Robust Persuasion of a Privately Informed Receiver

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Introduction

- This paper studies robust Bayesian persuasion of a privately informed receiver in which the sender has limited knowledge about the receiver's private information.
- Examples include:
  - rating agency v.s. investor,
  - school v.s. employer.
- When the sender does not know the receiver's source of private information,
  - does there exist an information disclosure rule that strictly benefits the sender?
  - if so, what is the sender's optimal disclosure rule?

Introduction

- Our model is built on Kamenica and Gentzkow (2011, KG henceforth).
- We model different sources of private information by different private information structures.
- The sender does not know from which information structure the receiver's private information is generated.
- Rather, the sender only knows what private information structures are possible.
- The sender has max-min expected utility function and thus he must design information that is robustly good to every possible receiver's private information structure.

Main Results

- When the sender has no knowledge about the receiver's private information, full information disclosure is optimal.
- When the sender's uncertainty about the receiver's private information vanishes, the sender can do almost as well as when the receiver does not have private information.
- In a 2×2 example, we fully characterize the sender's optimal information design.
  - In general, the sender's optimal information design may involve either finite many or a continuum of signals.

Related Literature

- Bayesian persuasion: Kamenica and Gentzkow (2011)
- Bayesian persuasion of a privately informed receiver: Rayo and Segal (2010), Section VI.A of Kamenica and Gentzkow (2011), Guo and Shmaya (2017), Kolotilin et al. (2017) and Kolotilin (2017).
- Foundation of max-min utility: Gilboa and Schmeidler (1989)
- Robust mechanism design under ambiguity aversion:
  - Frankel (2014),
  - Garrett (2014),
  - Carroll (2015),
  - Carrasco, Luz, Kos, Messner, Monteiro and Moreira (2017)

Outline

- Model setup
- Simplifying the problem
- Convexification lemma
- Value of persuasion under full and local ambiguity
- Optimal persuasion in the 2×2 example

Model

A 2×2 example

- Two states, $i = 1$ or $i = 2$.
- Two actions, $a = 1$ or $a = 2$.
- The receiver's ex post payoff

$$u(a, i) = \begin{cases} 1, & \text{if } a = i, \\ 0, & \text{if } a \neq i. \end{cases}$$

- The sender's ex post payoff

$$v(a, i) = \begin{cases} 1 & \text{if } a = 2, \\ 0 & \text{if } a = 1. \end{cases}$$

Model

A 2×2 example

- A belief $p \in [0,1]$: probability of state $i = 2$.
- So

$$a[p] = \begin{cases} 1, & \text{if } p < 1/2, \\ 2 & \text{if } p \geq 1/2. \end{cases}$$

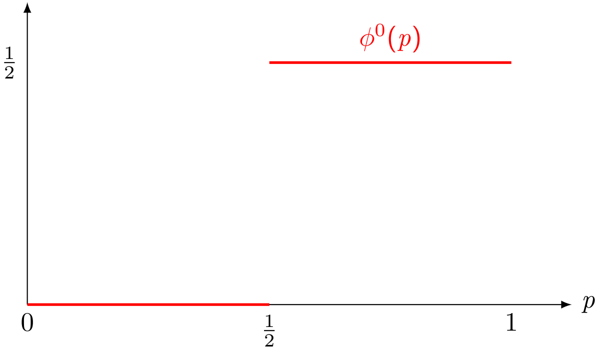


Figure 1: *Ex ante expected payoff*

Model

Receiver's private information

- The receiver receives a private signal.
- The signal is drawn from an information structure $I_r = (S_r, \mu_1, \cdots, \mu_N)$ ($N = 2$ is the number of states) consists of
  - signal space S_r, and
  - conditional distributions of signals, $\mu_i \in \Delta(S)$ for $i = 1, \ldots, N$.
- Given a common initial belief $p \in \Delta^{N-1}$ and I_r, each signal realization $s \in S_r$ leads to an updating of belief via Bayes rule

$$q^{I_r}(p, s) \equiv \left(\frac{p_1 f_1(s)}{\sum_i p_i f_i(s)}, \ldots, \frac{p_N f_N(s)}{\sum_i p_i f_i(s)} \right) \in \Delta^{N-1},$$

  where f_i is the R-N derivative of μ_i w.r.t. $\mu_0 \equiv (\sum_i \mu_i)/N$.
- $\mathcal{I}$: the set of all information structures

