The Reduced Form

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'Optimizing' the allocation of resources.

Parameters (called type) needed to determine an optimal allocation are private knowledge of agents who will consume the resources to be allocated.

Decision variables of optimization problem depends on the *entire* profile of types. If there are n agents and m possible types, this means at least $n \times m^n$ variables!

In a reduced form representation the relevant decision variables depend on type alone and *not* the entire profile of types.

1. Novel.
2. IC constraints not particularly relevant so can focus on reduced form representation.
3. Originally not analyzed using reduced form so one can compare approaches.
4. Not knife edge (see Yunan Li (2017).

Good to be allocated to agent with the highest value (private info).

Each agent would prefer to receive the good than not.

Transfers not permitted.

- n risk neutral agents
- Types are independent draws from $T = \{1, \ldots, m\}$
- $f_t > 0$ is probability that agent is of type t

For a cost $K > 1$, planner can verify an agent's type report.

Planner announces two functions whose argument is the profile of types reported.

Allocation rule: specifies what 'fraction' of the object goes to each agent as a function of profile of reported types.

Inspection rule: specifies probability that an agent will be inspected (conditional on being allocated) as a function of profile of reported types.

For simplicity assume 2 agents.

a is an **allocation rule**

$q_i(t, s)$ is probability good is allocated to agent i when agent 1 reports t and agent 2 reports s.

Feasibility:

$$q_1(t, s) + q_2(t, s) \leq 1 \ \forall t, s$$
$$q_i(t, s) \geq 0 \ \forall i, \ \forall t, s$$

Q_t^i is the **interim allocation probability** to agent i when she reports t.

$$Q_t^1 = \sum_{s \in T} f_s q_1(t, s)$$

$$Q_t^2 = \sum_{t \in T} f_t q_2(t, s)$$

An interim allocation probability Q is **implementable** if there exists a feasible allocation rule that corresponds to it.

Characterize the implementable Q's.

Suppose allocation rule is anonymous, i.e., does not depend on names.

$$Q_t^i = Q_t^j = Q_t$$

Q_t is **implementable** iff.

$$n \sum_{t \in S} f_t Q_t \leq 1 - (\sum_{i \notin S} f_t)^n \ \forall S \subseteq T.$$

$g(S) = 1 - (\sum_{i \notin S} f_t)^n$ is non-decreasing and submodular.

For the non-anonyous case:

$$\sum_{i=1}^{n} \sum_{t_i \in T_i'} Q_i(t_i) f_i(t_i) \leq \sum_{t \in \bigcup_i (T_i' \times T_{-i})} \prod_i f_i(t_i) = 1 - \prod_i \sum_{t_i \in T_i \setminus T_i'} f_i(t_i) \quad \forall \{T_i'\}_i \subseteq \cup_i T_i$$

$$Q_i(t_i) \geq 0 \quad \forall i \quad \forall t_i \in T_i$$

Let $E = \{1, 2, \ldots, n\}$ be a ground set. Real valued function g defined on subsets of E

is

- **non-decreasing** if $S \subseteq T \Rightarrow g(S) \leq g(T)$, and
- g is **submodular** if $\forall S \subset T$ and $j \notin T$:

$$g(T \cup j) - g(T) \leq g(S \cup j) - g(S)$$

Polymatroid:

$$P(g) = \{x \in \Re_+^n : \sum_{j \in S} x_j \le g(S) \quad \forall S \subseteq E\}$$

$$\max\{cx : x \in P(g)\}$$

$$c_1 \ge c_2 \ldots \ge c_k \ge 0 > c_{k+1} \ldots \ge c_n.$$

1. $S^0 = \emptyset$
2. $S^j = \{1, 2, \ldots, j\}$ for all $j \in E$.
3. $x_j = g(S^j) - g(S^{j-1})$ for $1 \le j \le k$
4. $x_j = 0$ for $j \ge k+1$.

So What?

