# Order on Types based on Monotone Comparative Statics

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## Comparative Statics

Comparative statics is one of the most important methodologies in Economics.

Parameter $\Rightarrow$ Optimal Solution or Equilibrium

e.g) Wealth $\Rightarrow$ Consumption

Cost $\Rightarrow$ Production in Cournot

Risk aversion $\Rightarrow$ Portfolio

Classical approach often appeals to implicit function theorem.

## Monotone Comparative Statics

Monotone comparative statics (MCS) is an approach that utilizes the order structure of the game in which whenever a parameter increases, the **set** of equilibria also increases.

- No concavity or differentiability is needed.
- Single-crossing / supermodularity (or strategic complementarity) play a big role.

**Our goal** is to enlarge the applicability of MCS.

## Example: Carlson and van Damme (1993)

| | | Player 2 | |
|---|---|---|---|
| | | Invest | Not Invest |
| Player 1 | Invest | $\theta, \theta$ | $\theta - 1, 0$ |
| | Not Invest | $0, \theta - 1$ | $0, 0$ |

where $\theta \in \mathbb{R}$ is the investment's profitability.

- $\theta > 1 \Rightarrow$ each player has a dominant strategy to invest.
- $\theta \in [0, 1] \Rightarrow$ two pure strategy NE
- $\theta < 0 \Rightarrow$ each player has a dominant strategy not to invest.

## Example: Carlson and van Damme (1993) Cont.

Suppose $\theta$ is common knowledge. As $\theta$ increases, the **set** of equilibrium investment levels increase. Hence, MCS holds.

Milgrom and Shannon (1994) derive a necessary and sufficient condition for MCS to hold. It is the single-crossing condition.

But, what if there is incomplete information about $\theta$?

## Example: Global Games

We use the same investment game but now assume the following:

- Each player $i$ observes $s_i \in \mathbb{R}$ as a noisy signal about $\theta$.

- There is a common prior on $(\theta, s_1, s_2)$.

- $i$'s posterior over $(\theta, s_{-i})$ upon observing $s_i$ is derived via Bayesian updating.

This is the setup often used in global games.

## MCS under Incomplete Information about $\theta$

Athey (2002) extends Milgrom and Shannon (1994):

Suppose the common prior on $(\theta, s)$ exhibits affiliation. Then, as $s_i$ increases, $i$'s equilibrium investment level increases.

To enlarge the applicability of MCS, we dispense with the assumptions Athey made.

- One-dimensional signal structure
- Common prior
- Bayesian updating

## What This Paper Does

We introduce an order on types: $t_i^{'}$ is higher than $t_i$ in the sense of **common certainty of optimism (CCO)** if $t_i^{'}$ is more optimistic that the news is good than $t_i$; $t_i^{'}$ is more optimistic that all are optimistic that the news is good than $t_i$, and so on ad infinitum.

- **Sufficiency**: If $t_i^{'}$ is higher than $t_i$ in the CCO order, $t_i^{'}$ takes a higher action than $t_i$ in **any** supermodular game.

- **Necessity**: There is a supermodular game in which $t_i^{'}$ is "not" higher than $t_i$ in the CCO order $\Rightarrow$ $t_i^{'}$ does "not" take a higher action than $t_i$. **This is our main theoretical contribution.**

# Lattice

Given a set $X$ and a partial order $\geq$: $\forall x, y \in X$, $x \vee y = \inf\{z \in X |\ z \geq x,\ z \geq y\}$ (join) and $x \wedge y = \sup\{z \in X |\ z \leq x,\ z \leq y\}$ (meet).

For $Y \subseteq X$, let $\bigvee Y \in X$ denote the least upper bound ("join") of $Y$, and $\bigwedge Y \in X$ denote the greatest lower bound ("meet") of $Y$.

A **lattice** is a set $X$ together with a partial order $\geq$ on $X$ such that the set is closed under meet and join operations.

A lattice $(X, \geq)$ is **complete** if every subset of $X$ has a meet and a join.

## Complete Info Supermodular Games

$g = \langle I, \prod_{i \in I} A_i, \Theta, (u_i)_{i \in I} \rangle$ denotes a **supermodular** game where

(i) $I = \{1, \ldots, I\}$: Set of Players;

(ii) $A_i$: $i$'s action space; complete metric lattice;

(iii) $\Theta$: a Polish parameter space; complete lattice;

(iv) $u_i : A \times \Theta \to \mathbb{R}$: $i$'s payoff function.

