Matching and Rematching with Commitment

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Workshop on Matching Theory, Search and Market Design

IMS, NUS

July 27, 2018

The Main Question

- Consider a senior level job matching market with many heterogeneous firms and workers. There are committed incumbent firms and workers.

- A committed agent can dissolve her partnership only if her partner wants to do so.

- How should we match workers and firms as well as possible without violating their commitments to their partners?

Outline

- Literature and Motivation
- Model and Basic Concepts
- Main Results and Algorithms

Literature and Motivation

- There is a huge volume of literature on matching theory starting with
  Gale and Shapley (1962)—Deferred Acceptance
  Shapley and Scarf (1974)—Top Trading Cycle

- Crawford and Knoer (1981), Kelso and Crawford (1982), Roth (1982), Blum, Roth and Rothblum (1997), Abdulkadiroğlu and Sönmez (1999), Roth, Sönmez and Ünver(2004), Hatfield and Milgrom (2005), Ostrovsky (2008), Kojima and Manea (2010), Combe, Tercieux and Terrier (2017), Sun and Yang (2016) etc.

Motivation

- Commitments exist in various forms and can influence people's behavior and affect the performance of the system involved. Commitments can be imposed by contract, by law, by custom, or by morals. For instance, universities with a tenure track system are committed to their tenured faculty members in the sense that they generally cannot fire a tenured faculty unless she/he is willing to leave. On the other hand, a tenured faculty can move rather freely without facing the same kind of commitment constraint.

Model and Basic Concepts

- $W = \{w_1, \cdots, w_s\}$: the set of all workers
  $F = \{f_1, \cdots, f_t\}$: the set of all firms
- Relations between firms and workers are governed by a finite set Σ of bilateral contracts.
- A typical contract α specifies a worker w_α and a firm f_α and the job description and the payment.
- A trivial contract $\alpha = w$ or f indicates that worker w does not have a job or firm f does not hire any worker.
- Each agent (worker or firm) x has strict preferences over the set of contracts $\Sigma(x)$ containing x.
- For $\Psi \subseteq \Sigma$, $\Psi(x)$: the set of contracts in Ψ in which agent x is involved.

Basic Concepts

- A contract $\alpha \in \Sigma(x)$ is acceptable to an agent $x \in W \cup F$ if $\alpha \succeq_x x$.

- A set Ψ of contracts is called a (one-to-one) matching if $\Psi(x)$ contains exactly one contract for every agent $x \in F \cup W$.

- At a matching Ψ, if $\Psi(w) \cap \Psi(f) \neq \emptyset$ for some firm f and some worker w, f and w will be called a partner of each other and described by $\mu_\Psi(w) = f$ and $\mu_\Psi(f) = w$; if a contract in Ψ involves only one agent x, the agent is a single or self-matched and will be described by $\mu_\Psi(x) = x$.

- The mapping μ_Ψ is a one-to-one mapping from the set $W \cup F$ onto itself of order two and is uniquely determined by the matching Ψ. μ_Ψ will be also called a matching.

Matching State and Commitment

- For a matching Ψ, $P(\Psi) = \{x \in W \cup F \mid \mu_\Psi(x) \neq x\}$: the set of agents who have a partner under Ψ.
- An agent $x \in P(\Psi)$ is committed if x can dissolve her partnership with $\mu_\Psi(x)$ only when her partner $\mu_\Psi(x)$ agrees to do so.
- $C(\Psi) = \{x \in P(\Psi) \mid x \text{ is committed}\}$: the set of agents who have commitments to their partners under Ψ.
- Let $V(\Psi) = (W \cup F) \setminus C(\Psi)$ represent the set of all free agents. A free agent $x \in P(\Psi)$ can rescind her relationship with her partner $\mu_\Psi(x)$.
- A matching state is described by a pair $(\Psi, C(\Psi))$ of matching Ψ and its commitment structure $C(\Psi)$.

Stability: Stable Matching State

- Given a matching state $(\Psi, C(\Psi))$, a chain of the state is an ordered sequence of an even number of distinct agents $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ with $K \geq 1$ such that $x_1, y_K \in V(\Psi)$ and $\mu_\Psi(y_k) = x_{k+1}$ for every $k = 1, 2, \cdots, K-1$.

- A cycle of the state is an ordered sequence of an even number of distinct agents $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ with $K \geq 1$ such that $\mu_\Psi(y_k) = x_{k+1}$ for every $k = 1, 2, \cdots, K$, where x_{K+1} becomes x_1 by convention.

Stability: Stable Matching State

- A matching state $(\Psi, C(\Psi))$ is individually rational if $\Psi(x) \succeq_x x$ for **every free agent** $x \in V(\Psi)$.
- A matching state $(\Psi, C(\Psi))$ is blocked by a chain or cycle $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$, if for each $k = 1, \cdots, K$, there is an acceptable contract α_k to both agents x_k and y_k such that $\alpha_k \succ_{x_k} \Psi(x_k)$ and $\alpha_k \succ_{y_k} \Psi(y_k)$.

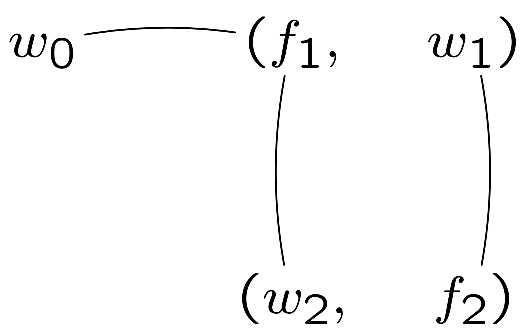
Definition 1: A matching state $(\Psi, C(\Psi))$ is stable if it is individually rational and not blocked by any its chain. It is strongly stable if it is individually rational and not blocked by any its chain or cycle.

Example 1 There are 3 workers w_0, w_1, w_2 and 2 firms f_1, f_2. There is no more than one potential contract between each pair of worker and firm. The preferences of each agent are given by:

$$\begin{array}{llll} \succ_{w_0}: & f_1, & w_0 & \\ \succ_{w_1}: & f_2, & f_1, & w_1 \\ \succ_{w_2}: & f_1, & f_2, & w_2 \end{array} \qquad \begin{array}{lllll} \succ_{f_1}: & w_0, & w_2, & w_1, & f_1 \\ \succ_{f_2}: & w_1, & w_2, & f_2 & \end{array}$$

Let $\mu = \begin{pmatrix} w_0, & w_1, & w_2 \\ w_0, & f_1, & f_2 \end{pmatrix}$ and $C(\mu) = \{w_1, w_2, f_1, f_2\}$.

Then, $(\mu, C(\mu))$ is stable, but is not strongly stable.
If $C'(\mu) = \{w_1, w_2, f_1\}$, then $(\mu, C'(\mu))$ is not stable.



The Matching Model

- Let $(\Psi^0, C(\Psi^0))$ denote the initial matching state of the market.
- This is a senior level market. Any agent $x = \mu_{\Psi^0}(x) = \mu^0(x)$ is an entrant and any agent $x \neq \mu^0(x)$ is an incumbent. Any agent in $C^0 = C(\Psi^0)$ is committed and any agent in $V^0 = V(\Psi^0) = (W \cup F) \setminus C^0)$ is free.
  Assumption: $x \succ_x \Psi^0(x) \Longrightarrow \mu^0(x) \in V^0$.
- We use $\mathcal{M} = (W, F, \Psi^0, C(\Psi^0), \succ)$ to denote our current matching model.
- Two basic questions: Given the model, what can be a desirable and natural outcome in terms of efficiency and stability? and how can such an outcome be reached?

Core

- W.r.t. our model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, a contract $\alpha \in \Sigma(x)$ is relatively acceptable to an agent $x \in W \cup F$ if $\alpha \succeq_x \Psi^0(x)$.
- Given a matching Ψ, an agent x is rematched if her current contract $\Psi(x)$ is different from her initial contract $\Psi^0(x)$ even if x has the same partner i.e., $\mu_\Psi(x) = \mu^0(x)$.
- A matching Ψ is feasible, if $\Psi(x)$ is acceptable or relatively acceptable for every agent $x \in W \cup F$, and is further acceptable for any $x \in V^0$, both acceptable and relatively acceptable for any x with $\mu^0(x) \in C^0$.

  "Feasibility" = "without violating the commitments"
- Observe that a feasible matching Ψ is defined by comparing with the initial matching state (Ψ^0, C^0).

Core

- A nonempty subset S of the set $W \cup F$ is a coalition.
- W.r.t. the initial matching state (Ψ^0, C^0), a coalition S is implementable if $x \in S \cap C^0$ implies her partner $\mu^0(x) \in S$.
- A coalition S improves upon a matching Ψ of the grand coalition $W \cup F$ if there exists a matching $\Phi(\subseteq \Sigma(S) = \cup_{x \in S} \Sigma(x))$ among workers and firms from the coalition S alone such that every x in S weakly prefers $\Phi(x)$ to $\Psi(x)$ and at least one agent $y \in S$ prefers $\Phi(y)$ to $\Psi(y)$.

Core

Definition 2: W.r.t. the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, a feasible matching Ψ is in the strict core and is called a strict core matching if it cannot be improved upon by any implementable coalition for the initial state (Ψ^0, C^0).

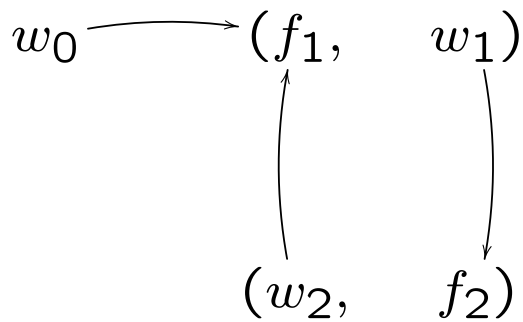
- Note that the set of all agents in a chain or a cycle of the initial state $(\mu^0, C(\mu^0))$ is an implementable coalition.

- A feasible matching Ψ is improved upon by a chain or cycle $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ of the initial state (Ψ^0, C^0) if for each $k = 1, \cdots, K$, there is a contract $\alpha_k \in \Sigma(x_k) \cap \Sigma(y_k)$ such that $\alpha_k \succeq_{x_k} \Psi(x_k)$ and $\alpha_k \succeq_{y_k} \Psi(y_k)$, and for some $k = 1, \cdots, K$, such that $\alpha_k \succ_{x_k} \Psi(x_k)$ or $\alpha_k \succ_{y_k} \Psi(y_k)$.

Lemma 1: If a feasible matching Ψ is improved upon by an implementable coalition S, then it must be improved upon by a chain or by a cycle of the initial state (μ^0, C^0).

Lemma 2: A feasible matching Ψ is in the strict core if it cannot be improved upon by any chain or any cycle of (μ^0, C^0).

Example 2: In the setting of Example 1,



The initial matching state (μ^0, C^0) is given by:

$$\mu^0 = \begin{pmatrix} w_0, & w_1, & w_2 \\ w_0, & f_1, & f_2 \end{pmatrix} \quad \text{and} \quad C^0 = \{w_1, w_2, f_1, f_2\}.$$

In this example, there is a unique strict core matching μ defined by

$$\mu = \begin{pmatrix} w_0, & w_1, & w_2 \\ w_0, & f_2, & f_1 \end{pmatrix}$$

Main Results and Algorithms

Theorem 1: In the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, there exists at least one stable matching state $(\Psi, C(\Psi))$ with a feasible matching Ψ and a commitment structure $C(\Psi) \subseteq C^0$.

