Pure-Strategy Equilibria in Bayesian Games

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Model

- Players: $I = \{1, 2, \ldots, n\}$.
- Actions: each player $i \in I$ has a finite action space X_i. Let $X = \prod_{1 \leq j \leq n} X_j$.
- Types: each player $i \in I$ has a type space T_i with a σ-algebra $\mathcal{T}_i$ on T_i. Let $T = \prod_{j=1}^{n} T_j$ and $\mathcal{T} = \otimes_{j=1}^{n} \mathcal{T}_j$.

Model

- Information structure: λ a probability measure on $(T, \mathcal{T})$ as the common prior, and λ_i the marginal probability measure of λ on T_i.
- Payoffs: each player $i \in I$ has a payoff function u_i from $X \times T$ to $\mathbb{R}_+$ such that $u_i(x, \cdot)$ is $\mathcal{T}$-measurable for each $x \in X$.

Strategy and Expected Payoff

- A behavioral strategy (resp. pure strategy) of player $i \in I$ is a measurable function from T_i to $\mathcal{M}(X_i)$ (resp. X_i).
- Given a behavioral strategy profile $f = (f_1, \ldots, f_n)$, player i's expected payoff is:

  $$U_i(f) = \int_T \int_X u_i(x,t) \prod_{j \in I} f_j(\mathrm{d}x_j | t_j) \lambda(\mathrm{d}t).$$

Bayesian Nash Equilibrium

A behavioral strategy profile $f^* = (f_1^*, f_2^*, \ldots, f_n^*)$ is a Bayesian Nash Equilibrium if for each $i \in I$,

$$U_i(f_i^*, f_{-i}^*) \geq U_i(g_i, f_{-i}^*)$$

holds for any behavioral strategy g_i of player i.

Equilibrium Existence in Behavioral Strategy?

- Harsanyi: when the joint type space T is finite, Bayesian Nash equilibrium exists in behavioral strategy.
- What happens if T is infinite?
- The answer is no in general! For example: Simon (2003).
- The key difference between the cases of finite types and infinite types is on the joint continuity of players' expected payoffs.

Absolutely Continuous Information

- Assume that λ is absolutely continuous with respect to the product probability measure $\bigotimes_{1\leq j\leq n}\lambda_j$ on $(T,\mathcal{T})$. Let q be the density function of λ with respect to $\bigotimes_{1\leq j\leq n}\lambda_j$.
- This assumption ensures the joint continuity for the expected payoffs.
- Milgrom-Weber (1985): Under the assumption of absolutely continuous information, Bayesian Nash equilibrium exists in behavioral strategy.

Equilibrium Existence in Pure Strategy?

- Dispersed information: for each $i \in I$, λ_i is atomless.
- Under absolutely continuous and dispersed information, does Bayesian Nash equilibrium exist in pure strategy?
- Radner and Rosenthal (1982) showed that the answer is no in general.
- This paper: to characterize the existence of pure-strategy Bayesian-Nash equilibria.

$\mathcal{G}$-atom

- Let $(S, \mathcal{S}, \mu)$ be a probability space, $\mathcal{G}$ a sub-σ-algebra of $\mathcal{S}$, a set $D \in \mathcal{S}$ with $\mu(D) > 0$.
- Let $\mathcal{G}^D$ and $\mathcal{S}^D$ be the restricted σ-algebras $\{D \cap D' \colon D' \in \mathcal{G}\}$ and $\{D \cap D' \colon D' \in \mathcal{S}\}$ on D.

$\mathcal{G}$-atom

- D is said to be a $\mathcal{G}$-atom if the restricted σ-algebras $\mathcal{G}^D$ and $\mathcal{S}^D$ are essentially the same; i.e., $\forall E \in \mathcal{S}^D$, $\exists E' \in \mathcal{G}^D$ such that $\mu(E \triangle E') = 0$.

- When $\mathcal{G}$ is the trivial σ-algebra $\{S, \emptyset\}$, D is a $\mathcal{G}$-atom iff for any $E \in \mathcal{S}$ with $E \subseteq D$, $\mu(E)$ is equal to $\mu(D)$ or 0; i.e., D is an atom in the usual sense.

