MŒGLIN'S EXPLICIT CONSTRUCTION OF LOCAL A-PACKETS

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Abstract. This note is a proceeding of the workshop "On the Langlands Program: Endoscopy and Beyond" held in National University of Singapore from 17 Dec. 2018 to 18 Jan. 2019. The purpose is to explain Mœglin's explicit constructions of A-packets both when the base field F is p-adic and when F is archimedean.

Introduction

To give a classification of discrete spectrum of automorphic forms, the notion of **A-parameters** is introduced by Arthur in 1980's. The local A-parameters are thus the "local factors" of the global classification. In this article, we focus on the local situation.

Let F be a local field of characteristic zero, and WD_F be the Weil–Deligne group of F, i.e.,

$$WD_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-archimedean,} \\ W_F & \text{if } F \text{ is archimedean,} \end{cases}$$

where W_F is the Weil group of F. For a quasi-split connected reductive algebraic group G over F, we denote by $\widehat{G}$ the complex dual group of G. A homomorphism $\psi\colon WD_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G} \rtimes W_F$ is an **A-parameter** for G if

1. ψ commutes the two projections $WD_F \times \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow W_F$ and $\widehat{G} \rtimes W_F \twoheadrightarrow W_F$;
2. $\psi(W_F)$ consists of semisimple elements;
3. $\psi|W_F$ is continuous;
4. $\psi(W_F)$ projects onto a relatively compact subset in $\widehat{G}$;
5. $\psi|\mathrm{SL}_2(\mathbb{C})$ is algebraic for each $\mathrm{SL}_2(\mathbb{C}) \subset WD_F \times \mathrm{SL}_2(\mathbb{C})$.

Two A-parameters are said to be **equivalent** if they are conjugate by an element in $\widehat{G}$. We define $\Psi(G)$ to be the set of equivalence classes of A-parameters for G. We say that $\psi \in \Psi(G)$ is **tempered** if the restriction of ψ to the last $\mathrm{SL}_2(\mathbb{C})$ is trivial, i.e., ψ factors through the projection $WD_F \times \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow WD_F$. We denote by $\Phi_{\mathrm{temp}}(G)$ the subset of $\Psi(G)$ consisting of tempered A-parameters. Associated with $\psi \in \Psi(G)$, we define the component group by $\mathcal{S}_\psi = \pi_0(Z_{\widehat{G}}(\mathrm{Im}(\psi))/Z(\widehat{G})^{W_F})$, which is a finite group. The set of equivalence classes of irreducible representations of $\mathcal{S}_\psi$ is denoted by $\mathrm{Irr}(\mathcal{S}_\psi)$.

Let $\mathrm{Irr}(G(F))$ be the set of equivalence classes of irreducible admissible representations of $G(F)$. We denote by $\mathrm{Irr}_{\mathrm{unit}}(G(F))$ (resp. $\mathrm{Irr}_{\mathrm{temp}}(G(F))$) the subset of $\mathrm{Irr}(G(F))$ consisting of equivalence classes of unitary (resp. tempered) representations of $G(F)$. In 2013, Arthur [Ar13] has completed the magnificent work on the classification of automorphic representations of symplectic and special orthogonal groups. The local main theorem in this work is as follows.

Theorem 0.1 ([Ar13, Theorem 2.2.1])**.** *Let G be split SO_{2n+1} or Sp_{2n} over F.*

(1) *For each* $\psi \in \Psi(G)$*, there is a finite multiset* Π_ψ *over* $\mathrm{Irr}_{\mathrm{unit}}(G(F))$ *with a map*
$$\Pi_\psi \to \mathrm{Irr}(\mathcal{S}_\psi),\ \pi \mapsto \langle \cdot, \pi \rangle_\psi$$
enjoying certain (twisted and standard) endoscopic character identities. We call Π_ψ *the* ***A-packet*** *for* $G(F)$ *associated with* ψ*.*

(2) *When* $\psi = \phi \in \Phi_{\mathrm{temp}}(G)$*, the A-packet* Π_ϕ *is in fact a subset of* $\mathrm{Irr}_{\mathrm{temp}}(G(F))$*. Moreover the map* $\Pi_\phi \ni \pi \mapsto \langle \cdot, \pi \rangle_\phi \in \mathrm{Irr}(\mathcal{S}_\phi)$ *is injective, which is bijective when* F *is non-archimedean. In addition,* $\Pi_\phi \cap \Pi_{\phi'} = \emptyset$ *for* $\phi \neq \phi'$*, and*
$$\mathrm{Irr}_{\mathrm{temp}}(G(F)) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G)} \Pi_\phi.$$

Theorem 0.1 (2) says that $\phi \in \Phi_{\mathrm{temp}}(G)$ together with $\eta \in \mathrm{Irr}(\mathcal{S}_\phi)$ classifies $\mathrm{Irr}_{\mathrm{temp}}(G(F))$. Let $\Phi(G)$ be the set of equivalence classes of **L-parameters** ϕ for G, i.e., homomorphisms $\phi\colon WD_F \to \widehat{G} \rtimes W_F$ such that

(1) ψ commutes the two projections $WD_F \twoheadrightarrow W_F$ and $\widehat{G} \rtimes W_F \twoheadrightarrow W_F$;
(2) $\psi(W_F)$ consists of semisimple elements;
(3) $\psi|W_F$ is continuous;
(4) $\psi|\mathrm{SL}_2(\mathbb{C})$ is algebraic if F is non-archimedean.

Using the Langlands classification, Theorem 0.1 (2) can be extended to $\phi \in \Phi(G)$, i.e., there exists a finite subset Π_ϕ of $\mathrm{Irr}(G(F))$ with an injective map $\Pi_\phi \ni \pi \mapsto \langle \cdot, \pi \rangle_\phi \in \mathrm{Irr}(\mathcal{S}_\phi)$, which is bijective when F is non-archimedean, such that
$$\mathrm{Irr}(G(F)) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi.$$
We call Π_ϕ the **L-packet** for $G(F)$ associated with ϕ. Therefore, L-packets Π_ϕ classify $\mathrm{Irr}(G(F))$.

On the other hands, A-packets Π_ψ associated with $\psi \in \Psi(G)$ are the "local components of global A-packets", and they do not classify $\mathrm{Irr}_{\mathrm{unit}}(G(F))$. For example,

- the map $\Pi_\psi \ni \pi \mapsto \langle \cdot, \pi \rangle_\psi \in \mathrm{Irr}(\mathcal{S}_\psi)$ is not injective nor surjective in general;
- there are $\psi, \psi' \in \Psi(G)$ such that $\Pi_\psi \cap \Pi_{\psi'} \neq \emptyset$ but $\psi \not\cong \psi'$;
- there exists $\pi \in \mathrm{Irr}_{\mathrm{unit}}(G(F))$ such that $\pi \notin \Pi_\psi$ for any $\psi \in \Psi(G)$.

Moreover, the A-packet Π_ψ is determined by endoscopic character identities, so that it is difficult to describe Π_ψ explicitly. In particular, it is an important problem to determine whether Π_ψ is multiplicity-free, i.e., a subset of $\mathrm{Irr}_{\mathrm{unit}}(G(F))$, or not.

Before Arthur, there are several works for "constructions of A-packets".

- Barbasch–Vogan (1985) [BV85] constructed a packet Π_ψ of unipotent representations when $F = \mathbb{C}$ and ψ is "unipotent" (of good parity). This packet is multiplicity-free. It was proven by Mœglin–Renard [MR17] that this packet coincides with Arthur's one.
- Adams–Johnson (1987) [AJ87] constructed a packet Π_ψ of cohomological representations when $F = \mathbb{R}$ and ψ is so-called "Adams–Johnson". It was proven by Arancibia–Mœglin–Renard [AMR] that this packet coincides with Arthur's one.
- Mœglin [Mœ17] and Mœglin–Renard [MRa, MRb] constructed Arthur's packet Π_ψ generally when $F = \mathbb{R}$ by using the Howe duality correspondence, cohomological

inductions, the translation principle, and irreducible parabolic inductions. However, since the translation functor is difficult, they have not yet obtained the multiplicity-free result.

- Using microlocal analysis of certain stratified complex varieties, Adams–Barbasch–Vogan (1992) [ABV92] constructed a packet Π_ψ^{ABV} for general ψ when $F = \mathbb{R}$. This packet is multiplicity-free, and coincides with Adams–Johnson packet Π_ψ when ψ is "Adams–Johnson". However, it is an open problem that whether Π_ψ^{ABV} is equal to Arthur's packet Π_ψ.
- Using Jacquet modules, Mœglin constructed the A-packets Π_ψ when F is p-adic (up to constructions of supercuspidal representations) in her consecutive works (e.g., [Mœ06, Mœ09a], etc.). In particular, she showed in [Mœ11] that the A-packets are multiplicity-free. For a detailed why the A-packets constructed by Mœglin agree with Arthur's ones, see also Xu's paper [X17b] in addition to the original papers of Mœglin.
- In a recent work [CFMMX], using a vanishing cycles functor of perverse sheaves on certain stratified complex varieties, Cunningham–Fiori–Mracek–Moussaoui–Xu constructed an "ABV packet" Π_ψ^{ABV} when F is p-adic. This is a p-adic analogue of the work of Adams–Barbasch–Vogan [ABV92]. This packet is multiplicity-free. The main conjecture in [CFMMX] is that Π_ψ^{ABV} would coincide with Arthur's packet Π_ψ.

The purpose of this note is to explain Mœglin's explicit constructions of A-packets both when F is p-adic and when F is archimedean. We will explain only the construction of the packet Π_ψ for $G = \mathrm{SO}_{2n+1}$ or $G = \mathrm{Sp}_{2n}$, but will not treat the proofs, the map $\pi \mapsto \langle \cdot, \pi \rangle_\psi$, or other groups. For these topics, we refer to relevant references.

In Part 1, we will explain the p-adic case along with a series of papers of Xu [X17a, X17b, X]. In §1, we fix notations for induced representations and Jacquet modules, and recall some basic results. For Mœglin's constructions of A-packets, we follow a filtration of A-parameters as follows:

$$\text{(elementary)} \subset \text{(having a DDR)} \subset \text{(of good parity)} \subset \text{(general)}.$$

These notions are defined in §2. In §3, we treat the case where the A-parameter ψ is elementary, together with a description of L-parameters for supercuspidal representations, which is one of main results in [X17a]. The case where ψ has a discrete diagonal restriction (DDR) is treated in §4. In particular, we determine the cardinality of the A-packet Π_ψ in this case. In §5, we treat the case where ψ is of good parity and the general case. In the general case, the A-packets are constructed by irreducible parabolic inductions. As a consequence, one can check that Π_ψ is multiplicity-free, which is the main conclusion in [X17b]. Unlike the case of DDR, in the case of good parity, we will construct representations which are irreducible or zero. In §6, we give an algorithm for the non-vanishing criterion of these representations, which was established in [X].