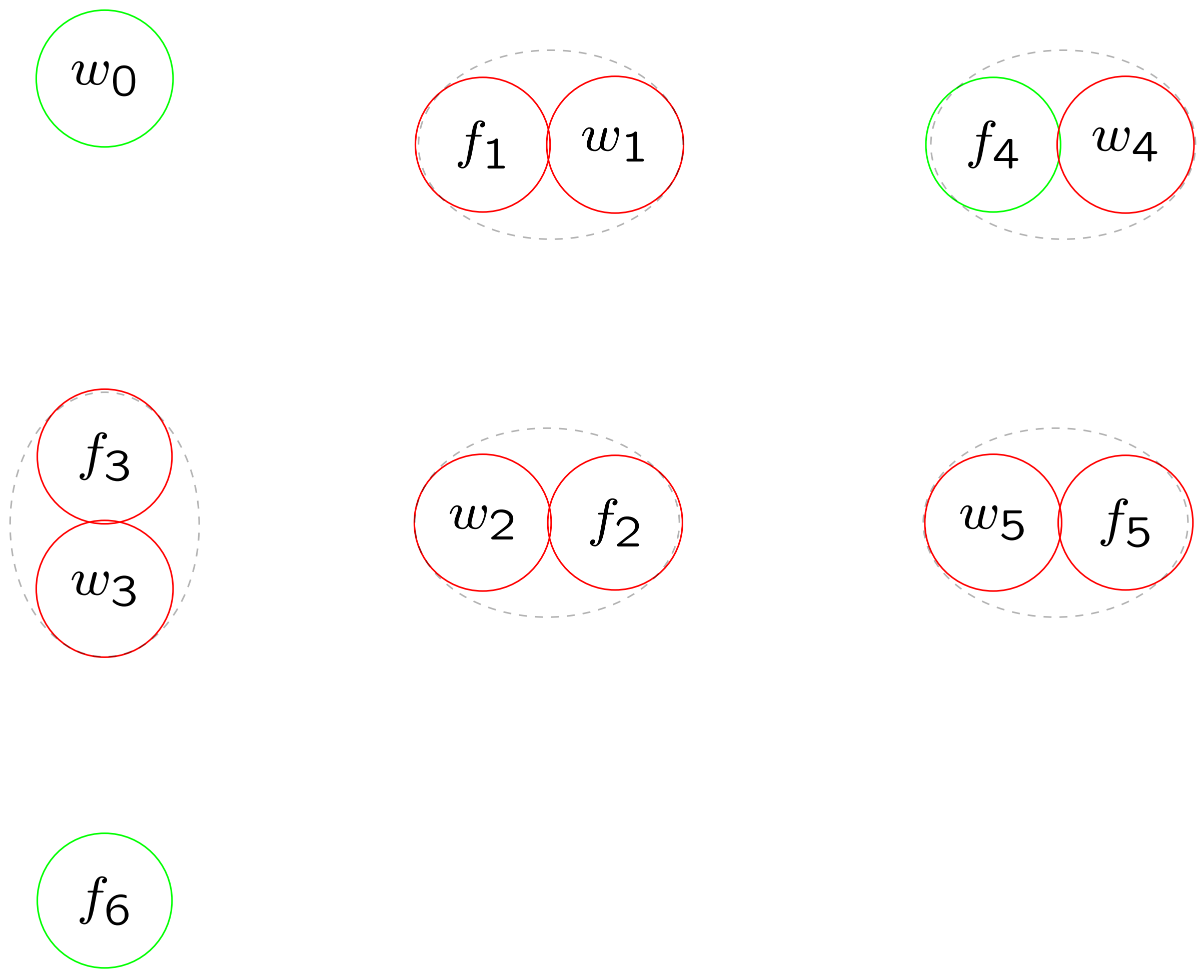
Theorem 2: The strict core of the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$ is not empty.

Theorem 3: In the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, there exists at least one strongly stable matching state $(\Psi, C(\Psi))$ such that Ψ is a strict core matching and $C(\Psi) \subseteq C^0$.

The Hybrid Procedure

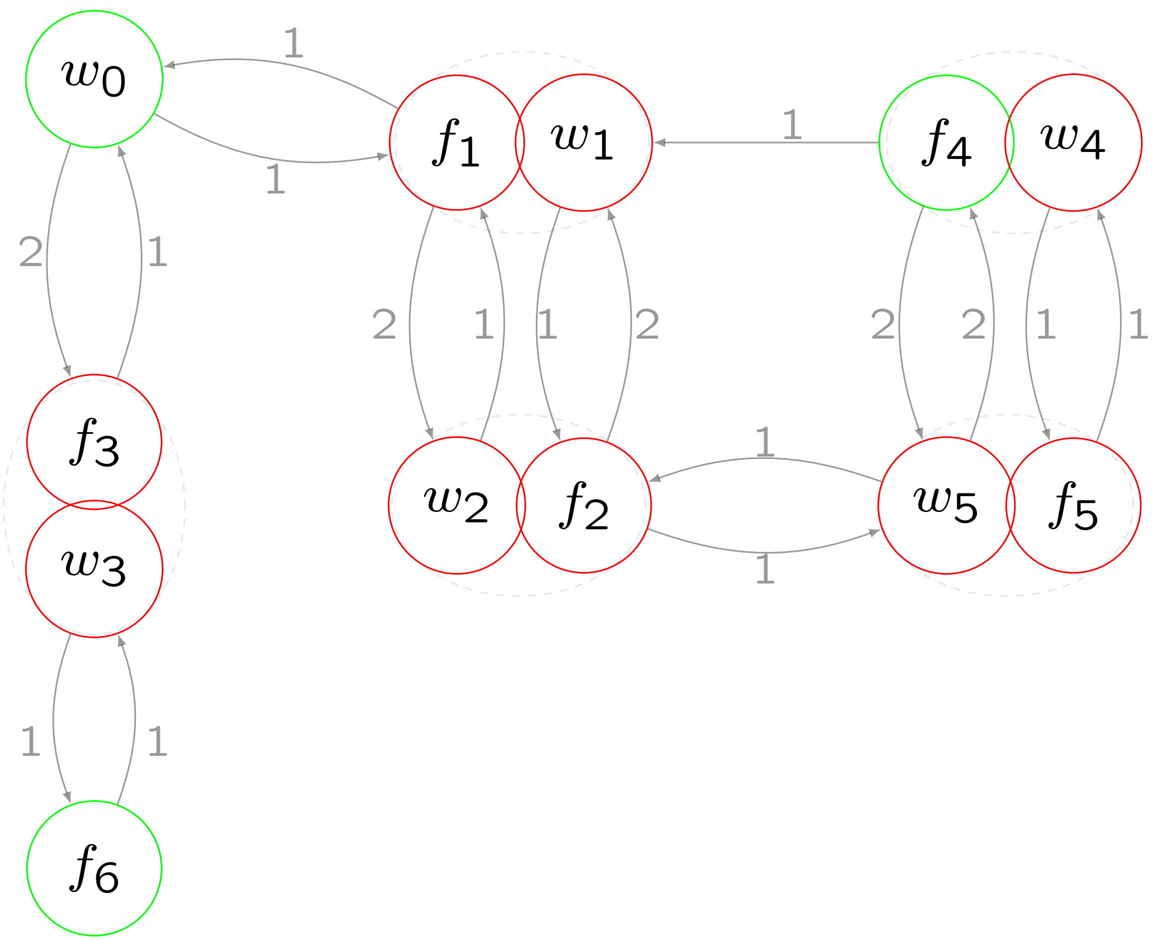
Provide a Hybrid Procedure (Generalized DA + Generalized TTC) for finding a strongly stable matching state with a core matching, and relaxing unnecessary commitments as many as possible.

An Example: The Hybrid Procedure = DA + TTC

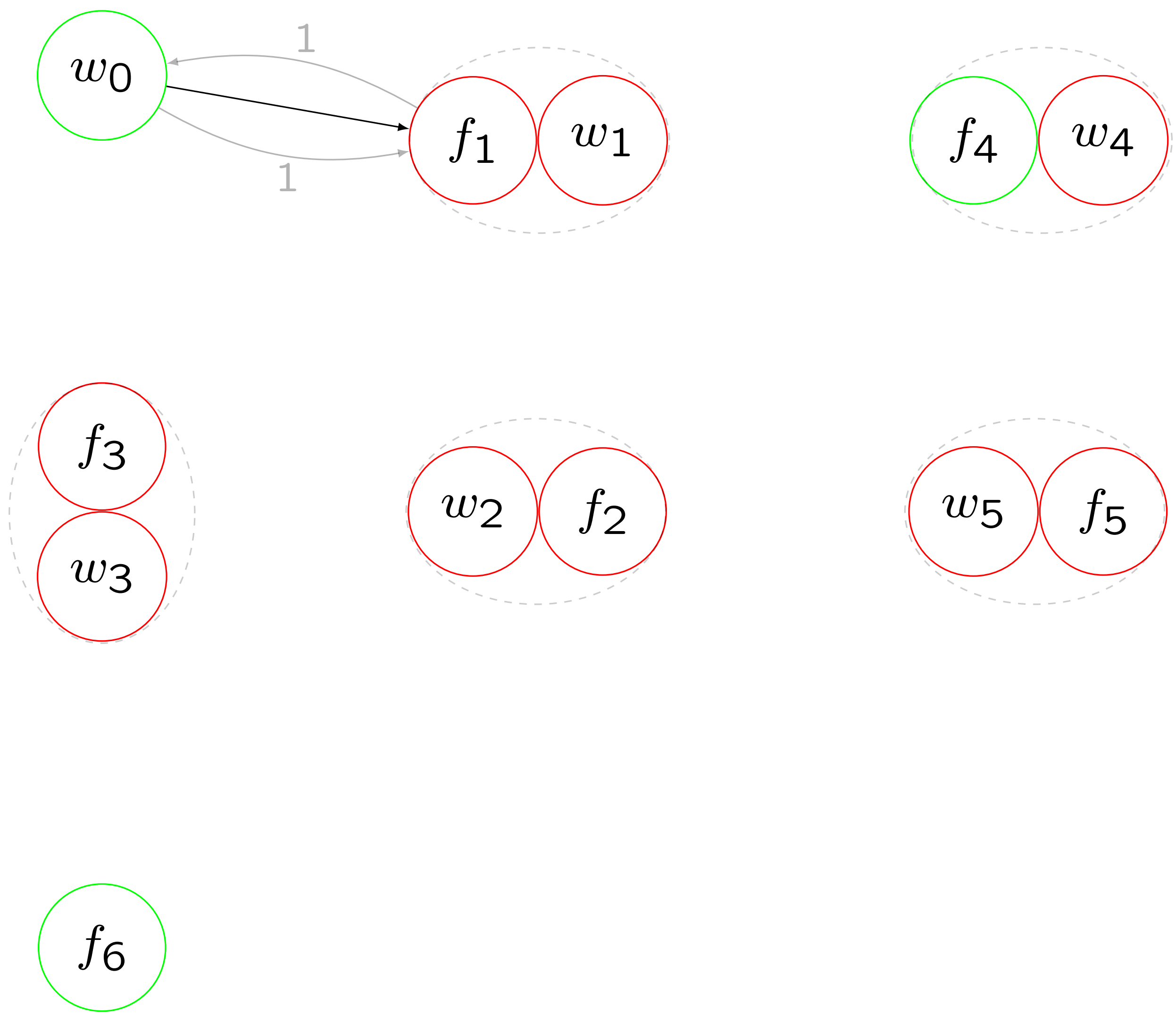


An Example: The Hybrid Procedure = DA + TTC

Agents' preferences: (no more than one contract between each pair of worker and firm)