Density Weighted Payoff

- Blanket assumption: the Bayesian game has absolutely continuous and dispersed information.
- The **density weighted payoff** of player j is:

$$w_j(x,t) = u_i(x,t) \cdot q(t).$$

Simple Coarser Inter-Player Information

Let $\mathcal{G}_i$ be a countably generated sub-σ-algebra of $\mathcal{T}_i$. Player i is said to have **simple coarser inter-player information** if $\mathcal{T}_i$ has no $\mathcal{G}_i$-atom under λ_i, and $w_l(x, \cdot, t_{-i})$ is $\mathcal{G}_i$-measurable for all $x \in X$, $t_{-i} \in T_{-i}$ and $l \neq i$.

Coarser Inter-Player Information

Player i is said to have **coarser inter-player information** if $\mathcal{T}_i$ has no $\mathcal{G}_i$-atom under λ_i, and for some positive integer J and each $l \neq i$,

$$w_l(x,t) = \sum_{1 \le j \le J} \left(w_l^j(x,t) \prod_{s \in I} \rho_s^j(t_s) \right),$$

where for $j = 1, \ldots, J$,

1. $w_l^j(x,\cdot)$ is $\otimes_{1 \le s \le n} \lambda_s$-integrable and $w_l^j(x,\cdot,t_{-i})$ is $\mathcal{G}_i$-measurable for all $x \in X$ and $t_{-i} \in T_{-i}$,
2. ρ_s^j is nonnegative and integrable on $(T_s, \mathcal{T}_s, \lambda_s)$.

Coarser Inter-Player Information

- $\mathcal{G}_i$ models player i's information flow to all other players, which describes the influence of player i's private information in other players' density weighted payoffs.
- A Bayesian game is said to have coarser inter-player information if each player has coarser inter-player information.
- The condition of coarser inter-player information means that each player's private information can fully influence her own density weighted payoff, but only partially influence other players' density weighted payoffs.

Remarks

- A Bayesian game with coarser inter-player information allows players' payoffs to be interdependent and types to be correlated.
- When players have independent types ($q \equiv 1$) and private values, the functions $w_j(x, \cdot, t_{-i}) = u_j(x, t_j), x \in X, t_{-i} \in T_{-i}, j \in I$, with $j \neq i$ do not depend on t_i.
  Hence, one can take $\mathcal{G}_i$ to be the trivial σ-algebra $\{T_i, \emptyset\}$. The condition of "coarser inter-player information" is trivially satisfied because λ_i is atomless.

Equilibrium Existence in Pure Strategies: Sufficiency

Theorem 1: Every Bayesian game with coarser inter-player information has a pure-strategy Bayesian-Nash equilibrium.

Theorem 1 covers the pure-strategy equilibrium existence results in some special cases as considered in Radner and Rosenthal (1982), Milgrom and Weber (1985), Khan, Rath and Sun (2006), Fu et al. (2007), Barelli and Duggan (2015).

Equilibrium Existence in Pure Strategies: Necessity

- Fix $n \geq 2$, and the player space $I = \{1, 2, \ldots, n\}$. For each $i \in I$, player i has type space $(T_i, \mathcal{T}_i, \lambda_i)$ and inter-player information $\mathcal{G}_i$ with $(T_i, \mathcal{G}_i, \lambda_i)$ being atomless.
- Let H_n be the collection of all Bayesian games with the player space I, private information spaces $\{(T_i, \mathcal{T}_i/\mathcal{G}_i, \lambda_i)\}_{i \in I}$.

Equilibrium Existence in Pure Strategies: Necessity

Theorem 2: If one of the following assumptions holds

1. any Bayesian game in H_n with type-irrelevant payoffs has a pure-strategy Bayesian-Nash equilibrium,
2. any Bayesian game in H_n with independent types has a pure-strategy Bayesian-Nash equilibrium,

then every Bayesian game in H_n has coarser inter-player information.

Purification and Closed Graph Property

- Purification: giving a Bayesian-Nash equilibrium in behavioral strategy, find an equivalent pure-strategy Bayesian-Nash equilibrium.

- Closed graph property: any sequence of pure-strategy equilibria for a sequence of Bayesian games converging to a limit game, should possess a subsequence which converges to a pure-strategy equilibrium of the limit game.

- Both properties can be characterized by the condition of coarser inter-player information.

Monotone Equilibrium and Continuous Choices

In Bayesian games with continuous choices and special (order) structures, the standard procedure for obtaining monotone pure-strategy equilibria (*e.g.*, Athey (2001), McAdams (2003), and Reny (2011)):

1. Construct a sequence of finite-action Bayesian games and show that there exists a monotone pure-strategy equilibrium in each game;
2. a limit argument is then applied to show that the sequence of monotone pure-strategy equilibria converges to a monotone pure-strategy equilibrium in the original Bayesian game.

