Perfect conditional ε-equilibria of multi-stage games with infinite sets of signals and actions

by

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Example 1. Problems from testing conditional rationality with positive probability in all events.

Date 1: Nature chooses independent θ_1 and θ_2, where $\theta_1 \in \{1,2\}$ has probability $\theta_1/3$ and $\theta_2 \sim U[0,1]$; player 1 chooses $x \in [0,1]$.

Date 2: Player 2 observes $s_2 = x$ if $\theta_1 = 1$ but observes $s_2 = \theta_2$ if $\theta_1 = 2$ and then chooses $y \in \{1,2\}$.

	$y = 1$	$y = 2$
[1/3]: $\theta_1 = 1$	1,1	0,0
[2/3]: $\theta_1 = 2$	1,0	0,1

- suppose for "conditional rationality tests," any observed event must be given positive probability when possible.
- then, e.g., $s_2 = 1/2$ will imply $\theta_1 = 1$ since $s_2 = 1/2$ has positive probability only when $(\theta_1, x) = (1,1/2)$.
- but then player 2 must always play $y = 1$,which is strictly dominated by always playing $y = 2$.

Perturb nature!

Multi-stage games

- Play occurs in a finite sequence of dates $t = 1, \ldots, T$.
- Date t : Nature chooses $a_{0t} \in A_{0t}$ according to $p_t(\cdot|a_{<t})$ and each player $i = 1,2, \ldots, I$ privately observes a signal s_{it} and then chooses an action from his action set A_{it}.
- $\sigma_{it}: A_{<t} \rightarrow S_{it}$ ={*signals for player i at date t*}
- $A = \times_{i=0}^{I} \times_{t=1}^{T} A_{it}$ ={*outcomes of the game*}.
- $u_i: A \rightarrow \mathbf{R}$ is player i's bounded vNM utility function.
- A *strategy for i at date t* is a function $b_{it}: S_{it} \rightarrow \Delta(A_{it})$.
- A *strategy* for i is $b_i = (b_{i1}, \ldots, b_{iT}) \in B_i$; $B = \times_{i=1}^{I} B_i$.

Conditional ε-equilibrium

- $c_i \in B_i$ is a *date-t continuation* of $b_i \in B_i$ iff $c_{ir} = b_{ir}$ for every $r < t$.

- For ε ≥ 0, $b \in B$ is a *conditional ε-equilibrium* iff $\forall it$ and $\forall Z \subseteq S_{it}$ satisfying $\text{Prob}(Z|b) > 0$,

$$U_i(c_i, b_{-i}|Z) \leq U_i(b|Z) + \varepsilon, \text{ for all date-}t \text{ continuations } c_i \text{ of } b_i.$$

***Example 2.* What about a "full support" topological approach?**

Date 1: Player 1 chooses $x \in [0,1]$.

Date 2: Player 2 observes $s_2 = x$ and then chooses $y \in [0,1]$.

Date 3: Nature chooses $\theta \in \{-1,1\}$, p(1) = p(-1) = 1/2.

Date 4: Player 3 observes $s_3 = \theta(x+y)$ and then chooses $z \in \{-1,1\}$.

$u_1 = u_2 = -\theta z$

$u_3 = \theta z$

- 1 and 2 do not want 3 to guess θ and they can prevent this by choosing $x = y = 0$.

- Hence, subgame perfection requires $x = y = 0$.

- But there are imperfect conditional ε-equilibria with full support.

  For example, 1 chooses $x \sim U[0,1]$, 2 chooses $y \sim U[0,1]$ no matter what action of player 1 she observes, and 3 chooses $z = 1$ if $s_3 > 0$ and chooses $z = -1$ if $s_3 < 0$.

- This imperfect conditional ε-equilibrium survives here because the conditional rationality of player 2 is not tested at the critical signal $s_2 = 0$.

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$$U_i(c_i, b_{-i}|Z) \leq U_i(b|Z) + \varepsilon, \text{ for all date-}t\text{ continuations } c_i \text{ of } b_i.$$

- For a strategy profile to be a *perfect* conditional ε-equilibrium, it must be possible, given any finite set of signals outside a negligible set, to perturb the players' strategies and nature's probability function arbitrarily slightly so that every signal in the finite set has positive probability and so that the perturbed strategy profile is a conditional ε-equilibrium in the game with nature's perturbed probability function.