An Example: WP-DA Procedure



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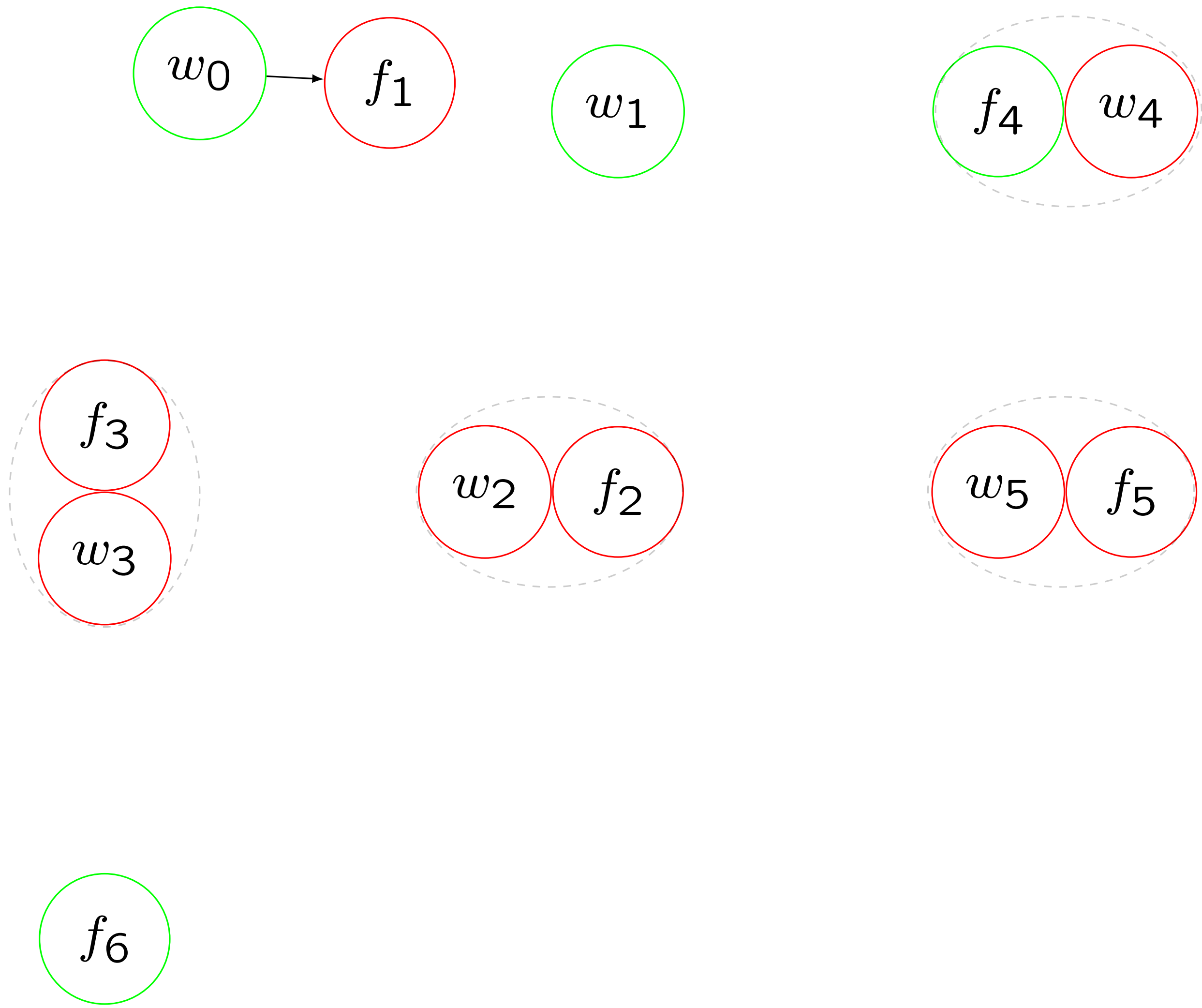

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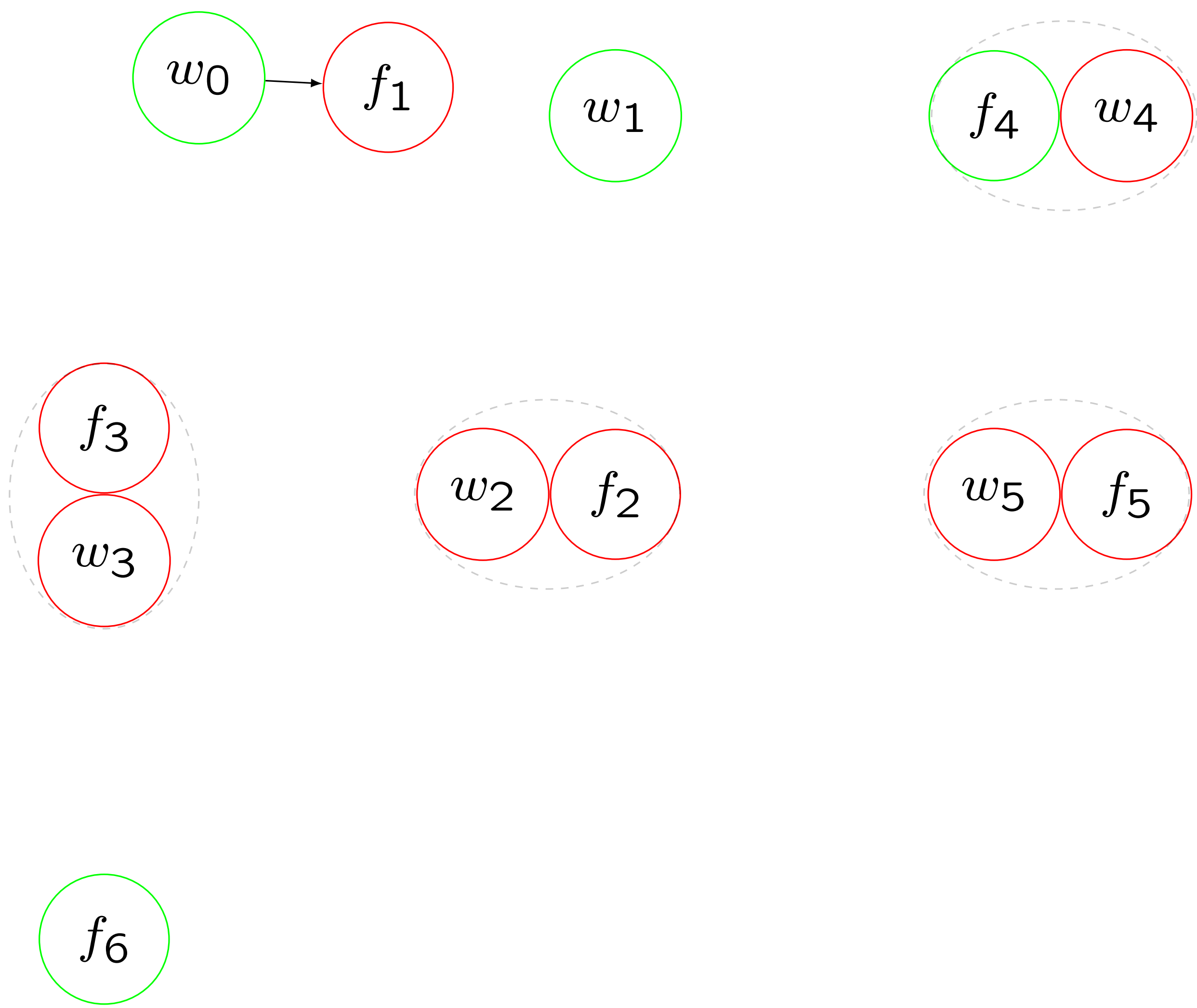

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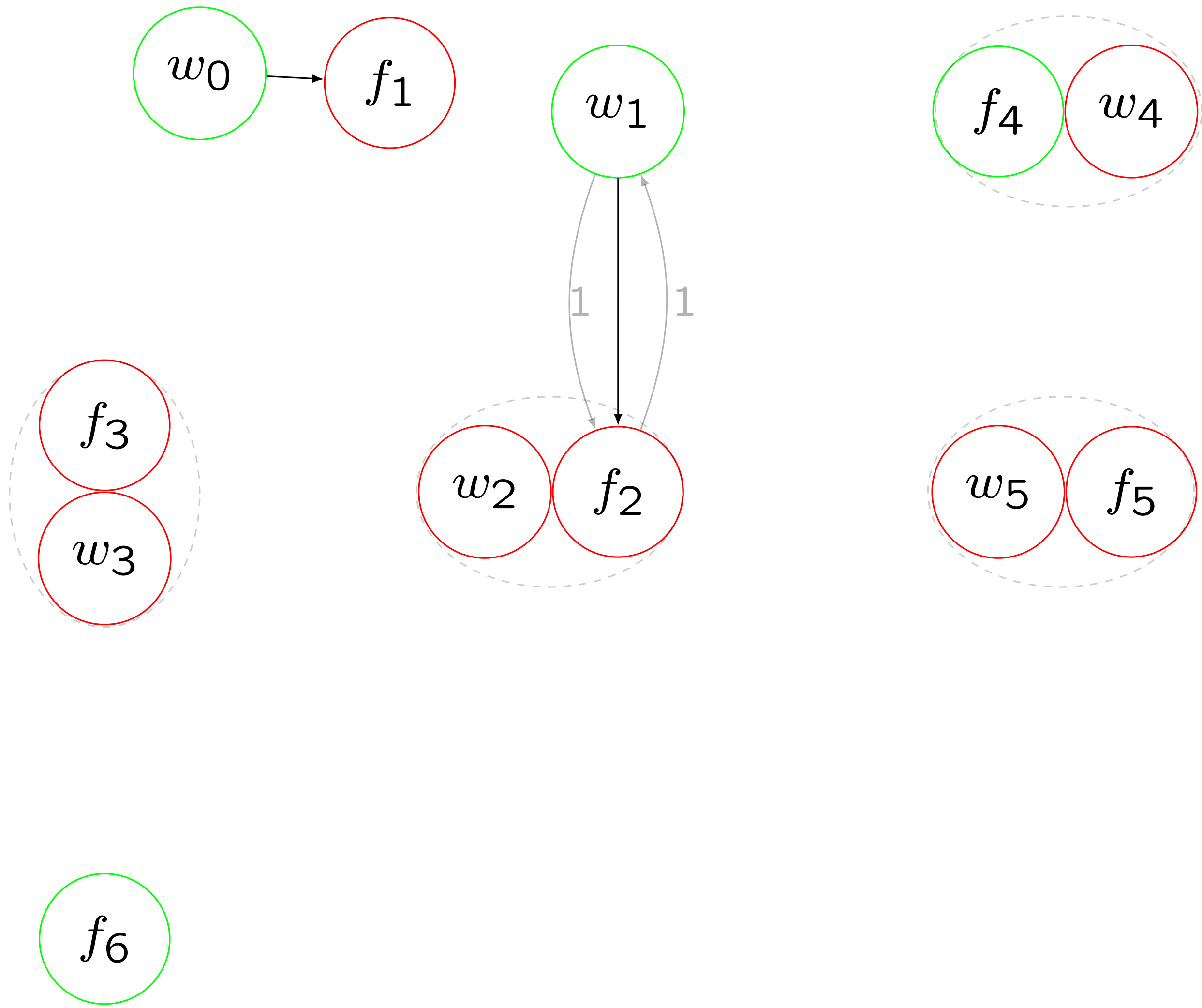