An All-Pay Auction: I

An all-pay auction G with interdependent values, multidimensional types and CARA preferences.

1. The set of bidders: $I = \{1, \ldots, n\}$.
2. Each bidder i has a private signal t_i from $T_i = \prod_{1 \leq l \leq n} [0, \bar{t}_i^l] \subseteq \mathbb{R}_+^n$.
3. The private signal t_i is affiliated, and is drawn according to λ_i with a density function q_i, independent of other bidders' signals.

An All-Pay Auction: II

1. Each bidder i submits a sealed bid b_i from $B_i = \{o\} \cup [\underline{b}_i, \bar{b}_i]$ with $\bar{b}_i > \underline{b}_i \geq 0$, where $o < 0$ represents the outside option, and $[\underline{b}_i, \bar{b}_i]$ is the set of feasible bids with the reserve price $\underline{b}_i$.
2. The constant risk aversion level of bidder i is $\alpha_i > 0$.
3. Given the final value r, the payoff of bidder i is

$$u_i(r) = \frac{1 - e^{-\alpha_i r}}{\alpha_i}.$$

An All-Pay Auction: III

The value of bidder i at the signal profile $(t_1, \ldots, t_n)$ is given by

$$v_i(t_i^i, t_{-i}^i) = t_i^i + \sum_{l \neq i} \kappa_l^i t_l^i + c_i,$$

where $\kappa_l^i \geq 0$ for $1 \leq l \leq n$ and $c_i \geq 0$.

1. The coordinate t_l^i of t_l is the value of bidder i;
2. c_i can be viewed as the initial wealth of bidder i.

If bidder i wins the good by bidding $b_i \neq o$, then her payoff is $u_i(v_i(t_i^i, t_{-i}^i) - b_i)$. If bidder i loses by bidding $b_i \neq 0$, then she gets $u_i(c_i - b_i)$. If bidder i chooses o, then she gets $u_i(c_i)$.

An All-Pay Auction: IV

Claim

The auction G has a pure-strategy monotone equilibrium.

1. We first construct a sequence of auctions $\{G^k\}_{k=1}^{\infty}$ with discretized bidding sets. For k sufficiently large, we verify that the condition of coarser inter-player information is satisfied in each G^k, and hence a pure-strategy equilibrium exists.
2. We then prove the existence of a monotone pure-strategy equilibrium in the auction G by following a limit argument.

Conditional expectations of correspondences

Let $(S, \mathcal{S}, \mu)$ be a probability space, $\mathcal{G}$ a sub-σ-algebra of $\mathcal{S}$, and F a correspondence from S to $\mathbb{R}^m$.

Define $\mathcal{I}_F^{(\mathcal{S},\mathcal{G})}$ to be the set

$\{\mathbb{E}(f|\mathcal{G})\colon f$ is an integrable selection of F on $(S, \mathcal{S}, \mu)\}$

Conditional expectations of correspondences

Conditional version of Lyapunov's Convexity Theorem [Sole (1972), Dynkin and Evstigneev (1976), also special cases in Maharam (1942), Jacobs (1978), etc]:

if $\mathcal{S}$ has no $\mathcal{G}$-atom, then for any $\mathcal{S}$-measurable, integrably bounded, closed valued correspondence F,

$$\mathcal{I}_F^{(\mathcal{S},\mathcal{G})} = \mathcal{I}_{\text{co}(F)}^{(\mathcal{S},\mathcal{G})}.$$

Conditional expectations of correspondences

1. The set $\mathcal{I}_F^{(\mathcal{S},\mathcal{G})}$ is convex for any correspondence F if and only if $\mathcal{S}$ has no $\mathcal{G}$-atom.

2. The set $\mathcal{I}_F^{(\mathcal{S},\mathcal{G})}$ is weakly compact (resp. weak* compact) in $L_p^{\mathcal{G}}(S, \mathbb{R}^m)$ when $1 \leq p < \infty$ (resp. $p = \infty$) for any p-integrably bounded and closed valued correspondence F if and only if $\mathcal{S}$ has no $\mathcal{G}$-atom.

3. Preservation of upper hemicontinuity is also equivalent to $\mathcal{S}$ having no $\mathcal{G}$-atom.

Thank you!