Model

Sender's information design problem

- The sender neither observes the receiver's signal nor is aware of from which source it is generated.
- The sender only and correctly knows the receiver's private IS is contained in a collection $\widehat{\mathcal{I}} \subseteq \mathcal{I}$.
- The sender designs $I_s = (S_s, \nu_1, \ldots, \nu_N)$.
- After receiving $s_r \in S_r$ and $s_s \in S_s$, the receiver's *posterior belief* is

$$q^{I_s}(q^{I_r}(p, s_r), s_s) \in \Delta^{N-1}$$

  and chooses $a\big[q^{I_s}(q^{I_r}(p, s_r), s_s)\big] \in A$ to maximize expected payoff.

Simplifying the sender's problem

Receiver's private belief distributions

- We focus on *belief based* $\widehat{\mathcal{I}}$:

$$\widehat{\mathcal{I}} = \left\{ I \in \mathcal{I} \middle| \operatorname{supp}(q^I(p)) \subseteq \widehat{\Delta} \right\},$$

  where $\widehat{\Delta} \subseteq \Delta^{N-1}$ is compact, convex and $p \in \widehat{\Delta}$.
- We write $\widehat{\mathcal{I}}(\widehat{\Delta}, p)$.
- Examples:
  - $\widehat{\Delta} = \{p\}$, no ambiguity;
  - $\widehat{\Delta} = \Delta^{N-1}$, full ambiguity;
  - $\widehat{\Delta} = \left\{ q \in \Delta^{N-1} \middle| q_i \geq \alpha_i, i = 1, \dots, N \right\}$, with $\alpha_i > 0$ and $\sum_i \alpha_i < 1$: bounded likelihood ratios.
- This assumption enables us to reformulate the receiver's IS as distribution of private beliefs:

$$\mathcal{G}(\widehat{\Delta}, p) \equiv \left\{ \nu \in \Delta(\Delta^{N-1}) \middle| \operatorname{supp}(\nu) \subseteq \widehat{\Delta} \text{ and } \int_{\Delta^{N-1}} q\nu(\mathrm{d}q) = p \right\}.$$

Simplifying the sender's problem

Sender's standard information structure

- Idea: pick a representative IS for each class of equivalent IS's.
- (Blackwell) An IS $I = (S, \mu_1, \ldots, \mu_N)$ is a *standard IS* if $S = \Delta^{N-1}$, and for all $i = 1, \ldots, N$,

$$\frac{\mathrm{d}\mu_i}{\mathrm{d}\mu_0}(s) = Ns_i, \ \mu_0 - a.s,$$

  where (recall) $\mu_0 = (\sum_i \mu_i)/N$.
- By definition, standard IS's are characterized by

$$\mathcal{F} \equiv \left\{ \mu \in \Delta(\Delta^{N-1}) \,\middle|\, \int_{\Delta^{N-1}} s_i \mu(\mathrm{d}s) = 1/N \text{ for } i = 1, \ldots, N \right\}.$$

- If I is standard, posterior belief is, for $p \in \Delta^{N-1}$ and $s \in S (= \Delta^{N-1})$,

$$q^I(p, s) \equiv \left(\frac{p_1 s_1}{\sum_i p_i s_i}, \ldots, \frac{p_N s_N}{\sum_i p_i s_i} \right) \in \Delta^{N-1}.$$

The sender's problem

A 2×2 example

- Let π be the common prior, and $\pi \in \widehat{\Delta} = [\alpha, \beta]$.
- A signal of a standard IS can be written as $(s, 1-s)$ where $s \in [0,1]$.
- A standard IS is characterized by a c.d.f. F over $[0,1]$ with $\int_{[0,1]} s \mathrm{d}F(s) = 1/2$.
- Hence,

$$\mathcal{F} = \left\{ \text{c.d.f } F \text{ over } [0,1] \,\middle|\, \int_{[0,1]} s \mathrm{d}F(s) = 1/2 \right\}.$$

