Random Mechanism Design on Multidimensional Domains

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Preliminaries: Preferences

- $I = \{1, \ldots, N\}, N \geq 2$: A finite set of voters;
- $A = \{a, b, c, \ldots\}, |A| \geq 3$: A finite set of alternatives;
- P_i: A preference, i.e., a linear order over A;
- $r_k(P_i)$: the kth ranked alternative in P_i;
- $\mathbb{D}$: The domain of preferences over A;
- $P \equiv (P_1, \ldots, P_N) \equiv (P_i, P_{-i}) \in \mathbb{D}^N$: A preference profile.

A domain $\mathbb{D}$ is **minimally rich** if for every $a \in A$, there exists $P_i \in \mathbb{D}$ with $r_1(P_i) = a$.

Definition

A **Deterministic Social Choice Function (DSCF)** is a map $f : \mathbb{D}^N \to A$.

Preliminaries: Random Social Choice Functions

Definition

A **Random Social Choice Function** (**RSCF**) is a map $\varphi : \mathbb{D}^N \to \Delta(A)$.

Definition

An RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **unanimous** if for all $a \in A$ and $P \in \mathbb{D}^N$,

$$[r_1(P_i) = a \text{ for all } i \in I] \Rightarrow [\varphi_a(P) = 1].$$

Definition (Gibbard, 1977)

An RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **strategy-proof** if for all $i \in I$; $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$, lottery $\varphi(P_i, P_{-i})$ first-order stochastically dominates lottery $\varphi(P_i', P_{-i})$ according to P_i, i.e.,

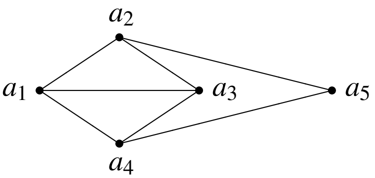
$$\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) \geq \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i', P_{-i}), \quad t = 1, \ldots, |A|.$$

Random dictatorships

Definition

An RSCF $\varphi^{RD} : \mathbb{D}^N \to \Delta(A)$ is a **random dictatorship** if there exists $\varepsilon_i \geq 0$ for each $i \in I$ with $\sum_{i\in I} \varepsilon_i = 1$ such that for all $P \in \mathbb{D}^N$ and $a \in A$, $\varphi^{RD}_a(P) = \sum_{i\in I:r_1(P_i)=a} \varepsilon_i$.

- A random dictatorship is unanimous and strategy-proof on any domains.
- A random dictatorship *never* admits compromise.
  For instance, let $r_1(P_1) = a$, $r_1(P_2) = b$ and $r_2(P_1) = r_2(P_2) = c$.
  However, $\varphi^{RD}_c(P_1, P_2) = 0$.
- Escape random dictatorships: Chatterji, Sen and Zeng (2014)



Top-separability

Assumption: Let $A = \times_{s \in M} A^s$ where M is finite with $|M| \geq 2$, and A^s is finite with $|A^s| \geq 2$ for all $s \in M$. We assume preferences satisfy **Top-separability**

$$\left[r_1(P_i) = (a^s)_{s \in M}\right] \Rightarrow [(a^s, z^{-s}) P_i (b^s, z^{-s}) \text{ for all } s \in M, b^s \neq a^s \text{ and } z^{-s} \in A^{-s}].$$

Definition

A domain is a **multidimensional domain** if all preferences are top-separable.

Every generalized dictatorship is strategy-proof if and only if all preferences are top-separable. Random generalized dictatorships however do not systematically admit compromise.

Two examples strengthening top-separability

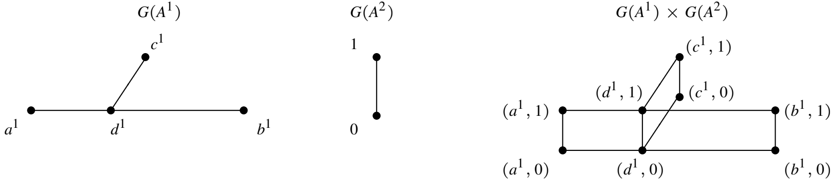
Definition (Le Breton and Sen, 2009)

A preference P_i is **separable** if there exists a (unique) marginal preference $[P_i]^s$ over A^s for each $s \in M$ such that for all $a, b \in A$, we have

$$\left[a^s [P_i]^s b^s \text{ and } a^{-s} = b^{-s}\right] \Rightarrow [a P_i b].$$

Definition (Barberà, Gul and Stacchetti, 1993)

For each $s \in M$, let all elements of A^s be located on a tree $G(A^s)$. A preference P_i is **multidimensional single-peaked** on the product of trees $\times_{s \in M} G(A^s)$ if for all distinct $x, y \in A$, we have $[x \in \langle r_1(P_i), y\rangle] \Rightarrow [x P_i y]$.



