On Stable and Efficient Mechanisms for Priority-based Allocation Problems

Kang Rong, Qianfeng Tang, and Yongchao Zhang (SUFE)

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- S : the set of schools (objects) to be assigned; each $s \in S$ has q_s seats
- I : the set of students (agents)
- P_i : the strict **preference** of student i on $S \cup \{\emptyset\}$
- $\mu : I \to S \cup \{\emptyset\}$ is an **assignment** if $|\mu^{-1}(s)| \leq q_s, \forall s$

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A **mechanism** φ maps every reported preference profile P to an assignment $\varphi(P)$

"Good" mechanisms

There are two known classes of **efficient and group strategy-proof (GSP)** mechanisms:

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- Top Trading Cycles (TTC)-Based Mechanisms, include serial dictatorships, priority-based TTC, hierarchical exchange rules, and trading cycles mechanisms.[1] They all follow endow-then-trade and have a recursive structure

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Our questions: What is driving the efficiency and GSP of these DA mechanisms? How are these DA and TTC related?

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Step 1. Each student applies to her most favorite school. Each school tentatively accepts the best students up to its quota and rejects the rest.

Step $k, k \geq 2$. Each rejected student applies to her next best school. Each school tentatively accepts the best from the accepted students and new applicants.

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$DA^{\succ,q}$ always produces the student-optimal stable assignment, but it is efficient only when $(\succ, q)$ is acyclic (Ergin, 2002)

A simple example

Suppose $S = \{s_1, s_2\}$, $I = \{1, 2, 3\}$, $q_{s_1} = 1$, and $q_{s_2} = 2$. Schools' priority lists and students' top preferences are as follows:

$\succ_{s_1}$	$\succ_{s_2}$
1	3
2	2
3	1
$\boxed{2}$, 3	$\boxed{1}$

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- $\{1, 2\}$ satisfies the following: assigning students in it to their favorite schools is not objectable by student 3, no matter what P_3 is

- We will (iteratively) identify sets like $\{1, 2\}$, then assign and remove them. We show that this process is solvable iff $DA^{\succ, q}$ is efficient

Related literature

(Efficient) DA: Gale and Shapley (1962), Ergin (2002), Kojima and Manea (2010)

TTC-based mechanisms: Shapley and Scarf (1974), Abdulkadiroglu and Sonmez (2003), Papai (2000), Pycia and Unver (2017)

Vary TTC for stability: Kesten (2004), Kesten (2006), Morrill (2015), Abdulkadiroglu et al. (2017)

Assignment criteria

Definition
Assignment ν Pareto dominates μ if $\forall i \in I, \nu(i) R_i \mu(i)$ and $\nu \neq \mu$.
An assignment is **(Pareto) efficient** if it is not Pareto dominated

Definition
φ is **group strategy-proof** if there is no $\emptyset \neq J \subset I$, P, and P'_J, such that $\forall i \in J, \varphi(P'_J, P_{-J})(i) R_i \varphi(P)(i)$ and for some $j \in J, \varphi(P'_J, P_{-J})(j) P_j \varphi(P)(j)$

Definition
At assignment μ, i's **priority at s is violated** if i desires s but someone with lower priority is assigned; μ is **fair** if there is no priority violation

Definition
An assignment μ is **stable** if it is fair and nonwasteful (all desired schools are fully assigned)

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A nonempty set of students $T \subset I$ is a **TFS** if $\forall i \in T$ and her favorite school $s \cup \{\emptyset\}$,

$$r_s(i) - |\{i' \in T : i' \succ_s i, i' \text{ favors } s' \neq s\}| \leq q_s.$$

where $r_s(i)$ denote i's rank at s;

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where $r_s(i)$ denote i's rank at s; or equivalently, if

$$|\{j \notin T : j \succ_s i\}| \leq q_s - |\{i' \in T : i' \text{ favors } s\}|.$$

Example

Suppose $I = \{1, 2, 3, 4, 5\}$, $S = \{s_1, s_2, s_3\}$, $q_{s_1} = 2$, and $q_{s_2} = q_{s_3} = 3$. Assume schools' priority lists and students' top preferences are as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
1	2	5
2	5	3
3	3	1
4	4	4
5	1	2
3, $\boxed{4}$, 5	$\boxed{1}$	$\boxed{2}$

TFS	
T_1	$\{1, 2, 3\}$
T_2	$\{1, 2, 4\}$
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- TTC involves trading among $1, 2$, and 5; its assignment is not fair