## Complete Info Supermodular Games Cont.

(v) $u_i(\cdot)$ is **supermodular** on $A_i$: $\forall \theta, a_{-i}, a_i, a_i^{'}$,

$$u_i(a_i \vee a_i^{'}, a_{-i}; \theta) + u_i(a_i \wedge a_i^{'}, a_{-i}; \theta) \geq u_i(a_i, a_{-i}; \theta) + u_i(a_i^{'}, a_{-i}; \theta).$$

and

(vi) $u_i(\cdot)$ has **increasing differences** in both $(a_i, a_{-i})$ and $(a_i, \theta)$: $\forall a_i, a_i' \in A_i$, $a_{-i}, a_{-i}^{'} \in A_{-i}$, and $\theta, \theta^{'} \in \Theta$, whenever $(a_{-i}, \theta) \geq (a'_{-i}, \theta')$, it follows that

$$u_i((a;\theta) \vee (a';\theta')) + u_i((a;\theta) \wedge (a';\theta')) \geq u_i(a;\theta) + u_i(a';\theta').$$

## Incomplete Information Supermodular Games

$(T_i, \mathscr{T}_i, \pi_i)_{i \in I}$ is a **type space** where

- $T_i$: $i$'s set of types;
- $\mathscr{T}_i$: a sigma-algebra over $T_i$; and
- $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$: $i$'s $\mathscr{T}_i$-measurable belief map.

$G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$ now describes an **incomplete-information** supermodular game.

## Belief Hierarchies induced by type $t_i$

$h^1(t_i) \in Z_i^1 = \Delta(\Theta)$: the set of player $i$'s **first-order beliefs**;

$h^2(t_i) \in Z_i^1 = \Delta(\Theta \times Z_{-i}^1)$: the set of $i$'s **second-order beliefs**;

$\vdots$

$h^k(t_i) \in Z_i^k = \Delta(\Theta \times Z_{-i}^1 \times \cdots \times Z_{-i}^{k-1})$: $i$'s $k$**th-order beliefs** where $k \geq 2$.

$\vdots$

Finally, $Z_i^\infty = \prod_{k=1}^\infty Z_i^k$: the set of $i$'s all coherent infinite belief hierarchies.

## First-Order Stochastic Dominance (FOSD)

Let $X$ be a Polish space endowed with a closed partial order $\succeq$.

A closed subset $Y \subseteq X$ is an **upper event** of $X$ if, $\forall y, z \in X$, $[y \in Y$ and $z \succeq y] \Rightarrow z \in Y$.

Let $U(X)$ denote the set of all upper events of $X$.

**Definition**: Let $\beta, \beta' \in \Delta(X)$. $\beta'$ (first-order) **stochastically dominates** $\beta$ (denoted $\beta' \succeq_{SD} \beta$) if $\beta'(Y) \geq \beta(Y)$ for any $Y \in U(X)$.

## Common Certainty of Optimism (CCO)

Suppose that (i) $t_i^{'}$ is more optimistic about $\Theta$ than $t_i$; (ii) $t_i^{'}$ is more optimistic about the optimism of other players about $\Theta$; (iii) $t_i^{'}$ is more optimistic about the optimism about the optimism of other players about $\Theta$ than $t_i$; and so on ad infinitum.

In such a case, we say that $t_i^{'}$ is at least high as $t_i$ in the order of **common certainty of optimism** and we denote it by $t_i^{'} \succeq_{CCO} t_i$. Formally:

**Definition**: $t_i^{'} \succeq_{CCO} t_i$ if $h^k(t_i^{'}) \succeq_{SD} h^k(t_i)$ for each $k \in \mathbb{N}$.

## Bayesian Nash Equilibrium (BNE)

Fix $G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$. $\sigma_i : T_i \to A_i$ denotes $i$'s $\mathscr{T}_i$-measurable **pure strategy**.

**Definition**: A strategy profile $\sigma^*$ is a (pure-strategy) **Bayesian Nash equilibrium** if, for each $i \in I$, $t_i \in T_i$, and $a_i \in A_i$,

$$\int_{\Theta \times T_{-i}} \left\{ u_i(\sigma_i^*(t_i), \sigma_{-i}^*(t_{-i}), \theta) - u_i(a_i, \sigma_{-i}^*(t_{-i}), \theta) \right\} d\pi_i(t_i)[\theta, t_{-i}] \geq 0.$$

$\Sigma^*$: the set of "all" BNE of $G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$.

It may be the case that $\Sigma^*$ is empty.

