# Equivalence of Stochastic and Deterministic Mechanisms

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# Optimal Auction Design (*Myerson (1981)*)

- The optimal auction is deterministic in Myerson's setting.

# Multidimensional Screening

- *McAfee and McMillan (1988)*
  - The optimality of deterministic mechanisms generalizes to multi-dimensional environments?
- This has been proved wrong with one agent; see, for example,
  - *Thanassoulis (2004)*
  - *Manelli and Vincent (2006, 2007)*
  - *Hart and Reny (2015)*
  - *Rochet and Thanassoulis (2015)*

# In This Paper

We show

the optimality of deterministic mechanisms

in remarkably general environments with multiple agents.

# Preview of Result

General social choice environment that has multiple agents, a finite set of alternatives, and independent and dispersed information

- For any BIC mechanism, there exists a deterministic mechanism that
  - i) BIC;
  - ii) the same interim expected allocation probabilities for all agents;
  - iii) the same interim expected utilities for all agents;
  - iv) the same ex ante social surplus.
- In environments with monetary transfers, the deterministic mechanism also guarantees
  - v) the same ex post transfers;
  - vi) the same expected revenue.

# Preview of Result

- Besides the standard social choice environments with linear utilities and independent, one-dimensional, private types,
    - a richer class of utility functions
    - multi-dimensional types
    - interdependent valuations
    - where monetary transfers are not feasible

- Example, *Jehiel and Moldovanu (2001).*

# The Optimality of Deterministic Mechanism

- Sharp contrast with the results in the screening literature.
- This is also surprising in view of the extant literature that studies the benefit of randomness (with multiple agents).
    - *Chawla, Malec, and Sivan (2015)* establish a constant factor upper bound for the benefit of randomness when the agents' values are independent. In the special case of multi-unit multi-item auctions, the revenue of any randomized mechanism is at most 33.75 times the revenue of the optimal deterministic mechanism.

# Deterministic Implementation of Stochastic Mechanisms

- The mechanism design literature essentially builds on the assumption that a mechanism designer can credibly commit to any outcome of a mechanism.
- Stochastic mechanism
  - A randomization device might not be available.
  - Imperfect commitment of the mechanism designer.

# Deterministic Implementation of Stochastic Mechanisms

> *"Ensuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced, because any deviation away from a given randomization can only be statistically detected once sufficiently many realizations of the contracts have been observed. . . . The enforcement of such stochastic mechanisms is thus particularly problematic."*
>
> *Laffont and Martimort (2002)*

Our result $\Longrightarrow$

*any mechanism, including any optimal mechanism (in terms of efficiency or revenue), can in fact be deterministically implemented,*

and thereby irons out the conceptual difficulties associated with stochastic mechanisms.

# Mechanism Equivalence

- Dominant-strategy mechanisms; see
  - *Manelli and Vincent (2010)*
  - *Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013)*).
- Symmetric auctions
  - See *Deb and Pai (2017)*.
- Deterministic mechanisms
  - This paper: Our result $\Longrightarrow$
    *the requirement of deterministic mechanisms is not restrictive in itself.*

# Notation

- $\mathcal{I} = \{1, 2, ..., I\}, \mathcal{K} = \{1, 2, ..., K\}$.
- $v_i \in V_i$, closed convex subset of finite dimensional Euclidean space.
- $V = V_1 \times V_2 \times ... \times V_I$ with generic element $v = (v_1, v_2, \ldots, v_I)$.
- $v_{-i} \in V_{-i} = \Pi_{j \neq i} V_j$.
- $\lambda$: common prior distribution on $V$.
  - Types are independent
  - $\lambda_i$ atomless

# Mechanism

- Restrict attention to direct mechanisms characterized by $K + I$ functions, $\{q^k(v)\}_{k \in \mathcal{K}}$ and $\{t_i(v)\}_{i \in \mathcal{I}}$.
  - where $q^k(v) \geq 0$ is the probability that alternative $k$ is implemented,
  - $t_i(v)$ is the transfer agent $i$ makes to the mechanism designer.
- Agent $i$'s gross utility in alternative $k$ is $u_i^k(v_i, v_{-i})$.