The constrained compromise property

Definition

An RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies **the constrained compromise property** if there exists $\hat{I} \subseteq I$ with $|\hat{I}| = \frac{N}{2}$ if N is even and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, such that given $P_i, P_j \in \mathbb{D}$, we have

$$\begin{bmatrix} r_1(P_i) \equiv (x^s, a^{-s}) \neq (y^s, a^{-s}) \equiv r_1(P_j) \text{ and} \\ r_2(P_i) = r_2(P_j) \equiv (z^s, a^{-s}) \text{ where } z^s \notin \{x^s, y^s\} \end{bmatrix} \Rightarrow \left[\varphi_{(z^s, a^{-s})}\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\right) > 0\right].$$

- The constrained compromise property focuses on non-assemblable compromise alternatives, and hence weakens the compromise property of Chatterji, Sen and Zeng (2016).

Question

Suppose a multidimensional domain admits a unanimous, strategy-proof RSCF which also satisfies the constrained compromise property: What can we infer about the structure of such a domain?

Adjacency

- Let $\Gamma(P_i, P'_i) = \{(a, b) \in A^2 | aP_ib \text{ and } bP'_ia\}$.
- Grandmont (1978), Monjardet (2009), Sato (2013) and Cho (2016).

P_i										P'_i
a	$\nearrow$	c		c		c		c	$\nearrow$	b
c		a		a	$\nearrow$	b		b		c
d		d	$\nearrow$	b		a	$\nearrow$	d		d
b		b		d		d		a		a
P_i^1		P_i^2		P_i^3		P_i^4		P_i^5		P_i^6

- Preferences P_i and P'_i are **adjacent**, denoted $P_i \sim P'_i$, if we have $\Gamma(P_i, P'_i) = \{(a, b)\}$ for some $a, b \in A$.

Adjacency$^+$

- After imposing top-separability, preferences P_i^2 and P_i^5 are excluded.

$$
\begin{array}{ccccccc}
P_i & & & & & & P_i' \\
a = (0,0) & \nearrow & c = (1,0) & & c = (1,0) & \nearrow & b = (1,1) \\
c = (1,0) & & a = (0,0) & \nearrow & b = (1,1) & & c = (1,0) \\
d = (0,1) & \nearrow & b = (1,1) & & a = (0,0) & \nearrow & d = (0,1) \\
b = (1,1) & & d = (0,1) & & d = (0,1) & & a = (0,0) \\
P_i^1 & & P_i^3 & & P_i^4 & & P_i^6
\end{array}
$$

- Besides adjacency (e.g., $P_i^3 \sim P_i^4$), multiple local switchings occurs simultaneously between P_i^1 and P_i^3: $\Gamma(P_i^1, P_i^3) = \big\{(a,c),(d,b)\big\}$.
- Preferences P_i and P_i' are **adjacent$^+$**, denoted $P_i \sim^+ P_i'$, if we have
  - (i) P_i and P_i' are separable preferences, and
  - (ii) $\Gamma(P_i, P_i') = \Big\{\big((a^s, z^{-s}),(b^s, z^{-s})\big)\Big\}_{z^{-s}\in A^{-s}}$ for some $s \in M$, $a^s, b^s \in A^s$.

Path

- A **path** $\{P_i^1, \ldots, P_i^t\}$ is a sequence of preferences such that $P_i^k \sim P_i^{k+1}$ or $P_i^k \sim^+ P_i^{k+1}$ for all $k = 1, \ldots, t-1$.
- Grandmont (1978): The notion of betweenness is stronger than a path as it requires the inclusion of all preferences between two preferences. Monjardet (2009), Sato (2013) and Cho (2016): Only adjacency.
- We introduce some parsimony in the lengths of these paths via the notion of a **connected$^+$ domain**.

Connectedness$^+$

Definition (The Interior$^+$ property)

Given $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i) \equiv a$, there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P'_i such that $r_1(P_i^k) = a$, $k = 1, \ldots, q$.