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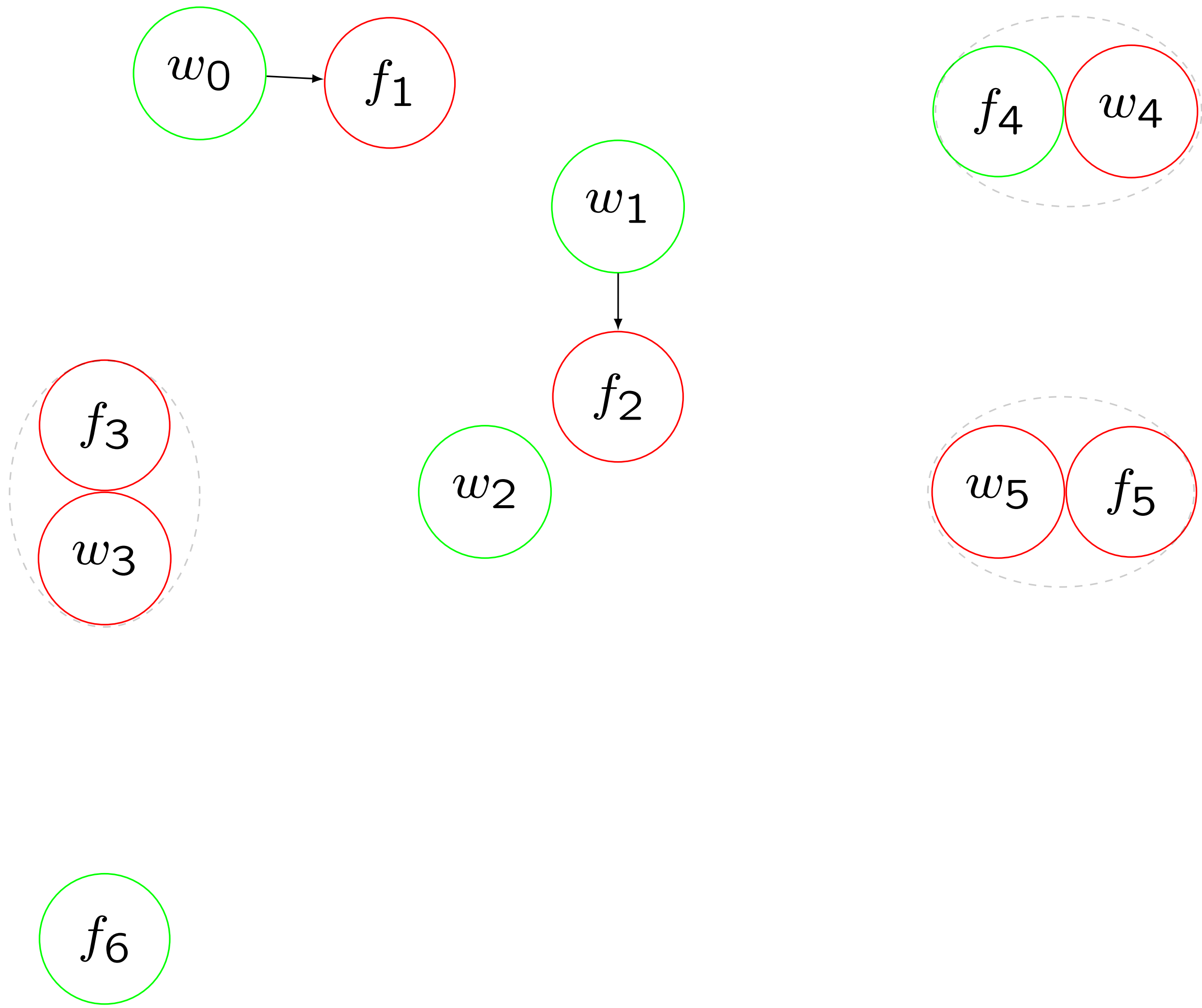

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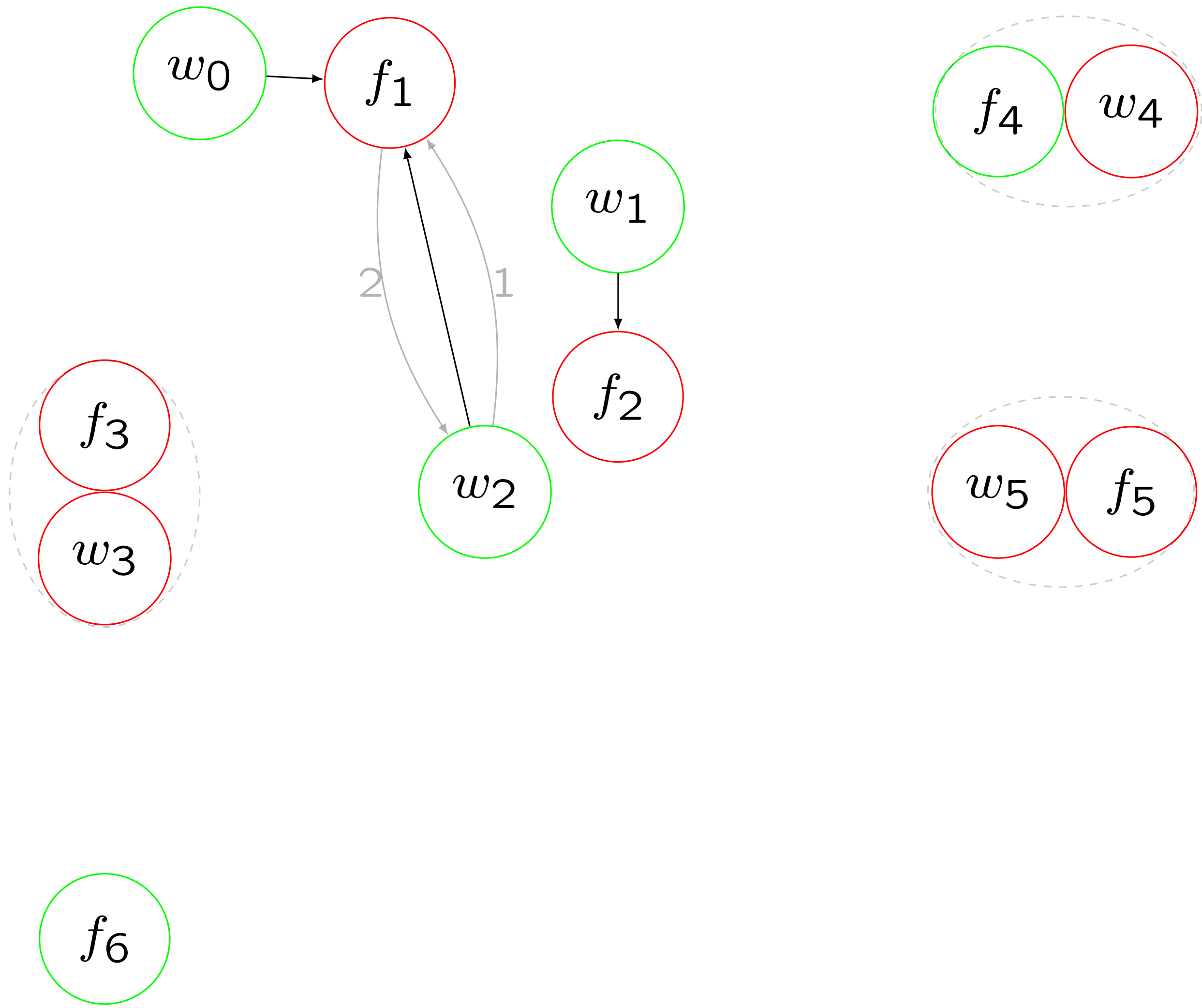

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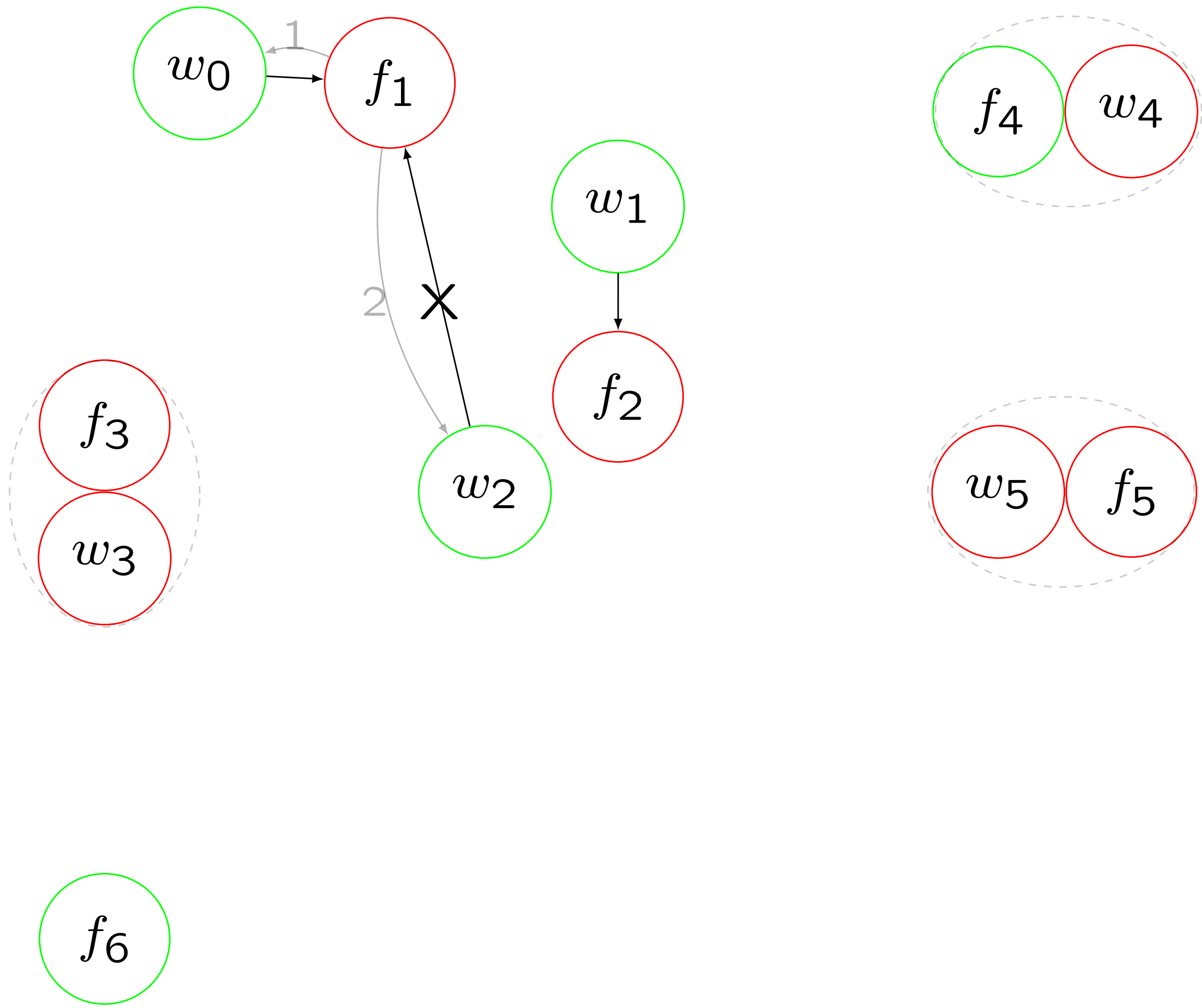

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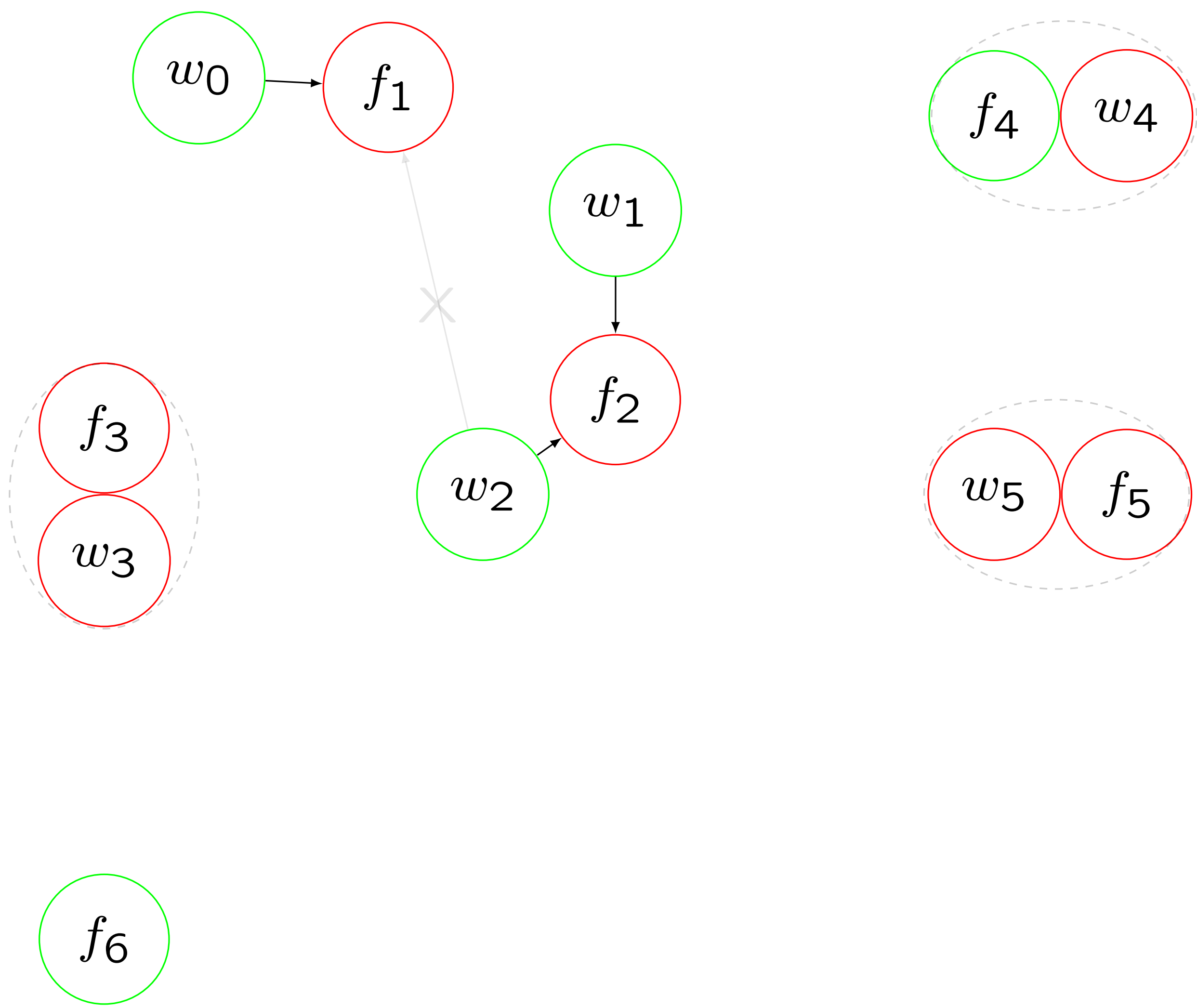

