m \rangle$ in ${}^*\mathbb{R}$ and $\epsilon_m \downarrow 0$, then we say that ϵ is infinitesimal because, for all $r > 0$, $\eta(\{m : 0 < \epsilon_m < r\}) = 1$, so $0 < \epsilon < r$.

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- For $E = \langle E_n \rangle$, ${}^*\mu(E) = \langle \mu(E_n) \rangle$, so domain of ${}^*\mu$ is large.

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- Let $f = \langle f_m \rangle$.

Claim: ${}^*\mu$ puts mass 1 on the infinitely steep polynomials.

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Will then analyze the **equilbria** of the games

$$^*\Gamma(\mu) := ((^*T, \sigma(^*\mathcal{T}), {}^{\circ *}\mu), \mathrm{st}_V(^*\mathbb{U}), \mathrm{st}_V(^*\mathcal{G})).$$

Approximate Equilibria

$T = [1, \infty)$, $\mathcal{T}$ is the (usual) Borel σ-field, and μ is a non-atomic, pfa probability on T with $\mu([t, \infty)) \equiv 1$. the common space of actions is $A = \{0, 1\}$, $\mathbb{U}$ is the closed unit ball in $C(A \times [0, 1])$ where $[0, 1]$ representing $\nu(a = 1)$.

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Example 1: $\mathcal{G}(t) = a \cdot (\frac{1}{t} - \nu)$.

- If $\nu_a > 0$ is equilibrium, then $a^* = 1$ is only a best response for t in the null set $(0, 1/\nu_a]$ — $[\nu_a > 0] \Rightarrow [\nu_a = 0]$.
- If $\nu_a = 0$ is equilibrium, then for all $t \in T$, $\frac{1}{t} > \nu_a$, so everyone should (apparently) play the action 1, making $\nu_a = 1$.
- For ϵ-equilibria, any tiny set of people play $a = 1$.

But the Equilibria Involve

$V(a, \nu) := -a \cdot \nu$, $\mathcal{G}(t) = a \cdot \frac{1}{t} + V(a, \nu)$, for any $\delta > 0$, we have

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If $\mu(\{t : \mathcal{G}(t) = V\}) = 1$, then equilibria have $\mu(\{t : a(t) = 0\}) = 1$.

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$\mathcal{G}(t) = a \cdot u(t, \nu)$ where

$$u(t,\nu) = \begin{cases} 1 & \text{if } \nu \leq \frac{1}{2}, \\ 1 - t(\nu - \frac{1}{2}) & \text{if } \frac{1}{2} \leq \nu \leq \frac{1}{2} + \frac{2}{t}, \text{ and} \\ -1 & \text{if } \frac{1}{2} + \frac{2}{t} \leq \nu. \end{cases}$$

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To represent steepness $= \infty$, the domain, $\Delta(\{0,1\}) = [0,1]$, must expand.

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- $[\nu > \frac{1}{2}] \Rightarrow [\mu(\{t : \frac{1}{2} + \frac{2}{t} < \nu_a\}) = 1]$. A mass 1 set of players loses utility of 1 by playing $a = 1$, so ϵ-best responses must put mass at least $1 - \epsilon$ on $a = 0$.

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Equilibrium involves everyone with $t < (\leq) t_c$ playing $a = 1$ where $F_1(t_c) = \frac{1}{2} + \frac{1}{t_c}$,

Equilibria with ${}^*\mu$

Now replace the spaces with their nonstandard extensions and analyze the dependence of equilibrium distributions of actions and utilities. Example, let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ be an infinite integer. For any Borel $E \subset [1, \infty)$, $\mu_1(E) := {}^{\circ *}Unif[N, N+1]({}^*E)$ with cdf is $F_1(t) = x$ for $N \leq x \leq N+1$.

Equilibrium involves everyone with $t < (\leq) t_c$ playing $a = 1$ where $F_1(t_c) = \frac{1}{2} + \frac{1}{t_c}$, using the quadratic formula on $t_c = \frac{1}{2} + \frac{1}{t_c}$ yields

$$t_c = \frac{1}{2}\left[(N + \tfrac{1}{2}) + \sqrt{(N + \tfrac{1}{2})^2 + 4}\right],$$

which involves $t_c/(N + \frac{1}{2}) = 1 + \epsilon$ for an $\epsilon \simeq 0$.

Observations

- Agents in $[N, t_c]$, who have mass (a positive infinitesimal greater than) $\frac{1}{2}$, play $a = 1$, and their utility is distributed uniformly on $[0, 1]$, agents in $(t_c, N + 1]$ play $a = 0$ and receive utility 0. No strategy in the original game achieves this joint distribution of actions and utilities.

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- Related, $\nu = \frac{1}{2} + 1/t_c$ is NOT an element of $[0, 1]$, it is an element of ${}^*[0, 1]$. To find the equilibrium, the domain of the utility functions, $\{0, 1\} \times [0, 1]$, was extended.

Equilibrium Outcomes Depend on μ

Now suppose μ_2 the weak* standard part of $\frac{1}{4}U[0, N] + \frac{3}{4}U[0, N^2]$ for infinite N. Can solve for exact cutoff t_c, it satisfies $t_c/(N + \frac{1}{3}N^2) \simeq 1$.

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Equilibrium outcomes: just over half of the agents, those in $[0, t_c]$ play $a = 1$, the rest play $a = 0$. Playing $a = 0$ yields utility 0. Half of the $a = 1$ agents receive utility 1 and half of them have utility uniformly distributed on $[0, 1]$.

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Again, no strategy in the original game achieves this joint distribution of outcomes and actions.

Examples

Fishburn (1970). A society's preference ordering, $\succsim_S$ satisfies Arrow assumptions iff for some pfa point mass η we have

$$[x \succsim_S y] \Leftrightarrow (\exists E \subset T)[\eta(E) = 1 \text{ and } E = \{t \in T : x \succsim_t y\}].$$

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Khan et al. (2016). Pfa population measures $\Rightarrow$ some population games have no equilibria. Missing agents and their utility functions.

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- Can substitute compact Hausdorff spaces for the pieces of $\Gamma^*(\mu)$.
- The compactification of e.g. the unit ball in $C([0,1])$ is an incredibly cool Hausdorff space.

Anything Else?

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