Learning by Matching

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Background

- Two-sided markets:
  - Marriage market
  - Job market
  - College admission market
  - School choice
  - . . .

Complete Information Assumption

Assumption: Information is complete (CI), i.e.,

Every agent's characteristics and preferences are common knowledge.

Outline

1. Incorporate firm-specific info by means of partitional information structure
2. Path to stability
3. Proof

Related Literature

1. One-to-one job market:
   Shapley and Shubik (1971), Crawford and Knoer (1981), Chen et al. (2016), Liu et al. (2014). . .

2. Incomplete information:
   Roth (1989), Chakraborty et al. (2010), Liu et al. (2014) (LMPS), Bikhchandani (2017), Pomatto (2015). . .

3. Path to stability:
   Knuth (1976), Roth and Vande Vate (1990), Kojima and Ünver (2008), Klaus and Klijn (2007), Chen et al. (2010, 2016), Fujishige and Yang (2016). . .

The Model

Agents

- Agents
  - $I \ni i$: a finite set of workers.
  - $J \ni j$: a finite set of firms.
- Types
  - $\mathbf{w} : I \to W$, where W is finite.
  - $\mathbf{f} : J \to F$, where F is finite. $\mathbf{f}$ is public information.
  - $\Omega \subset W^{|I|}$: a set of possible type assignment functions.

Values and Payoffs

- Values for match (w, f)
  - worker premuneration value: $\nu_{wf} \in \mathbb{R}$.
  - firm premuneration value: $\phi_{wf} \in \mathbb{R}$.
  - surplus of the match: $\nu_{wf} + \phi_{wf}$.
- Payoffs
  - $\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p$ for the worker.
  - $\phi_{\mathbf{w}(i),\mathbf{f}(j)} - p$ for the firm.

Allocation

- matching: $\mu : I \to J \cup \emptyset$, one-to-one on $\mu^{-1}(J)$.
- payment scheme: $\mathbf{p}$ associated with a matching function μ.
  - $\mathbf{p}_{i,\mu(i)} \in \mathbb{R}$ for each $i \in I$.
  - $\mathbf{p}_{\mu^{-1}(j),j} \in \mathbb{R}$ for each $j \in J$.
  - $\mathbf{p}_{\emptyset j} = \mathbf{p}_{i\emptyset} = 0$.
- $\mathscr{A} \ni (\mu, \mathbf{p})$: the set of all allocations.
  - $(\mu, \mathbf{p})$is observable for all agents.

Information

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  - $\mathbf{w} \in \Omega \subset W^{|I|}$.

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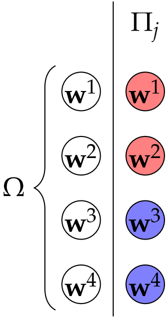
- Π_j: Information Partition of a firm $j \in J$.
  - Π_j is a partition of Ω.
  - $\mathbf{w}' \in \Pi_j(\mathbf{w})$:
    Firm j thinks $\mathbf{w}'$ is possible when $\mathbf{w}$ is true.



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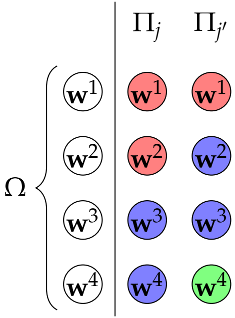
- $\Pi := (\{\Pi_j\}_{j \in J})$.
- Complete info: every partition cell is a singleton.



State of the Market

A state of the matching market, $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, specifies

- an allocation $(\mu, \mathbf{p})$;
- a type assignment function $\mathbf{w}$; and
- a partition profile Π.

Stability

Requirement 1 of Stability: Individual Rationality

Definition 1

A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *is said to be* ***individually rational*** *if*

$$\nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \geq 0 \text{ for all } i \in I \text{ and}$$

$$\phi_{\mathbf{w}(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j} \geq 0 \text{ for all } j \in J.$$

Requirement 2 of Stability: No Blocking

- Following LMPS, 'a firm cares about the worst case of worker if she does not know his true type.'