The sender's problem

A 2×2 example

- Abusing notation, private belief $(1-p, p)$ and signal $(s, 1-s)$ lead to posterior belief

$$\Big(\frac{(1-p)s}{(1-p)s+p(1-s)}, \frac{p(1-s)}{(1-p)s+p(1-s)}\Big).$$

- The receiver takes action $a = 2$ if and only if

$$\frac{p(1-s)}{(1-p)s+p(1-s)} \geq \frac{1}{2} \iff s \leq p.$$

- Thus the sender's *contingent payoff* from F when the receiver's private belief is p can be written as

$$\begin{aligned} \phi^F(p) &= \int_{[0,p]} \Big[(1-p)s + p(1-s)\Big] \mathrm{d}F(s) \\ &= pF(p) + (1-2p) \int_{[0,p]} s \mathrm{d}F(s). \end{aligned}$$

The sender's problem

A 2×2 example

- We assume the sender evaluates an information structure by its worst case expected payoff:

$$V^F(\widehat{\Delta}, \pi) = \inf_{\nu \in \mathcal{G}(\widehat{\Delta}, \pi)} \int_{\Delta^{N-1}} \phi^F(p) \nu(\mathrm{d}p)$$

- The sender's information design problem becomes

$$V(\widehat{\Delta}, \pi) = \max_{F \in \mathcal{F}} V^F(\widehat{\Delta}, \pi).$$

- Information design problem is transformed into the problem of choosing a probability measure subject to the mean constraints.

Convexification

- For $\widehat{\Delta}$, let $\mathrm{co}_{\widehat{\Delta}}\phi^F : \widehat{\Delta} \to \mathbb{R}$ be the *largest convex function* below $\phi^F|_{\widehat{\Delta}}$.
- Notice $\mathrm{co}_{\widehat{\Delta}}\phi^F$ depends on $\widehat{\Delta}$.

Lemma 1

For any $\widehat{\Delta}$, $\pi \in \widehat{\Delta}$ and $F \in \mathcal{F}$,

$$V^F(\widehat{\Delta}, \pi) = \inf_{\nu \in \mathcal{G}(\widehat{\Delta}, \pi)} \int_{\Delta^{N-1}} \phi^F(p)\nu(\mathrm{d}p) = \mathrm{co}_{\widehat{\Delta}}\phi^F(\pi).$$

Convexification

Proof of Lemma 1.

Because $\mathrm{co}_{\widehat{\Delta}}\phi^F \le \phi^F|_{\widehat{\Delta}}$ by definition, we know

$$\begin{aligned} V^F(\widehat{\Delta},\pi) &\ge \inf_{\nu\in\mathcal{G}(\widehat{\Delta},\pi)} \int_{\widehat{\Delta}} \mathrm{co}_{\widehat{\Delta}}\phi^F(p)\nu(\mathrm{d}p) \\ &\ge \inf_{\nu\in\mathcal{G}(\widehat{\Delta},\pi)} \mathrm{co}_{\widehat{\Delta}}\phi^F\Big(\int_{\widehat{\Delta}} p\nu(\mathrm{d}p)\Big) = \mathrm{co}_{\widehat{\Delta}}\phi^F(\pi), \end{aligned}$$

where the second inequality comes from Jensen's inequality. On the other hand, because $\widehat{\Delta}$ is convex, it is well known that $\mathrm{co}_{\widehat{\Delta}}\phi^F$ can be expressed as

$$\mathrm{co}_{\widehat{\Delta}}\phi^F(p) = \inf\left\{ \sum_{k=1}^{N+1} \lambda^k\phi^F(p^k) \;\middle|\; \begin{array}{l} p^k\in\widehat{\Delta},\ \lambda^k\ge 0 \text{ for } k=1,\dots,K, \\ \sum_{k=1}^{N+1}\lambda^k = 1 \text{ and } \sum_{k=1}^{N+1}\lambda^k p^k = p \end{array} \right\}.$$

Thus, it is immediate that $V^F(\widehat{\Delta},\pi) \le \mathrm{co}_{\widehat{\Delta}}\phi^F(\pi)$. $\square$