In Part 2, we will explain the archimedean case. Both in the cases where $F = \mathbb{R}$ and $F = \mathbb{C}$, the general packets Π_ψ are constructed by irreducible parabolic inductions from packets for parameters of good parity as in the p-adic case. In §8, we explain the complex case along with Mœglin–Renard [MR17]. For ψ of good parity, the packet $\Pi_\psi = \Pi_\psi^{\mathrm{BV}}$ is constructed by assigning infinitesimal characters and wavefront sets. We also give a more explicit description of Π_ψ^{BV} by Barbasch [B89]. In §9, we explain the real case. According to Mœglin [Mœ17] and Mœglin–Renard [MRa, MRb], one should follow a filtration of A-parameters as follows:

$$(\text{unipotent}) \subset (\text{very regular}) \subset (\text{of good parity}) \subset (\text{general}).$$

The packets are constructed by using the theta correspondence, cohomological inductions, the translation principle, and irreducible parabolic inductions. However, since these techniques seem not to be so explicit, we only explain the case where ψ is "Adams–Johnson", which is a spacial case of very regular parameters. In this case, the packet Π_ψ is constructed by derived functor modules $A_{\mathfrak{q}}(\lambda)$ with λ in the good range.

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Contents

Part 1. The non-archimedean case

In Part 1, we explain the theory of A-packets in the non-archimedean case. Let F be a non-archimedean local field of characteristic zero. We denote by W_F the Weil group of F. The norm map $|\cdot|\colon W_F \to \mathbb{R}^\times$ is normalized so that $|\mathrm{Frob}| = q^{-1}$, where $\mathrm{Frob} \in W_F$ is a fixed (geometric) Frobenius element, and $q = q_F$ is the cardinality of the residual field of F.

Each irreducible representation ρ of W_F of dimension d is identified with the irreducible supercuspidal representation of $\mathrm{GL}_d(F)$ via the local Langlands correspondence for GL_d. For each integer d, the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d is denoted by S_d. We denote by $S_a \boxtimes S_b$ the outer tensor product, which is an irreducible representation of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, and by $S_a \otimes S_b$ the inner tensor product, which is a representation of $\mathrm{SL}_2(\mathbb{C})$ such that

$$S_a \otimes S_b \cong S_{a+b-1} \oplus S_{a+b-3} \oplus \cdots \oplus S_{|a-b|+1}.$$

For a representation Π of some group, we write s.s.(Π) for the semisimplification of Π.

1. Induced representations and Jacquet modules

In this section, we recall some results on induced representations and Jacquet modules.

1.1. The case of GL_N.

Let $P = MN$ be a standard parabolic subgroup of $\mathrm{GL}_N(F)$, i.e., P contains the Borel subgroup consisting of upper half triangular matrices. Then the Levi subgroup M is isomorphic to $\mathrm{GL}_{d_1}(F) \times \cdots \times \mathrm{GL}_{d_r}(F)$ with $d_1 + \cdots + d_r = N$. For smooth representations $\tau_1, \dots, \tau_r$ of $\mathrm{GL}_{d_1}(F), \dots, \mathrm{GL}_{d_r}(F)$, respectively, we denote the normalized induced representation by

$$\tau_1 \times \cdots \times \tau_r \coloneqq \mathrm{Ind}_P^{\mathrm{GL}_N(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r).$$

A **segment** is a symbol $[x, y]$, where $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}$. We identify $[x, y]$ with the set $\{x, x-1, \dots, y\}$ if $x \geq y$, and $\{x, x+1, \dots, y\}$ if $x \leq y$, so that $\#[x, y] = |x - y| + 1$. Let ρ be an irreducible (unitary) supercuspidal representation of $\mathrm{GL}_d(F)$. Then the normalized induced representation

$$\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y$$

of $\mathrm{GL}_{d(|x-y|+1)}(F)$ has a unique irreducible subrepresentation, which is denoted by

$$\langle \rho; x, \dots, y \rangle.$$

If $x \geq y$, this is called a **Steinberg representation** and is denoted by

$$|\cdot|^{\frac{x+y}{2}} \mathrm{St}(\rho, x - y + 1),$$

which is a discrete series representation of $\mathrm{GL}_{d(x-y+1)}(F)$. When $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, we write $\mathrm{St}_d = \mathrm{St}(\mathbf{1}_{\mathrm{GL}_1(F)}, d) = \langle (d-1)/2, (d-3)/2, \dots, -(d-1)/2 \rangle$. If $x < y$, this is called a **Speh representation** and is denoted by

$$|\cdot|^{\frac{x+y}{2}} \mathrm{Sp}(\rho, y - x + 1).$$

For example, if $\rho = \mu$ is a unitary character (i.e., $d = 1$) and $x < y$, then $\langle \mu; x, \dots, y \rangle = \mu|\det_{y-x+1}|^{(x+y)/2}$ is a character of $\mathrm{GL}_{y-x+1}(F)$, where we denote by $\det_k$ the determinant character of $\mathrm{GL}_k(F)$.

Definition 1.1. *Let $[x, y]$ and $[x', y']$ be two segments.*

(1) *When* $(x-y)(x-y') \geq 0$, *we say that* $[x,y]$ *and* $[x',y']$ *are* ***linked*** *if* $[x,y] \not\subset [x',y']$, $[x',y'] \not\subset [x,y]$ *as sets, and* $[x,y] \cup [x,y']$ *is also a segment.*
(2) *When* $(x-y)(x'-y') < 0$, *we say that* $[x,y]$ *and* $[x',y']$ *are* ***linked*** *if* $[y,x]$ *and* $[x',y']$ *are linked, and* $x, y \notin [x',y']$ *and* $x', y' \notin [x,y]$.

The linked-ness gives an irreducibility criterion for induced representations.

Theorem 1.2 (Zelevinsky [Z80, Theorems 4.2, 9.7], Mœglin–Waldspurger [MW89])**.** *Let* $[x,y]$ *and* $[x',y']$ *be segments, and let* ρ *and* ρ' *be irreducible unitary supercuspidal representations of* $\mathrm{GL}_d(F)$ *and* $\mathrm{GL}_{d'}(F)$*, respectively. Then the induced representation*

$$\langle \rho; x, \ldots, y \rangle \times \langle \rho'; x', \ldots, y' \rangle$$

is irreducible unless $[x,y]$ *are* $[x',y']$ *are linked, and* $\rho \cong \rho'$.

For a partition $(k_1, \ldots, k_r)$ of k, we denote by $\mathrm{Jac}_{(k_1,\ldots,k_r)}$ the normalized Jacquet functor of representations of $\mathrm{GL}_k(F)$ with respect to the standard maximal parabolic subgroup $P = MN$ with $M \cong \mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F)$. The Jacquet module of $\langle \rho; x, \ldots, y \rangle$ with respect to a maximal parabolic subgroup is computed by Zelevinsky.

Proposition 1.3 ([Z80, Propositions 3.4, 9.5])**.** *Let* ρ *be an irreducible (unitary) supercuspidal representation of* $\mathrm{GL}_d(F)$*. Suppose that* $x \neq y$ *and set* $k = d(|x-y|+1)$*. Then* $\mathrm{Jac}_{(k_1,k_2)}(\langle \rho; x, \ldots, y \rangle) = 0$ *unless* $k_1 \equiv 0 \bmod d$*. If* $k_1 = dm$ *with* $1 \leq m \leq |x-y|$*, we have*

$$\mathrm{Jac}_{(k_1,k_2)}(\langle \rho; x, \ldots, y \rangle) = \langle \rho; x, \ldots, x - \epsilon(m-1) \rangle \boxtimes \langle \rho; x - \epsilon m, \ldots, y \rangle,$$

where $\epsilon \in \{\pm 1\}$ *is given so that* $\epsilon(x-y) > 0$.

Let $\mathcal{R}_N$ be the Grothendieck group of the category of smooth representations of $\mathrm{GL}_N(F)$ of finite length. By the semisimplification, we identify the objects in this category with elements in $\mathcal{R}_N$. Equivalence classes of irreducible smooth representations of $\mathrm{GL}_N(F)$ form a $\mathbb{Z}$-basis of $\mathcal{R}_N$. Set $\mathcal{R} = \oplus_{N \geq 0} \mathcal{R}_N$. The induction functor gives a product

$$m \colon \mathcal{R} \otimes \mathcal{R} \to \mathcal{R},\ \tau_1 \otimes \tau_2 \mapsto \mathrm{s.s.}(\tau_1 \times \tau_2).$$

This product makes $\mathcal{R}$ an associative commutative ring. On the other hand, the Jacquet functor gives a coproduct

$$m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$$

which is defined by the $\mathbb{Z}$-linear extension of

$$\mathrm{Irr}(\mathrm{GL}_N(F)) \ni \tau \mapsto \sum_{k=0}^{N} \mathrm{s.s.Jac}_{(k,N-k)}(\tau).$$

Then m and m^* make $\mathcal{R}$ a graded Hopf algebra, i.e., $m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ is a ring homomorphism.