- ▶ For economic applications goal is not merely to solve the optimization problem but identify some of its qualitative properties.

- ▶ Polymatroid optimizations problems are valuable because they admit a simple greedy solution.

- ▶ Allow one to handle certain kinds of additional constraints like budget and quota constraints.

Suppose Q is implementable.

How do we recover corresponding feasible allocation rule a?

Trick from stochastic scheduling. Consider extreme point Q's.

Treat Q_t's as priorities.

In any profile allocate to agent whose type t has highest priority.

Type space of agent 1 is $\{t, t'\}$ and of agent 2 is $\{s, s'\}$.

$$q_1(t, s) + q_2(t, s) \leq 1$$

$$q_1(t', s) + q_2(t', s) \leq 1$$

$$q_1(t', s') + q_2(t', s') \leq 1$$

$$q_1(t, s') + q_2(t, s') \leq 1$$

$$f_s q_1(t, s) + f_{s'} q_1(t, s') = Q_t^1$$

$$f_s q_1(t', s) + f_{s'} q_1(t', s') = Q_{t'}^1$$

$$f_t q_2(t, s) + f_{t'} q_2(t', s) = Q_s^2$$

$$f_t q_2(t, s') + f_{t'} q_2(t', s') = Q_{s'}^2$$

Algebraic: eliminate the $a(\cdot, \cdot)$ variables.

Geometric: determine the projection of the polyhedron into the Q space.

Digression # 1

$K = \{(x, z) \in \Re^n \times \Re^p : Ax + Bz = b\}$.

$Proj_x(K)$ is the set of vectors $x \in \Re^n$ such that there exists $z \in \Re^p$ such that $(x, z) \in K$.

Theorem
Let $C_K = \{u \in \Re^m : uB = 0\}$. Then,

$$proj_x(K) = \{x \in \Re^n : uAx \leq ub \; \forall u \in C_K\}.$$

Not necessary to use every $u \in C_K$, sufficient to use extreme rays/generators of C_K.

Type space of agent 1 is $\{t, t'\}$ and of agent 2 is $\{s, s'\}$.