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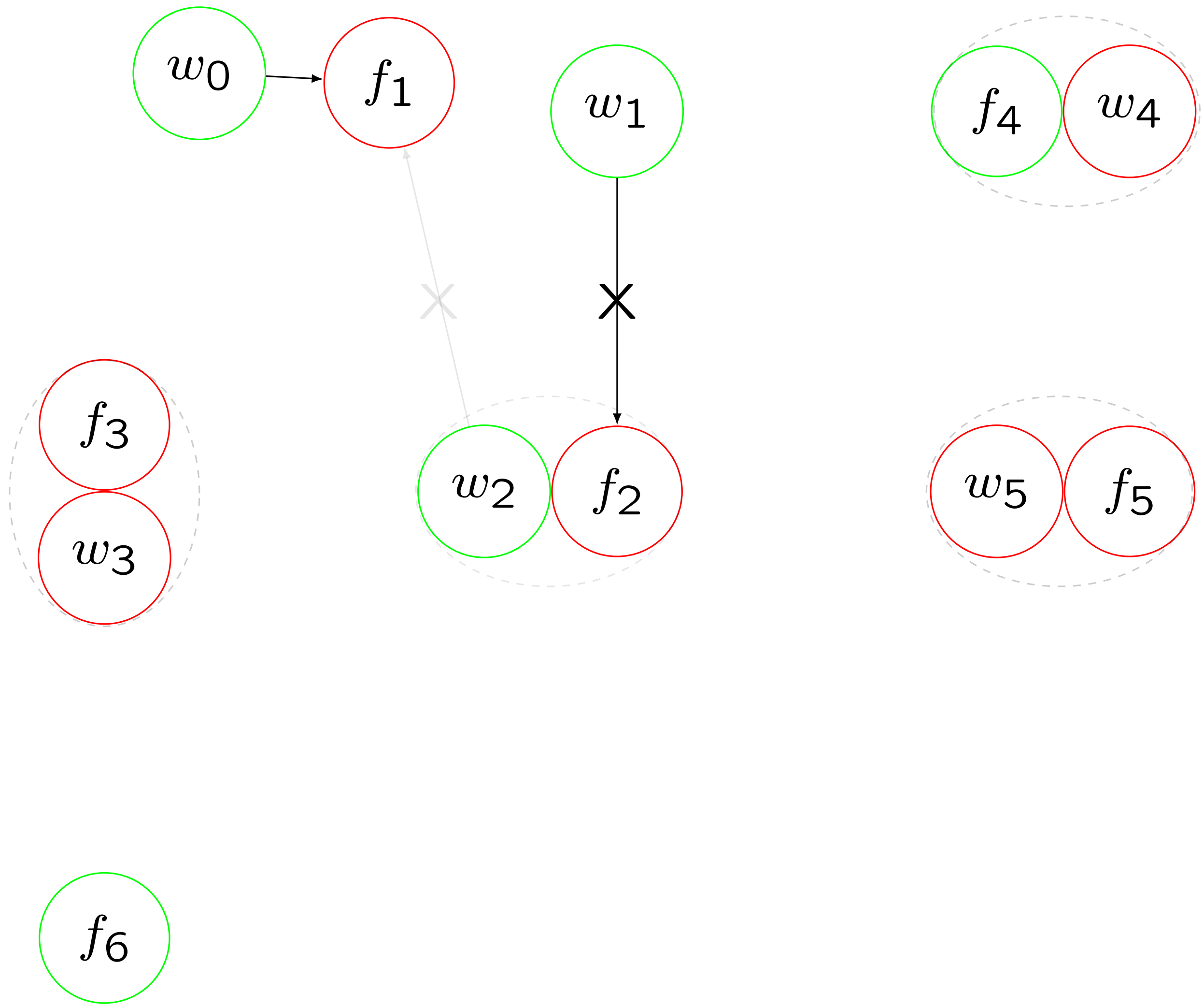

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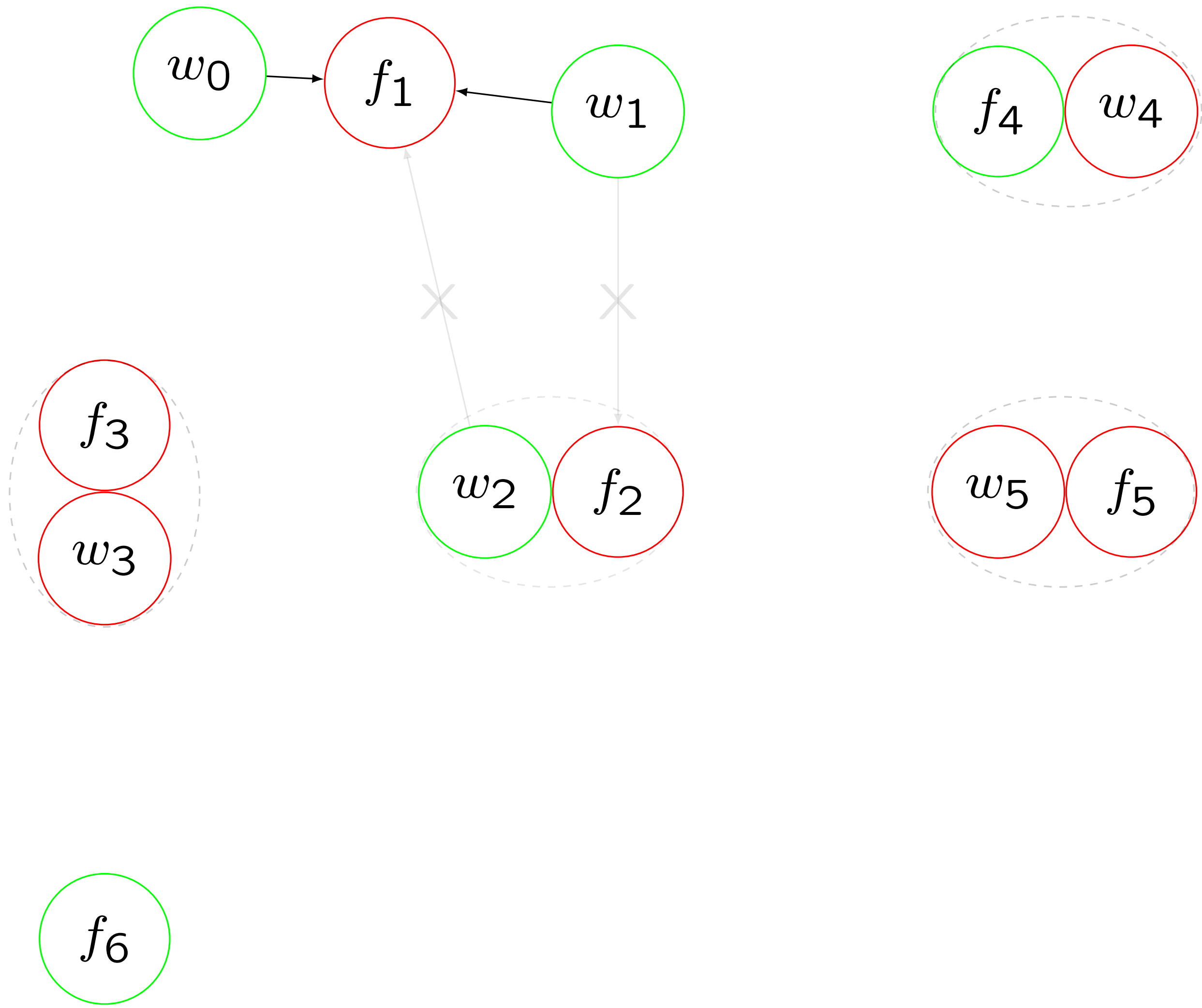

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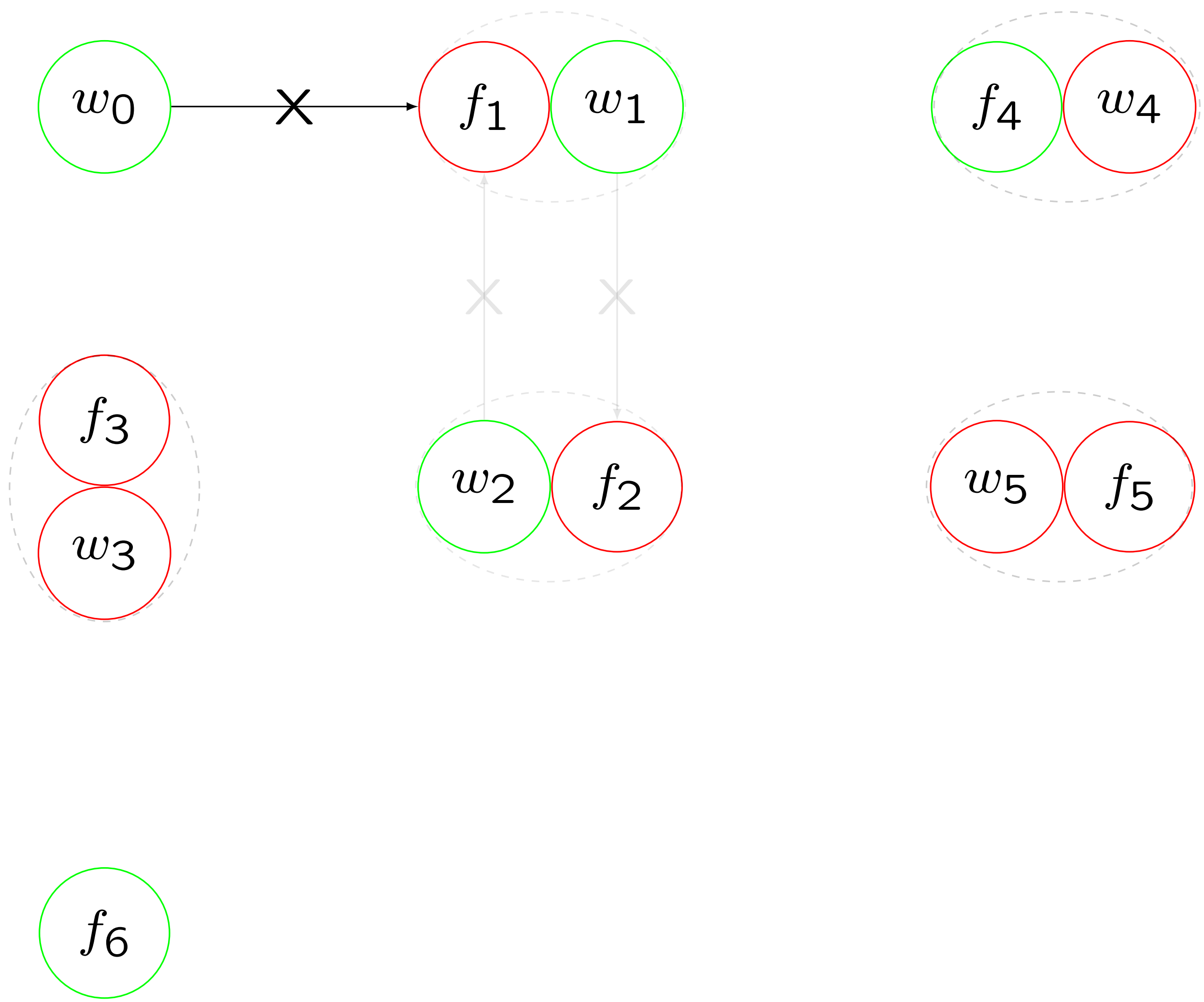