Definition 2

A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *is said to be **blocked** if there exists a worker-firm pair* (i, j) *and a payment* $p \in \mathbb{R}$ *such that*

$$v_{\mathbf{w}(i),\mathbf{f}(j)} + p > v_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ and}$$
$$\phi_{\mathbf{w}'(i),\mathbf{f}(j)} - p > \phi_{\mathbf{w}'(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j}$$

for all $\mathbf{w}' \in \Pi_j(\mathbf{w})$

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for all $\mathbf{w}' \in \Pi_j(\mathbf{w})$ *satisfying*

$$\nu_{\mathbf{w}'(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)}.$$

Consistency: A firm can observe the type of her own employee, if any.

$$\forall \mathbf{w}' \in \Pi_j(\mathbf{w}), \mathbf{w}'(\mu^{-1}(j)) = \mathbf{w}(\mu^{-1}(j)).$$

Example 1

- One worker α with possible types $w = -1$ (true) and $w' = 1$.
  Two firms a and b. Firms' type: $f_a = 1$ and $f_b = -1$.
  Values: $\nu_{wf} = \phi_{wf} = wf$.

- Allocation: No firm is matched with the worker.

- $\Pi_a = \{\{w\}, \{w'\}\}$ and
  $\Pi_b = \{\{w, w'\}\}$.

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- (α, a) is a blocking pair at w' but not at w, i.e., $N_a = \{\{w\}, \{w'\}\}$.

- 'The state is not blocked by firm a' $\implies$ firm b can learn N_a, i.e.,

$$\Pi_b \vee N_a = \{\{w\}, \{w'\}\}.$$

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$$\underbrace{\text{The fact of IR and no blocking}}_{\text{1. Partition Representation}}\ \underbrace{\text{provides no information}}_{\text{2. Information Aggregation}}\ \text{to agents.}$$

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1. Given a state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$, let $N^{(\mu,\mathbf{p},\Pi)}$ be a partition of Ω:

   $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}') = N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}'')$ if and only if either neither $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ nor $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ is blocked or both of them are blocked.

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2. Aggregating two pieces of information $\rightarrow$ Join of two partitions.

- Inferences: $[H_{\mu,\mathbf{p}}(\Pi)]_j := N^{(\mu,\mathbf{p},\Pi)} \vee \Pi_j, \forall j \in J$, i.e.,

$$[H_{\mu,\mathbf{p}}(\Pi)]_j(\mathbf{w}') := \Pi_j(\mathbf{w}') \cap N^{(\mu,\mathbf{p},\Pi)}(\mathbf{w}'), \forall \mathbf{w}' \in \Omega, \forall j \in J.$$

Stability

Definition 3

A state $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *is said to be* ***stable*** *if*

1. *it is individually rational,*
2. *it is not blocked by any pair, and*
3. Π *is a fixed point of* $H_{\mu,\mathbf{p}}$*, i.e.* $H_{\mu,\mathbf{p}}(\Pi) = \Pi$*.*

Learning and Blocking

Consider $(\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$ where Π and $(\mu, \mathbf{p})$ are common knowledge.

- The state is not blocked:

$$\Pi \longrightarrow H_{\mu,\mathbf{p}}(\Pi).$$

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- The state is blocked by $(i, j; p)$.
  Extra information described by $B^{(\mu,\mathbf{p},\Pi;i,j;p)}$:

  $B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}') = B^{(\mu,\mathbf{p},\Pi;i,j;p)}(\mathbf{w}'')$ if and only if either $(i, j; p)$ blocks both $(\mu, \mathbf{p}, \mathbf{w}', \Pi)$ and $(\mu, \mathbf{p}, \mathbf{w}'', \Pi)$ or neither.

$$\forall j',\ \Pi_{j'} \longrightarrow \Pi_{j'} \vee B^{(\mu,\mathbf{p},\Pi;i,j;p)}$$

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$$\forall j', \ \Pi_{j'} \longrightarrow \Pi_{j'} \vee B^{(\mu,\mathbf{p},\Pi;i,j;p)}$$

  State updating: $(\mu', \mathbf{p}', \mathbf{w}^*, \Pi') \xleftarrow{(i,j;p)} (\mu, \mathbf{p}, \mathbf{w}^*, \Pi)$, if

  - $(i, j; p)$ is satisfied in the new state, and
  - for all $j' \neq j$, $\Pi'_{j'} = \Pi_{j'} \vee B^{(\mu,\mathbf{p},\Pi;i,j;p)}$.