# Mechanism

- Write $Q_i^k(v_i) = E_{v_{-i}}(q^k(v_i, v_{-i}))$ and $T_i(v_i) = E_{v_{-i}}(t_i(v_i, v_{-i}))$.
- Agent i's interim expected utility is

$$U_i(v_i) = \int_{V_{-i}} \left[ \sum_{1 \le k \le K} u_i^k(v_i, v_{-i}) q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \right] \lambda_{-i}(dv_{-i})$$

$$= \int_{V_{-i}} \left[ \sum_{1 \le k \le K} u_i^k(v_i, v_{-i}) q^k(v_i, v_{-i}) \right] \lambda_{-i}(dv_{-i}) - T_i(v_i)$$

# Deterministic Mechanism

**Definition**

A mechanism $(q, t)$ is deterministic if for all type profiles, the mechanism implements some alternative $k$ for sure. That is, for all $v \in V, q^k(v) = 1$ for some $1 \leq k \leq K$.

# Mechanism Equivalence

**Definition**

Two mechanisms $(q, t)$ and $(\tilde{q}, \tilde{t})$ are equivalent if and only if they deliver

1. the same interim expected allocation probabilities;
2. the same interim expected utilities for all agents; and
3. the same ex ante expected social surplus for the mechanism designer.

# Illustrating Example

- A single-unit auction with two bidders.
- $(v_1, v_2)$ uniformed distributed on the square $[0,1]^2$.
- Consider the following stochastic mechanism $(q,t)$, where

$$q_1(v_1,v_2) = v_1, \quad q_2(v_1,v_2) = 1 - q_1(v_1,v_2).$$

# Illustrating Example

The interim expected probability of bidder 1 getting the object is

$$\int_0^1 q_1(v_1, v_2)\, \mathrm{d}v_2 = \int_0^1 v_1\, \mathrm{d}v_2 = v_1$$

for all $v_1$, and the interim expected probability of bidder 2 getting the object is

$$\int_0^1 q_2(v_1, v_2)\, \mathrm{d}v_1 = \int_0^1 (1 - v_1)\, \mathrm{d}v_1 = \frac{1}{2}$$

for all $v_2$.

# Illustrating Example

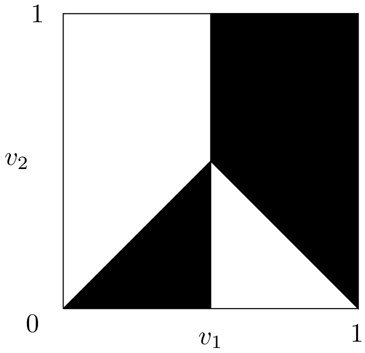


Figure: Bidder 1 is allocated the object in the shaded region, and bidder 2 is allocated the object in the unshaded region.

# Interim Expected Allocation Probabilities

**Theorem**

*For any allocation rule $q$, there exists a deterministic allocation rule $\hat{q}$ such that $q$ and $\hat{q}$ induce the same interim expected allocation probabilities for all agents; that is, for all $i \in \mathcal{I}$ and all $v_i \in V_i$,*

$$\mathbb{E}(\hat{q}|v_i) = \mathbb{E}(q|v_i). \tag{1}$$

# Mutual Purification

Theorem

*Let $h$ be a function from $V$ to $\mathbb{R}^N_{++}$ for some positive integer $N$. For any allocation rule $q$, there exists a deterministic allocation rule $\hat{q}$ such that for all $i \in \mathcal{I}$ and all $v_i \in V_i$,*

$$\mathbb{E}(\hat{q}h_j|v_i) = \mathbb{E}(qh_j|v_i) \tag{2}$$

*for all $1 \leq j \leq N$.*

# Separable Payoff

## Definition

For all $i \in \mathcal{I}$, agent $i$ is said to have separable payoff if for any $k \in \mathcal{K}$ and type profile $v \in V$,

$$u_i^k(v) = \sum_{1 \leq m \leq M} w_{i,m}^k(v_i)\, r_{i,m}^k(v_{-i}).$$

# Main Result

**Theorem**

*Suppose that for each agent $i \in \mathcal{I}$, her payoff function is separable. For any BIC mechanism $(q, t)$, there exists a deterministic mechanism $(\hat{q}, t)$ such that*