Convexification

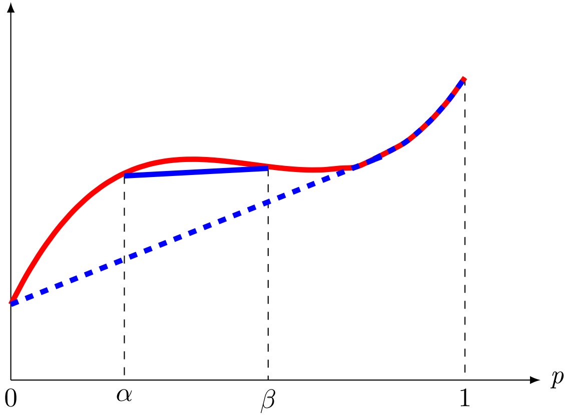


Figure 2: An illustration of convexification on different domains

The value of persuasion

- The sender can *benefit from persuasion* for $(\widehat{\Delta}, \pi)$ if the sender is strictly better off by designing some IS than not supplying any information.
- Define $\phi^0 : \Delta^{N-1} \to \mathbb{R}$ as

$$\phi^0(p) \equiv \sum_i p_i v\big(a[p], i\big),\ \forall p \in \Delta^{N-1}.$$

- The sender can benefit from persuasion for $(\widehat{\Delta}, \pi)$ if and only if

$$V(\widehat{\Delta}, \pi) > \mathrm{co}_{\widehat{\Delta}}\phi^0(\pi).$$

The value of persuasion

Full ambiguity

- Suppose $\widehat{\Delta} = \Delta^{N-1}$.
- Let γ^i be the belief that puts probability 1 on state i.

Proposition 1

If $\widehat{\Delta} = \Delta^{N-1}$, then full information disclosure is optimal for any prior $\pi \in \Delta^{N-1}$. In this case, the sender's value is

$$V(\widehat{\Delta}, \pi) = \sum_{i=1}^{N} \pi_i v\big(a[\gamma^i], i\big), \ \forall \pi \in \Delta^{N-1}.$$

- Intuition: no matter what IS the sender designs, the sender's worst case payoff can not be more than $\sum_{i=1}^{N} \pi_i v\big(a[\gamma^i], i\big)$. By fully revealing the states, the sender can guarantee himself this much.

The value of persuasion

Full ambiguity

Proposition 2

Let $\widehat{\Delta} = \Delta^{N-1}$*. The followings are equivalent:*

(i) *There exists* $\pi \in \Delta^{N-1}$*, such that* $\phi^0(\pi) < \sum_i \pi_i v\big(a[\gamma^i], i\big)$*.*

(ii) *The sender benefits from persuasion for some* $\pi \in \Delta^{N-1}$*.*

(iii) *The sender benefits from persuasion for all* $\pi \in \mathrm{int}\Delta^{N-1}$ *where* $\mathrm{int}\Delta^{N-1}$ *is the set of all interior points of* Δ^{N-1}*.*

- Idea: suppose $\phi^0(\pi) < \sum_i \pi_i v\big(a[\gamma^i], i\big)$. Any $\pi' \in \mathrm{int}\Delta^{N-1}$ is a convex combination of π and at most $N-1$ γ^i's, say $\gamma^1, \ldots, \gamma^{N-1}$. In other words, when the prior is π', the receiver might has a private IS that leads to private beliefs π, $\gamma^1, \ldots, \gamma^{N-1}$. Thus the sender gets strictly lower payoff if he does not provide information than full information disclosure.

The value of persuasion

Full ambiguity

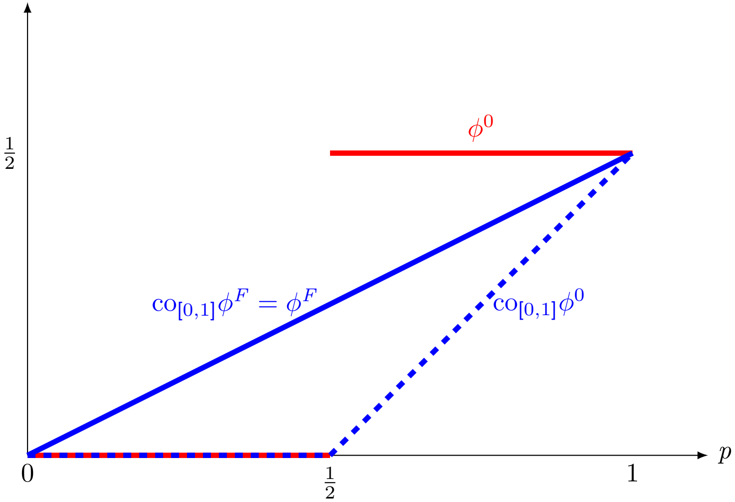


Figure 3: Full ambiguity

The value of persuasion

Local ambiguity

- The sender knows that the receiver's private information source can only provide coarse information and thus her possible private beliefs are all close to the common prior.

