Weak Stability and Pareto Efficiency in School Choice

Qianfeng Tang and Yongchao Zhang
Shanghai University of Finance and Economics

Workshop on Matching, Search and Market Design
Institute for Mathematical Sciences, NUS

Two-sided matching

Initiated by the classical work of Gale and Shapley (1962) on marriage market

Applications include school choice, college admission, medical residency program, ...

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We use the language of school choice problems

- assign students in $I = \{i_1, \ldots, i_n\}$ to schools in $S = \{s_1, s_2, \ldots, s_m\}$
- each student i has a strict preference P_i over $S \cup \{\emptyset\}$
- each school s has a strict priority list $\succ_s$ over I
- a matching is a function $\mu : I \to S \cup \{\emptyset\}$ such that $|\mu^{-1}(s)| \leq q_s, \forall s$

Stability (Gale and Shapley, 1962)

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- i desires s but someone with lower priority is assigned to s; or
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A matching is stable if it has no blocking pairs

Implicit assumption: a blocking pair (who prefer each other) can freely rematch, without considering the consequence to others' assignments

Relaxing stability: A new perspective

If a student and a school form a blocking pair for an unstable matching, to object the current matching, they need to propose a better alternative–a "more stable" matching that matches them

Deferred acceptance algorithm (Gale and Shapley, 1962)

For each school choice problem, the (student-proposing) DA operates as follows:

Step 1 Each student applies to her most favorite school. Each school tentatively accepts the best students up to its capacity and rejects the rest.

Step $k, k \geq 2$ Each rejected student applies to her next best school. Each school tentatively accepts the best from the accepted students and new applicants

Stop when no student is rejected

DA produces the student-optimal stable matching which Pareto dominates all other stable matchings for students

Motivating example

Consider schools s_1, s_2, s_3, each has one seat, and students $1, 2, 3, 4$. Below are the priorities/preferences/DA procedure:

s_1	s_2	s_3	1	2	3	4
4	2	3	s_1	s_1	s_1	s_2
1	3	4	$\boxed{\emptyset}$	$\boxed{s_2}$	s_2	s_3
2	4	$\vdots$		$\vdots$	$\boxed{s_3}$	$\boxed{s_1}$
3	$\vdots$					

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1	3	4	$\boxed{\emptyset}$	$\boxed{s_2}$	s_2	s_3
2	4	⋮		⋮	$\boxed{s_3}$	$\boxed{s_1}$
3	⋮					

s_1	s_2	s_3	$\emptyset$
$\boxed{1}$, 2, 3	4		
	$\boxed{2}$, 3, 4		
		$\boxed{3}$, 4	
1, $\boxed{4}$			
$\boxed{4}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$

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s_1	s_2	s_3	$\emptyset$
$\boxed{1}$, 2, 3	4		
	$\boxed{2}$, 3, 4		
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Two unstable improvements of DA:

	s_1	s_2	s_3	$\emptyset$	Blocking pairs
μ_1	2	3	4	1	$\{(1, s_1)\}$
μ_2	2	4	3	1	$\{(1, s_1), (3, s_2)\}$

Weak stability

Denote the set of blocking pairs of μ by $B(\mu)$; say that ν is more stable than μ if $B(\nu) \subseteq B(\mu)$

Observation: $(3, s_2) \in B(\mu_2)$ can propose a more stable matching μ_1 that matches them

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Definition

A matching is weakly stable if none of its blocking pairs, if exists, can be matched by a more stable matching

Or equivalently, if matching any of its blocking pairs inevitably creates new blocking pairs

An overview of our work

We introduce weak stability, study its properties, and reveal some novel structure of matchings via Kesten's EADAM

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Other weakening of stability: Kesten (2004), Cantala and Papai (2014), Alcade and Romeiro-Medina (2015), Klijn and Masso (2003), Ehlers (2007), Dur et al. (2015), Kloosterman and Troyan (2017), etc.

Facts

Fact

All stable matchings are weakly stable

Fact

Matchings more stable than any weakly stable matching are also weakly stable

Fact

TTC, DA-TTC, and the Boston mechanism are not weakly stable

Efficiency-adjusted DA mechanism

EADAM (Kesten, 2010; Tang and Yu, 2014) endogenously relaxes stability under constraint to improve student's welfare

A consenting constraint is a set $C \subset I \times S$.
Meaning: If $(i, s) \in C$, then (i, s) consent to give up their rights to block

EADAM iteratively removes Pareto unimprovable student's consented applications and reruns DA

EADAM

For any problem $(P, \succ)$ with consenting constraint C :

Round 0 Run DA for the problem $(P, \succ)$
Round $k, k \geq 1$ This round consists of three steps:

- Identify the underdemanded schools at the round-$(k-1)$ DA matching, then settle and remove the assignments at these schools
- If i is removed, desires s and $(i, s) \notin C$, truncate $\succ_s$ from i
- Rerun DA

Stop when all schools are removed

Denote the EADAM outcome under constraint C by $EA^C(P, \succ)$

Example revisited

Suppose $(1, s_1) \in C$.