Perfect conditional ε-equilibrium

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- $Z \in \mathcal{M}(S_{it})$ is *negligible* iff $Prob(Z|b) = 0, \forall b \in B$.
- Let $S = \cup S_{it}$ be the set of all signals in the game.
- $Z \subset S$ is negligible iff $Z \cap S_{it}$ is negligible for every it.

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- $\hat{p}$ is a δ-*perturbation* of p iff $|\hat{p}_t(C|a_{<t}) - p_t(C|a_{<t})| \leq \delta$, $\forall a_{<t} \in A_{<t}$, $\forall C \in \mathcal{M}(A_{0t})$, $\forall t$.

- $\hat{b} \in B$ is a δ-*perturbation* of $b \in B$ iff $|\hat{b}_{it}(C|s_{it}) - b_{it}(C|s_{it})| \leq \delta$, $\forall s_{it} \in S_{it}$, $\forall C \in \mathcal{M}(A_{it})$, $\forall it$.

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- Let $\Gamma(\hat{p})$ denote the *perturbed game* in which nature's probability function is $\hat{p}$ instead of p.

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- $b \in B$ is a *perfect conditional ε-equilibrium* iff $\exists$ngbl $N \subseteq S$ s.t. $\forall$finite $Z \subseteq S \backslash N$, $\forall \delta > 0$, there are δ-perturbations $\hat{b}$ of b and $\hat{p}$ of p s.t., in $\Gamma(\hat{p})$, $\hat{b}$ gives every $s \in Z$ positive probability and $\hat{b}$ is a conditional ε-equilibrium.

Fact. In finite games, a strategy profile is part of a sequential equilibrium iff it is the limit of a sequence of perfect conditional ε-equilibria as ε tends to zero.

(Can define conditional belief systems that are "finitely consistent" and that make a given perfect conditional ε-equilibrium sequentially ε-rational. Can use beliefs to test plausibility as in KW.)

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There are many perfect conditional ε-equilibria. For example,

- Player 1 chooses $x \sim U[0,1]$ and player 2 always chooses $y = 2$.
- For any finite set of signals $Z \subseteq [0,1]$ for player 2, perturb θ_2 to be uniform on Z with small positive probability.

***Example 2.* What about a "full support" topological approach?**

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- But this imperfect equilibrium is not a perfect conditional ε-equilibrium because it is not a conditional ε-equilibrium in any perturbation that gives positive probability to $s_2 = 0$.

Subgame perfection

- For any date t, $a_{<t} \in A_{<t}$ is a *subgame* iff $\sigma_{it}^{-1}(\sigma_{it}(a_{<t})) = \{a_{<t}\}, \forall i$.

- $b \in B$ is a *subgame perfect ε-equilibrium* iff $\forall it$, ∀subgames $a_{<t}$ outside a negligible set,

  $U_i(c_i, b_{-i}|a_{<t}) \leq U_i(b|a_{<t}) + \varepsilon$, for all date-$t$ continuations c_i of b_i.

Fact. Every perfect conditional ε-equilibrium is a subgame perfect ε-equilibrium.

Perfect Conditional Equilibrium Distributions

- $b \in B$ is a *perfect conditional ε-equilibrium* iff ∃ngbl $N \subseteq S$ s.t. ∀finite $Z \subseteq S\backslash N$, ∀δ>0, there are δ-perturbations $\hat{b}$ of b and $\hat{p}$ of p s.t., in $\Gamma(\hat{p})$, $\hat{b}$ gives every $s \in Z$ positive probability and $\hat{b}$ is a conditional ε-equilibrium.

- A *perfect conditional equilibrium distribution* is any $\mu \in [0,1]^{\mathcal{M}(A)}$ s.t.,

$$\mu(H) = \lim_{\alpha} \text{Prob}(H|b^{\alpha}), \forall H \in \mathcal{M}(A),$$

where $\{b^{\alpha}\}$ is a net of perfect conditional ε_{α}-equilibria, and $\lim_{\alpha} \varepsilon_{\alpha} = 0$.

Fact. In finite games, a strategy profile is part of a sequential equilibrium iff its outcome distribution is a perfect conditional equilibrium distribution.