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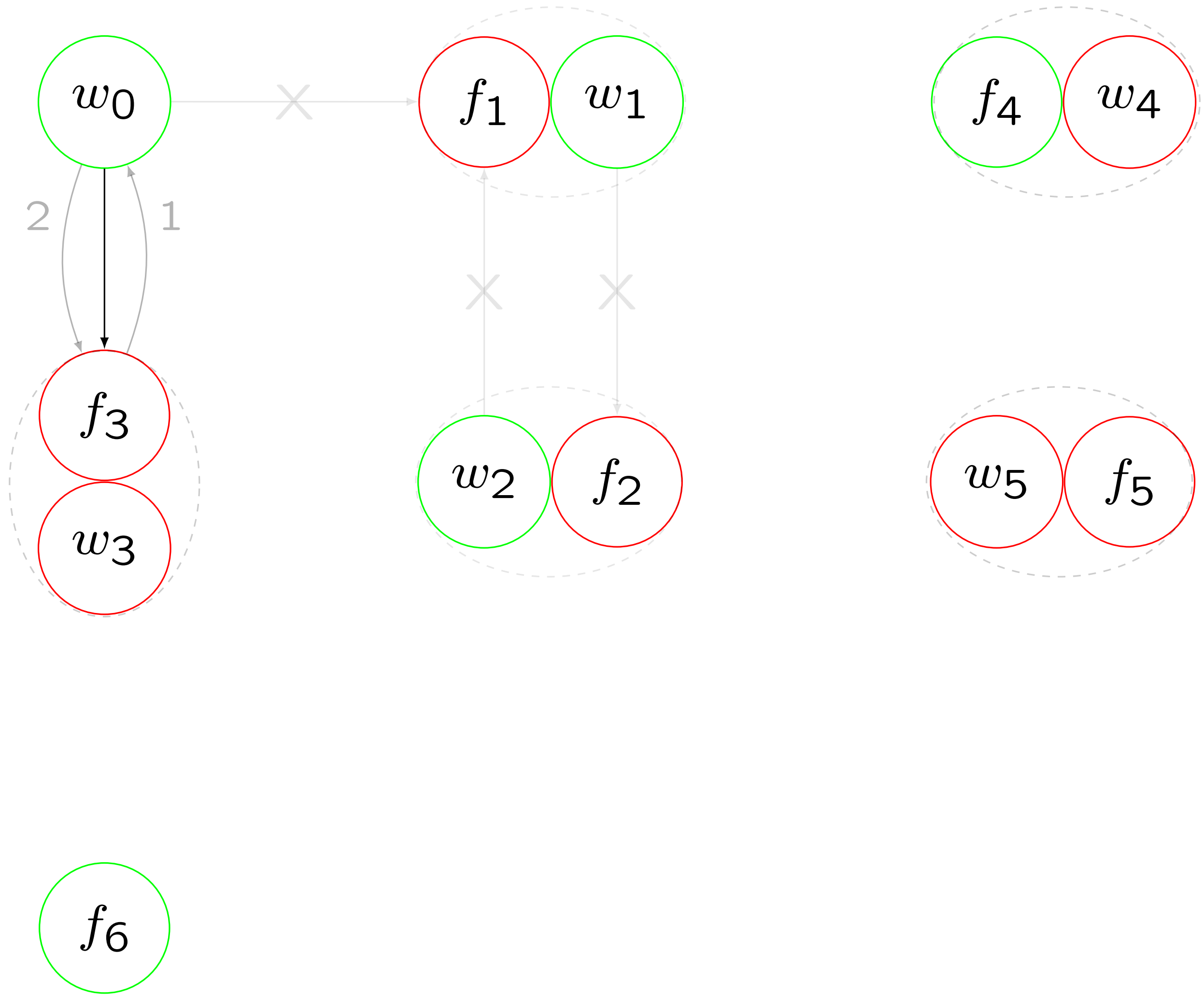

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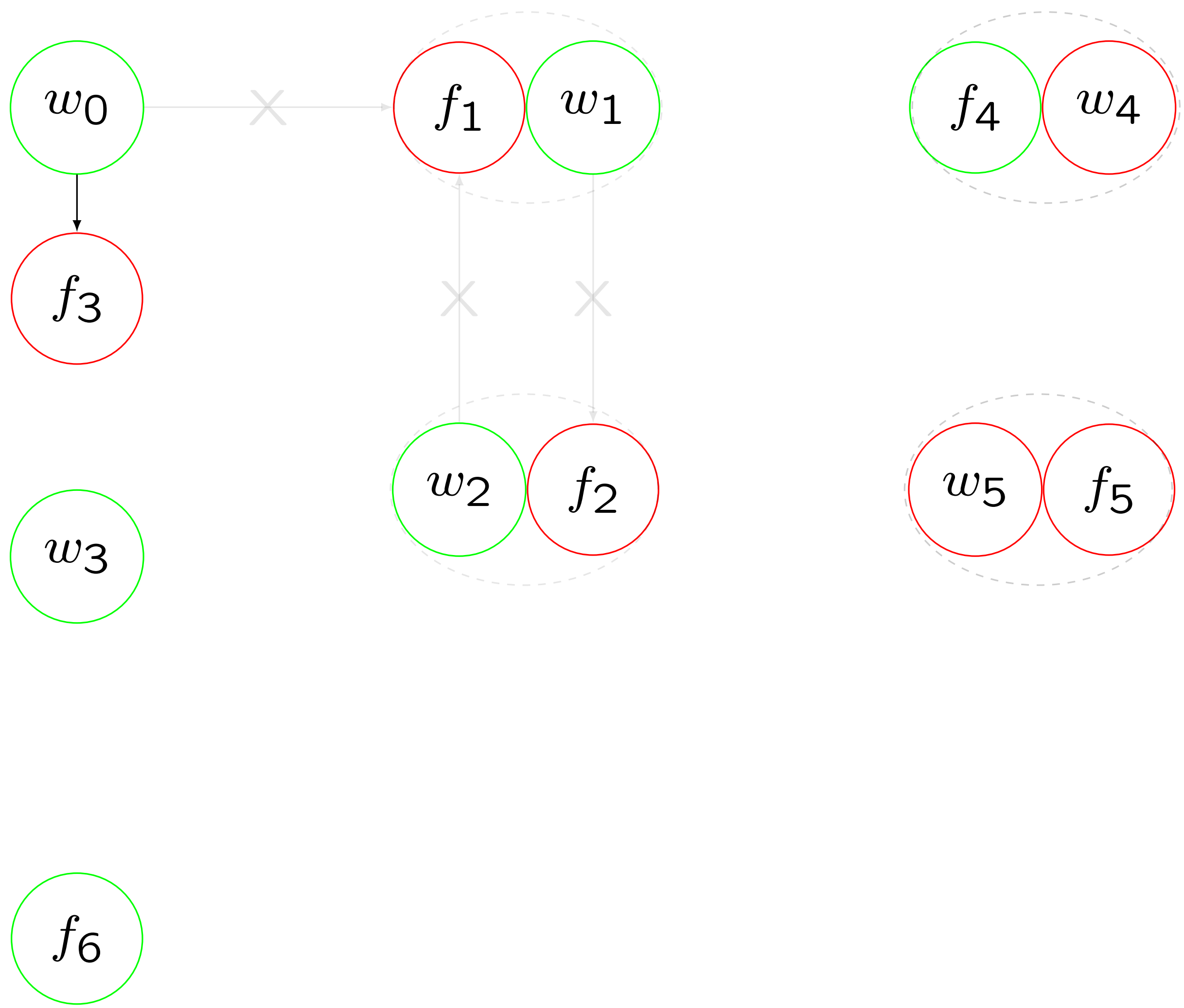

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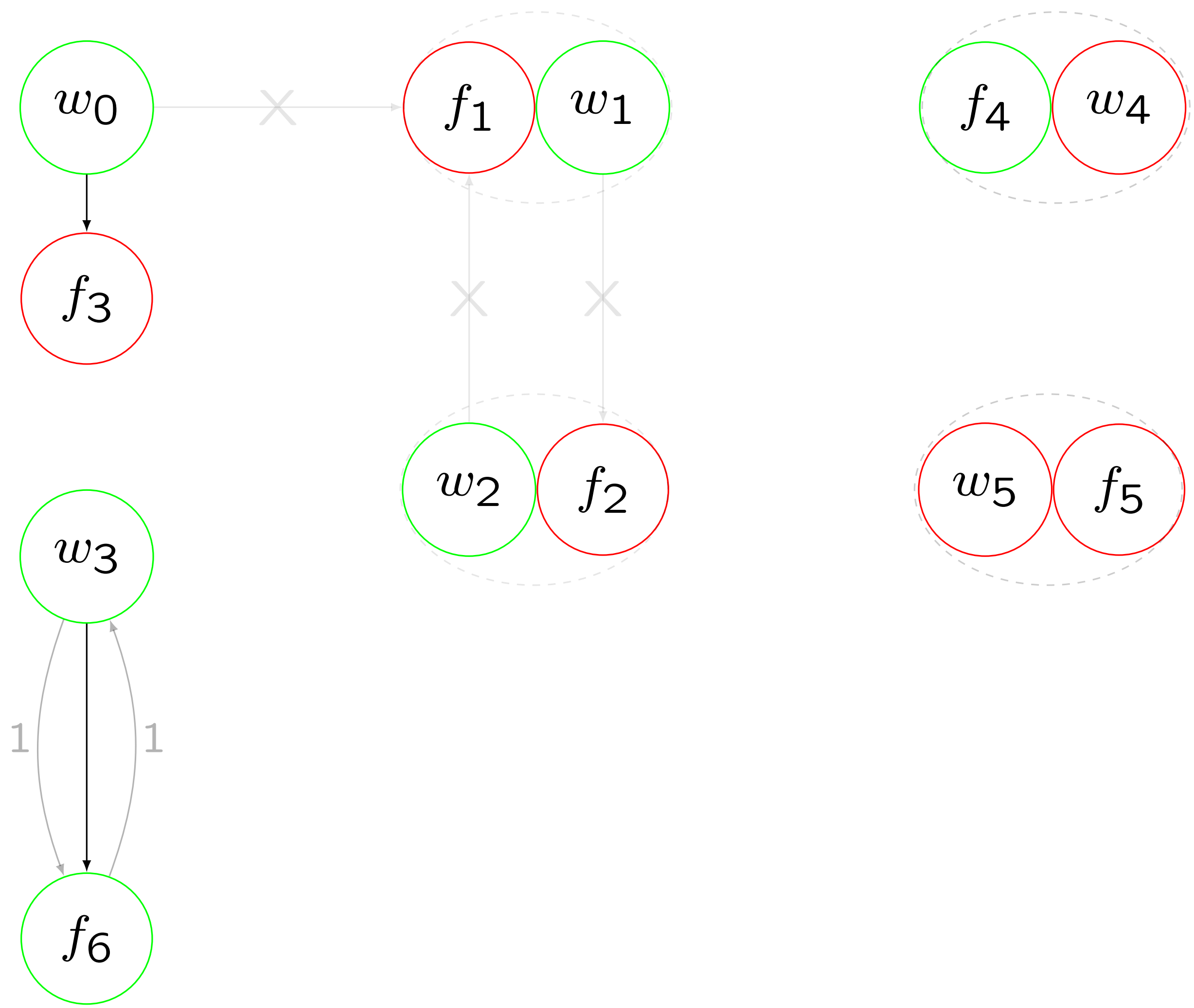