1.2. **The cases of** SO_{2n+1} **and** Sp_{2n}**.** Next, we set G_n to be split SO_{2n+1} or Sp_{2n}, i.e., G_n is the split algebraic group of type B_n or C_n. Fix a Borel subgroup of $G_n(F)$. Let $P = MN$ be a standard parabolic subgroup of $G_n(F)$. Then the Levi part M is of the form $\mathrm{GL}_{d_1}(F) \times \cdots \times \mathrm{GL}_{d_r}(F) \times G_{n_0}(F)$ with $d_1 + \cdots + d_r + n_0 = n$. For a smooth representation $\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0$ of M, we denote the normalized induced representation by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi_0 = \mathrm{Ind}_P^{G_n(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0).$$

On the other hand, for a smooth representation π of $G_n(F)$, we denote the normalized Jacquet module with respect to P by

$$\mathrm{Jac}_P(\pi),$$

and its semisimplification by $\mathrm{s.s.Jac}_P(\pi)$. When $r = 1$, i.e., $M \cong \mathrm{GL}_d(F) \times G_{n-d}(F)$ and

$$\mathrm{s.s.Jac}_P(\pi) = \bigoplus_{i \in I} \tau_i \boxtimes \pi_i,$$

for a fixed irreducible supercuspidal unitary representation ρ of $\mathrm{GL}_d(F)$ and for a real number x, we set

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi) = \bigoplus_{\substack{i \in I \\ \tau_i \cong \rho|\cdot|^x}} \pi_i.$$

This is a representation of $G_{n-d}(F)$. Also, for $\rho_1, \dots, \rho_r$ and for $x_1, \dots, x_r \in \mathbb{R}$, we set

$$\mathrm{Jac}_{\rho_1|\cdot|^{x_1}, \dots, \rho_r|\cdot|^{x_r}}(\pi) = \mathrm{Jac}_{\rho_r|\cdot|^{x_r}} \circ \cdots \circ \mathrm{Jac}_{\rho_1|\cdot|^{x_1}}(\pi).$$

Now suppose that an irreducible representation π of $G_n(F)$ is a subrepresentation (resp. a quotient) of an induced representation $\tau \rtimes \pi_0$ with irreducible representation $\tau \boxtimes \pi_0$ of $M \cong \mathrm{GL}_d(F) \times G_{n-d}(F)$. Then (using the contragredient and the MVW functors if necessary), the Frobenius reciprocity implies that $\mathrm{s.s.Jac}_P(\pi)$ contains $\tau \boxtimes \pi_0$ (resp. $\widetilde{\tau} \boxtimes \pi_0$). In particular:

Lemma 1.4. *Let $\tau \otimes \pi_0$ be an irreducible representation of $M \cong \mathrm{GL}_d(F) \times G_{n-d}(F)$. If $\mathrm{s.s.Jac}_P(\tau \rtimes \pi_0)$ contains $\tau \boxtimes \pi_0$ (resp. $\widetilde{\tau} \boxtimes \pi_0$) with multiplicity one, then the induced representation $\tau \rtimes \pi_0$ has a unique irreducible subrepresentation (resp. a unique irreducible quotient).*

We will use this technique (or its variant) to construct A-packets.

Let

$$\mathcal{R}(G) = \bigoplus_{n \geq 0} \mathcal{R}(G_n)$$

be the direct sum of the Grothendieck groups $\mathcal{R}(G_n)$ of the categories of smooth representations of $G_n(F)$ of finite length. The parabolic induction defines a module structure

$$\rtimes \colon \mathcal{R} \otimes \mathcal{R}(G) \to \mathcal{R}(G),$$

and the Jacquet functor defines a comodule structure

$$\mu^* \colon \mathcal{R}(G) \to \mathcal{R} \otimes \mathcal{R}(G)$$

by

$$\mathrm{Irr}(G_n(F)) \ni \pi \mapsto \sum_{d=0}^{n} \mathrm{s.s.Jac}_{P_d}(\pi),$$

where $P_d = M_d N_d$ is the standard parabolic subgroup of $G_n(F)$ with the Levi factor $M_d \cong \mathrm{GL}_d(F) \times G_{n-d}(F)$.

The contragredient functor $\tau \mapsto \widetilde{\tau}$ defines an automorphism $\sim \colon \mathcal{R} \to \mathcal{R}$ in a natural way. Let $s \colon \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ be the homomorphism defined by $\sum_i \tau_i \otimes \tau'_i \mapsto \sum_i \tau'_i \otimes \tau_i$.

One can compute the Jacquet modules of induced representations by the following formula.

Theorem 1.5 (Tadić [T95]). *Consider the composition*
$$M^* = (m \otimes \mathrm{id}) \circ (\sim \otimes m^*) \circ s \circ m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}.$$
Then for a standard parabolic subgroup $P = MN$ of $G_n(F)$ with $M \cong \mathrm{GL}_d(F) \times G_{n-d}(F)$, and for an admissible representation $\tau \boxtimes \pi$ of M, we have
$$\mu^*(\tau \rtimes \pi) = M^*(\pi) \rtimes \mu^*(\pi).$$

1.3. **Aubert involution.** For $\pi \in \mathcal{R}(G_n)$, we define $D_{G_n}(\pi) \in \mathcal{R}(G_n)$ by
$$D_{G_n}(\pi) = \sum_{P=MN} (-1)^{\dim A_M} \mathrm{Ind}_P^{G_n(F)}(\mathrm{Jac}_P(\pi)) \in \mathcal{R}(G_n),$$
where $P = MN$ runs over all standard parabolic subgroups of $G_n(F)$, and A_M is the maximal split central torus of M.

Theorem 1.6 ([Au95]). *The operator D_{G_n} on $\mathcal{R}(G_n)$ has the following properties:*

(1) $D_{G_n} \circ \sim = \sim \circ D_{G_n}$;
(2) $D_{G_n}^2 = \mathrm{id}$.
(3) *When $\tau_i = \langle \rho_i; x_i, \dots, y_i \rangle$ for $i = 1, \dots, r$,*
$$D_{G_n}(\tau_1 \times \cdots \times \tau_r \rtimes \pi_0) = \hat{\tau}_1 \times \cdots \times \hat{\tau}_r \rtimes D_{G_{n_0}}(\pi_0),$$
where $\hat{\tau}_i = \langle \rho_i; y_i, \dots, x_i \rangle$.
(4) *If $P = MN$ with $M \cong \mathrm{GL}_d(F) \times G_{n-d}(F)$, and if $\pi \in \mathcal{R}(G_n)$ satisfies*
$$\mathrm{s.s.Jac}_P(\pi) \cong \sum_i \tau_i \boxtimes \pi_0,$$
then
$$\mathrm{s.s.Jac}_P(D_{G_n}(\pi)) \cong \sum_i \widetilde{\tau}_i \boxtimes D_{G_{n_0}}(\pi_0).$$
In particular,
$$D_{G_{n_0}}\left(\mathrm{Jac}_{\rho|\cdot|^x}(\pi)\right) \cong \mathrm{Jac}_{\widetilde{\rho}|\cdot|^{-x}}\left(D_{G_n}(\pi)\right).$$
(5) *If π is an irreducible representation of $G_n(F)$, then there exists a sign $\epsilon \in \{\pm 1\}$ such that $\hat{\pi} = \epsilon \cdot D_{G_n}(\pi)$ is also an irreducible representation of $G_n(F)$.*
(6) *If π is an irreducible supercuspidal representation, then $\hat{\pi} \cong \pi$.*

For an irreducible representation π of $G_n(F)$, we call the irreducible representation $\hat{\pi}$ the **Aubert involution** of π.

2. A-parameters

In this section, we review Arthur's theory for A-packets.

2.1. **The case of** GL_N. Fix a (geometric) Frobenius element $\mathrm{Frob} \in W_F$. A homomorphism
$$\psi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$$
is called a **representation** of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ if

- $\psi(\mathrm{Frob}) \in \mathrm{GL}_N(\mathbb{C})$ is semisimple;
- $\psi|W_F$ is smooth, i.e., has an open kernel;
- $\psi|\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is algebraic.

An **A-parameter** for $\mathrm{GL}_N(F)$ is a representation $\psi\colon W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{GL}_N(\mathbb{C})$ such that $\psi(W_F)$ is bounded. Two A-parameters are said to be **equivalent** if they are equivalent as representations.

For an A-parameter ψ for $\mathrm{GL}_N(F)$, one can associate an irreducible unitary representation τ_ψ of $\mathrm{GL}_N(F)$ as follows: When ψ is irreducible, it is decomposed into a tensor product

$$\psi=\rho\boxtimes S_a\boxtimes S_b,$$

where ρ is an irreducible bounded representation of W_F. Then we set τ_ψ to be the unique irreducible subrepresentation $\mathrm{Sp}(\mathrm{St}(\rho,a),b)$ of

$$\left\langle\rho;\frac{a-1}{2},\dots,-\frac{a-1}{2}\right\rangle|\cdot|^{-\frac{b-1}{2}}\times\cdots\times\left\langle\rho;\frac{a-1}{2},\dots,-\frac{a-1}{2}\right\rangle|\cdot|^{\frac{b-1}{2}}.$$

If $b=1$, it is the Steinberg representation $\mathrm{St}(\rho,a)=\langle\rho;(a-1)/2,\dots,-(a-1)/2\rangle$, and if $a=1$, it is the Speh representation $\mathrm{Sp}(\rho,b)=\langle\rho;-(b-1)/2,\dots,(b-1)/2\rangle$. It is easy to check that τ_ψ is also the unique irreducible subrepresentation of

$$\left\langle\rho;-\frac{b-1}{2},\dots,\frac{b-1}{2}\right\rangle|\cdot|^{\frac{a-1}{2}}\times\cdots\times\left\langle\rho;-\frac{b-1}{2},\dots,\frac{b-1}{2}\right\rangle|\cdot|^{-\frac{a-1}{2}}.$$

In general, ψ can be decomposed into a direct sum

$$\psi=\psi_1\oplus\cdots\oplus\psi_r,$$

where $\psi_1,\dots,\psi_r$ are irreducible representations of $W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$. We set

$$\tau_\psi=\tau_{\psi_1}\times\cdots\times\tau_{\psi_r},$$

which is irreducible by Theorem 1.2.

2.2. **The case of SO_{2n+1} and Sp_{2n}.** An **A-parameter** for SO_{2n+1} is a symplectic representation

$$\psi\colon W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{Sp}_{2n}(\mathbb{C})$$

of $W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$ such that $\psi(W_F)$ is bounded. Similarly, an **A-parameter** for Sp_{2n} is an orthogonal representation

$$\psi\colon W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{SO}_{2n+1}(\mathbb{C})$$

of $W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$ such that $\psi(W_F)$ is bounded. For $G_n=\mathrm{SO}_{2n+1}$ or $G_n=\mathrm{Sp}_{2n}$, we set $\Psi(G_n)$ to be the set of equivalence classes of A-parameters for G_n. We say that $\psi\in\Psi(G_n)$ is **tempered** if $\psi|\{1\}\times\{\mathbf{1}_2\}\times\mathrm{SL}_2(\mathbb{C})$ is trivial. We denote by $\Phi_{\mathrm{temp}}(G_n)$ the subset of $\Psi(G_n)$ consisting of tempered A-parameters.