Proposition 3

Suppose every $a \in \widehat{A}$ uniquely maximizes the receiver's expected payoff for some belief. Then for all $\pi \in \mathrm{int}\Delta^{N-1}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\widehat{\Delta} \subseteq O(\pi, \delta)$, $V(\widehat{\Delta}, \pi) > V(\{\pi\}, \pi) - \varepsilon$. As a result, for any $\pi \in \mathrm{int}\Delta^{N-1}$ and decreasing sequence of $\{\widehat{\Delta}_n\}_{n \geq 1}$ such that $\bigcap_n \widehat{\Delta}_n = \{\pi\}$, $\lim_n V(\widehat{\Delta}_n, \pi) = V(\{\pi\}, \pi)$.

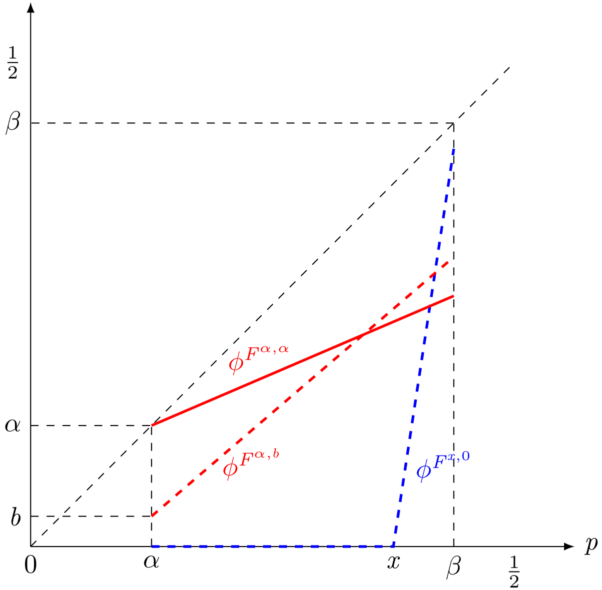
A 2×2 example

Linear-contingent-payoff standard IS

- For general $\alpha < \beta < 1$, the analysis is much more difficult.
- The key idea is to construct *linear-contingent-payoff standard IS*: $F \in \mathcal{F}$ that leads to (piecewise) linear contingent payoff ϕ^F over $[\alpha, \beta]$.
- We first construct such standard IS's and then show that they are the optimal information design for the sender.

A 2×2 example

Linear-contingent-payoff standard IS



A 2×2 example

Linear-contingent-payoff standard IS

$$F^{\alpha,\alpha}(s) = \begin{cases} 0 & \text{if } s \in [0, \alpha), \\ \frac{1}{2(1-\alpha)} & \text{if } s \in [\alpha, 1), \\ 1 & \text{if } s = 1. \end{cases} \tag{1}$$

is the optimal persuasion rule in KG given common prior α.

A 2×2 example

Linear-contingent-payoff standard IS

$$F^{x,0}(s) \equiv \begin{cases} 0, & \text{if } s \in [0, x), \\ a^{x,0}\left[(1-2x) - \sqrt{\frac{x(1-x)}{s(1-s)}}(1-2s)\right], & \text{if } s \in [x, \beta), \\ a^{x,0}\left[(1-2x) - \sqrt{\frac{x(1-x)}{\beta(1-\beta)}}(1-2\beta)\right], & \text{if } s \in [\beta, 1), \\ 1 & \text{if } s = 1. \end{cases} \tag{2}$$

with

$$a^{x,0} = \frac{1}{2(1-x)\left[1 - \sqrt{\frac{x(1-\beta)}{(1-x)\beta}}\right]}$$