Regular projective games

A multi-stage game Γ is a *regular projective game* iff there is a finite index set J and sets A_{nrj} such that for every player i and date t,

(R.1) $A_{it} = \times_{j\in J} A_{itj}$, $A_{0t} = \times_{j\in J} A_{0tj}$,

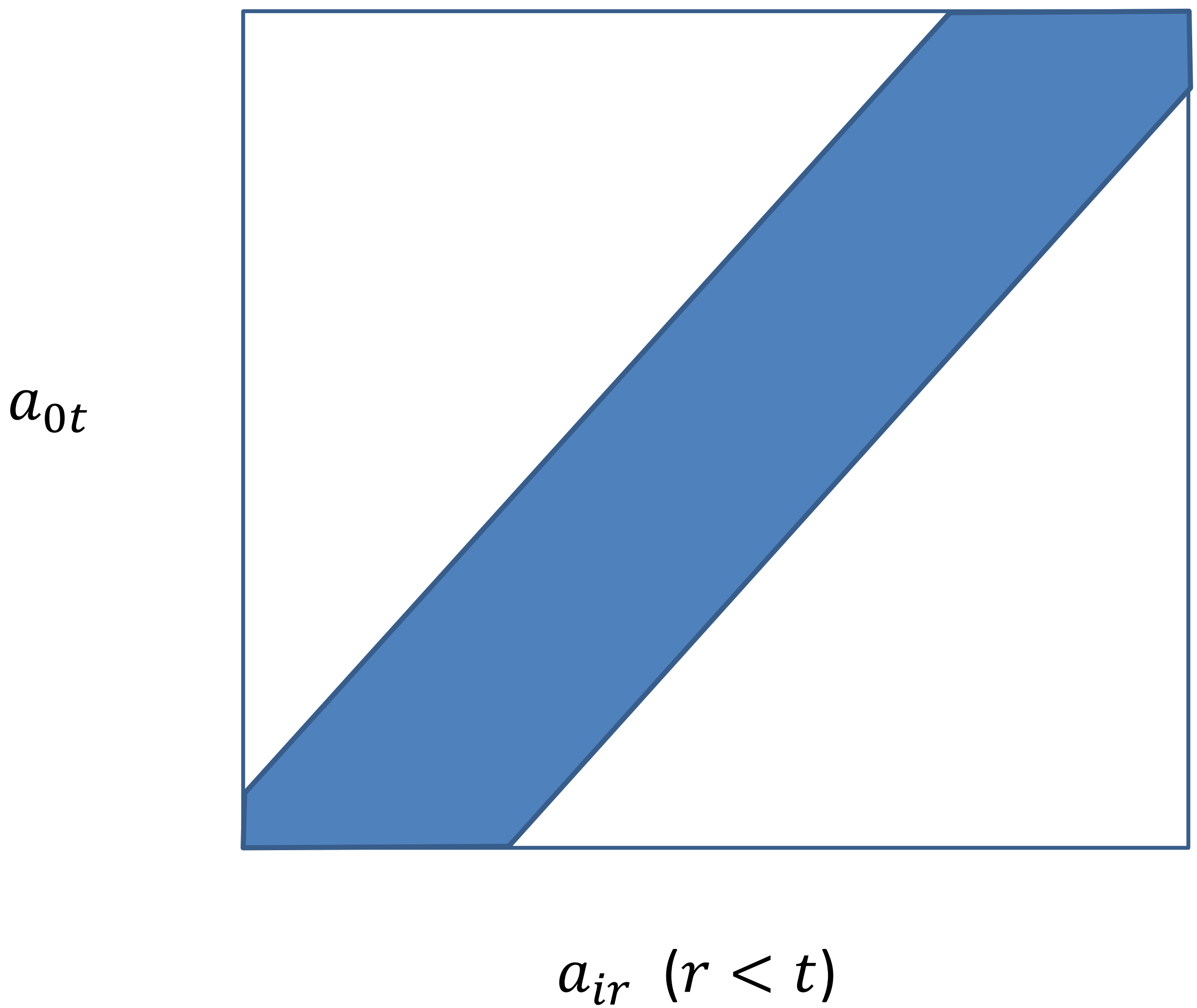
(R.2) there is a set $M_{it} \subseteq \{0,1,\ldots,I\} \times \{1,\ldots,t-1\} \times J$ such that $\sigma_{it}(a_{<t}) = (a_{nrj})_{nrj\in M_{it}}$.