For $\psi\in\Psi(\mathrm{SO}_{2n+1})$ (resp. $\psi\in\Psi(\mathrm{Sp}_{2n})$), we can decompose

$$\psi=m_1\psi_1+\cdots+m_r\psi_r+\psi'+\psi'^\vee,$$

where $\psi_1,\dots,\psi_r$ are distinct irreducible symplectic (resp. orthogonal) representations of $W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$, and ψ' is a sum of irreducible representations which are not symplectic (resp. not orthogonal). We define the **component group** A_ψ of ψ by

$$A_\psi=\bigoplus_{i=1}^{r}(\mathbb{Z}/2\mathbb{Z})\alpha_i\cong(\mathbb{Z}/2\mathbb{Z})^r.$$

Namely, A_ψ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank r and $\{\alpha_1, \dots, \alpha_r\}$ is a basis of A_ψ with α_i associated to ψ_i. We set

$$z_\psi = \sum_{i=1}^{r} m_i \alpha_i \in A_\psi$$

and call z_ψ the **central element** in A_ψ. Then $\mathcal{S}_\psi = \pi_0(Z_{\widehat{G}}(\mathrm{Im}(\psi))/Z(\widehat{G})^{W_F})$ is canonically isomorphic to $A_\psi / \langle z_\psi \rangle$.

As explained in Theorem 0.1, for $\psi \in \Psi(G_n)$, there is an A-packet Π_ψ, which is a finite multiset over $\mathrm{Irr}_{\mathrm{unit}}(G_n(F))$, together with a map

$$\Pi_\psi \to \widehat{A_\psi}, \ \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

such that $\langle z_\psi, \pi \rangle_\psi = 1$ for any $\pi \in \Pi_\psi$. These are characterized by certain (twisted and standard) endoscopic character identities. Mœglin constructed A-packets explicitly, and showed that they are multiplicity-free (see, e.g., [Mœ06, Mœ09a, Mœ11], etc.). Since Mœglin's A-packets satisfy the endoscopic character identities (see also [X17b]), they coincide with Arthur's ones. Consequently, we obtain the following deep result.

Theorem 2.1 ([Mœ11], [X17b, Theorem 8.12])**.** *The A-packet Π_ψ is multiplicity-free, i.e., Π_ψ is a subset of $\mathrm{Irr}_{\mathrm{unit}}(G_n(F))$.*

The purpose of Part 1 is to review Mœglin's construction of A-packets. It is carried out through several stages.

Let

$$\Delta\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}), \ (w, \alpha) \mapsto (w, \alpha, \alpha)$$

be the diagonal map. For $\psi \in \Psi(G_n)$, we set $\psi_d = \psi \circ \Delta$ to be the diagonal restriction of ψ.

Definition 2.2. *Let $G_n = \mathrm{SO}_{2n+1}$ or $G_n = \mathrm{Sp}_{2n}$. We define a chain*

$$\Psi(G_n) \supset \Psi_{\mathrm{gp}}(G_n) \supset \Psi_{\mathrm{DDR}}(G_n) \supset \Psi_{\mathrm{el}}(G_n)$$

as follows:

(1) *$\psi \in \Psi_{\mathrm{gp}}(G_n)$ if ψ is a sum of irreducible symplectic (resp. orthogonal) representations when $G_n = \mathrm{SO}_{2n+1}$ (resp. $G_n = \mathrm{Sp}_{2n}$). In this case, we say that ψ is **of good parity**.*

(2) *$\psi \in \Psi_{\mathrm{DDR}}(G_n)$ if ψ is of good parity and the diagonal restriction $\psi_d = \psi \circ \Delta$ is multiplicity-free. In this case, we say that ψ has a **discrete diagonal restriction (DDR)**.*

(3) *$\psi \in \Psi_{\mathrm{el}}(G_n)$ if ψ has a DDR and*

$$\psi = \bigoplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$$

*with $\min\{a_i, b_i\} = 1$ for any $i \in I$. In this case, we say that ψ is **elementary**.*

3. The case of elementary parameters

In this section, we construct A-packets Π_ψ for elementary A-parameters $\psi \in \Psi_{\mathrm{el}}(G_n)$. See [X17a] and [X17b, §6] for more precision. Before the construction, we review the case of general linear groups GL_N.

3.1. **The case of GL_N.** Let ψ be an A-parameter for $\mathrm{GL}_N(F)$. Assume in this subsection that ψ is elementary, i.e., $\psi \cong \oplus_{i\in I}\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$ with $\min\{a_i, b_i\} = 1$ for any $i \in I$ and $\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \not\cong \rho_j \boxtimes S_{a_j} \boxtimes S_{b_j}$ for $i \neq j$. Recall that one can associate an irreducible unitary representation τ_ψ of $\mathrm{GL}_N(F)$. We construct τ_ψ once more.

If $a_i = b_i = 1$ for any $i \in I$, then $\tau_\psi = \times_{i\in I}\rho_i$. Now fix an irreducible representation ρ of W_F such that $\rho \subset \psi|W_F$. For $\rho \boxtimes S_a \subset \psi_d$, we set

$$\delta_{\rho,a,\psi} = \begin{cases} 1 & \text{if } \rho \boxtimes S_a \boxtimes \mathbf{1} \subset \psi, \\ -1 & \text{otherwise.} \end{cases}$$

Let a_0 be the smallest integer such that $\rho \boxtimes S_{a_0} \subset \psi_d$. If $a_0 > 1$, then τ_ψ is the unique irreducible subrepresentation of

$$\rho|\cdot|^{\delta_0 \frac{a_0-1}{2}} \times \tau_{\psi'} \times \rho|\cdot|^{-\delta_0 \frac{a_0-1}{2}},$$

where ψ' is an elementary A-parameter such that

$$\psi'_d = \psi_d - \rho \boxtimes S_{a_0} + \rho \boxtimes S_{a_0-2}$$

and

$$\delta_0 \coloneqq \delta_{\rho,a_0,\psi} = \delta_{\rho,a_0-2,\psi'}.$$

If $a_0 = 1$ and there exists the next smallest integer a_1 such that $\rho \boxtimes S_{a_1} \subset \psi_d$, then τ_ψ is the unique irreducible subrepresentation of

$$\left\langle \rho; \delta_1 \frac{a_1-1}{2}, \dots, 0 \right\rangle \times \tau_{\psi'} \times \left\langle \rho; 0, \dots, -\delta_1 \frac{a_1-1}{2} \right\rangle,$$

where ψ' is an elementary A-parameter such that

$$\psi'_d = \psi_d - \rho \boxtimes (\mathbf{1} \oplus S_{a_1})$$

and $\delta_1 \coloneqq \delta_{\rho,a_1,\psi}$.

3.2. **L-parameters for supercuspidal representations.** Let G_n be split SO_{2n+1} or Sp_{2n}. Fix an elementary A-parameter $\psi \in \Psi_{\mathrm{el}}(G_n)$ and a character $\varepsilon \in \widehat{A_\psi}$ such that $\varepsilon(z_\psi) = 1$. When $\rho \boxtimes S_a \subset \psi_d$, we set $\varepsilon(\rho, a) = \varepsilon(\alpha_{\rho,a}) \in \{\pm 1\}$, where $\alpha_{\rho,a} \in A_\psi$ is the element associated to $\rho \boxtimes S_a \boxtimes S_1$ or $\rho \boxtimes S_1 \boxtimes S_a$.

Definition 3.1. (1) *If $\rho \boxtimes S_a \subset \psi_d$ with $a > 1$, set*

$$\delta_{\rho,a,\psi} = \begin{cases} 1 & \text{if } \rho \boxtimes S_a \boxtimes \mathbf{1} \subset \psi, \\ -1 & \text{if } \rho \boxtimes \mathbf{1} \boxtimes S_a \subset \psi. \end{cases}$$

When $a = 1$, we set $\delta_{\rho,a,\psi} = 1$.

(2) *Define $\mathcal{T}_{\rho,\psi,\varepsilon}$ to be the set containing 0 and all integers $a > 0$ satisfying the following conditions:*
   - **(chain condition):** $\rho \boxtimes S_k \subset \psi_d$ *for any* $k \leq a$ *with* $k \equiv a \bmod 2$*;*
   - **(alternating condition):** $\varepsilon(\rho, k) = -\varepsilon(\rho, k+2)$ *for* $0 < k < a$ *with* $k \equiv a \bmod 2$*;*
   - **(initial condition):** *if* $a \equiv 0 \bmod 2$ *so that* $\rho \boxtimes S_2 \subset \psi_d$*, then* $\varepsilon(\rho, 2) = -1$.

(3) *Set*

$$\begin{aligned} b_{\rho,\psi,\varepsilon} &= \max \mathcal{T}_{\rho,\psi,\varepsilon}, \\ a_{\rho,\psi,\varepsilon} &= \min\{a > b_{\rho,\psi,\varepsilon} \mid \rho \boxtimes S_a \subset \psi_d\}, \\ \delta_{\rho,\psi,\varepsilon} &= \delta_{\rho,a_{\rho,\psi,\varepsilon},\psi}. \end{aligned}$$

Remark that if $a, a' \in \mathcal{T}_{\rho,\psi,\varepsilon}$ with $a, a' \neq 0$, then $a \equiv a' \bmod 2$. When $\{a > b_{\rho,\psi,\varepsilon} \mid \rho \boxtimes S_a \subset \psi_d\} = \emptyset$, we understand $a_{\rho,\psi,\varepsilon} = \infty$.

Now we set $\Phi_2(G_n) = \Psi_{\mathrm{el}}(G_n) \cap \Phi_{\mathrm{temp}}(G_n)$. For $\phi \in \Phi_2(G_n)$ and $\varepsilon \in \widehat{A_\phi}$ with $\varepsilon(z_\phi) = 1$, we denote by $\pi(\phi, \varepsilon)$ the unique element in Π_ϕ corresponding to ε via the bijection $\Pi_\phi \to (A_\phi / \langle z_\phi \rangle)\widehat{\ }$. It is known that $\pi(\phi, \varepsilon)$ is a discrete series representation of $G_n(F)$. The following theorem is a criterion when $\pi(\phi, \varepsilon)$ is supercuspidal.

Theorem 3.2 (Mœglin, Xu [X17a]). *Let $\phi \in \Phi_2(G_n)$ and $\varepsilon \in \widehat{A_\phi}$ with $\varepsilon(z_\phi) = 1$. Then $\pi(\phi, \varepsilon) \in \Pi_\phi$ is supercuspidal if and only if $a_{\rho,\phi,\varepsilon} = \infty$ for any $\rho \subset \phi|W_F$.*

This is one of main results in [X17a]. In the rest of this subsection, we give a sketch of the proof of this theorem.