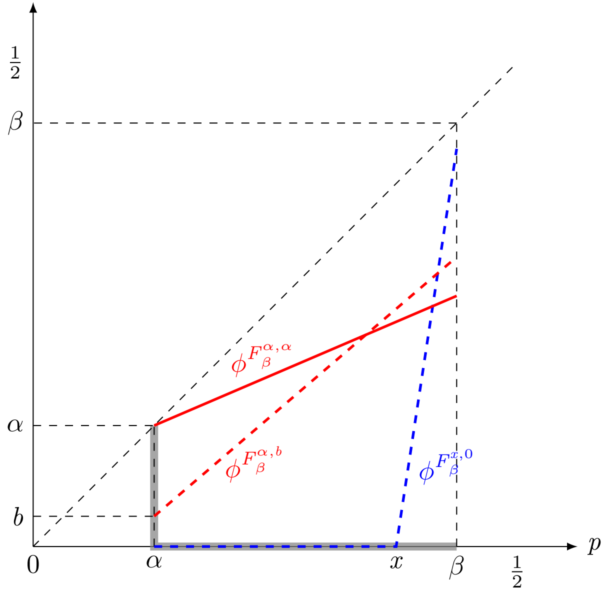
A 2×2 example

Two technical concerns

- Does a linear-contingent-payoff standard IS always exist?
  Answer: Not necessarily. Depending on (α, β), we need to carefully choose x, the starting point where the payoff becomes positive, and b, the jump of payoff at x.
- Is it true that the optimal value is the upper envelope of all feasible linear contingent payoff functions?
  Answer: Yes!

A 2×2 example

Possible ranges of (x, b)



A 2×2 example

Possible ranges of (x, b)

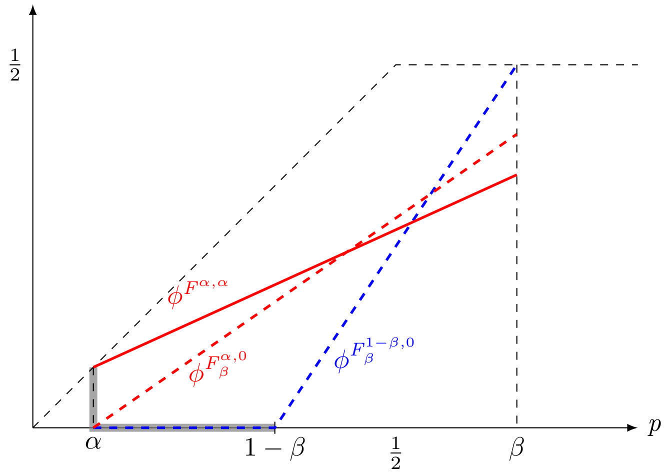


Figure 6: An illustration of $\phi^{F_\beta^{x,b}}$ when $\alpha < 1-\beta < \frac{1}{2} < \beta$

A 2×2 example

Possible ranges of (x, b)

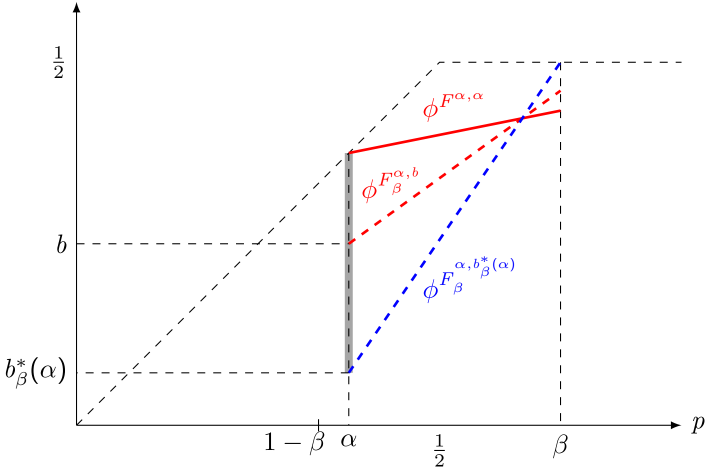


Figure 7: An illustration of $\phi^{F_\beta^{x,b}}$ when $1-\beta \leq \alpha < \frac{1}{2} < \beta$

A 2×2 example

Optimal value

The optimality of the upper envelope of all feasible linear contingent payoff functions is established by showing:

Lemma 2

Let $f : [u, v] \to \mathbb{R}$ be a function. Suppose f is bounded from below. Then for all $x \in (u, v)$, we have

$$\mathrm{co}_{[u,v]} f(x) = \sup_{\substack{\text{linear } \ell:[u,v]\to\mathbb{R}, \\ \ell \leq f}} \ell(x). \tag{3}$$