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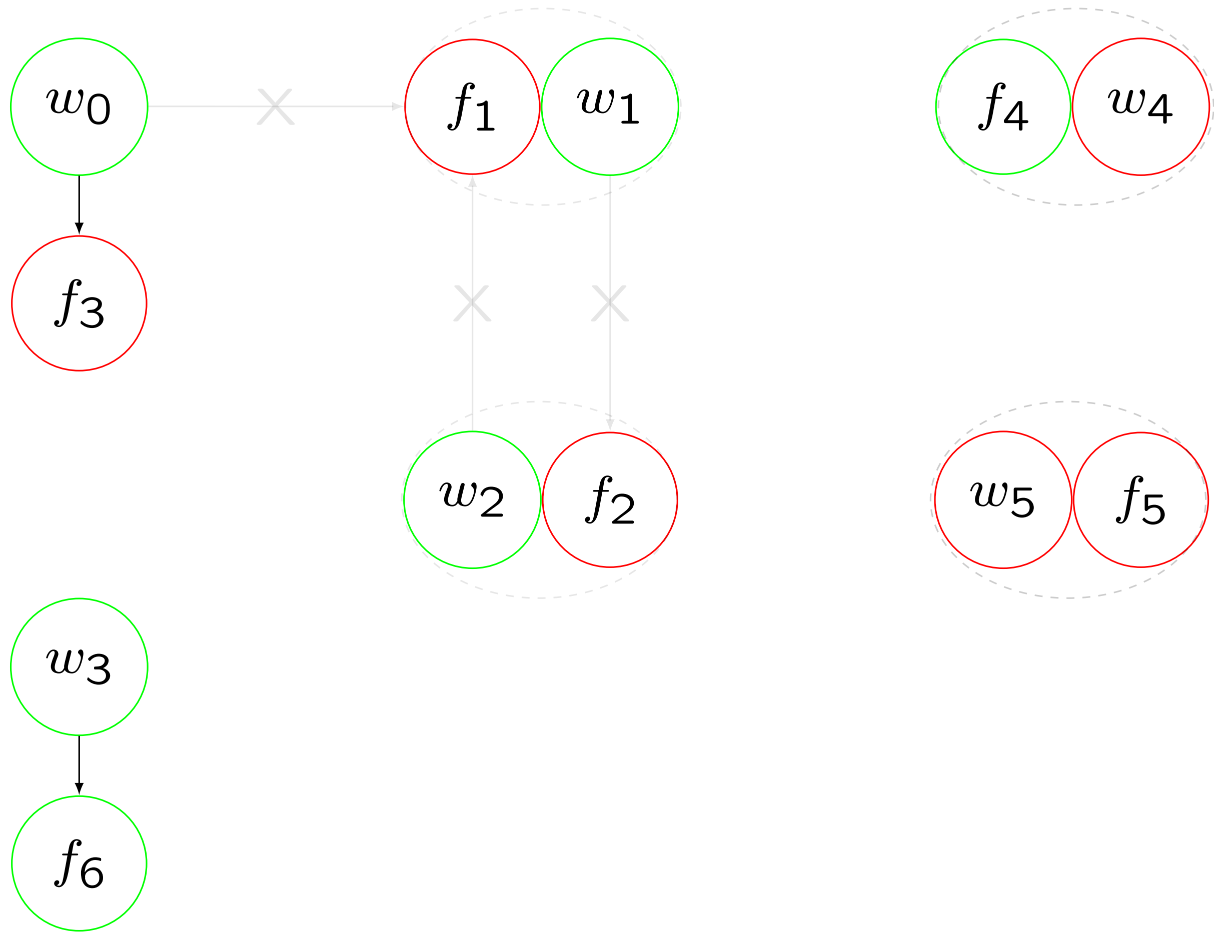

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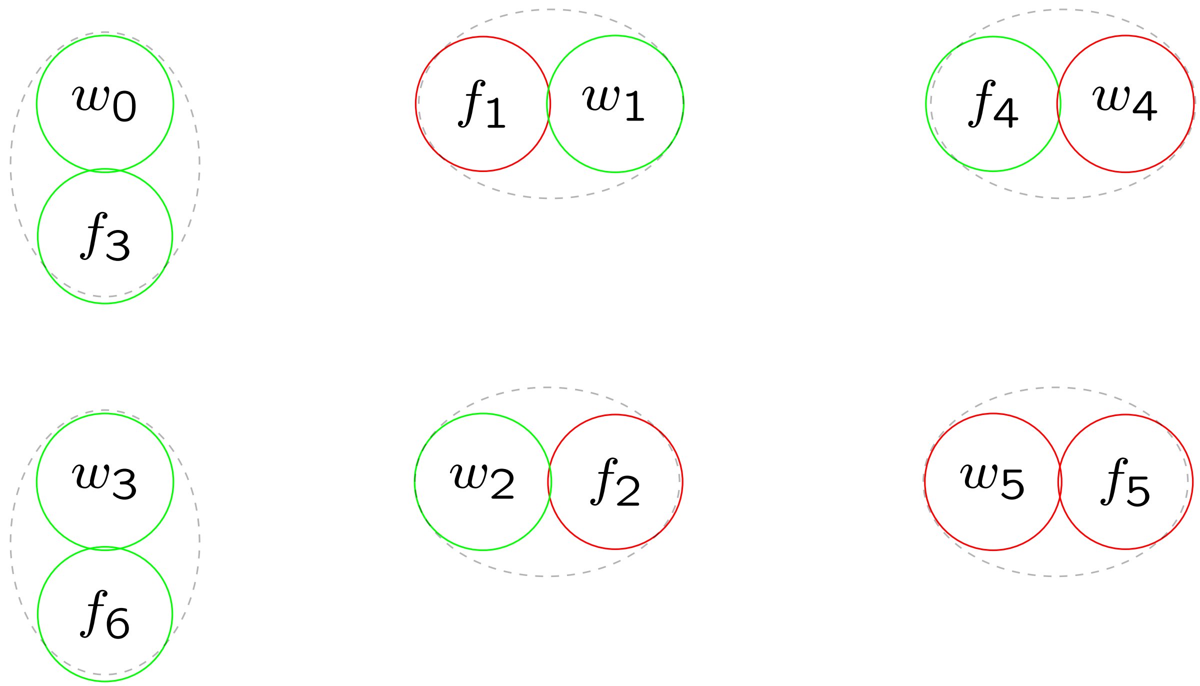
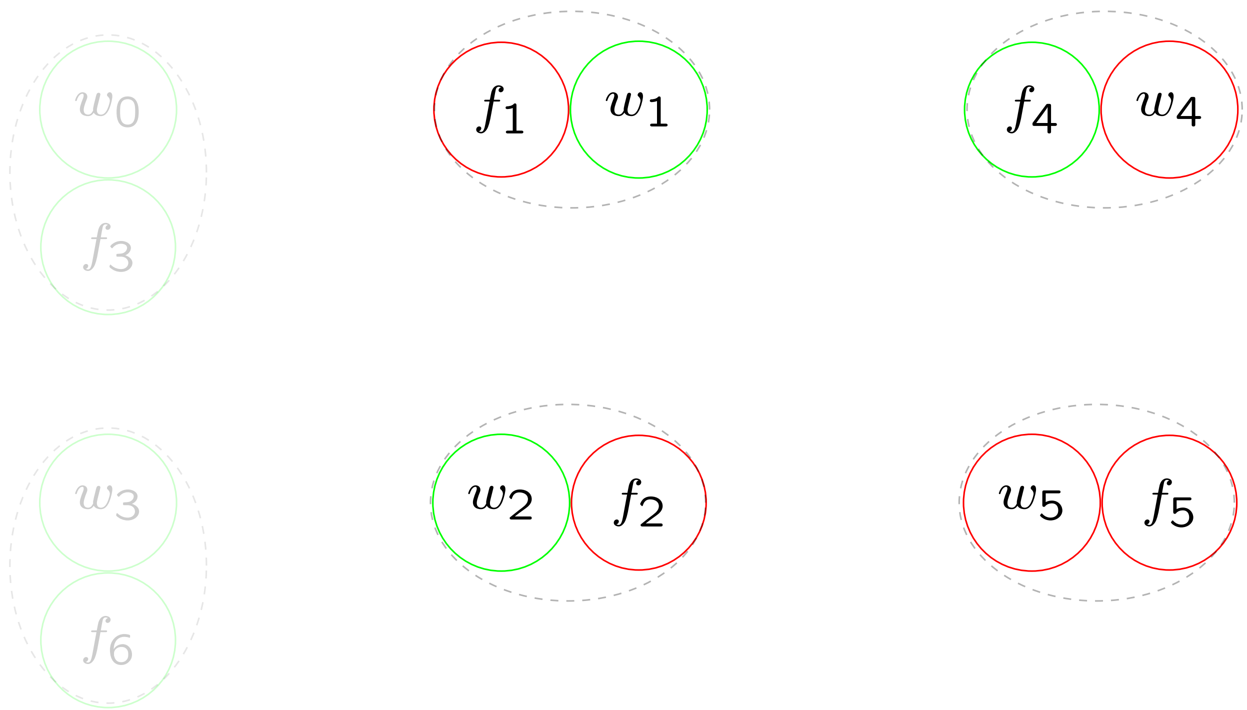


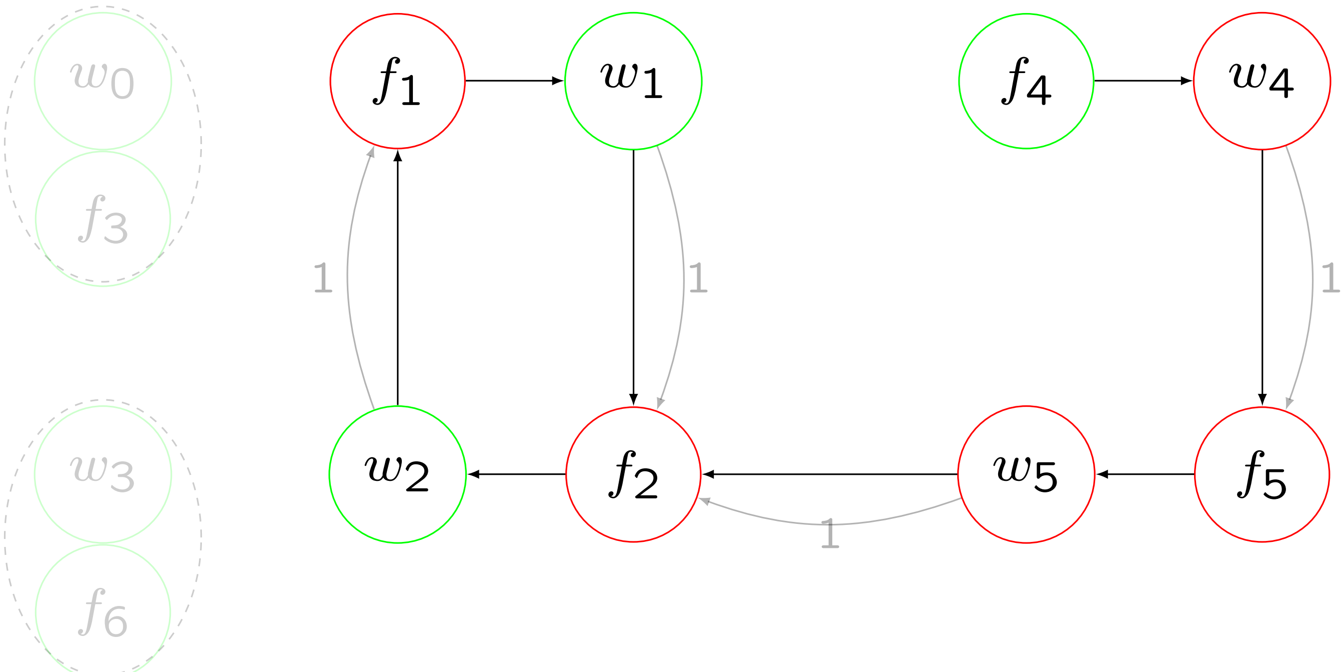
The outcome of WP-DA procedure:

