Perfect Equilibria in Large Games

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Outline

Perfect equilibrium

- Selten (1975) introduced (trembling hand) perfect equilibrium to restrict the set of Nash equilibria in finite games.
- This refinement precludes weakly dominated actions by requiring some notion of neighborhood robustness to small perturbations.

Literature review

- Selten (1975):
  finite players, finite actions.
- How to formulate perfect equilibrium in other environments?
- Simon and Stinchcombe (1995):
  finite-player games with infinitely many actions.
- Rath (1994, 1998):
  games with a continuum of players and finite actions.
- Games with a continuum of players and infinite many actions: this project.

Basic setup of large games

- Player space $(I, \mathcal{F}, \lambda)$: an atomless probability space.
- Player i's action set A_i: a subset of a compact metric space A.
- Action correspondence $\mathcal{A}\colon i \mapsto A_i$: compact-valued.
- Mixed action correspondence $\mathcal{M}(\mathcal{A})\colon i \mapsto \mathcal{M}(A_i)$: compact-valued.

Strategies in large games

- Pure strategy profile:

  a measurable function $g\colon I \to A$ such that $g(i) \in A_i$ a.e.

$\Rightarrow$ a measurable selection of $\mathcal{A}$.

- Behavioral strategy profile:

  a measurable function $g\colon I \to \mathcal{M}(A)$ such that $g(i) \in \mathcal{M}(A_i)$ a.e.

$\Rightarrow$ a measurable selection of $\mathcal{M}(\mathcal{A})$.

Societal summaries

- Given a pure strategy profile g, the societal summary is λg^{-1}.
- Given a behavioral strategy profile g, the societal summary is defined as the Gelfand integral $\int_I g(i)\,\mathrm{d}\lambda(i)$.
- $\mathcal{D} = \{\lambda g^{-1} \mid g \text{ is a measurable selection of } \mathcal{A}\}$.

$\Rightarrow$ $\int_I g(i)\,\mathrm{d}\lambda(i) \in \mathcal{D}$.

- $\mathcal{D}$: the set of societal summaries.

Basic setup of large games (cont.)

- Each player's payoff continuously depends on her own actions as well as on societal summaries.
- $\mathrm{Gr} = \{(i, a) \in I \times A \mid a \in A_i\}$.

  the graph of the action correspondence $\mathcal{A}$.
- Large game: $G\colon\ \mathrm{Gr} \times \mathcal{D} \to \mathbb{R}$ such that $G(i, \cdot, \cdot)\colon A_i \times \mathcal{D} \to \mathbb{R}$ is continuous for each $i \in I$.
- Measurability of G:
  - $\mathcal{A}$ is measurable,
  - for each $\mu \in \mathcal{D}$, $G(\cdot, \mu)$ is measurable.

Nash equilibria in large games

- A pure strategy profile g is said to be a pure strategy Nash equilibrium if for λ-almost all $i \in I$,

$$u_i(g(i), \lambda g^{-1}) \geq u_i(a, \lambda g^{-1}) \text{ for all } a \in A_i.$$

Trembling strategies

- To captures the notion of trembling strategies, we consider full-support Borel probability measures on A_i.
- Two ways to measure the extents of trembling:
  - the strong metric ρ^s on $\mathcal{M}(A)$,

$$\rho^s(\mu, \nu) = \sup\{|\mu(B) - \nu(B)| \mid B \in \mathcal{B}(A)\},$$

  - the weak metric ρ^w on $\mathcal{M}(A)$,

$$\rho^w(\mu, \nu) = \inf\{\varepsilon > 0 \mid \mu(B) \leq \nu(B^\varepsilon) + \varepsilon, \nu(B) \leq \mu(B^\varepsilon) + \varepsilon\}.$$

ε-perfect equilibria

- $\mathrm{Br}_i(\int_I g^\varepsilon) = \arg\max_{a \in A_i} u_i\big(a, \int_I g^\varepsilon\big) \subseteq A_i$.
- $\mathcal{M}\big(\mathrm{Br}_i(\int_I g^\varepsilon)\big)$: set of best behavioral strategies given $\int_I g^\varepsilon$.
- A measurable behavioral strategy profile with full support g^ε is said to be a strong ε-perfect equilibrium if for λ-almost all $i \in I$,

$$\rho^s\Big(g^\varepsilon(i), \mathcal{M}\big(\mathrm{Br}_i(\int_I g^\varepsilon)\big)\Big) = \inf_{\mu \in \mathcal{M}(\mathrm{Br}_i(\int_I g^\varepsilon))} \rho^s(g^\varepsilon(i), \mu) < \varepsilon.$$