Learning-Blocking Path

A learning-blocking path is a sequence of states $\{(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)\}_{l=0}^{L}$ s.t. for any two adjacent states $(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)$ and $(\mu^{l+1}, \mathbf{p}^{l+1}, \mathbf{w}^*, \Pi^{l+1})$,

- if $(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)$ is not blocked,

  then $(\mu^{l+1}, \mathbf{p}^{l+1}) = (\mu^l, \mathbf{p}^l)$ and $\Pi^{l+1} = H_{\mu^l, \mathbf{p}^l}(\Pi^l)$;

- if $(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)$ is blocked,

  then $(\mu^{l+1}, \mathbf{p}^{l+1}, \mathbf{w}^*, \Pi^{l+1}) \xleftarrow{(i,j;p)} (\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)$,

  where $(i, j; p)$ is a blocking combination for $(\mu^l, \mathbf{p}^l, \mathbf{w}^*, \Pi^l)$.

Main Result

Theorem 1

Suppose payments permitted in the job market are all integers.
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Then the random learning-blocking path starting from an arbitrary state converges with probability one to a stable state.

Theorem 3

$(\mu, \mathbf{p}, \mathbf{w})$ *is an incomplete-info. stable outcome in the sense of LMPS*
if and only if
there exists a partition profile Π *such that* $(\mu, \mathbf{p}, \mathbf{w}, \Pi)$ *is stable.*

Proof of Theorem 1

Initial state: $(\mu, \mathbf{p}, \mathbf{t}^*, \Pi)$, assumed to be IR.

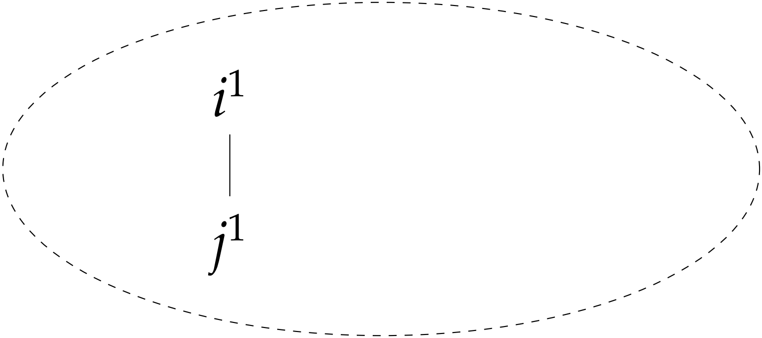
$$(\mu, \mathbf{p}, \mathbf{t}^*, \Pi) : \begin{cases} \textit{Blocked} \\ \textit{Not blocked}, \quad (\mu, \mathbf{p}, \mathbf{t}^*, H_{\mu,\mathbf{p}}(\Pi)) : \begin{cases} \textit{Blocked} \\ \textit{Not blocked} \end{cases} \quad \dots \end{cases}$$

Finite time: blocked OR stable.

Proof of Theorem 1

Initial state: $(\mu, \mathbf{p}, \mathbf{t}^*, \Pi)$ is blocked, where (i^1, j^1) is a blocking pair.

A new state: $(\mu', \mathbf{p}', \mathbf{t}^*, \Pi')$.



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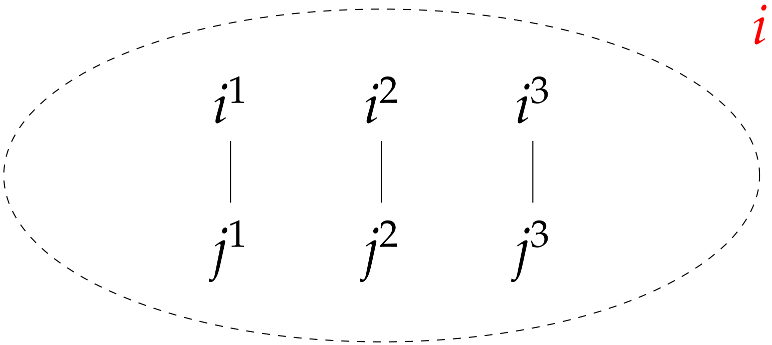
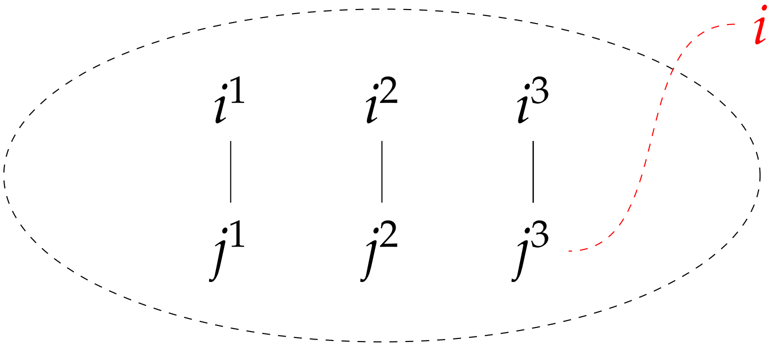