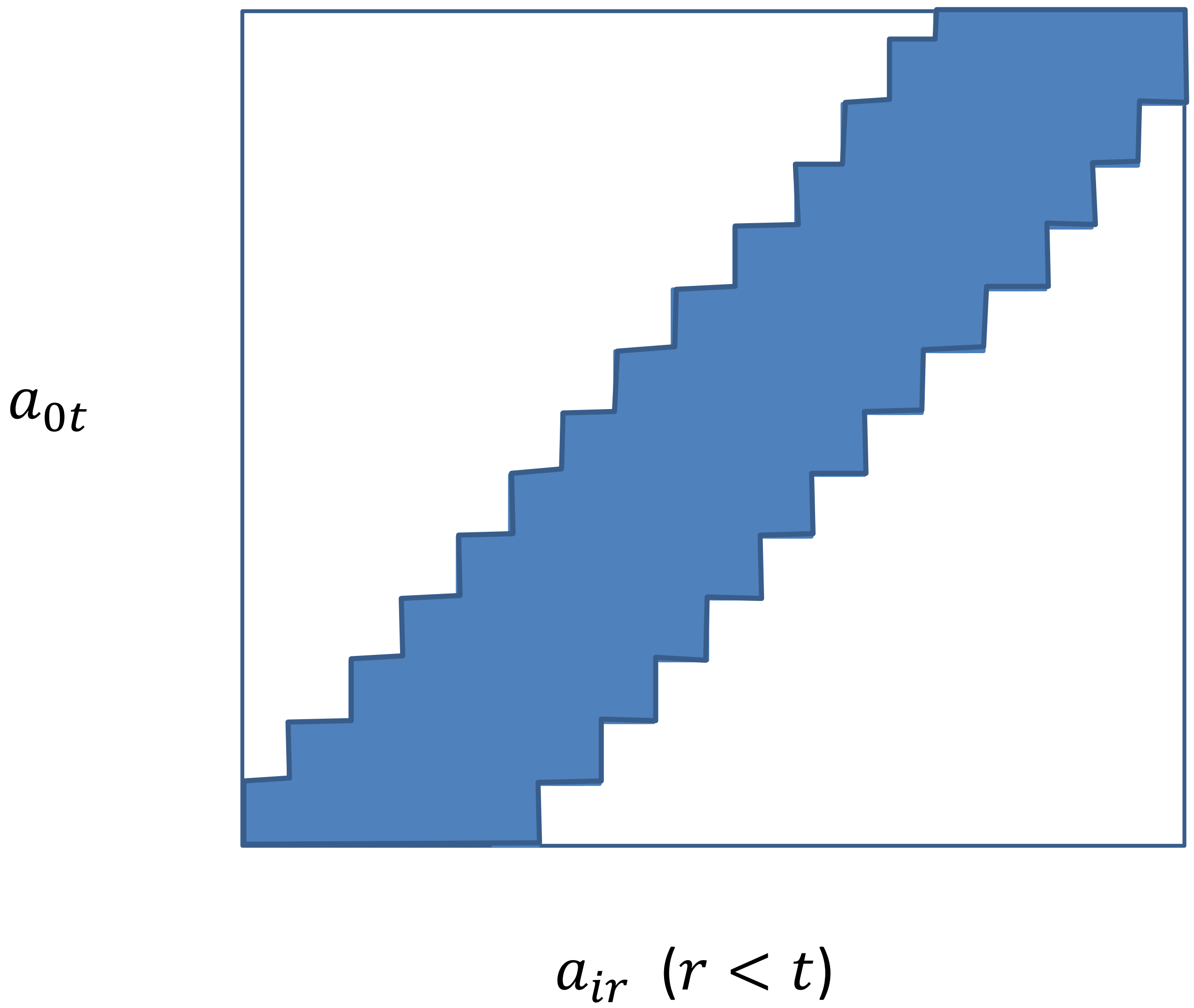
(R.3) A_{0tj} and A_{itj} are nonempty compact metric spaces $\forall j\in J$, and all spaces, including all products, are given their Borel sigma-algebras,

(R.4) $u_i: A \to \mathbf{R}$ is continuous,

(R.5) $p_t(C|a_{<t}) = \int_C f_t(a_{0t}|a_{<t})\Pi_{j\in J}\rho_{0tj}(da_{0tj})$ $\forall C\in\mathcal{B}(A_{0t})$, $\forall a_{<t}\in A_{<t}$, where $f_t: A_{0t} \times A_{<t} \to [0,\infty)$ is the product of a positive continuous function and a nonnegative function that is measurable w.r.t. a finite product partition of $A_{0t} \times A_{<t}$, and where $\rho_{0tj} \in \Delta(A_{0tj})$ $\forall j\in J$.