First, we explain the following proposition.

Proposition 3.3 ([X17a, Proposition 3.1]). *Suppose that $\pi(\phi, \varepsilon) \in \Pi_\phi$ is supercuspidal. Fix an irreducible bounded representation ρ of W_F. If $\phi \supset \rho \boxtimes S_k$ with $k > 2$, then $\phi \supset \rho \boxtimes S_{k-2}$.*

Proof. Recall that ρ is identified with the corresponding irreducible supercuspidal unitary representation of $\mathrm{GL}_d(F)$. For $s \in \mathbb{C}$, we consider the usual unnormalized intertwining operators

$$\begin{aligned} J_{\overline{P}|P}(s) &\colon \mathrm{Ind}_P^{G_{n+d}(F)}(\rho|\cdot|^s \boxtimes \pi) \to \mathrm{Ind}_{\overline{P}}^{G_{n+d}(F)}(\rho|\cdot|^s \boxtimes \pi), \\ J_{P|\overline{P}}(s) &\colon \mathrm{Ind}_{\overline{P}}^{G_{n+d}(F)}(\rho|\cdot|^s \boxtimes \pi) \to \mathrm{Ind}_P^{G_{n+d}(F)}(\rho|\cdot|^s \boxtimes \pi), \end{aligned}$$

where $P = MN$ is the standard parabolic subgroup of $G_{n+d}(F)$ with Levi part $M \cong \mathrm{GL}_d(F) \times G_n(F)$, and $\overline{P} = M\overline{N}$ is the opposite parabolic subgroup to P. These operators are defined by the meromorphic continuations of Jacquet integrals which converge when s belongs to a certain open set in $\mathbb{C}$. Since $\rho \boxtimes \pi$ is supercuspidal, it is known as a theorem of Harish-Chandra that both $J_{\overline{P}|P}(s)$ and $J_{P|\overline{P}}(s)$ are holomorphic for $\mathrm{Re}(s) \neq 0$.

On the other hand, there is a rational function $\mu(s)$ such that

$$J_{\overline{P}|P}(s) \circ J_{P|\overline{P}}(s) = \mu(s)^{-1}.$$

The function $\mu(s)$ is called the **Plancherel measure**. By results of Arthur [Ar13] and Shahidi [S90], it is known that the Prancherel measure $\mu(s)$ is given by the product of gamma factors:

$$\mu(s) = \gamma(s, \rho \otimes \widetilde{\phi}, \psi_F)\gamma(-s, \widetilde{\rho} \otimes \phi, \psi_F^{-1})\gamma(2s, R \circ \rho, \psi_F)\gamma(-2s, R \circ \widetilde{\rho}, \psi_F^{-1})$$

up to a positive constant. Here, ψ_F is a fixed non-trivial additive character of F, and

$$R = \begin{cases} \mathrm{Sym}^2 & \text{if } G_n = \mathrm{SO}_{2n+1}, \\ \wedge^2 & \text{if } G_n = \mathrm{Sp}_{2n}. \end{cases}$$

The ambiguity comes from the choices of Haar measures on N and $\overline{N}$ to define Jacquet integrals. In fact, one can choose these Haar measures using ψ_F so that the above equation actually holds.

Suppose now that $\phi \supset \rho \boxtimes S_k$ with $k > 2$. Then in $s \in \mathbb{R}$ with $s > 1/2$,

- $\gamma(2s, R \circ \rho, \psi_F)$ and $\gamma(-2s, R \circ \widetilde{\rho}, \psi_F^{-1})$ are holomorphic and nonzero;
- $\gamma(s, \rho \otimes \widetilde{\phi}, \psi_F)$ is nonzero;
- $\gamma(-s, \widetilde{\rho} \otimes \phi, \psi_F^{-1})$ is holomorphic and has a zero at $s = (k-1)/2$.

Since $\mu(s)$ is nonzero at $s = (k-1)/2$, the gamma factor $\gamma(s, \rho \otimes \widetilde{\phi}, \psi_F)$ must have a pole at $s = (k-1)/2$. This occurs only when $\phi \supset \rho \boxtimes S_{k-2}$. □

Now fix an irreducible bounded representation ρ of W_F, and a real number x. If $\phi \supset \rho \boxtimes S_{2x+1}$ and $x > 0$, define $\phi_- \in \Phi_2(G_{n-d})$ with $d = \dim(\rho)$ by

$$\phi_- = \phi - \rho \boxtimes S_{2x+1} \oplus \rho \boxtimes S_{2x-1}.$$

The following lemma follows from a compatibility of twisted endoscopic character identities and Jacquet modules.

Lemma 3.4 ([X17a, Lemma 7.2]). *We have*

$$\mathrm{Jac}_{\rho|\cdot|^x}\left(\bigoplus_{\pi \in \Pi_\phi} \pi\right) = \begin{cases} \bigoplus_{\pi_- \in \Pi_{\phi_-}} \pi_- & \text{if } \phi \supset \rho \boxtimes S_{2x+1} \text{ and } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, a compatibility of standard endoscopic character identities and Jacquet modules gives the following more precise result:

Lemma 3.5 ([X17a, Lemma 7.3]). *Suppose that $\phi \in \Phi_2(G_n)$ satisfies that $\phi \supset \rho \boxtimes S_{2x+1}$. Set $\phi_- = \phi - \rho \boxtimes S_{2x+1} \oplus \rho \boxtimes S_{2x-1}$.*

(1) *If $x > 1/2$ and $\rho \boxtimes S_{2x-1} \not\subset \phi$, then $\pi(\phi_-, \varepsilon) = \mathrm{Jac}_{\rho|\cdot|^x} \pi(\phi, \varepsilon)$ for any $\varepsilon \in \widehat{A_{\phi_-}} \cong \widehat{A_\phi}$.*

(2) *If $x > 1/2$ and $\rho \boxtimes S_{2x-1} \subset \phi$, then $A_\phi = A_{\phi_-} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{\rho,2x+1}$, where $\alpha_{\rho,a}$ is the element in A_ϕ corresponding to $\rho \boxtimes S_a$. Define a surjection $A_\phi \twoheadrightarrow A_{\phi_-}$ by $\alpha_{\rho,2x+1} \mapsto \alpha_{\rho,2x-1}$ and by identity on A_{ϕ_-}. This gives an injection $\widehat{A_{\phi_-}} \hookrightarrow \widehat{A_\phi}$. Then $\mathrm{Jac}_{\rho|\cdot|^x} \pi(\phi, \varepsilon) = 0$ unless $\varepsilon \in \widehat{A_{\phi_-}}$, i.e.,*

$$\varepsilon(\rho, 2x+1)\varepsilon(\rho, 2x-1) = 1.$$

In this case, $\pi(\phi_-, \varepsilon) = \mathrm{Jac}_{\rho|\cdot|^x} \pi(\phi, \varepsilon)$.

(3) *If $x = 1/2$, then $A_\phi = A_{\phi_-} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{\rho,2}$. Let $A_\phi \twoheadrightarrow A_{\phi_-}$ be the projection, and $\widehat{A_{\phi_-}} \hookrightarrow \widehat{A_\phi}$ be the induced injection. Then $\mathrm{Jac}_{\rho|\cdot|^x} \pi(\phi, \varepsilon) = 0$ unless $\varepsilon \in \widehat{A_{\phi_-}}$, i.e.,*

$$\varepsilon(\rho, 2) = 1.$$

In this case, $\pi(\phi_-, \varepsilon) = \mathrm{Jac}_{\rho|\cdot|^x} \pi(\phi, \varepsilon)$.

Let $\phi \in \Phi_2(G_n)$ and $\varepsilon \in \widehat{A_\phi}$ such that $\varepsilon(z_\phi) = 1$. By Lemmas 3.4 and 3.5, $a_{\rho,\phi,\varepsilon} = \infty$ for any $\rho \subset \phi|W_F$ if and only if $\mathrm{Jac}_{\rho|\cdot|^x} \pi(\phi, \varepsilon) = 0$ for any ρ and $x \in \mathbb{R}$. This condition is equivalent that $\pi(\phi, \varepsilon)$ is supercuspidal. Hence we obtain Theorem 3.2.

3.3. The case of SO_{2n+1} and Sp_{2n}.

Now let $\psi \in \Psi_{\mathrm{el}}(G_n)$. We construct the elements in the A-packet Π_ψ by induction on n. The following is the main theorem in the elementary case.

Theorem 3.6 (Mœglin). *For $\psi \in \Psi_{\mathrm{el}}(G_n)$ and $\varepsilon \in \widehat{A_\psi}$ with $\varepsilon(z_\psi) = 1$, we can construct a representation $\pi(\psi, \varepsilon)$ of $G_n(F)$ as the following manner. Then $\pi(\psi, \varepsilon)$ is irreducible, $\pi(\psi, \varepsilon) \not\cong \pi(\psi, \varepsilon')$ if $\varepsilon \neq \varepsilon'$, and $\pi(\psi, \varepsilon)$ satisfies the following basic properties:*

(Jacquet module): *If $\mathrm{Jac}_{\rho|\cdot|^x}(\pi) \neq 0$ for some $x \in \mathbb{R}$, then there exists $a > b_{\rho,\psi,\varepsilon}$ with $\rho \boxtimes S_a \subset \psi_d$ such that*

$$x = \delta_{\rho,a,\psi} \frac{a-1}{2}.$$

(Non-unitary irreducibility): *For $x \in (1/2)\mathbb{Z}$ with $x \geq 1/2$, if $x \neq 1/2$ and $\rho \boxtimes S_{2x-1} \not\subset \psi_d$, or if $0 \leq 2x-1 < b_{\rho,\psi,\varepsilon}$, then $\rho|\cdot|^x \rtimes \pi(\psi,\varepsilon)$ is irreducible.*

(Unitary reducibility): *Suppose that there exists an odd integer a such that $\rho \boxtimes S_a \not\subset \psi_d$. Then $\rho \rtimes \pi(\psi,\varepsilon)$ is irreducible if $\rho \subset \psi_d$, and is semisimple of length 2 without multiplicities otherwise. Moreover, if σ is an irreducible subrepresentation of $\rho \rtimes \pi(\psi,\varepsilon)$ in the both cases, then $\rho \times \cdots \times \rho \rtimes \sigma$ is irreducible.*

Construction: Let $\psi \in \Psi_{\mathrm{el}}(G_n)$ and $\varepsilon \in \widehat{A_\psi}$ with $\varepsilon(z_\psi) = 1$.