$$q_1(t,s) + q_2(t,s) \leq 1$$

$$q_1(t',s) + q_2(t',s) \leq 1$$

$$q_1(t',s') + q_2(t',s') \leq 1$$

$$q_1(t,s') + q_2(t,s') \leq 1$$

$$f_s q_1(t,s) + f_{s'} q_1(t,s') = Q_t^1$$

$$f_s q_1(t',s) + f_{s'} q_1(t',s') = Q_{t'}^1$$

$$f_t q_2(t,s) + f_{t'} q_2(t',s) = Q_s^2$$

$$f_t q_2(t,s') + f_{t'} q_2(t',s') = Q_{s'}^2$$

$$f_t f_s q_1(t, s) + f_t f_s q_2(t, s) \leq f_t f_s$$

$$q_1(t', s) + q_2(t', s) \leq 1$$

$$q_1(t', s') + q_2(t', s') \leq 1$$

$$q_1(t, s') + q_2(t, s') \leq 1$$

$$f_s q_1(t, s) + f_{s'} q_1(t, s') = Q_t^1$$

$$f_s q_1(t', s) + f_{s'} q_1(t', s') = Q_{t'}^1$$

$$f_t q_2(t, s) + f_{t'} q_2(t', s) = Q_s^2$$

$$f_t q_2(t, s') + f_{t'} q_2(t', s') = Q_{s'}^2$$

$$f_t f_s q_1(t, s) + f_t f_s q_2(t, s) \le f_t f_s$$

$$f_{t'} f_s q_1(t', s) + f_{t'} f_s q_2(t', s) \le f_{t'} f_s$$

$$f_{t'} f_{s'} q_1(t', s') + f_{t'} f_{s'} q_2(t', s') \le f_{t'} f_{s'}$$

$$f_t f_{s'} q_1(t, s') + f_t f_{s'} q_2(t, s') \le f_t f_{s'}$$

$$f_s q_1(t, s) + f_{s'} q_1(t, s') = Q_t^1$$

$$f_s q_1(t', s) + f_{s'} q_1(t', s') = Q_{t'}^1$$

$$f_t q_2(t, s) + f_{t'} q_2(t', s) = Q_s^2$$

$$f_t q_2(t, s') + f_{t'} q_2(t', s') = Q_{s'}^2$$

$$f_t f_s q_1(t, s) + f_t f_s q_2(t, s) \le f_t f_s$$

$$f_{t'} f_s q_1(t', s) + f_{t'} f_s q_2(t', s) \le f_{t'} f_s$$

$$f_{t'} f_{s'} q_1(t', s') + f_{t'} f_{s'} q_2(t', s') \le f_{t'} f_{s'}$$

$$f_t f_{s'} q_1(t, s') + f_t f_{s'} q_2(t, s') \le f_t f_{s'}$$

$$f_t f_s q_1(t, s) + f_t f_{s'} q_1(t, s') = f_t Q_t^1$$

$$f_s q_1(t', s) + f_{s'} q_1(t', s') = Q_{t'}^1$$

$$f_t q_2(t, s) + f_{t'} q_2(t', s) = Q_s^2$$

$$f_t q_2(t, s') + f_{t'} q_2(t', s') = Q_{s'}^2$$

$$f_t f_s q_1(t, s) + f_t f_s q_2(t, s) \leq f_t f_s$$

$$f_{t'} f_s q_1(t', s) + f_{t'} f_s q_2(t', s) \leq f_{t'} f_s$$

$$f_{t'} f_{s'} q_1(t', s') + f_{t'} f_{s'} q_2(t', s') \leq f_{t'} f_{s'}$$

$$f_t f_{s'} q_1(t, s') + f_t f_{s'} q_2(t, s') \leq f_t f_{s'}$$

$$f_t f_s q_1(t, s) + f_t f_{s'} q_1(t, s') = f_t Q_t^1$$

$$f_{t'} f_s q_1(t', s) + f_{t'} f_{s'} q_1(t', s') = f_{t'} Q_{t'}^1$$

$$f_t f_s q_2(t, s) + f_{t'} f_s q_2(t', s) = f_s Q_s^2$$

$$f_t f_{s'} q_2(t, s') + f_{t'} f_{s'} q_2(t', s') = f_{s'} Q_{s'}^2$$

$$x_i(u, v) = f_u f_v q_i(u, v).$$

$$\mathbf{x_1(t, s)} + \mathbf{x_2(t, s)} \le f_t f_s$$

$$x_1(t', s) + x_2(t', s) \le f_{t'} f_s$$

$$x_1(t', s') + x_2(t', s') \le f_{t'} f_{s'}$$

$$x_1(t, s') + x_2(t, s') \le f_t f_{s'}$$

$$\mathbf{x_1(t, s)} + x_1(t, s') = f_t Q_t^1$$

$$x_1(t', s) + x_1(t', s') = f_{t'} Q_{t'}^1$$

$$\mathbf{x_2(t, s)} + x_2(t', s) = f_s Q_s^2$$

$$x_2(t, s') + x_2(t', s') = f_{s'} Q_{s'}^2$$

Hitchcock-Koopmans Transportation problem

S a set of supply nodes, where $i \in S$ has supply s_i.

D a set of demand nodes, where $j \in D$ has demand d_j.

E set of edges (i, j) with $i \in S$ and $j \in D$.

Let x_{ij} be the flow from $i \in S$ to $j \in D$ through $(i,j) \in E$.

A flow x is feasible if

$$\sum_{j \in D:(i,j)\in E} x_{ij} \leq s_i \ \forall i \in S$$

$$\sum_{i \in S:(i,j)\in E} x_{ij} = d_j \ \forall j \in D$$

$$x_{ij} \geq 0 \ \forall(i,j) \in E$$

Does a feasible flow x exist?