Round-0

s_1	s_2	s_3	$\emptyset$
$\boxed{1}$, 2, 3	4		
	$\boxed{2}$, 3, 4		
		$\boxed{3}$, 4	
1, $\boxed{4}$			
$\boxed{4}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$

Example revisited

Suppose $(1, s_1) \in C$.

Round-0

s_1	s_2	s_3	$\emptyset$
$\boxed{1}, 2, 3$	4		
	$\boxed{2}, 3, 4$		
		$\boxed{3}, 4$	
$1, \boxed{4}$			
$\boxed{4}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$

Example revisited

Suppose $(1, s_1) \in C$.

Round-0

s_1	s_2	s_3	$\emptyset$
$\boxed{1}, 2, 3$	4		
	$\boxed{2}, 3, 4$		
		$\boxed{3}, 4$	
$1, \boxed{4}$			
$\boxed{4}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$

Round-1

s_1	s_2	s_3
$\boxed{2}, 3$	4	
	$\boxed{3}, 4$	
$\boxed{2}$	$\boxed{3}$	$\boxed{4}$

Example revisited

Suppose $(1, s_1) \in C$.

Round-0

s_1	s_2	s_3	$\emptyset$
$\boxed{1}$, 2, 3	4		
	$\boxed{2}$, 3, 4		
		$\boxed{3}$, 4	
1, $\boxed{4}$			
$\boxed{4}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$

Round-1

s_1	s_2	s_3
$\boxed{2}$, 3	4	
	$\boxed{3}$, 4	
$\boxed{2}$	$\boxed{3}$	$\boxed{4}$

$$\Rightarrow EA^{B(\mu_2)}(P, \succ) = EA^{B(\mu_1)}(P, \succ) = \mu_1$$

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Definition

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Theorem

For any consenting constraint C, $EA^C(P, \succ)$ is weakly stable and self-constrained optimal

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It's known that when C is large enough, $EA^C(P, \succ)$ is Pareto efficient

Main theorem

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The following are equivalent:
(i) μ is weakly stable and self-constrained efficient;
(ii) μ is self-constrained optimal;
(iii) $\mu = EA^{B(\mu)}(P, \succ)$.

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A matching μ is self-constrained efficient if it is not Pareto dominated by any matching more stable than it

Theorem

The following are equivalent:
(i) μ is weakly stable and self-constrained efficient;
(ii) μ is self-constrained optimal;
(iii) $\mu = EA^{B(\mu)}(P, \succ)$.

Key intermediate result: if $\mu \neq EA^{B(\mu)}(P, \succ)$, then it is blocked by it

Summary

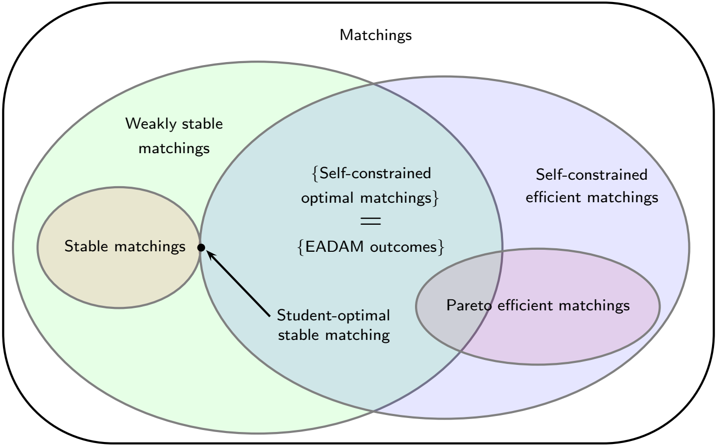

Figure: Relationships among different categories of matchings when the student-optimal stable matching (SOSM) is not Pareto efficient.

Thank You

Discussion: vNM stable set

Th von Neumann-Morgenstern (vNM) stable set is a set of matchings V that satisfies:
(i) Internal stability: if $\mu, \mu' \in V$, then μ does not block μ'; and
(ii) External stability: every matching $\nu \notin V$ is blocked by some $\mu \in V$.

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Theorem
All matchings in the vNM stable set are weakly stable

The converse is not true. Further connections are to be explored