Perfect equilibria

A pure (resp. behavioral) strategy profile g is said to be a pure (resp. behavioral) strategy strong perfect equilibrium if there exists a sequence of behavioral strategy profiles $\{g^n\}_{n\in\mathbb{Z}_+}$ and a sequence of positive constants $\{\varepsilon_n\}_{n\in\mathbb{Z}_+}$ such that

1. each g^n is a strong ε_n-perfect equilibrium with $\varepsilon_n \to 0$ as n goes to infinity,
2. for λ-almost all $i \in I$, $g(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ (resp. $\operatorname{Ls} g^n(i)$),
3. $\lim\limits_{n\to\infty} \int_I g^n(i)\, \mathrm{d}\lambda(i) = \lambda g^{-1}$ (resp. $\int_I g(i)\, \mathrm{d}\lambda(i)$).

Negative result

- Property: A perfect equilibrium is a Nash equilibrium.
- Khan et al. (1997) provided a large game (with infinitely many actions), which does not have a pure strategy Nash equilibrium.

$\Rightarrow$ A perfect equilibrium does not exist either.

Nowhere equivalence

- Player space: $(I, \mathcal{F}, \lambda)$.
- $\mathcal{G}$ is a countably-generated sub-σ-algebra of $\mathcal{F}$.
  We assume that $(I, \mathcal{G}, \lambda)$ is a complete probability space.
- We also assume that the game G is $\mathcal{G}$-measurable:
  - the corresponding action correspondence $\mathcal{A}$ is $\mathcal{G}$-measurable,
  - $G(\cdot, \mu)$ is $(\mathcal{G} \otimes \mathcal{B}(A))^{\mathrm{Gr}}$-measurable for each $\mu \in \mathcal{D}$.
- $\mathcal{G}$ can be viewed as the σ-algebra generated by the mapping specifying the individual characteristics (payoff functions and action sets).
  $(I, \mathcal{G}, \lambda)$: the characteristic type space.

Nowhere equivalence (cont.)

- For any non-negligible subset $D \in \mathcal{F}$, the restricted probability space $(D, \mathcal{G}^D, \lambda^D)$ is defined as follows:
  - $\mathcal{G}^D$ is the σ-algebra $\{D \cap D' \mid D' \in \mathcal{G}\}$
  - λ^D is the probability measure re-scaled from the restriction of λ to $\mathcal{G}^D$.
- He-Sun-Sun (2017 TE): The σ-algebra $\mathcal{F}$ is said to be nowhere equivalent to the sub-σ-algebra $\mathcal{G}$, if for every non-negligible subset $D \in \mathcal{F}$, $\mathcal{F}^D$ and $\mathcal{G}^D$ are not the same:

  there exists an $\mathcal{F}$-measurable subset D_0 of D such that $\lambda(D_0 \triangle D_1) > 0$ for any $D_1 \in \mathcal{G}^D$, where $D_0 \triangle D_1$ denotes the symmetric difference $(D_0 \setminus D_1) \cup (D_1 \setminus D_0)$.

Nowhere equivalence (cont.)

- Nowhere equivalence: given any non-trivial collection of players, when the player space and the characteristic type space are restricted to this collection, the former contains the latter strictly in terms of measure spaces.
- By distinguishing the player space from the characteristic type space, the condition of nowhere equivalence allows the heterogeneity that different players with the same characteristics (payoff and action set) to select different optimal (pure) actions, which in turn guarantees the existence of pure strategy equilibria.

Existence

- Theorem:

  Every $\mathcal{G}$-measurable large game G has an $\mathcal{F}$-measurable pure strategy strong perfect equilibrium if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.
- The "only if" part follows from Theorem 2 in He-Sun-Sun (2017): Every $\mathcal{G}$-measurable large game G has an $\mathcal{F}$-measurable pure strategy Nash equilibrium if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

Existence: Proof of "if"

- Lemma: Every $\mathcal{G}$-measurable large game G has a $\mathcal{G}$-measurable strong ε-perfect equilibrium.
    - Fixed-point theorem for correspondences.
- The existence of behavioral strategy perfect equilibria.
- The existence of pure strategy perfect equilibria.

Weakly dominated perfect equilibrium

- A perfect equilibrium strategy can be weakly dominated.
- Let the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ be the player space and let the set $A = [0, 1]$ be the common action space.
- The common payoff function is $u(a, \mu) = a \cdot \rho^w(\mu, \eta)$, where η is the uniform distribution on $[0, 1]$ (i.e., the Lebesgue measure).
- Perfect equilibrium: each player i chooses the strategy i.