For each $T \subseteq D$ let $N(T) = \{i \in S : (i,j) \in E \& j \in T\}$.

A feasible flow exists iff for all $T \subseteq D$:

$$\sum_{i \in N(T)} s_i \geq \sum_{j \in T} d_j.$$

$$x_1(t, s) + x_2(t, s) \leq f_t f_s$$

$$x_1(t', s) + x_2(t', s) \leq f_{t'} f_s$$

$$x_1(t', s') + x_2(t', s') \leq f_{t'} f_{s'}$$

$$x_1(t, s') + x_2(t, s') \leq f_t f_{s'}$$

$$x_1(t, s) + x_1(t, s') = f_t Q_t^1$$

$$x_1(t', s) + x_1(t', s') = f_{t'} Q_{t'}^1$$

$$x_2(t, s) + x_2(t', s) = f_s Q_s^2$$

$$x_2(t, s') + x_2(t', s') = f_{s'} Q_{s'}^2$$

- Q_t is interim allocation probability to type $t \in T$.
- $1 - c(t)$ is the probability of checking a report of type t conditional on the good being allocated to a type t.
- Total value less the cost of inspection is

$$\sum_{t=1}^{m} f_t t Q_t - K \sum_{t=1}^{m} f_t Q_t [1 - c(t)]$$

$$= \sum_{t \in T} f_t Q_t [t - K + K c(t)]$$

$$\max \sum_{t \in T} f_t Q_t [t - K + Kc(t)]$$

$$\text{s.t. } Q_t \geq Q_s c(s) \ \forall t, s \in T \tag{1}$$

$$0 \leq c(t) \leq 1 \ \forall t \in T \tag{2}$$

$$n \sum_{t \in S} f_t Q_t t \leq 1 - (\sum_{t \notin S} f_t)^n = g(S) \ \forall S \subseteq T \tag{3}$$

$$Q_t \geq Q_s c(s) \ \Rightarrow Q_s \leq \frac{Q_t}{c(s)}$$

Never good to inspect $t = 1$ because $K > 1$. So, $c(1) = 1$.

Therefore, $Q_1 \leq Q_t \ \forall t \in T$.

$$c(s) \leq \frac{Q_t}{Q_s} \ \forall t \ \Rightarrow c(s) \leq \frac{Q_1}{Q_s}.$$

$$\max \sum_{t \in T} f_t Q_t [t - K + Kc(t)]$$

$$\text{s.t. } c(t) \leq \frac{Q_1}{Q_t} \ \forall t \in T$$

$$Q_t \geq Q_1 \ \forall t \in T$$

$$0 \leq c(t) \leq 1 \ \forall t \in T$$

$$n \sum_{t \in S} f_t Q_t \leq g(S) \ \forall S \subseteq T$$

$c(t) = \min\{\frac{Q_1}{Q_t}, 1\} = \frac{Q_1}{Q_t}.$

$$\max \sum_{t=1}^{m} f_t Q_t[t-K] + K Q_1$$

$$\text{s.t. } Q_t \geq Q_1 \ \forall t \in T$$

$$n \sum_{t \in S} f_t Q_t \leq g(S) \ \forall S \subseteq T$$

1. $Q_t = x_t + Q_1$ for all $t \geq 2$
2. $H(S) = g(S) - n Q_1 \sum_{i \in S} f_i$.
3. H is submodular.
4. For $Q_1 \leq \min_S \frac{g(S)}{n \sum_{t \in S} f_t}$, H is monotone.

$$\left(\sum_{t=1}^{m} t f_t\right) Q_1 + \max \sum_{t=2}^{m} f_t x_t [t - K]$$

$$\text{s.t. } n \sum_{t \in S} f_t x_t \leq H(S) \ \forall S \subseteq T \setminus \{1\}$$

One more change of variables: $z_t = f_t x_t$ for all $t \geq 2$.

