The very odd lattice of cumulative distributions

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Outline of talk

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- Decomposable sets
- The odd property
- The open question
- The Tourky-Meneghel theorem
- Outline of proof of Tourky-Meneghel
- Application: a fixed point theorem
- Application to games.

The lattice of cumulative distributions

Let C be a compact subset of $[0,1]$ that contains $0,1$. Let $X(C)$ be the set of cumulative distributions on with support that is a subset of $[0,1]$ and values in C. That is, the set of upper semi-continuous functions f from $[0,1]$ to C satisfying $f(0)=0$ and $f(1)=1$. Viewing it as a subset of L_1 we see the following:

1. As a subset of L_1 it is norm compact.
2. It is a lattice under the usual ordering ordering.
3. If f first order stochastically dominates g, $g \geq f$, then the order interval $\{g \geq h \geq f\}$ is path connected.
4. It is a continuous lattice.

Decomposable sets

Let $f, g\colon [0,1] \to \mathbb{R}$ be measurable functions and $E \subseteq [0,1]$ be a measurable set. We write $f_E g$ for the function

$$f_E g(a) = \begin{cases} f(a) & \text{if } a \in E\,, \\ g(a) & \text{otherwise}\,. \end{cases}$$

A subset D of L_1 is *decomposable* if $f, g \in D$ implies $f_E g \in D$ for all measurable $E \in [0,1]$.

Intesection of $X(C)$ with decomposable stes

If D is a closed decomposable set, and $D \cap X(C)$ is non empty, then $D \cap X(C)$ has the following properties:

1. Norm compact.
2. It is a lattice under the usual ordering ordering.
3. If f first order stochastically dominates g, $g \geq f$, then the order interval $\{g \geq h \geq f\}$ is path connected.
4. It is a continuous lattice.

The odd property

If D is a closed decomposable set, and $D \cap X(C)$ is non empty, then the following holds true:

1. $X(C)$ is a norm compact absolute retract.
2. $D \cap X(C)$ is a norm compact absolute retract.

Once we observe that $D \cap X(C)$ has the lattice properties, there is nothing new in proving this. It has been understood in lattice theory for a very long time.

The Question

Characterise the sets Y in L_1 such that

1. Y is a norm-compact absolute retract.
2. If D is any set that is closed and decomposable and $D \cap Y$ is nonempty, then $D \cap Y$ is an absolute retract.

I'll call such sets "OAR" Absolute Retracts. (OAR stands for OAR Absolute Retracts. :-))

Why are we interested

1. It is interesting.
2. If Y is a "OAR" Absolute Retract and $F\colon Y \to L_1$ is a closed, decomposable valued correspondence such that $F(x) \cap Y$ is nonempty for all $x \in Y$, then F has a fixed point in Y.
3. Decomposable valued mappings arise in many applications.

The Tourky-Meneghel theorem

Theorem: If X is a set that is compact in L_∞, then there is a compact set $Y \subseteq L_1$, with $X \subset Y$, such that Y is an "OAR" Absolute Retract.

The fixed point question

Q: Suppose that D is a closed decomposable subset of L_1 and $F\colon D \to D$ is a non-empty valued decomposable-valued mapping. If there is a (pointwise) compact set $X \subseteq D$ such that $F(f) \cap X \neq \emptyset$ for all $f \in D$, then does there exist f^* satisfying $f^* \in F(f^*)$?

A: Prior literature—yes when D is compact, but that means it is a singleton (obvious)
The Meneghel-Tourky theorem answers this in the affirmative.

A fixed point version of the Meneghel-Tourky theorem

Meneghel-Tourky: If X is a compact set in L_∞, then there is a compact set $Y \subseteq L_1$, $X \subset Y$, satisfying:

- If $F\colon Y \to L_1$ is a closed, decomposable-valued correspondence satisfying $F(f) \cap Y \neq \emptyset$, then F has a fixed point in Y.
- If $F\colon L_1 \to L_1$ is a closed, decomposable-valued correspondence satisfying $F(f) \cap X \neq \emptyset$, then F has a fixed point in Y.

Back to monotone mappings

Let C be a compact subset of $[0, 1]$, X be the set of all constant functions $f : [0, 1] \to C$.

1. X is compact in L_∞.
2. Y of Meneghel-Tourky is the set of monotone functions $g : [0, 1] \to C$.
3. Y is "AOR" Absolute Retract, and X need not be an Absolute Retract.

Outline of proof

Let X be a compact metric space. Let $C \subset [0, 1]$ be the Cantor ternary set. By the Hahn-Mazurkiewicz theorem there exists a continuous function such that $f : C \to X$ is onto.

So set of constant functions $f : [0, 1] \to C$ maps to X. The set of monotone functions $g : [0, 1] \to C$ maps back to the desired $Y \supset X$.