Definition (The Exterior$^+$ property)

Given $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P'_i)$, and $a, b \in A$ with aP_ib and aP'_ib, there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P'_i such that aP_i^kb, $k = 1, \ldots, q$. In particular, when $r_1(P_i) \equiv (a^s, z^{-s}) \neq (b^s, z^{-s}) \equiv r_1(P'_i)$, the path $\{P_i^k\}_{k=1}^q$ satisfies the **non-detour property**, i.e., $r_1(P_i^k) \in (A^s, z^{-s})$, $k = 1, \ldots, q$.

A **connected$^+$ domain**: A multidimensional domains satisfying the Interior$^+$ Property and the Exterior$^+$ Property.

Connectedness$^+$: Inclusions

- The top-separable domain
- The separable domain
- The multidimensional single-peaked domain
- The intersection of the separable domain and the multidimensional single-peaked domain
- The union of the separable domain and the multidimensional single-peaked domain(s)

Connectedness^{+}: Exclusions

- The complete domain (Gibbard, 1973)
- The single-peaked domain (Moulin, 1980; Demange, 1982)
- The single-dipped domain (Barberà, Berga and Moreno, 2012)
- Single-crossing domains (Saporiti, 2009; Carroll, 2012)
- The lexicographically separable domain (Chatterji, Roy and Sen, 2012)

Characterization of multidimensional single-peakedness

Theorem

Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. If it admits a unanimous and strategy-proof RSCF satisfying the constrained compromise property, it is multidimensional single-peaked.
Conversely, a multidimensional single-peaked domain admits a unanimous and strategy-proof RSCF satisfying the constrained compromise property.

- Multidimensional domains were excluded by Chatterji, Sen and Zeng (2016). Furthermore, we endogenize the tops-only property here and work with a weaker notion of the compromise property.

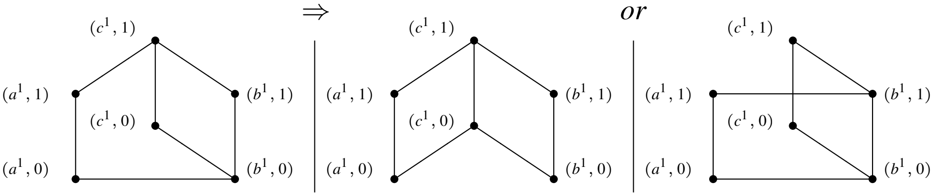
Deterministic Social Choice Functions

Theorem

Let domain $\mathbb{D}$ be minimally rich and connected^{+}. If it admits a unanimous, anonymous and strategy-proof DSCF, it is multidimensional single-peaked.

- Generalize the results in Chatterji, Sanver and Sen (2013) and Chatterji and Massó (2018): No restriction on the number of voters, endogenize the tops-only property, and recover full single-peakedness.
- Nehring and Puppe (2007), Bogomolnaia (1998).

For instance,



Step 1 The constrained compromise property implies
$\varphi\big((a^1, 1), (b^1, 1)\big) = \alpha e_{(a^1,1)} + (\beta - \alpha) e_{(c^1,1)} + (1 - \beta) e_{(b^1,1)}$, where $0 \le \alpha < \beta \le 1$.

Since $(a^1, 1) \sim^+ (a^1, 0)$, from profile $\big((a^1, 1), (b^1, 1)\big)$ to $\big((a^1, 0), (b^1, 1)\big)$, strategy-proofness implies $\varphi_{(c^1,1)}\big((a^1, 0), (b^1, 1)\big) + \varphi_{(c^1,0)}\big((a^1, 0), (b^1, 1)\big) = \beta - \alpha$.

Step 2 Since $(a^1, 0) \sim^+ (b^1, 0)$, unanimity and strategy-proofness imply
$\varphi_{(a^1,0)}\big((a^1, 0), (b^1, 0)\big) + \varphi_{(b^1,0)}\big((a^1, 0), (b^1, 0)\big) = 1$.

Since $(b^1, 0) \sim^+ (b^1, 1)$, from profile $\big((a^1, 0), (b^1, 0)\big)$ to $\big((a^1, 0), (b^1, 1)\big)$, strategy-proofness implies $\varphi_{(c^1,1)}\big((a^1, 0), (b^1, 1)\big) + \varphi_{(c^1,0)}\big((a^1, 0), (b^1, 1)\big) = 0$. A contradiction to tops-onlyness.