$$\left(\sum_{t=1}^{m} t f_t\right) Q_1 + \max \sum_{t=2}^{m} z_t[t - K]$$

$$\text{s.t. } n \sum_{t \in S} z_t \leq H(S) \ \forall S \subseteq T \setminus \{1\}$$

1. Set $z_t = 0$ for all $t \leq K$. Therefore $Q_t = Q_1$.
2. $c(t) = 1$ for all $t \leq K$.
3. There is a cutoff, λ so that in any profile of types, award the object to the agent with the highest type provided it exceeds λ.
4. Inspect their report with positive probability. The probability of inspection rises with t.
5. If all reported types fall below the cutoff, randomize equally between all types below λ and don't inspect.

Sequence of boxes, $1, \ldots, n$.

Inspect boxes one at a time in given.

Box i contains a random number t_i with probability $f_i(t_i)$.

Upon inspecting a box, one can reject the draw and move to the next box or keep the draw and stop.

Maximize the expected value of number selected.

$z_i(t_i) = Pr[\text{choose } t_i | 1, \ldots, i-1 \text{ not chosen}]$.

Note independence of $(t_1, \ldots, t_{i-1})$.

Only constraint that such variables should satisfy is $0 \leq z_i(t_i) \leq 1 \quad \forall i \quad \forall t_i$.

Let $f_{1:i-1}(t_{1:i-1}) = \prod\limits_{j=1}^{i-1} f_i(t_i)$.

$Q_i(t_i) = Pr[\text{choose } t_i] = z_i(t_i) \sum\limits_{t_{1:i-1}} \prod\limits_{j=1}^{i-1} (1 - z_j(t_j)) f_{1:i-1}(t_{1:i-1})$.

Secretary Problem (cardinal version)

$$z_i(t_i) = \frac{Q_i(t_i)}{\sum\limits_{t_{1:i-1}} \prod\limits_{j=1}^{i-1} (1 - z_j(t_j)) f_{1:i-1}(t_{1:i-1})}.$$

CLAIM (by induction):

$$z_i(t_i) = \frac{Q_i(t_i)}{1 - \sum\limits_{j=1}^{i-1} \sum\limits_{t_j} Q_j(t_j) f_j(t_j)}$$

$$Q_i(t_i) + \sum_{j=1}^{i-1} \sum_{t_j} Q_j(t_j) f_j(t_j) \leq 1 \quad \forall i \quad \forall t_i$$

$$Q_i(t_i) \geq 0 \quad \forall i \quad \forall t_i$$

Prophet Inequality

Optimal Offline

$$\max_Q \sum_i \sum_{t_i} t_i Q_i(t_i) f_i(t_i)$$

$$s.t. \sum_i \sum_{t_i' \geq t_i} Q_i(t_i') f_i(t_i') \leq 1 - \prod_{i=1}^n F_i(t_i) \quad \forall t$$

$$Q_i(t_i) \geq 0 \quad \forall i \quad \forall t_i$$

Optimal Online

$$\max_Q \sum_i \sum_{t_i} t_i Q_i(t_i) f_i(t_i)$$

$$s.t. \ Q_i(t_i) + \sum_{j=1}^{i-1} \sum_{t_j} Q_j(t_j) f_j(t_j) \leq 1 \quad \forall i \quad \forall t_i$$

$$Q_i(t_i) \geq 0 \quad \forall i \quad \forall t_i$$

Theorem

The optimal online objective function value is at least $1/2$ of the optimal offline objective function value.

Prophet Inequality

Let Q^* be optimal off-line solution. Set $Q_i(t) = \frac{Q_i^*(t)}{2}$.