Remark. Since distinct players can observe the same coordinate of Nature, regular projective games need not satisfy the information diffuseness condition of Milgrom-Weber (1985) .


a_{0t}
a_{ir} $(r < t)$



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Existence

- $b \in B$ is a *perfect conditional ε-equilibrium* iff ∃ngbl $N \subseteq S$ s.t. ∀finite $Z \subseteq S\backslash$N, ∀δ>0, there are δ-perturbations $\hat{b}$ of b and $\hat{p}$ of p s.t., in Γ($\hat{p}$), $\hat{b}$ gives every $s \in Z$ positive probability and $\hat{b}$ is a conditional ε-equilibrium.

- A *perfect conditional equilibrium distribution* is any $\mu \in [0,1]^{\mathcal{M}(A)}$ s.t.,

$$\mu(H) = \lim_{\alpha} \text{Prob}(H | b^{\alpha}), \forall H \in \mathcal{M}(A),$$

where $\{b^{\alpha}\}$ is a net of perfect conditional ε_{α}-equilibria, and $\lim_{\alpha} \varepsilon_{\alpha} = 0$.

Theorem. Let Γ be any regular projective game. Then, (i) for every ε > 0, Γ possesses at least one perfect conditional ε-equilibrium, and (ii) Γ possesses at least one perfect conditional equilibrium distribution.

Perfect conditional ε-equilibrium.

- For a strategy profile to be a perfect conditional ε-equilibrium, it must be possible, given any finite set of signals outside a negligible set, to perturb the players' strategies and nature's probability function arbitrarily slightly so that every signal in the finite set has positive probability and so that the perturbed strategy profile is a conditional ε-equilibrium in the game with nature's perturbed probability function.

Properties of perfect conditional ε-equilibria.

- they are subgame perfect ε-equilibria
- players with the same information behave as if they have the same beliefs
- in finite multi-stage games, the set of their limits as $\varepsilon \to 0$ coincides with the set of sequential equilibrium strategy profiles
- they can be shown to exist in a large class of infinite multi-stage games that we call *regular projective games*

***Example*. Spurious signaling in naïve finite approximations**

Date 1: Nature chooses $\theta \in \{1,2\}$, $p(\theta=1) = 1/4$, and

Player 1 chooses $x \in [0,1]$.

Date 2: Player 2 observes $s_2 = x^\theta$, then chooses $y \in \{1,2\}$.

	$y = 1$	$y = 2$
[1/4]: $\theta = 1$	1,1	0,0
[3/4]: $\theta = 2$	1,0	0,1

1's payoff is zero in any Nash equilibrium.

But if player 1 is restricted to ***any large finite subset*** F of his action space [0,1], he ***must*** obtain $u_1 \geq 1/4$ in any SPE; since when s_2 is the highest action in F less than 1, player 2 must respond with $y = 1$ since the state must be $\theta = 1$.

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[1/4]: $\theta = 1$	1,1	0,0
[3/4]: $\theta = 2$	1,0	0,1

There are many perfect conditional ε-equilibria. For example,

- Player 1 chooses $x = 0$ and player 2 always chooses $y = 2$.
- For any finite set of signals $Z \subseteq [0,1]$ for player 2, perturb player 1's strategy toward $\tilde{b}_1$, where $\tilde{b}_1(x) = 3\tilde{b}_1(\sqrt{x}) > 0$ for all x in the square-root closure of Z (i.e., $\cup_{z \in Z} \{z, \sqrt{z}, \sqrt{\sqrt{z}}, \ldots\}$).

***Example*. Effects of correlated perturbations of nature**

Date 1: Nature chooses $\theta_1, \theta_2 \sim U[-1,3]$.

Date 2: Player 1 observes $s_1 = \theta_1$ and chooses $x \in \{-1,1\}$.

Date 3: Player 2 observes $s_2 = x$ and chooses $y \in \{-1,1\}$.