Elaboration: Necessity (con.)

5. Every separable preference is multidimensional single-peaked on $\times_{s \in M} G(A^s)$:
   A consequence of the constrained compromise property since each marginal preference is driven to be single-peaked.

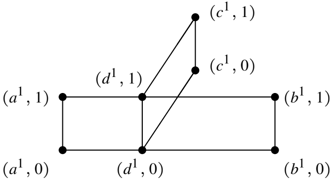
6. Every preference is multidimensional single-peaked on $\times_{s \in M} G(A^s)$:
   A consequence of connectedness^{+}.

   Suppose that P_i is not multidimensional single-peaked, e.g., $(x^s, z^{-s}) \in \langle r_1(P_i), (y^s, z^{-s}) \rangle$ but $(y^s, z^{-s}) P_i (x^s, z^{-s})$.

P_i	$\cdots$	P_i^k		P_i^{k+1}	$\cdots$	P_i'
(a^s, a^{-s})		(a^s, a^{-s})	$\neq$	b		(y^s, z^{-s})
$\vdots$		$\vdots$		$\vdots$		$\vdots$
(y^s, z^{-s})		(y^s, z^{-s})		$\cdot$		$\cdot$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
(x^s, z^{-s})		(x^s, z^{-s})		$\cdot$		$\cdot$
$\vdots$		$\vdots$		$\vdots$		$\vdots$

Elaboration: Sufficiency

1. Let $I = \{1, 2\}$. The multidimensional single-peaked domain $\mathbb{D}_{MSP}$.



2. A projection rule:
   Fix a *threshold* $z \in A$.
   Given $P_1, P_2 \in \mathbb{D}_{MSP}$, assume $r_1(P_1) = x$ and $r_1(P_2) = y$.
   Then, $f^z(P_i, P_j) = \pi\big(z, \langle x, y \rangle\big)$.

Elaboration: Sufficiency (con.)

3. **A mixed projection rule**: a mixture of all projection rules
   Let $\lambda^z > 0$ for all $z \in A$ and $\sum_{z \in A} \lambda^z = 1$.

   $$\text{For all } P_1, P_2 \in \mathbb{D}_{MSP},\ \varphi(P_1, P_2) = \sum_{z \in A} \lambda^z f^z(P_1, P_2).$$

4. A mixed projection rule is unanimous and strategy-proof, and satisfies the constrained compromise property. Moreover, a mixed projection rule also satisfies the compromise property of Chatterji, Sen and Zeng (2016).

Indispensability of unanimity

- Consider the top-separable domain $\mathbb{D}_{TS}$.
- A two-voter *point voting scheme* $\varphi : \mathbb{D}^2_{\text{TS}} \to \Delta(A)$ introduced by Barberà (1979) is strategy-proof and satisfies the constrained compromise property:
  - Fix $(\alpha_1, \alpha_2, \ldots, \alpha_{|A|}) \in \mathbb{R}_+^{|A|}$ such that $\alpha_1 > 0$, $\alpha_2 > 0$ and $\sum_{k=1}^{|A|} \alpha_k = \frac{1}{2}$.
  - Given $P_i, P_j \in \mathbb{D}_{TS}$, if $a = r_s(P_i)$ and $a = r_t(P_j)$, then $\varphi(P_i, P_j) = \alpha_s + \alpha_t$.

Indispensability of strategy-proofness

- Consider the top-separable domain $\mathbb{D}_{TS}$.
- A two-voter DSCF $f : \mathbb{D}^2_{TS} \to A$

$$f(P_i, P_j) = \begin{cases} a & \text{if } r_1(P_i) \neq r_1(P_j) \text{ and } r_2(P_i) = r_2(P_j) \equiv a; \\ r_1(P_i) & \text{otherwise.} \end{cases}$$

is unanimous and satisfies the constrained compromise property.

Indispensability of the constrained compromise property

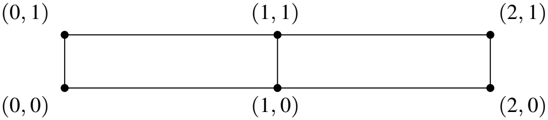
- Consider the top-separable domain $\mathbb{D}_{TS}$.
- A generalized random dictatorship is unanimous and strategy-proof.