(1) *the same interim expected allocation probability;*

(2) *the same ex ante expected social surplus;*

(3) *the same interim expected utilities; and*

(4) $(\hat{q}, t)$ *is BIC.*

# Assumptions: Counterexamples

- The requirement of multiple agents
- Independence across agents
- Atomless distribution
- Separable payoff

# Remarks

Implications:

- Ex post payment, which further implies revenue
- No monetary transfers
- Coalitional version of the result

What else:

- Symmetric mechanism
- Construction of approximately equivalent mechanisms

# Conclusion

- For any BIC mechanism, there exists an equivalent deterministic mechanism
- Implications
  - The optimality of deterministic mechanisms.
  - Deterministic implementation of stochastic mechanisms.
  - The requirement of deterministic mechanisms is not restrict in itself.

# Sketch of Proof: Step 1

The proof proceeds in three steps.

For any $q \in \Upsilon$, let

$$\Upsilon_q = \{q' \in \Upsilon : \mathbb{E}(q'|v_i) = \mathbb{E}(q|v_i) \text{ for all } i \in \mathcal{I} \text{ and } \lambda_i\text{-almost all } v_i \in V_i\}.$$

1. The set $\Upsilon_q$ is nonempty, convex, and weakly compact.
2. $\Upsilon_q$ admits an extreme point.

# Sketch of Proof: Step 2

1. The aim is to show that all the extreme points of $\Upsilon_q$ are deterministic allocation rules.
2. We shall show that if an allocation $q'$ is not deterministic, then $\exists \bar{q},\ \bar{\bar{q}} \in \Upsilon_q$ such that

$$q' = \frac{1}{2}(\bar{q} + \bar{\bar{q}}).$$

$\implies$ $\exists$ a deterministic allocation rule $\tilde{q} \in \Upsilon_q$.

# Sketch of Proof: Step 2

If an allocation $q'$ is not deterministic, then there exists

(1) $0 < \delta < 1$;

(2) a Borel measurable set $D \subseteq V$ with $\lambda(D) > 0$; and

(3) indices $j_1, j_2$

such that

$$\delta \leq q'^{j_1}(v),\ q'^{j_2}(v) \leq 1 - \delta$$

for any $v \in D$.

# Sketch of Proof: Step 2

1. Consider the following system of equations where $\alpha \in L_\infty(D, \mathbb{R})$ are the unknown:

$$\int_{D_{-i}(v_i)} \alpha(v_i, v_{-i})\, \lambda_{-i}(\mathrm{d}v_{-i}) = 0. \tag{3}$$

   for all $i \in \mathcal{I}$ and $v_i \in D_i$, where $D_{-i}(v_i) = \{v_{-i} : (v_i, v_{-i}) \in D\}$.

2. $\lambda_i$ is atomless for all $i \in \mathcal{I} \implies$ a nontrivial bounded solution $\alpha$ with $|\alpha| \le \delta$.
3. Since $\alpha$ is defined on $D$, we extend the domain of $\alpha$ to $V$ by setting $\alpha(v) = 0$ whenever $v \notin D$.

Construct $\bar{q}$ and $\bar{\bar{q}}$ as follows: for all $v \in V$,

$$\bar{q}(v) = q'(v) + \alpha(v)(e_{j_1} - e_{j_2});$$
$$\bar{\bar{q}}(v) = q'(v) + \alpha(v)(e_{j_2} - e_{j_1}).$$

where $e_{j_1}$ and $e_{j_2}$ are the standard basis vectors.

# Sketch of Proof: Step 3

1. Step (1) and Step (2) together imply that there exists $\tilde{q} \in \Upsilon_q$ that is deterministic at $\lambda$-almost all $v \in V$. Note that $\tilde{q}$ is not necessarily deterministic at all $v \in V$.
2. Furthermore, such $\tilde{q}$ induces the same interim expected allocation probabilities for all $i \in \mathcal{I}$ and $\lambda$-almost all $v_i \in V_i$, but not for all $v_i \in V_i$.
3. In this step, we construct a deterministic allocation rule $\hat{q}$ such that

$$\mathbb{E}(q'|v_i) = \mathbb{E}(q|v_i)$$

for all $i \in \mathcal{I}$ and $v_i \in V_i$, by modifying $\tilde{q}$ on sets of measure zero.