Proof of Theorem 1

$$\alpha = i$$



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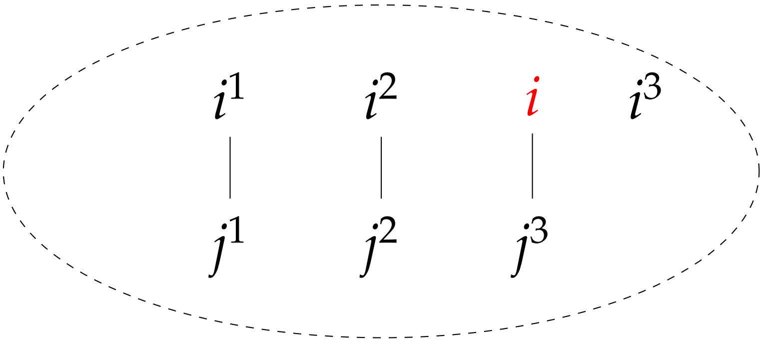

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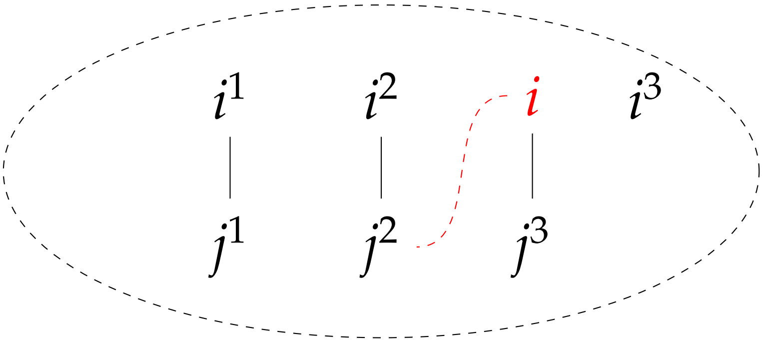

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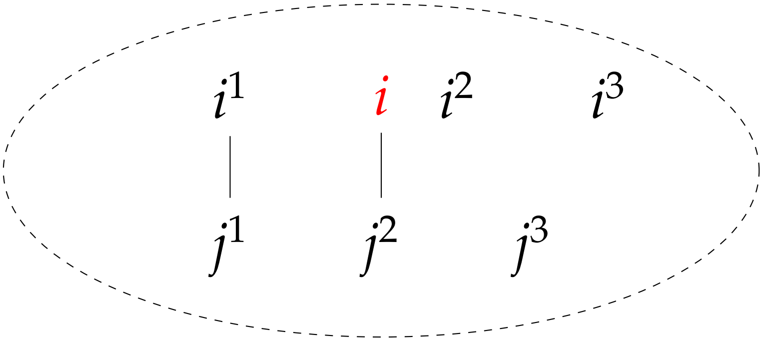

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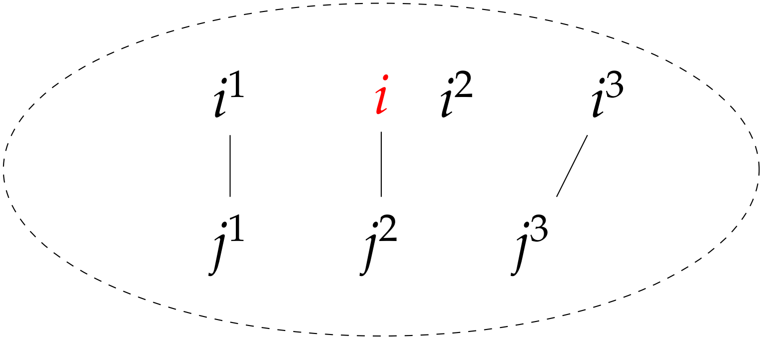