Indisspensability of top-separability

- Let $A = \times_{s \in M} A^s$ where $|A^s| = 2$ for all $s \in M$.
- The complete domain satisfies the Interior$^+$ and Exterior$^+$ properties.
- A random dictatorship is unanimous and strategy-proof, and satisfies the constrained compromise property *vacuously*.

Indispensability of minimal richness

- Let $A = A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$. Specify domain $\mathbb{D}_{\text{MSP}}$ on $G(A^1) \times G(A^2)$ below. Remove all preferences with peak $(2, 0)$ or $(2, 1)$, i.e., let $\hat{\mathbb{D}} = \{P_i \in \mathbb{D}_{MSP} | r_1(P_i) \neq (2, 0) \text{ and } r_1(P_i) \neq (2, 1)\}$. Add a new preference $\bar{P}_i$.

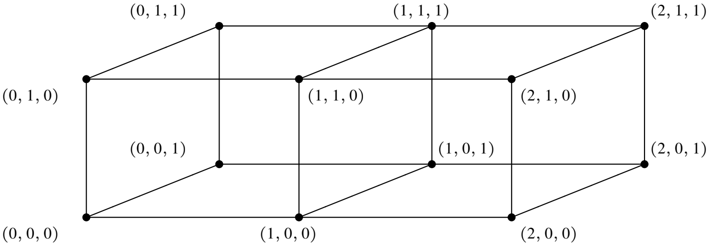


$\bar{P}_i$
(0, 0)
(0, 1)
(1, 0)
(2, 0)
(2, 1)
(1, 1)

- Domain $\mathbb{D} = \hat{\mathbb{D}} \cup \{\bar{P}_i\}$ is connected$^+$ but never multidimensional single-peaked.
- A two-voter mixed projection rule associating positive weights to all projectors other than $(2, 0)$ and $(2, 1)$ is unanimous and strategy-proof and satisfies the constrained compromise property *vacuously*.

Indispensability of paths in connectedness⁺

- Let $A = A^1 \times A^2 \times A^3$, $A^1 = \{0, 1, 2\}$ and $A^2 = A^3 = \{0, 1\}$. Specify domain $\mathbb{D}_{MSP}$ on $G(A^1) \times G(A^2) \times G(A^3)$ below. Moreover, add a new preference $\bar{P}_i$.



$\bar{P}_i$

(0, 0, 0)
(1, 0, 0)
(2, 0, 0)
(0, 1, 0)
(1, 1, 0)
(2, 1, 0)
(0, 0, 1)
(1, 0, 1)
(2, 0, 1)
(0, 1, 1)
(2, 1, 1)
(1, 1, 1)

- Domain $\mathbb{D} = \mathbb{D}_{MSP} \cup \{\bar{P}_i\}$ is minimally rich and top-separable, but never multidimensional single-peaked.
- Domain $\mathbb{D}$ satisfies the Interior⁺ property but violates the Exterior⁺ property since there exists no path connecting $\bar{P}_i$ and a preference with peak $(2, 1, 1)$ along which $(2, 1, 1)$ always ranks above $(1, 1, 1)$.
- A two-voter mixed projection rule associating positive weight to every projector other than $(1, 1, 1)$ and $(2, 1, 1)$ is unanimous and strategy-proof, and satisfies the constrained compromise property.

Why do we adopt randomization?

Theorem

Let domain $\mathbb{D}$ be minimally rich and connected$^+$. If it admits a unanimous, anonymous and strategy-proof DSCF, it is multidimensional single-peaked.

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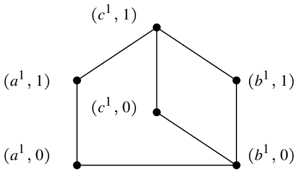
Let domain $\mathbb{D}$ be minimally rich and connected^{+}. If it admits a unanimous, anonymous and strategy-proof DSCF, it is multidimensional single-peaked.

- Generalize the results in Chatterji, Sanver and Sen (2013) and Chatterji and Massó (2018): No restriction on the number of voters, endogenize the tops-only property, and recover the full single-peakedness.
- Elicit a product of tree $\times_{s\in M}G(A^s)$.



- Four cases:

  $f\big((a^1,0),(b^1,1)\big) = (a^1,0)$.