(1) If $a_{\rho,\psi,\varepsilon} = \infty$ for all ρ, then we define $\pi(\psi,\varepsilon) = \pi(\psi_d,\varepsilon)$, which is supercuspidal by Theorem 3.2.

(2) If $a_{\rho,\psi,\varepsilon} > b_{\rho,\psi,\varepsilon} + 2$ or $b_{\rho,\psi,\varepsilon} = 0$, we define $\pi(\psi,\varepsilon)$ to be the unique irreducible subrepresentation of
$$\rho|\cdot|^{\delta\frac{a-1}{2}} \rtimes \pi(\psi',\varepsilon'),$$
where $a = a_{\rho,\psi,\varepsilon}$, $\delta = \delta_{\rho,\psi,\varepsilon}$, and (ψ',ε') is given so that
$$\psi'_d = \psi_d - \rho \boxtimes S_a + \rho \boxtimes S_{a-2}, \quad \delta_{\rho,a-2,\psi'} = \delta_{\rho,a,\psi},$$
and $\varepsilon'(\rho, a-2) = \varepsilon(\rho,a)$.

(3) If $a_{\rho,\psi,\varepsilon} = b_{\rho,\psi,\varepsilon} + 2$, we divide into three cases.

(a) If $b_{\rho,\psi,\varepsilon}$ is even and nonzero, then we define $\pi(\psi,\varepsilon)$ to be the unique irreducible subrepresentation of
$$\left\langle \rho; \delta\frac{a-1}{2}, \dots, \delta\frac{1}{2} \right\rangle \rtimes \pi(\psi_-,\varepsilon_-),$$
where $a = a_{\rho,\psi,\varepsilon}$, $\delta = \delta_{\rho,\psi,\varepsilon}$, and (ψ_-,ε_-) is given so that
$$(\psi_-)_d = \psi_d - \rho \boxtimes S_a, \quad \delta_{\rho,\alpha,\psi_-} = \begin{cases} -\delta & \text{if } \alpha \leq b_{\rho,\psi,\varepsilon}, \\ \delta_{\rho,\alpha,\psi} & \text{otherwise,} \end{cases}$$
and
$$\varepsilon_-(\rho,\alpha) = \begin{cases} -\varepsilon(\rho,\alpha) & \text{if } \alpha \leq b_{\rho,\psi,\varepsilon}, \\ \varepsilon(\rho,\alpha) & \text{otherwise.} \end{cases}$$

(b) If $b_{\rho,\psi,\varepsilon}$ is odd and $b_{\rho,\psi,\varepsilon} \neq 1$, then we define $\pi(\psi,\varepsilon)$ to be the unique common irreducible subrepresentation of
$$\left\langle \rho; \delta\frac{a-1}{2}, \dots, 0 \right\rangle \rtimes \pi(\psi_-,\varepsilon_-)$$
and
$$\left\langle \rho; \delta\frac{a-1}{2}, \dots, -\delta\frac{b-1}{2} \right\rangle \rtimes \pi(\psi',\varepsilon'),$$
where $a = a_{\rho,\psi,\varepsilon}$, $b = b_{\rho,\psi,\varepsilon}$, $\delta = \delta_{\rho,\psi,\varepsilon}$, and $(\psi_-,\varepsilon_-), (\psi',\varepsilon')$ are given so that
$$(\psi')_d = \psi_d - \rho \boxtimes S_a - \rho \boxtimes S_b, \qquad \delta_{\rho,\alpha,\psi'} = \delta_{\rho,\alpha,\psi},$$
$$(\psi_-)_d = \psi_d - \rho \boxtimes S_a - \rho, \qquad \delta_{\rho,\alpha,\psi_-} = \begin{cases} -\delta & \text{if } \alpha \leq b_{\rho,\psi,\varepsilon}, \\ \delta_{\rho,\alpha,\psi} & \text{otherwise,} \end{cases}$$

and $\varepsilon' = \varepsilon | A_\psi$, and

$$\varepsilon_-(\rho, \alpha) = \begin{cases} -\varepsilon(\rho, \alpha) & \text{if } \alpha \leq b_{\rho,\psi,\varepsilon}, \\ \varepsilon(\rho, \alpha) & \text{otherwise.} \end{cases}$$

(c) If $a_{\rho,\psi,\varepsilon} = 3$ and $b_{\rho,\psi,\varepsilon} = 1$, we have $(\psi_-, \varepsilon_-) = (\psi', \varepsilon')$ in the notation (b). By (Unitary reducibility), $\sigma = \rho \rtimes \pi(\psi', \varepsilon')$ is semisimple of length 2, and hence we can write $\sigma = \pi_+ \oplus \pi_-$ according to the following two cases.

(i) When $\psi_d \not\supset \rho \boxtimes S_\alpha$ for any $\alpha > 3$, we fix arbitrary parametrization in σ, and we define $\pi(\psi, \varepsilon)$ to be the unique irreducible subrepresentation of

$$\rho|\cdot|^{\delta_{\rho,3,\psi}} \rtimes \pi_\zeta$$

with $\zeta = \varepsilon(\rho, 3)\delta_{\rho,3,\psi}$.

(ii) When $\psi_d \supset \rho \boxtimes S_\alpha$ for some $\alpha > 3$, we can specify the parametrization in σ as follows. Set (ψ'', ε'') so that

$$(\psi'')_d = (\psi') - \rho \boxtimes S_{a'} + \rho, \quad \delta_{\rho,1,\psi''} = \delta' = \delta_{\rho,a',\psi'}$$

with $a' = a_{\rho,\psi',\varepsilon'}$, and $\varepsilon''(\rho, 1) = \varepsilon'(\rho, a')$. Put

$$\Pi = \rho \times \left\langle \rho; \delta' \frac{a'-1}{2}, \ldots, \delta' \right\rangle \rtimes \pi(\psi'', \varepsilon''),$$

$$\sigma_q = \left\langle \rho; \delta' \frac{a'-1}{2}, \ldots, 0 \right\rangle \rtimes \pi(\psi'', \varepsilon''),$$

$$\sigma_s = \left\langle \rho \times \left\langle \rho; \delta' \frac{a'-1}{2}, \ldots, \delta' \right\rangle \right\rangle \rtimes \pi(\psi'', \varepsilon''),$$

where $\langle \rho \times \langle \rho; \delta'(a'-1)/2, \ldots, \delta' \rangle \rangle$ is the unique irreducible subrepresentation of $\rho \times \langle \rho; \delta'(a'-1)/2, \ldots, \delta' \rangle$. Hence there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \sigma_s \arrow[r] & \Pi \arrow[r] & \sigma_q \arrow[r] & 0. \\ & & \sigma \arrow[u, hook] & & \end{tikzcd}$$

We set $\pi_+ = \sigma \cap (\text{s.s.}\sigma_q)$ and $\pi_- = \sigma \cap (\text{s.s.}\sigma_s)$. Then we define $\pi(\psi, \varepsilon)$ to be the unique irreducible subrepresentation of

$$\rho|\cdot|^{\delta_{\rho,3,\psi}} \rtimes \pi_\zeta$$

with $\zeta = \varepsilon(\rho, a_{\rho,\psi',\varepsilon'})\delta_{\rho,\psi',\varepsilon'}\delta_{\rho,3,\psi}$.

Remark 3.7. (1) *The representation $\pi(\psi, \varepsilon)$ can be constructed by using the* ***generalized Aubert involution****. This is defined by a combination of induction functors and Jacquet functors (see* [X17b, §6.2]*). By proving a compatibility of the generalized Aubert involution and twisted endoscopic character identities, Xu* [X17b, §6] *showed that*

$$\Pi_\psi = \left\{ \pi(\psi, \varepsilon) \;\middle|\; \varepsilon \in \widehat{A_\psi},\ \varepsilon(z_\psi) = 1 \right\}.$$

(2) *For* $\psi = \oplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\mathrm{el}}(G_n)$*, define its dual* $\hat{\psi} \in \Psi_{\mathrm{el}}(G_n)$ *by* $\hat{\psi} = \oplus_{i \in I} \rho_i \boxtimes S_{b_i} \boxtimes S_{a_i}$*. Then the component group* $A_{\hat{\psi}}$ *is canonically isomorphic to* A_ψ*. By* [Mœ06, Theorem 5] *and* [X17b, Theorem 6.10]*, the (usual) Aubert involution of* $\pi(\psi, \varepsilon) \in \Pi_\psi$ *is given by*

$$\hat{\pi}(\psi, \varepsilon) = \pi(\hat{\psi}, \varepsilon) \in \Pi_{\hat{\psi}}.$$

(3) *When* $\psi = \phi \in \Phi_2(G_n)$*, the map* $\Pi_\phi \ni \pi \to \langle \cdot, \pi \rangle_\phi \in \widehat{A_\phi}$ *is given by* $\pi(\phi, \varepsilon) \mapsto \varepsilon$*. However, for general* $\psi \in \Psi_{\mathrm{el}}(G_n)$*, the character* $\langle \cdot, \pi(\psi, \varepsilon) \rangle_\psi$ *does not coincide with* ε*. According to* [X17b, Theorem 6.21]*, one can define a character* $\varepsilon_\psi^{M/MW} \in \widehat{A_\psi}$ *explicitly such that* $\langle \cdot, \pi(\psi, \varepsilon) \rangle_\psi = \varepsilon \varepsilon_\psi^{M/MW}$*.*

We give an example. When $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, we write $\langle \rho; x, \dots, y \rangle = \langle x, \dots, y \rangle$.