## Lattice Structure of the set of BNE

We call $\underline{\sigma} \in \Sigma^*$ the **least** equilibrium if, for each $\sigma^* \in \Sigma^*$, $i$, and $t_i$, we have $\sigma_i^*(t_i) \succeq_{A_i} \underline{\sigma}_i(t_i)$,

and similarly, call $\bar{\sigma} \in \Sigma^*$ the **greatest** equilibrium if, for each $\sigma^* \in \Sigma^*$, $i \in I$, and $t_i$, we have $\overline{\sigma}_i(t_i) \succeq_{A_i} \sigma_i^*(t_i)$.

In addition, $\Sigma^*$ has the following lattice structure: for any $\sigma^* \in \Sigma^*, i \in I$, and $t_i$, we have that $\bar{\sigma}_i(t_i) \succeq_{A_i} \sigma_i^*(t_i) \succeq_{A_i} \underline{\sigma}_i(t_i)$.

Due to this structure, we only focus on the least equilibrium in the rest of the analysis.

## The Least Interim Correlated Rationalizability (ICR)

Let $A_i^0[t_i] = A_i$ and $\underline{a}_i^0[t_i] = \bigwedge A_i^0[t_i]$.

$$\underline{A}_i^1(t_i) = \arg\max_{a_i \in A_i^0(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}_{-i}^0(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^1(t_i) = \bigwedge \underline{A}_i^1(t_i)$.

We assume that $\underline{a}_i^1(\cdot)$ is a measurable mapping and $\underline{A}_i^1(t_i)$ is a complete sublattice.

$\Rightarrow \underline{a}_i^1(t_i) \in \underline{A}_i^1(t_i)$.

By supermodularity, any $a_i$ such that $a_i \not\geq_{A_i} \underline{a}_i^1(t_i)$ is a never-best response against $\underline{a}_{-i}^0(\cdot)$.

## The Least Interim Correlated Rationalizability (ICR) Cont.

By induction, for each $k \geq 1$,

$$\underline{A}_i^{k+1}(t_i) = \text{arg} \max_{a_i \in A_i^k(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}^k_{-i}(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^{k+1}(t_i) = \bigwedge \underline{A}_i^{k+1}(t_i)$.

Again, we assume $\underline{a}_i^{k+1}(\cdot)$ is a measurable mapping and $\underline{A}_i^{k+1}(t_i)$ is a complete sublattice.

$\Rightarrow \underline{a}_i^{k+1}(t_i) \in \underline{A}_i^{k+1}(t_i)$.

By supermodularity, any $a_i$ such that $a_i \not\geq_{A_i} \underline{a}_i^{k+1}(t_i)$ is a never-best response against $\underline{a}^k_{-i}(\cdot)$.

## The Least Interim Correlated Rationalizability (ICR) Cont.

Finally, define

$$\underline{a}_i^\infty(t_i) \;=\; \bigvee\{\underline{a}_i^1(t_i), \underline{a}_i^2(t_i), \ldots\}.$$

$A_i$ is a complete lattice $\Rightarrow \underline{a}_i^\infty(t_i) \in A_i$.

if $\underline{a}_i^\infty(t_i)$ is a best response to $\underline{a}_{-i}^\infty(\cdot) \Rightarrow \underline{\sigma}$ defined by $\underline{\sigma}_i(t_i) = \underline{a}_i^\infty(t_i)$ constitutes an equilibrium.

By construction, $\underline{\sigma}$ must be the least equilibrium of the game.

## Characterization of the Least Equilibrium

Therefore,

**Proposition**: Assume that, for each $i, t_i$, and $k \geq 1$, (i) $\underline{A}_i^k(t_i)$ is a complete sublattice, (ii) $\underline{a}_i^k(\cdot) = \bigwedge \underline{A}_i^k(\cdot)$ is a measurable mapping, and (iii) $\underline{a}_i^\infty(t_i)$ is a best response to $\underline{a}_{-i}^\infty(\cdot)$. Then, $\underline{\sigma}$ defined by $\underline{\sigma}_i(t_i) = \underline{a}_i^\infty(t_i)$ for each $i$ and $t_i$ constitutes the least equilibrium.

Van Zandt and Vives (2007) propose more primitive assumptions for the existence of the least equilibrium: (i) $A_i$ is a compact metric lattice; (ii) $u_i(\cdot)$ is bounded, continuous in $a_i$ and measurable in $\theta$; and (iii) $\pi_i(\cdot)$ is measurable.

## Sufficiency of Common Certainty of Optimism for MCS

**Theorem**: Let $G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$ be an incomplete information supermodular game that satisfies: for each $i \in I$, $t_i \in T_i$, and $k \geq 1$, (i) $\underline{A}_i^k(t_i)$ is a complete sublattice; (ii) $\underline{a}_i^k(\cdot) = \bigwedge \underline{A}_i^k(\cdot)$ is a measurable mapping; and (iii) $\underline{a}_i^\infty(t_i)$ is a best response to $\underline{a}_{-i}^\infty$.