TFS vs. TTC

- Both identify students who should be assigned their favorite schools
- Both are decisive: their formations are independent of others' preferences
- TFS is justified by fairness, while TTC by trading (endowments or priorities)
- When $q_s = 1, \forall s$, TFS reduces to the union of TTCs

Non-existence of TFS

Modify the first example by reducing q_{s_2} to one:

$\succ_{s_1}$	$\succ_{s_2}$
1	3
2	1
3	2
2, 3	1

P_1	P_2	P_3
s_2	s_1	s_1
$\emptyset$	s_2	s_2

Even though for the P above, $DA^{\succ,q}(P)$ is an efficient assignment, TFS does not exist at $(\succ, q; P)$

Properties of TFS

Proposition

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(i) implies the existence of a maximal TFS, if any TFS exists; (ii) shows that assigning by TFS has a form of consistency

Finding TFS

For any school choice problem $(\succ, q; P)$, an **(iterative) elimination process** operates as follows:

Step 1. Let each student apply to her favorite school. Then let each school select the best applicants up to its quota; the rest are eliminated

Step $t, t \geq 2$. For each $s \in S$, operate the following: let **all** students who have ever been eliminated apply to s. Then let s select the best applicants among new applicants and accepted students; the rest are eliminated.
Stop when no new students are eliminated

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Proposition

The set of students who survive the elimination process, if nonempty, is the maximal TFS. If it is empty, then no TFS exists

TFS algorithm

For each preference profile P of students, $TFS^{\succ,q}$ operates as follows:

Step 1. If $(\succ, q; P)$ has no TFS, stop. Otherwise, find a TFS, assign and remove it

Step t, $t \geq 2$. Repeat Step 1 on the subproblem that remains

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Definition

A priority structure $(\succ, q)$ is **TFS-solvable** if at any P, $TFS^{\succ,q}$ produces a complete assignment

Main result

Theorem

$(\succ, q)$ *is TFS-solvable if and only if* $DA^{\succ,q}$ *is Pareto efficient*

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Moreover, if $(\succ, q)$ is TFS-solvable, then $TFS^{\succ,q}(\cdot) = DA^{\succ,q}(\cdot)$

Ergin's characterization

Definition (Ergin, 2002)

$(\succ, q)$ is **acyclic** if $\nexists$ distinct schools s_1, s_2 and distinct students i, j, k such that: (i) $i \succ_{s_1} j \succ_{s_1} k \succ_{s_2} i$; and (ii) $\exists$ disjoint $I_{s_1}, I_{s_2} \subset I \backslash \{i, j, k\}$ such that $I_{s_1} \succ_{s_1} j, I_{s_2} \succ_{s_2} i, |I_{s_1}| = q_{s_1} - 1$, and $|I_{s_2}| = q_{s_2} - 1$

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Theorem (Ergin, 2002)

$(\succ, q)$ is acyclic iff $DA^{\succ, q}$ is Pareto efficient (or GSP, or consistent)

Corollary

$(\succ, q)$ is TFS-solvable if and only if it is acyclic

Implications

- When $(\succ, q)$ is acyclic, the TFS-decomposition of $DA^{\succ,q}$ makes its efficiency, GSP, and consistency more intuitive

- TFS reveals that efficient DA mechanisms also have an iterative removal structure, as the TTC-like mechanisms

- Efficient DA vs. other good mechanisms reduces to TFS vs. TTC

Extend TFS?

We do not yet know how to properly extend (weaken) TFS for priority structures that are not acyclic

This relates to finding a maximally stable good mechanism. Attempts include Kesten's ETTC and Morrill's Clinch-and-Trade, but the goal has not been achieved (Abdulkadiroglu et. al, 2017)

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A natural way is to remove TFS if it exists and otherwise remove TTC. However, such a mechanism is not strategy-proof

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	P_1	P_2	P_3	P_4
1	3	4	s_2	s_3	s_1	s_1
2	2	3	$\vdots$	s_1	$\vdots$	s_3
3	1	2		s_2		s_2
4	4	1				
3, 4	1	2				

Conclusion

- We discover a hidden structure, TFS, from existing DA mechanisms; it is the only known alternative to TTC
- TFS improves our understanding of efficient and GSP mechanisms
- When schools have substitutable (or more general) priority, the definition of TFS can be naturally extended
- We don't know how to properly extend the TFS algorithm to cyclic priority structures to design new mechanisms

Thank you.