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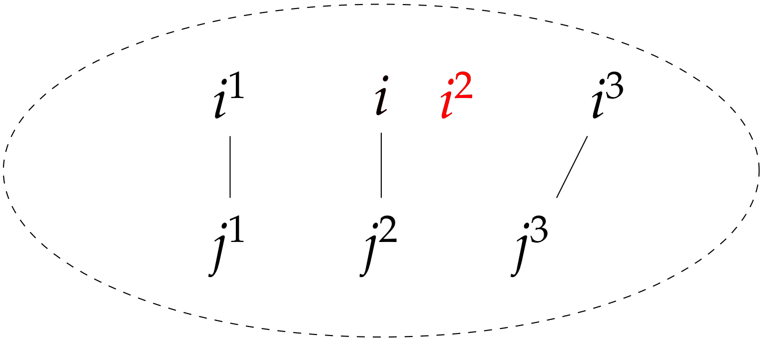

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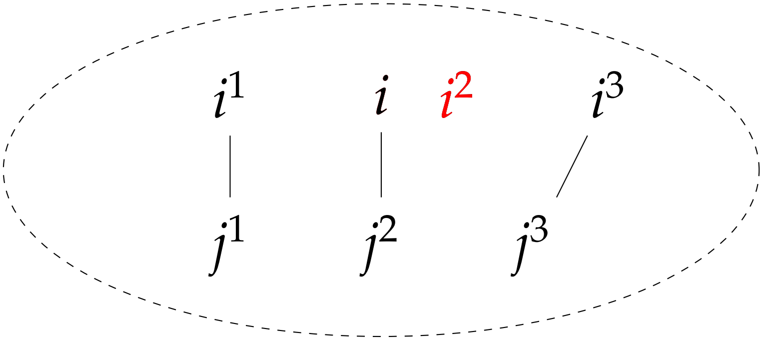

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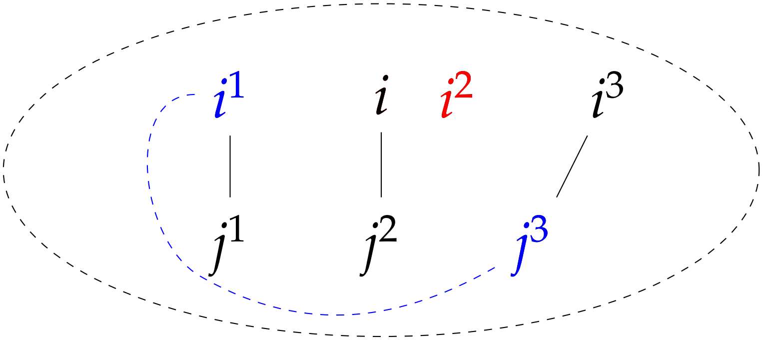

When tracking stops: the set contains no blocking pair
OR there is one more direct observation.

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A partial answer:

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  Values: $\nu_{wf} = |wf|$ and $\phi_{wf} = wf$.
- Status quo: no match and $\Pi_b = \{\{w_\beta, w'_\beta\}\}$.
- The status quo is
  - incomplete-information stable but
  - not efficient (not complete-information stable).

Conclusion

1. Stability with one-sided incomplete information.

   i Describes firms' information by firm specific and flexible partitions.

   ii Makes (II) stability a natural extension of (CI) stability.
   Isolates the role played by information (requirement 3).

   iii Allows for natural definition of stability with *two*-sided (II). CH2017.

2. Path to stability.

   i Describes information updating along a blocking path.

   ii Shows the convergence of Learning-Blocking Paths.

   iii Robustness of convergence w.r.t. learning pattern.

3. Connection with LMPS's stability notions.

   i Generates the same set of stable allocations as LMPS.

   ii Different conceptual starting points: one state V.S. a set of outcomes.

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