a stable matching state with less commitments

An Example: WP-TTC Procedure



An Example: WP-TTC Procedure



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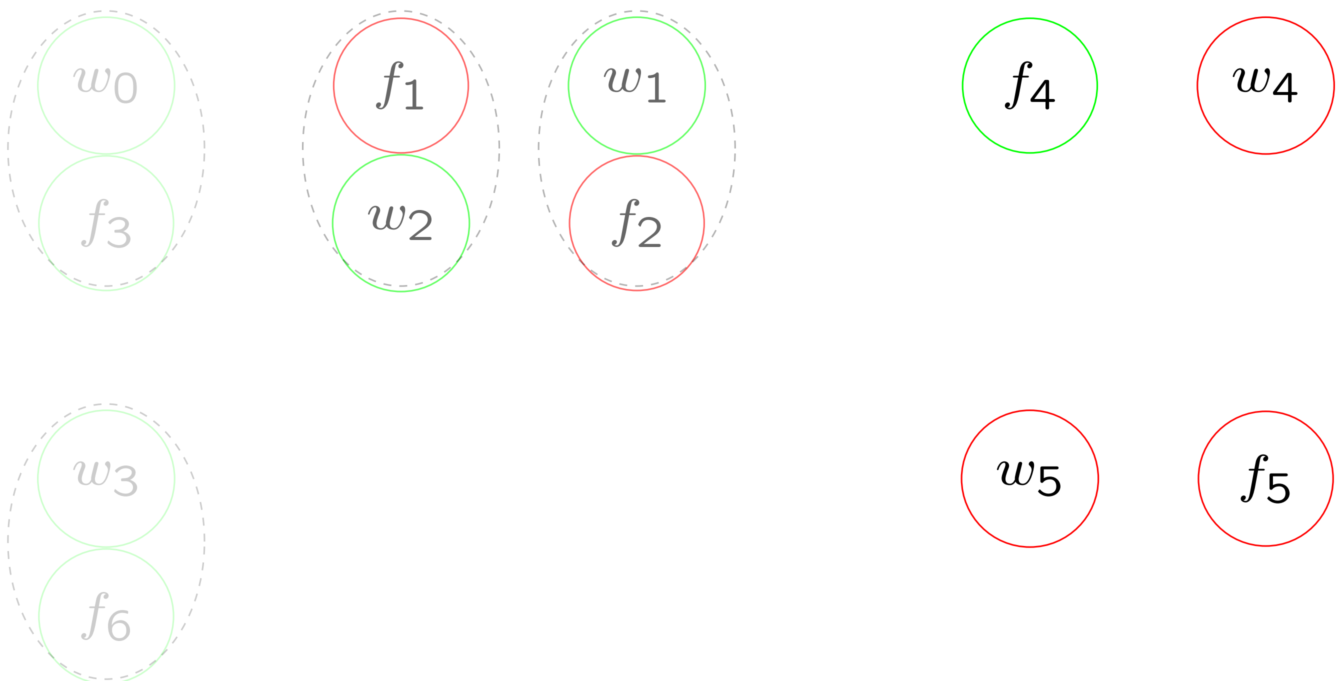

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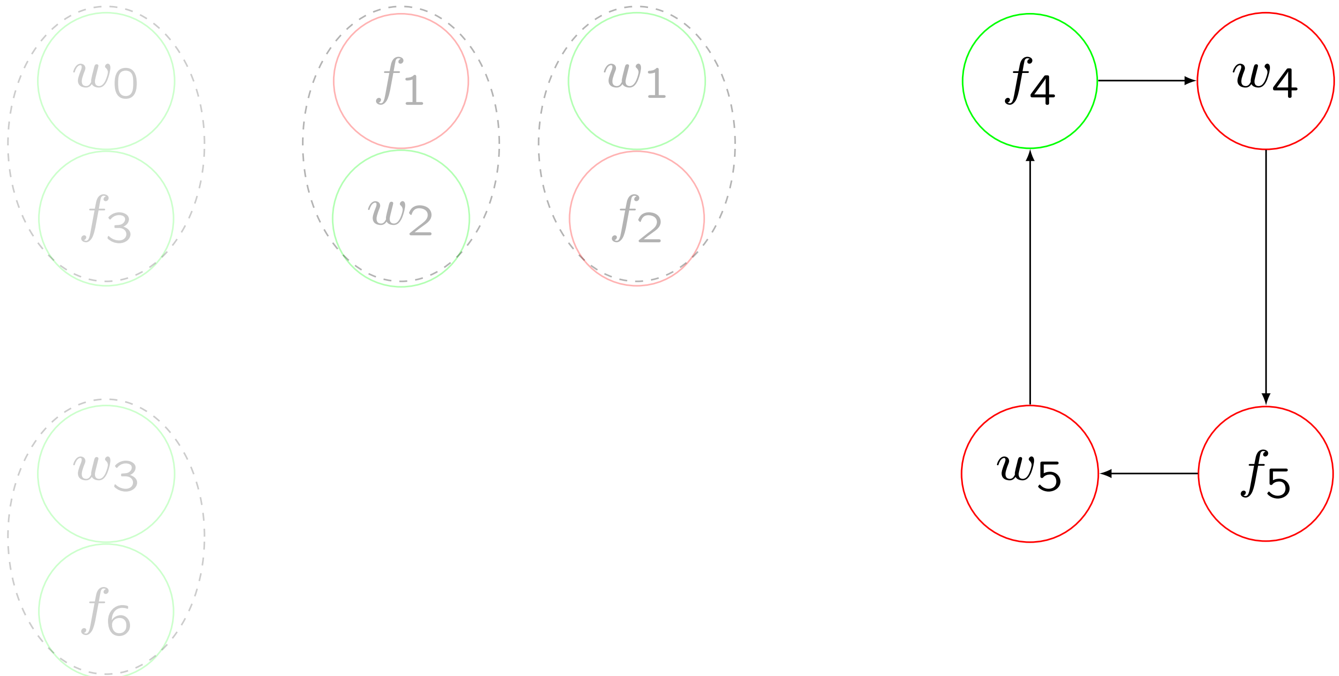

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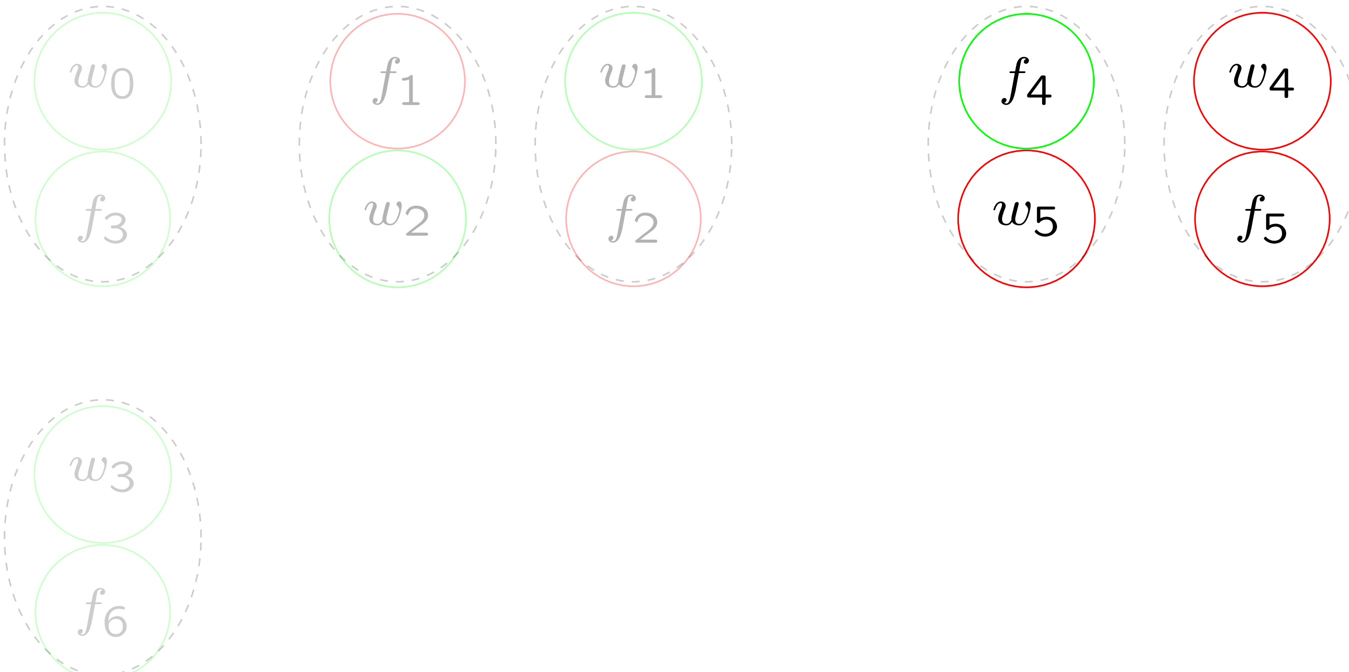

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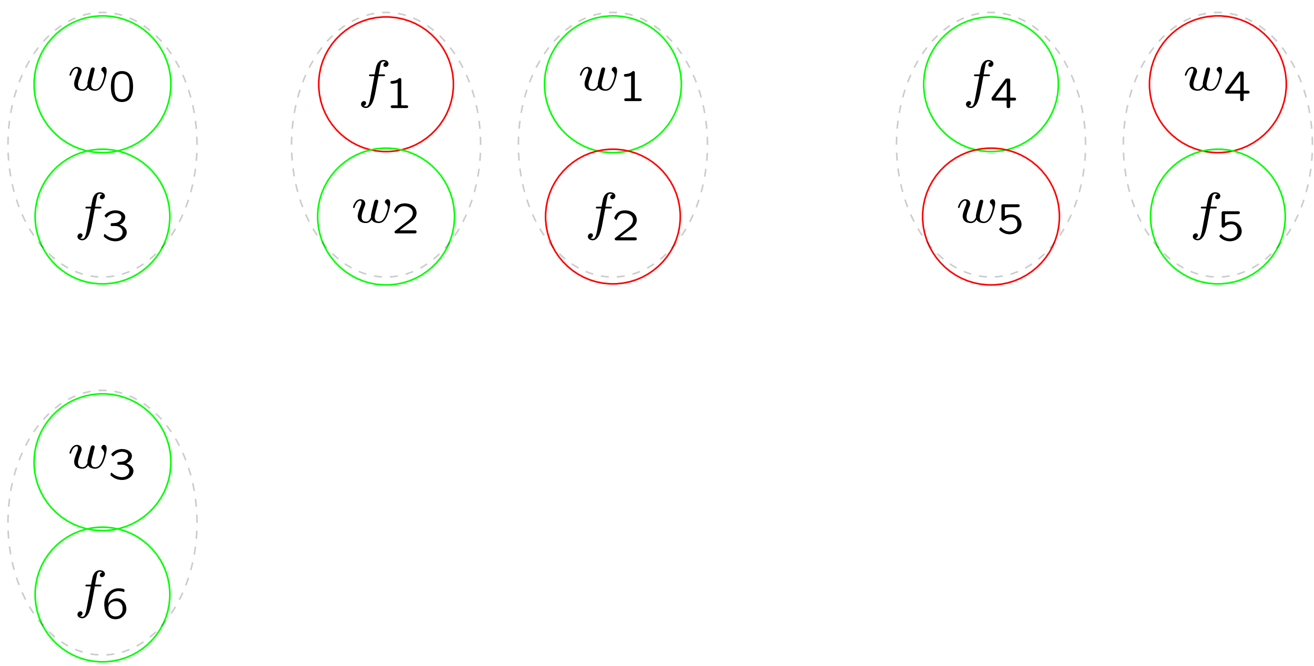

The final outcome: a strongly stable matching state

with a core matching and less commitments

Conclusions

- Introduce commitment into matching model formally, define the concept of matching state and stable matching state.
- Consider a matching model starting from a given initial matching state.
- Define the concept core matching.
- Provide a hybrid procedure for finding a strongly stable matching state with a core matching, and relaxing unnecessary commitments as many as possible.

Thank you!