A 2×2 example

Optimal value

Lemma 3

Fix $0 \leq \alpha < \beta < 1$ and $\alpha < \frac{1}{2}$. For any $F \in \mathcal{F}$ and linear $\ell : [\alpha, \beta] \to \mathbb{R}$ such that $\ell \leq \phi^F|_{[\alpha,\beta]}$, there exists a feasible linear contingent payoff $\phi^{F_\beta^{x,b}}$ such that

$$\ell(p) \leq \phi^{F_\beta^{x,b}}(p), \ \forall p \in [\alpha, \beta].$$

A 2×2 example

Optimal value

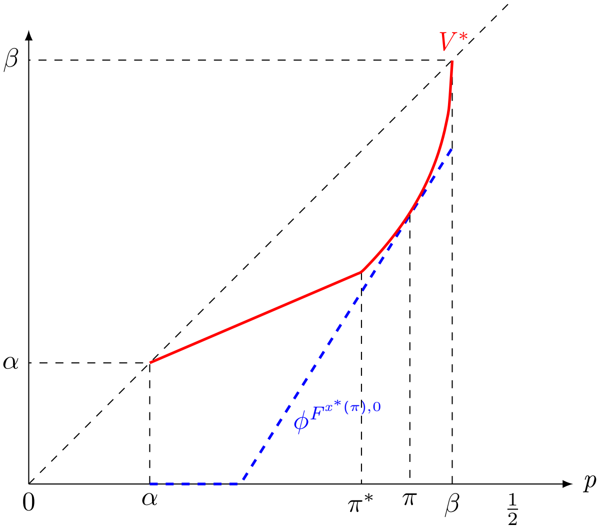


Figure 8: The value function V^* when $\alpha < \beta < \frac{1}{2}$

A 2×2 example

Optimal value

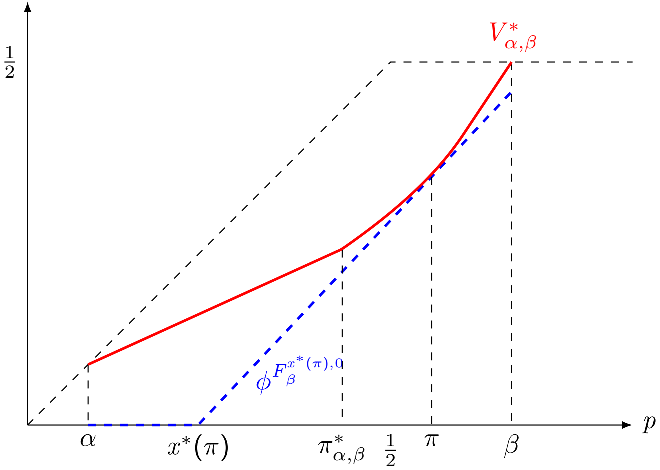


Figure 9: The value function V^* when $\alpha < 1 - \beta < \frac{1}{2} < \beta$

A 2×2 example

Optimal value

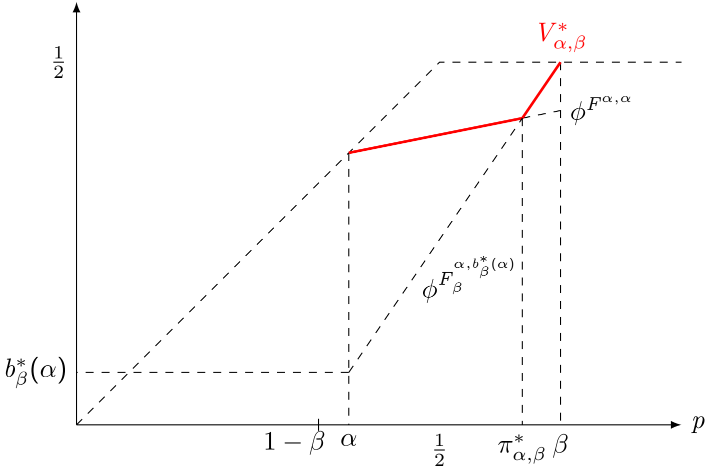


Figure 10: The value function V^* when $1-\beta \leq \alpha < \frac{1}{2} < \beta$

Conclusions

- New ingredients of our model:
  - Bayesian persuasion with robustness concerns;
  - Full ambiguity v.s. local ambiguity
- Develop a new convexification method to characterize the optimal robust persuasion
- The sender is more likely to gain from persuasion
- Fully characterize the optimal robust persuasion in a 2×2 example

Optimality of linear contingent payoff