$u_1 = xy$

$u_2 = \theta_2 y$

- Since no player observes θ_2 and $E(\theta_2) > 0$, player 2 should always choose $y = 1$, regardless of the action of player 1 that she observes.
- So player 1 should choose $x = 1$ regardless of the θ_1 that he observes. The only sensible equilibrium payoffs are $u_1 = u_2 = 1$. (?)
- But consider $b_1(\theta_1) = -1$ iff $\theta_1 \neq -1$ and $b_2(x) = -x$. This yields the payoff vector $(u_1, u_2) = (-1,1)$
- These strategies are supported by perturbing nature to give small positive probability to the event $\{(\theta_1,\theta_2)=(-1,-1)\}$.

(Can eliminate this equilibrium if Nature's states are perturbed *independently*.)

***Example*. Effects of "far" perturbations of nature**

Date 1: Nature chooses $(\theta_1,\theta_2)\in [0,1]^2$. With probability ½, the coordinates are independent and uniform on [0,1], and with probability ½ the coordinates are equal and uniform on [0,1].

Date 2: Player 1 observes $s_1 = \theta_1$ and chooses $x\in\{-1,1\}$. Player 2 observes $s_2 = x$ and chooses $y\in\{-1,1\}$.

$u_1 = xy$,

$u_2 = y(1/3 - |\theta_1-\theta_2|)$.

Player 2 should choose $y = 1$ iff she expects $|\theta_1-\theta_2|$ to be less than 1/3.

Player 1 wants to choose an action that player 2 will match.

Since for every θ_1, θ_2 is equally likely to be equal to θ_1 (in which case $|\theta_1-\theta_2| =0$) as to be uniform on [0,1] (in which case $E|\theta_1-\theta_2| \leq ½$), player 2 should expect $|\theta_1-\theta_2|$ to be no greater than ¼, regardless of 1's strategy. So player 2 should choose $y = 1$.

Thus all sensible equilibria give probability 1 to $(x,y) = (1,1)$. (?)

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Player 1 wants to choose an action that player 2 will match.

Thus all sensible equilibria give probability 1 to $(x,y) = (1,1)$. (?)

But consider (b_1,b_2) where $b_1(\theta_1) = -1$ iff $\theta_1 \neq 1$, and $b_2(x) = -x$.

(b_1,b_2) gives probability 1 to $(x,y) = (-1,1)$, and is supported in a perfect conditional ε-equilibrium by the perturbation of Nature that never perturbs θ_2 but that with probability ε perturbs the distribution of θ_1 so that it is a mass point on $\theta_1 = 1$. With this perturbation, player 2 equates $x = 1$ with $\theta_1 =1$ and therefore expects the value of $|\theta_1-\theta_2|$ to be ½.

Can eliminate this equilibrium if Nature's states are perturbed only to *nearby* states.

Local perturbations of nature

Suppose that nature's date-t states can be written as $A_{0t} = \times_{j\in J} A_{0tj}$ and that each A_{0tj} is a metric space.

For any δ ≥ 0, a probability function $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_T)$ is a *local* δ-perturbation of nature's probability function $p = (p_1, \ldots, p_T)$ iff $\forall t \leq T$, $\hat{p}_t$ is of the form

$$\hat{p}_t(C|a_{<t}) = \int_{A_{0t}} \prod_{j\in J} \phi_{tj}\left(C_j \middle| a_{0tj}\right) p_t(da_{0t}|a_{<t}), \quad \forall C = \times_{j\in J} C_j \subseteq A_{0t}, \forall a_{<t}$$

for any transition probabilities $\phi_{tj}: A_{0tj} \to \Delta(A_{0tj})$ that satisfy

$$\phi_{tj}\left(a_{0tj}\middle|a_{0tj}\right) \geq 1 - \frac{\delta}{\#J} \text{ and } \phi_{tj}\left(Ball_\delta(a_{0tj})\middle|a_{0tj}\right) = 1, \forall a_{0tj} \in A_{0tj}\ \forall j \in J\ .$$

Remark. For each date t, $\hat{p}_t$ works as follows. First, a provisional date-t state $a_{0t} = (a_{0tj})_{j\in J}$ is chosen according to $p_t(\cdot\,|\,a_{<t})$. Then, independently for each coordinate j, the j-th coordinate is either unchanged (with probability 1- δ/#J), or, (with probability δ/#J) it is replaced with some element of A_{0tj} that is within δ of a_{0tj}. So, in local perturbations, nature's state is only changed slightly and with small probability.