Example 3.8. *Suppose that* $\psi \in \Psi_{\mathrm{el}}(\mathrm{SO}_7)$ *such that* $\psi_d = S_4 \oplus S_2$*. Fix* $\varepsilon \in \widehat{A_\psi}$ *with* $\varepsilon(z_\psi) = 1$*. For* $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$*, set* $\delta_a = \delta_{\rho, a, \psi}$ *and* $\varepsilon_a = \varepsilon(\rho, a)$*. Then* $\pi(\psi, \varepsilon)$ *is given as follows.*

(1) *If* $(\delta_4, \delta_2) = (+, +)$ *and* $(\varepsilon_4, \varepsilon_2) = (+, +)$*, then* $\pi_1 = \pi(\psi, \varepsilon)$ *is discrete series but not supercuspidal. It is a subrepresentation of*

$$|\cdot|^{\frac{1}{2}} \rtimes \pi(S_4) \hookrightarrow |\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{3}{2}} \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)},$$

where $\pi(S_4) \in \Pi_{S_4}$ *is a discrete series representation of* $\mathrm{SO}_5(F)$*.*

(2) *If* $(\delta_4, \delta_2) = (+, +)$ *and* $(\varepsilon_4, \varepsilon_2) = (-, -)$*, then* $\pi_2 = \pi(\psi, \varepsilon)$ *is discrete series but not supercuspidal. It is a subrepresentation of*

$$\left\langle \frac{3}{2}, \frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} \hookrightarrow \left\langle \frac{3}{2}, \frac{1}{2} \right\rangle \times |\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}.$$

(3) *If* $(\delta_4, \delta_2) = (+, -)$ *and* $(\varepsilon_4, \varepsilon_2) = (+, +)$*, then* $\pi_3 = \pi(\psi, \varepsilon)$ *is non-tempered and is the unique subrepresentation of*

$$|\cdot|^{-\frac{1}{2}} \rtimes \pi(S_4) \hookrightarrow |\cdot|^{-\frac{1}{2}} \times |\cdot|^{\frac{3}{2}} \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)},$$

where $\pi(S_4) \in \Pi_{S_4}$ *is a discrete series representation of* $\mathrm{SO}_5(F)$*. Hence* $\pi_3 \in \Pi_{\phi_\psi}$ *with* $\phi_\psi = |\cdot|^{\frac{1}{2}} \oplus S_4 \oplus |\cdot|^{-\frac{1}{2}}$*.*

(4) *If* $(\delta_4, \delta_2) = (+, -)$ *and* $(\varepsilon_4, \varepsilon_2) = (-, -)$*, then by definition,* $\pi_4 \cong \pi_2$ *so that* $\pi_4 \in \Pi_{\psi_d}$ *is discrete series.*

(5) *If* $(\delta_4, \delta_2) = (-, +)$ *and* $(\varepsilon_4, \varepsilon_2) = (+, +)$*, then we claim that* $\pi_5 = \pi(\psi, \varepsilon)$ *is non-tempered and is the unique subrepresentation of*

$$|\cdot|^{-\frac{3}{2}} \rtimes \sigma,$$

where σ *is tempered and is the direct summand of* $\mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ *such that* $\mathrm{Jac}_{|\cdot|^{\frac{1}{2}}}(\sigma)$ *does not contain* $\pi(S_2)$*, where* $\pi(S_2) \in \Pi_{S_2}$ *is a discrete series representation of* $\mathrm{SO}_3(F)$*. Hence the* L*-parameter of* π_5 *is*

$$|\cdot|^{\frac{3}{2}} + S_2 + S_2 + |\cdot|^{-\frac{3}{2}}.$$

Indeed, by the construction,

$$\pi_5 \hookrightarrow |\cdot|^{\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} \cong |\cdot|^{-\frac{3}{2}} \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}.$$

Consider the following diagram of two exact sequences

$$
\begin{tikzcd}
 & & 0 & & \\
 & & \mathbf{1}_{\mathrm{GL}_2(F)} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \arrow[u] & & \\
0 \arrow[r] & |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} \arrow[r] & |\cdot|^{\frac{1}{2}} \times |\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \arrow[r] \arrow[u] & |\cdot|^{\frac{1}{2}} \rtimes \pi(S_2) \arrow[r] & 0. \\
 & & \mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \arrow[u] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}
$$

Since $\pi_5 \hookrightarrow |\cdot|^{\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}$*, we have* $\mathrm{Jac}_{|\cdot|^{\frac{1}{2}}}(\pi_5) \neq 0$*. This implies that there is no nonzero homomorphism* $\pi_5 \to |\cdot|^{-\frac{3}{2}} \times \mathbf{1}_{\mathrm{GL}_2(F)} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ *since* $\mathrm{Jac}_{|\cdot|^{\frac{1}{2}}}(\mathbf{1}_{\mathrm{GL}_2(F)} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}) = 0$*. Hence there is a representation* σ *of* $\mathrm{SO}_5(F)$ *which is a common subrepresentation of* $|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}$ *and* $\mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ *such that* $\pi_5 \hookrightarrow |\cdot|^{-\frac{3}{2}} \rtimes \sigma$*. Computing Jacquet modules, we see that*

$$
\begin{aligned}
\mathrm{s.s.}(|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}) \cap (\mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}) &\neq 0, \\
\mathrm{s.s.}(|\cdot|^{\frac{1}{2}} \rtimes \pi(S_2)) \cap (\mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}) &\neq 0.
\end{aligned}
$$

Since $\mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ *is a direct sum of two irreducible tempered representations,* $\mathrm{s.s.}(|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}) \cap (\mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)})$ *is irreducible. Hence* σ *must be irreducible. Moreover, since* $\sigma \hookrightarrow |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}$*, we see that* $\mathrm{Jac}_{|\cdot|^{\frac{1}{2}}}(\sigma)$ *does not contain* $\pi(S_2)$.

(6) *If* $(\delta_4, \delta_2) = (-,+)$ *and* $(\varepsilon_4, \varepsilon_2) = (-,-)$*, then* $\pi_6 = \pi(\psi, \varepsilon)$ *is non-tempered and is the unique subrepresentation of*

$$|\cdot|^{-\frac{3}{2}} \times |\cdot|^{-\frac{1}{2}} \rtimes \pi(S_2) \hookrightarrow |\cdot|^{-\frac{3}{2}} \times |\cdot|^{-\frac{1}{2}} \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)},$$

where $\pi(S_2) \in \Pi_{S_2}$ *is a discrete series representation of* $\mathrm{SO}_3(F)$*. Hence* $\pi_6 \in \Pi_{\phi_\psi}$ *with* $\phi_\psi = |\cdot|^{\frac{3}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus S_2 \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}}$.

(7) *If* $(\delta_4, \delta_2) = (-,-)$ *and* $(\varepsilon_4, \varepsilon_2) = (+,+)$*, then* $\pi_7 = \pi(\psi, \varepsilon)$ *is non-tempered and is the unique subrepresentation of*

$$|\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_5(F)} \hookrightarrow \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \times |\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}.$$

Hence $\pi_7 \in \Pi_{\phi_\psi}$ *with* $\phi_\psi = |\cdot|^{\frac{3}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}}$.

(8) *If* $(\delta_4, \delta_2) = (-,-)$ *and* $(\varepsilon_4, \varepsilon_2) = (-,-)$*, then by definition,* $\pi_8 \cong \pi_6$ *so that* π_8 *is non-tempered but* $\pi_8 \notin \Pi_{\phi_\psi}$ *with* $\phi_\psi = |\cdot|^{\frac{3}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}}$.

4. The case of discrete diagonal restriction

In this section, we construct A-packets Π_ψ for A-parameters $\psi \in \Psi_{\mathrm{DDR}}(G_n)$ with discrete diagonal restrictions. See [X17b, §7] for more precision. Before the construction, we review the case of general linear groups GL_N.

4.1. **The case of** GL_N**.** Let ψ be an A-parameter for $\mathrm{GL}_N(F)$. Assume in this subsection that the diagonal restriction $\psi_d = \psi \circ \Delta$ is multiplicity-free. Recall that one can associate an irreducible unitary representation τ_ψ of $\mathrm{GL}_N(F)$. When $\psi \supset \rho \boxtimes S_a \boxtimes S_b$ with $\min\{a,b\} > 1$, we see that τ_ψ is the unique irreducible subrepresentation of

$$\left\langle \rho; \frac{a-b}{2}, \dots, -\zeta\frac{a+b-2}{2} \right\rangle \times \tau_{\psi'} \times \left\langle \rho; \zeta\frac{a+b-2}{2}, \dots, -\frac{a-b}{2} \right\rangle,$$

where $\zeta \in \{\pm 1\}$ such that $\zeta(a-b) \geq 0$, and

$$\psi' = \begin{cases} \psi - \rho \boxtimes S_a \boxtimes S_b \oplus \rho \boxtimes S_a \boxtimes S_{b-2} & \text{if } \zeta = +1, \\ \psi - \rho \boxtimes S_a \boxtimes S_b \oplus \rho \boxtimes S_{a-2} \boxtimes S_b & \text{if } \zeta = -1. \end{cases}$$

4.2. **The case of** SO_{2n+1} **and** Sp_{2n}**.** Let G_n be split SO_{2n+1} or Sp_{2n}. Recall that $\psi \in \Psi(G_n)$ has a discrete diagonal restriction if ψ is of good parity and $\psi_d = \psi \circ \Delta$ is multiplicity-free. Note that

$$S_a \otimes S_b = S_{a+b-1} \oplus S_{a+b-3} \oplus \cdots \oplus S_{|a-b|+1}.$$

Write

$$\psi = \bigoplus_{i=1}^{r} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}.$$

For $i = 1, \dots, r$, we set

$$d_i = \min\{a_i, b_i\}, \quad \zeta_i = \begin{cases} 1 & \text{if } a_i > b_i, \\ -1 & \text{if } a_i < b_i. \end{cases}$$

When $a_i = b_i$, we choose $\zeta_i \in \{\pm 1\}$ arbitrarily.

Definition 4.1. *Let $\psi = \oplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\mathrm{DDR}}(G_n)$ be an A-parameter with DDR, and d_i and ζ_i be as above.*

(1) *Define Σ_ψ to be the set of pairs $(\underline{l}, \underline{\eta})$ such that*
   - *$\underline{l} = (l_1, \dots, l_r) \in \mathbb{Z}^r$ such that $0 \leq l_i \leq d_i/2$;*
   - *$\underline{\eta} = (\eta_1, \dots, \eta_r) \in \{\pm 1\}^r$ such that*

$$\prod_{i=1}^{r} \eta_i^{d_i} (-1)^{[d_i/2]+l_i} = 1.$$

Here, $[x]$ denotes the greatest integer which is not larger than x.

(2) *Define an equivalence relation $\sim_\psi$ on Σ_ψ by*

$$(\underline{l}, \underline{\eta}) \sim_\psi (\underline{l}', \underline{\eta}') \iff \underline{l} = \underline{l}' \quad \text{and} \quad \eta_i = \eta_i' \text{ unless } l_i = \frac{d_i}{2}.$$

(3) *Define $\varepsilon_{\underline{l},\underline{\eta}} \in \widehat{A_\psi}$ by*

$$\varepsilon_{\underline{l},\underline{\eta}}(\alpha_i) = \eta_i^{d_i} (-1)^{[d_i/2]+l_i},$$

where $\alpha_i \in A_\psi$ is the element corresponding to $\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$.