Then, $t_i' \succeq_{CCO} t_i \Rightarrow \underline{\sigma}_i(t_i') \succeq_{A_i} \underline{\sigma}_i(t_i)$.

## Necessity of CCO: Optimism-Elicitation Game

This is our main result.

**Theorem**: There is a supermodular game with the property that, for any player $i \in I$ and two types $t_i, t_i^{'}$, we have that $t_i^{'} \succeq_{CCO} t_i$ if and only if $\underline{\sigma}_i(t_i^{'}) \succeq_{A_i} \underline{\sigma}_i(t_i)$, where $\underline{\sigma}$ is the least equilibrium of this supermodular game.

## Flavor of the Proof: a Single Agent Case

**Step 1**: Any upper set on $\Theta$ can be approximated by a countable set.

Each $U_n$ denotes an upper set such that the closure of $\bigcup_{n=1}^{\infty} U_n$ is equivalent to the set of all upper sets.

**Step 2**: The agent's strategy is $\beta : U_n \mapsto [0,1]$ and $\beta$ is monotone: $U_n \subseteq U_m \Rightarrow \beta(U_n) \leq \beta(U_m)$.

$\beta$ is defined as a capacity rather than a probability measure so that $B = \{\beta : U_n \mapsto [0,1] | \ \beta \text{ is monotone}\}$ constitutes a complete lattice. If we choose a topology on $B$ right, we can make $B$ a compact metric space.

**Step 3**: The agent's payoff function using a strategy $\beta$ in state $\theta$ is

$$u(\beta, \theta) = \sum_{n=1}^{\infty} \left[ \beta(U_n) \mathbf{1}_{U_n}(\theta) - \frac{\beta(U_n)^2}{2} \right] \mu(U_n),$$

where $\mathbf{1}_{U_n}$ denotes the indicator function and $\mu$ is a full support distribution over all $\{U_n\}$.

**Step 4**: It is always optimal to choose the truthful probability assessment of $U_n$.

## Flavor of How to Extend to the Multiple Players Case

Set $X^1 = \Theta$; $X^2 = (\Delta(X^1))^{I-1}$; and $X^k = (\Delta(X^1 \times \cdots \times X^{k-1}))^{I-1}$ for each $k \geq 3$, where $I$ stands for the number of players.

Finally, define $X^\infty = \prod_{k=1}^{\infty} X^k$.

**Step I**: Any upper set over $X^k$ can be approximated by a countable set.

Each $U_n^{(k)}$ denotes an upper set on $X^k$ such that the closure of $\bigcup_{n=1}^{\infty} U_n^{(k)}$ is equivalent to the set of all upper sets on $X^k$.

**Step II**: Each agent's strategy $\beta = (\beta^k)_{k=1}^{\infty}$ is such that $\beta^k : U_n^{(k)} \mapsto [0,1]$.

$\beta^k$ is monotone: $U_n^{(k)} \subseteq U_m^{(k)} \Rightarrow \beta^k(U_n^{(k)}) \leq \beta^k(U_m^{(k)})$

**Step III**: Each agent's payoff function using strategy $\beta$ in state $x \in X^{\infty}$ is

$$u(\beta, x) = \sum_{k=1}^{\infty} \delta^{k-1} \left[ \sum_{n=1}^{\infty} \left[ \beta^k(U_n) \mathbf{1}^k_{U_n^{(k)}}(x^k) - \frac{(\beta^k(U_n^{(k)}))^2}{2} \right] \mu^k(U_n) \right],$$

where $0 < \delta < 1$; $x^k$ is the restriction of $x$ to $X^k$; $\mathbf{1}^k_{U_n^{(k)}}$ denotes the indicator function on $X^k$; and $\mu^k$ is a full support distribution over all $\{U_n^{(k)}\}$.

**Step IV**: The unique rationalizable strategy profile leads to each agent's choosing the truthful probability assessment of $U_n^{(k)}$.

So, $\underline{\sigma}_i(t_i) = \bar{\sigma}_i(t_i)$.

## Summary

- This paper introduces an order on types by which MCS is valid in all supermodular games with incomplete information.

- We fully characterize this order in terms of **common certainty of optimism**: $t_i^{'}$ is higher than $t_i$ if $t_i^{'}$ is more optimistic that the news is good for all than $t_i$; $t_i^{'}$ is more optimistic that all are more optimistic that the news is good for all than $t_i$, and so on ad infinitum.

- Our work-in-progress investigates all possible orders on types induced by stochastic dominance and shows that our CCO order is the maximal one.