If Q is not feasible there exists agent i and type s such that

$$Q_i(s) + \sum_{j=1}^{i-1} \sum_{t_j} Q_j(t_j) f_j(t_j) > 1 \Leftrightarrow$$

$$Q_i^*(s) + \sum_{j=1}^{i-1} \sum_{t_j} Q_j^*(t_j) f_j(t_j) > 2$$

On the other hand we know that $Q_i^*(s) \leq 1$. Additionally, by feasibility of Q^*

$$\sum_{j=1}^{i-1} \sum_{t_j} Q_j^*(t_j) f_j(t_j) \leq 1$$

Q_k^i is the probability of allocating the item to agent i conditioned on the event that type k was realized.

c_k^i is the probability that agent i will NOT be inspected when he declares type k.

$$\max_{Q,c} \sum_{i=1}^{n} \sum_{k=1}^{m_i} f_k Q_k^i [t_k - K(1 - c_k^i)]$$

$$s.t. \ Q_k^i \geq Q_l^i c_l^i \quad \forall i \quad \forall k, l$$

$$c_k^i \leq 1 \quad \forall i \quad \forall k$$

$$Q_k^i + \sum_{j=1}^{i-1} \sum_{l=1}^{m_j} f_l Q_l^j \leq 1 \quad \forall i \quad \forall k$$

$$Q, c \geq 0$$

On-line Inspection

Wasteful to inspect the lowest type, so $c_1^i = 1$.

Incentive compatibility:

$$Q_1^i \leq Q_k^i \quad \forall k > 1$$

$$\max_{Q,c} \sum_{i=1}^{n} \sum_{k=1}^{m_i} f_k Q_k^i [t_k - K(1 - c_k^i)]$$

$$s.t. \ c_k^i \leq \frac{Q_1^i}{Q_k^i} \quad \forall i \quad \forall k, l$$

$$c_k^i \leq 1 \quad \forall i \quad \forall k$$

$$Q_k^i + \sum_{j=1}^{i-1} \sum_{l=1}^{m_j} f_l Q_l^j \leq 1 \quad \forall i \quad \forall k$$

$$Q, c \geq 0$$

Set $c_k^i = \frac{Q_1^i}{Q_k^i}$ and rewrite :

$$\max_Q \sum_{i=1}^{n} \sum_{k=1}^{m} f_k Q_k^i [t_k - K] + K \sum_{i=1}^{n} Q_1^i$$

$$s.t. \ Q_k^i + \sum_{j=1}^{i-1} \sum_{l=1}^{m} f_l Q_l^j \leq 1 \quad \forall i \quad \forall k$$

$$Q_1^i \leq Q_k^i \quad \forall i \quad \forall k \geq 2$$

$$Q \geq 0$$

Set $Q_k^i = Q_1^i + x_k^i$ and $Q_1^i = \alpha_i$, and we get the following linear program:

$$\max_{\alpha, x} \sum_{i=1}^{n} \sum_{k=1}^{m} f_k(t_k - K)x_k^i + \sum_{k=1}^{m} f_k t_k \sum_{i=1}^{n} \alpha_i$$

$$s.t. \sum_{j=1}^{i} \alpha_i + x_k^i + \sum_{j=1}^{i-1} \sum_{l=1}^{m} f_l x_l^j \leq 1 \quad \forall i \quad \forall k$$

$$x_1^i = 0 \quad \forall i$$

$$\alpha, x \geq 0$$

Each agent i has a one dimensional type θ^i drawn independently from a common distribution F (with density f) over $\{1, 2, \ldots, m\}$.

Agent i's value from consuming the object when in state $j \in S^i$ is $v^i(j|\theta^i)$.

$$v^i(j|\theta^i) = \theta^i + r_j^i.$$

State of project i is of the form (θ^i, j).

When a project i is in state (θ^i, j) it cannot transition to a state (σ^i, k) for $k \in S^i$ and $\sigma^i \neq \theta^i$.

Let $x_j^i(\theta^i)$ be the expected discounted number of times agent i is awarded the object when she is in state (θ^i, j).

Assume the second component of the state, r_j^i is known to the seller. Equivalently, agent i will truthfully report r_j^i when she is in state (θ^i, r_j^i).