  $f\big((a^1,0),(b^1,1)\big) = (b^1,1)$.

  $f\big((a^1,0),(b^1,1)\big) = (b^1,0)$.

  $f\big((a^1,0),(b^1,1)\big) \notin \{(a^1,0),(b^1,1),(b^1,0)\}$.

- Loosely speaking, all these four cases are covered *simultaneously* in the random setting.

Summary

- Generalized random dictatorships and top-separability.
- Connectedness$^+$ and a characterization of multidimensional single-peakedness in both random and deterministic settings.
- The characterization of multidimensional single-peakedness remains robust to the voting under constraints.

Generalized random dictatorships

Assumption: Let $A = \times_{s \in M} A^s$ where M is finite with $|M| \geq 2$, and A^s is finite with $|A^s| \geq 2$ for all $s \in M$.

- For each $s \in M$, a voter $i^s \in I$ is fixed. A voter sequence: $\underline{i} \equiv (i^s)_{s \in M}$.
- A *generalized dictatorship*: For instance, fix voter sequence $(1, 2)$. Let $r_1(P_1) = (a^1, a^2)$ and $r_1(P_2) = (b^1, b^2)$, we have $f^{\underline{i}}(P_1, P_2) = (a^1, b^2)$.

Definition

An RSCF $\varphi^{GRD} : \mathbb{D}^N \to \Delta(A)$ is a **generalized random dictatorship** if there exists $\gamma(\underline{i}) \geq 0$ for each $\underline{i} \in I^N$ with $\sum_{\underline{i} \in I^N} \gamma(\underline{i}) = 1$ such that for all $P \in \mathbb{D}^N$,

$$\varphi^{GRD}(P) = \sum_{\underline{i} \in I^N} \gamma(\underline{i}) f^{\underline{i}}(P)$$

A GRD does not admit non-assemblable compromise

- For instance, assume $\gamma(\underline{i}) > 0$ for all $\underline{i} \in I^N$.
- Let $r_1(P_1) = (x^1, x^2)$, $r_1(P_2) = (y^1, y^2)$ and $r_2(P_1) = r_2(P_2) = (x^1, y^2)$. Thus, the compromise alternative (x^1, y^2) can be assembled via voter sequence $(1, 2)$ at (P_1, P_2), i.e., $f^{(1,2)}(P_1, P_2) = (x^1, y^2)$. Hence, $\varphi^{GRD}_{(x^1,y^2)}(P_1, P_2) > 0$.
- Let $r_1(P_1) = (x^1, a^2)$, $r_1(P_2) = (y^1, a^2)$ and $r_2(P_1) = r_2(P_2) = (z^1, a^2)$. Thus, the compromise alternative (z^1, a^2) is unable to be assembled via any voter sequence at (P_1, P_2), i.e., $f^{\underline{i}}(P_1, P_2) \neq (z^1, a^2)$ for all $\underline{i} \in I^2$. Hence, $\varphi^{GRD}_{(z^1,a^2)}(P_1, P_2) = 0$

\{}begin{frame}{Elaboration: Necessity}\{}rm
\{}begin{enumerate} \{}item[1.] Two preferences disagreeing on peaks are never adjacent: \{}\{}A consequence of top-separability.\{}medskip
\{}item[2.] Every unanimous and strategy-proof RSCF satisfies \{}textbf{the tops-only property}: given $P, P' \{}in \{}mathbb{D}^{N}$, \{}begin{center} $\{}big[r_{1}(P_{i}) = r_{1}(P_{i}')\{}; \{}textrm{for all}\{}; i \{}in I\{}big] \{}Rightarrow [\{}varphi(P) = \{}varphi(P')]$: \{}end{center}
A consequence of connectedness\{}textsuperscript{+}. \{}medskip
Degenerate $P_{i} \{}sim^{+} P_{i}'$ with $r_{1}(P_{i})\{}equiv a \{}neq b \{}equiv r_{1}(P_{i}')$ to $a \{}sim^{+} b$.\{}medskip
\{}item[3.] If $|A^s| = 2$ for all $s \{}in M$, top-separability implies multidimensional single-peakedness immediately.\{}medskip
\{}item[4.] If $|A^s| > 2$ for some $s \{}in M$, we elicit a product of tree $\{}times_{s \{}in M}G(A^{s})$: A consequence of the constrained compromise property.
\{}end{enumerate}
\{}end{frame}