  $g^\varepsilon(i) = (1 - \varepsilon)\delta_i + \varepsilon\eta$.
- The strategy i adopted by player i is weakly dominated.

Admissibility

- A strategy is admissible if it puts no mass on the set of weakly dominated strategies.
- Simon and Stinchcombe (1995):

  finite-player game with infinitely many actions.

  an admissible perfect equilibrium may fail to exist.
- Rath (1994, 1998):

  large game with finite actions.

  there is a perfect equilibrium that is not admissible.

Admissibility (Cont.)

- In large games with infinitely many actions, an admissible perfect equilibrium may fail to exist.
- Let the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ be the player space and let the set $A = [0, \frac{1}{2}]$ be the common action space. The common payoff function for each player i is $u(a_i, \xi) = \int_0^{\frac{1}{2}} v(a_i, y)\, \mathrm{d}\xi(y)$ when player i's action is a_i and the societal summary is ξ, where $v(\cdot, \cdot)$ is a continuous function on $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ given by:

$$v(x, y) = \begin{cases} x, & \text{if } x \leq \frac{1}{2}y, \\ \frac{y(1-x)}{2-y}, & \text{if } \frac{1}{2}y < x. \end{cases}$$

$f(i) \equiv 0$ is the unique perfect equilibrium, which is dominated.

$\Rightarrow$ The existence and the admissibility may not be compatible in large games with infinitely many actions.

Limit admissibility

- Simon and Stinchcombe (1995)
- Limit admissibility:

  A strategy is limit admissible if it puts no mass on the interior of the set of weakly dominated strategies.
- In finite-player games with infinitely many actions:

  there exists a limit admissible perfect equilibrium.
- In large games: there exists a limit admissible perfect equilibrium.

  finite actions: Rath (1998).

  infinitely many actions: this paper.

Limit admissibility (Cont.)

- Rath (1998): it is impossible that every perfect equilibrium is limit admissible in large games (with finite actions).
- In a large game with finite actions, a "limit admissible strategy" is indeed an "admissible strategy".

$\Rightarrow$ This formulation of PE is not ideal.

ε-perfect* equilibria

- $\widehat{\int_I g^\varepsilon}$ is a perturbation of $\int_I g^\varepsilon$.

  $\widehat{\int_I g^\varepsilon}$ is a full-support probability measure on $\mathcal{D}$, with at least $(1-\varepsilon)$ weight on $\int_I g^\varepsilon$.
- $\mathrm{Br}_i(\widehat{\int_I g^\varepsilon}) = \arg\max_{a \in A_i} \int_{\mathcal{D}} u_i(a,\tau)\,\mathrm{d}\widehat{\int_I g^\varepsilon}(\tau)$.
- A measurable behavioral strategy profile with full support g^ε is said to be a strong ε-perfect* equilibrium if for λ-almost all $i \in I$,

$$\rho^s\left(g^\varepsilon(i), \mathcal{M}\left(\mathrm{Br}_i(\widehat{\int_I g^\varepsilon})\right)\right) = \inf_{\mu \in \mathcal{M}(\mathrm{Br}_i(\widehat{\int_I g^\varepsilon}))} \rho^s(g^\varepsilon(i), \mu) < \varepsilon.$$

- The perturbation $\widehat{\cdot}\colon \tau \mapsto \widehat{\tau}$ is continuous.

Perfect* equilibria

- The "limit" of a sequence of ε-perfect* equilibria.
- Property: A perfect* equilibrium is a Nash equilibrium.
- Theorem: Every $\mathcal{G}$-measurable large game G has an $\mathcal{F}$-measurable pure strategy strong perfect* equilibrium if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

Admissibility result

- Θ_i: the set of weakly dominated actions.
- Lemma: For each full-support probability measure ζ on $\mathcal{D}$, $\mathrm{Br}_i(\zeta) \subseteq \Theta_i^c$, where $\mathrm{Br}_i(\zeta) = \arg\max_{a \in A_i} \int_{\mathcal{D}} u_i(a, \tau) \, \mathrm{d}\zeta(\tau)$.
- Theorem: For each perfect* equilibrium g, $g(i) \in \overline{\Theta_i^c}$ a.e.

Summary and discussion

- Formulation of perfect* equilibria for large games with infinitely many actions.
- It is a refinement of Nash equilibrium.
- Existence and limit admissibility.
- Proper equilibrium.

Question and Answer

Thank you