Let $p^i(\theta^i)$ denote the total expected payment of agent i when she reports type θ^i.

Let $X^i(\theta^i)$ denote the total discounted number of times agent i receives the object when she reports type θ^i. Observe that $X^i(\theta^i) = \sum_{j \in S^i} x_j^i(\theta^i)$.

Bayesian incentive compatibility:

$$\theta^i X^i(\theta^i) + \sum_{j \in S^i} r_j^i x_j^i(\theta^i) - p^i(\theta^i) \geq \theta^i X^i(\sigma^i) + \sum_{j \in S^i} r_j^i x_j^i(\sigma^i) - p^i(\sigma^i) \ \forall \sigma^i \neq \theta^i. \quad (4)$$

Interchanging θ^i and σ^i yields:

$$\sigma^i X^i(\sigma^i) + \sum_{j \in S^i} r_j^i x_j^i(\sigma^i) - p^i(\sigma^i) \geq \sigma^i X^i(\theta^i) + \sum_{j \in S^i} r_j^i x_j^i(\theta^i) - p^i(\theta^i) \ \forall \theta^i \neq \sigma^i. \quad (5)$$

Adding (4) to (5) produces

$$(X^i(\theta^i) - X^i(\sigma^i))(\theta^i - \sigma^i) \geq 0. \quad (6)$$

Bayesian incentive compatibility is equivalent to monotonicity of $X^i(\cdot)$.

$$p^i(\theta^i) \leq \theta^i X^i(\theta^i) - \sum_{\sigma < \theta^i} X^i(\sigma) + \sum_{j \in S^i} r^i_j x^i_j(\theta^i).$$

The problem of finding the revenue maximizing mechanism can be formulated as:

$$\max \sum_{i \in N} \sum_{\theta^i} f(\theta^i) p^i(\theta^i) \tag{7}$$

$$\text{s.t. } X^i(\theta^i) \leq X^i(\sigma^i) \ \forall i, \ \forall \theta^i \leq \sigma^i \tag{8}$$

$$p^i(\theta^i) \leq \theta^i X^i(\theta^i) - \sum_{\sigma < \theta^i} X^i(\sigma) + \sum_{j \in S^i} r^i_j x^i_j(\theta^i) \ \forall i, \ \theta^i \tag{9}$$

$$X^i(\theta^i) = \sum_{j \in S^i} x^i_j(\theta^i) \ \forall i, \ \theta^i \tag{10}$$

$$\{x^i_j(\theta^i)\}_{i \in N, \theta^i} \in E \tag{11}$$

$$\max \sum_{i \in N} \sum_{\theta^i} f(\theta^i)[\theta^i - \frac{1 - F(\theta^i)}{f(\theta^i)}]X^i(\theta^i) + \sum_{i \in N} \sum_{\theta^i} f(\theta^i) \sum_{j \in S^i} r_j^i x_j^i(\theta^i) \qquad (12)$$

$$\text{s.t. } X^i(\theta^i) \leq X^i(\sigma^i) \ \forall i, \ \forall \theta^i \leq \sigma^i \qquad (13)$$

$$X^i(\theta^i) = \sum_{j \in S^i} x_j^i(\theta^i) \ \forall i, \ \theta^i \qquad (14)$$

$$\{x_j^i(\theta^i)\}_{i \in N, \theta^i} \in E \qquad (15)$$

Eliminating the $X^i(\theta^i)$ variables and dropping the monotonicity constraint yields the following relaxation:

$$\max \sum_{i \in N} \sum_{j \in S^i} \sum_{\theta^i} f(\theta^i)[\theta^i - \frac{1 - F(\theta^i)}{f(\theta^i)} + r_j^i]x_j^i(\theta^i) \ (MAB_\theta) \qquad (16)$$

$$\text{s.t.} \ \{x_j^i(\theta^i)\}_{i \in N, \theta^i} \in E \qquad (17)$$