Lemma 4.2. *The cardinality of the set $\Sigma_\psi/\sim_\psi$ of equivalence classes is equal to*

$$\begin{cases} \frac{1}{2}\prod_{i=1}^{r}(d_i+1) & \text{if } d_i \text{ is odd for some } i, \\ \frac{1}{2}\left(\prod_{i=1}^{r}(d_i+1)+(-1)^{\frac{1}{2}(d_1+\cdots+d_r)}\right) & \text{if } d_i \text{ is even for any } i. \end{cases}$$

Proof. For $\epsilon \in \{\pm 1\}$, we define Σ_ψ^ϵ by a similar set to Σ_ψ changing the second condition with

$$\prod_{i=1}^{r}\eta_i^{d_i}(-1)^{[d_i/2]+l_i} = \epsilon,$$

and set

$$a^\epsilon_{(d_1,\dots,d_r)} = \#\Sigma^\epsilon_\psi/\sim_\psi .$$

Hence $a^+_{(d_1,\dots,d_r)} = \#\Sigma_\psi/\sim_\psi$.

If d_i is odd for some i, then $a^+_{(d_1,\dots,d_r)} = a^-_{(d_1,\dots,d_r)}$ so that

$$a^\epsilon_{(d_1,\dots,d_r)} = \frac{1}{2}\left(a^+_{(d_1,\dots,d_r)} + a^-_{(d_1,\dots,d_r)}\right) = \frac{1}{2}\prod_{i=1}^{r}(d_i+1).$$

Suppose that d_i is even for any i. Set $a^+_\emptyset = 1$ and $a^-_\emptyset = 0$. Then for $r \geq 1$, we have

$$\begin{pmatrix} a^+_{(d_1,\dots,d_r)} \\ a^-_{(d_1,\dots,d_r)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} d_r+1+(-1)^{d_r/2} & d_r+1-(-1)^{d_r/2} \\ d_r+1-(-1)^{d_r/2} & d_r+1+(-1)^{d_r/2} \end{pmatrix}\begin{pmatrix} a^+_{(d_1,\dots,d_{r-1})} \\ a^-_{(d_1,\dots,d_{r-1})} \end{pmatrix}.$$

Note that there exists $P \in \mathrm{GL}_2(\mathbb{C})$ such that for any $\alpha, \beta \in \mathbb{C}$,

$$P\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}P^{-1} = \begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha-\beta \end{pmatrix}.$$

Thus,

$$\begin{aligned} P\begin{pmatrix} a^+_{(d_1,\dots,d_r)} \\ a^-_{(d_1,\dots,d_r)} \end{pmatrix} &= \begin{pmatrix} d_r+1 & 0 \\ 0 & (-1)^{d_r/2} \end{pmatrix} P \begin{pmatrix} a^+_{(d_1,\dots,d_{r-1})} \\ a^-_{(d_1,\dots,d_{r-1})} \end{pmatrix} \\ &= \begin{pmatrix} \prod_{i=1}^r(d_i+1) & 0 \\ 0 & \prod_{i=1}^r(-1)^{d_i/2} \end{pmatrix} P \begin{pmatrix} a^+_\emptyset \\ a^-_\emptyset \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{pmatrix} a^+_{(d_1,\dots,d_r)} \\ a^-_{(d_1,\dots,d_r)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \prod_{i=1}^r(d_i+1)+(-1)^{\frac{1}{2}(d_1+\cdots+d_r)} & * \\ * & * \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We obtain the lemma. □

Now for $(\underline{l}, \underline{\eta}) \in \Sigma_\psi$, we define $\pi(\psi, \underline{l}, \underline{\eta})$ to be the unique irreducible subrepresentation of

$$\bigtimes_{i=1}^{r}\left(|\cdot|^{-\zeta_i\frac{d_i-l_i}{2}}\mathrm{Sp}(\mathrm{St}(\rho_i, \frac{a_i+l_i}{2}+\zeta_i\frac{a_i-l_i}{2}), \frac{b_i+l_i}{2}-\zeta_i\frac{b_i-l_i}{2})\right) \rtimes \pi(\psi', \varepsilon'),$$

where

- ψ' is an elementary parameter given so that
$$(\psi')_d = \bigoplus_{i=1}^{r} \bigoplus_{c_i} \rho_i \boxtimes S_{c_i}$$
with c_i running over all integers such that
$$|a_i - b_i| + 2l_i + 1 \leq c_i \leq a_i + b_i - 2l_i - 1 \quad \text{and} \quad c_i \equiv a_i + b_i - 1 \bmod 2,$$
and $\delta_{\rho_i, c_i, \psi'} = \zeta_i$;
- $\varepsilon' \in \widehat{A_{\psi'}}$ is given so that
$$\varepsilon'(\rho_i, c_i) = \eta_i \cdot (-1)^{\frac{c_i-1}{2} - |\frac{a_i-b_i}{2}| - l_i}.$$

We recall that $\mathrm{Sp}(\mathrm{St}(\rho, a), b)$ denotes the irreducible unitary representation $\tau_{\rho \boxtimes S_a \boxtimes S_b}$. Also, we remark that when ψ is elementary, any $(\underline{l}, \underline{\eta}) \in \Sigma_\psi$ satisfies that $l_i = 0$, and $\pi(\psi, \underline{l}, \underline{\eta}) = \pi(\psi, \varepsilon_{\underline{l}, \underline{\eta}})$ with $\varepsilon_{\underline{l}, \underline{\eta}}(\alpha_i) = \eta_i$.

Theorem 4.3. *Let $\psi \in \Psi_{\mathrm{DDR}}(G_n)$ be an A-parameter having a DDR. Then for $(\underline{l}, \underline{\eta}), (\underline{l}', \underline{\eta}') \in \Sigma_\psi$,*
$$\pi(\psi, \underline{l}, \underline{\eta}) \cong \pi(\psi, \underline{l}', \underline{\eta}') \iff (\underline{l}, \underline{\eta}) \sim_\psi (\underline{l}', \underline{\eta}').$$
The A-packet Π_ψ is given by
$$\Pi_\psi = \left\{ \pi(\psi, \underline{l}, \underline{\eta}) \;\middle|\; (\underline{l}, \underline{\eta}) \in \Sigma_\psi / \sim_\psi \right\}.$$
In particular, $\#\Pi_\psi = \#(\Sigma_\psi / \sim_\psi)$, which is given explicitly in Lemma 4.2.

Remark 4.4. *Let $\psi \in \Psi_{\mathrm{DDR}}(G_n)$. One can define a character $\varepsilon_\psi^{M/MW} \in \widehat{A_\psi}$ such that*
$$\left\langle \cdot, \pi(\psi, \underline{l}, \underline{\eta}) \right\rangle_\psi = \varepsilon_{\underline{l}, \underline{\eta}} \cdot \varepsilon_\psi^{M/MW}.$$

Example 4.5. (1) *Consider $\psi = S_3 \boxtimes S_2 \in \Psi_{\mathrm{DDR}}(\mathrm{SO}_7)$. Then $\Sigma_\psi / \sim_\psi = \{(1, \pm 1)\}$ is a singleton. The representation $\pi(\psi, (1, \pm 1))$ is the unique irreducible subrepresentation of*
$$|\cdot|^{-\frac{1}{2}} \langle 1, 0, -1 \rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}.$$
Hence it is non-tempered, and belongs to Π_{ϕ_ψ} with $\phi_\psi = |\cdot|^{\frac{1}{2}} S_3 \oplus |\cdot|^{-\frac{1}{2}} S_3$.

(2) *Consider $\psi = S_2 \boxtimes S_3 \in \Psi_{\mathrm{DDR}}(\mathrm{SO}_7)$. Then $\Sigma_\psi / \sim_\psi = \{(1, \pm 1)\}$ is a singleton. The representation $\pi(\psi, (1, \pm 1))$ is the unique irreducible subrepresentation of*
$$|\cdot|^{\frac{1}{2}} \langle -1, 0, 1 \rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}.$$
Since $|\cdot|^{\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ is irreducible, we have
$$\begin{aligned}
\pi(\psi, (1, \pm 1)) &\hookrightarrow |\cdot|^{\frac{1}{2}} \langle -1, 0, 1 \rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\
&\hookrightarrow |\cdot|^{-\frac{1}{2}} \times |\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\
&\cong |\cdot|^{-\frac{1}{2}} \times |\cdot|^{\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\
&\cong |\cdot|^{-\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}.
\end{aligned}$$

Consider the following diagram of exact sequences:

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \pi(S_2) \arrow[r] \arrow[u] & \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \arrow[r] \arrow[u] & \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & |\cdot|^{-\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \rtimes \pi(S_2) \arrow[r] \arrow[u] & |\cdot|^{-\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \arrow[r] \arrow[u] & |\cdot|^{-\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \pi(S_2) \arrow[r] \arrow[u] & \left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \arrow[r] \arrow[u] & \left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} \arrow[r] \arrow[u] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Since $\pi(\psi,(1,\pm1)) \hookrightarrow |\cdot|^{-\frac{1}{2}} \times |\cdot|^{-\frac{3}{2}} \times |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$, *we see that* $\pi(\psi,(1,\pm1))$ *is a subrepresentation of one of* $\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \pi(S_2)$, $\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}$, $\left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \pi(S_2)$, *or* $\left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}$. *Since* $\mathrm{Jac}_{|\cdot|^{-\frac{1}{2}},|\cdot|^{\frac{1}{2}},|\cdot|^{\frac{3}{2}}}(\pi(\psi,(1,\pm1))) \neq 0$ *but*

$$\mathrm{Jac}_{|\cdot|^{-\frac{1}{2}},|\cdot|^{\frac{1}{2}},|\cdot|^{\frac{3}{2}}}\left(\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \pi(S_2)\right) = \mathrm{Jac}_{|\cdot|^{-\frac{1}{2}},|\cdot|^{\frac{1}{2}},|\cdot|^{\frac{3}{2}}}\left(\left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}\right) = 0,$$

we see that $\pi(\psi,(1,\pm1)) \not\subset \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \pi(S_2)$ *and* $\pi(\psi,(1,\pm1)) \not\subset \left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}$. *Suppose that* $\pi(\psi,(1,\pm1)) \subset \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)}$. *Note that*

$$\begin{aligned}
\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} &\hookrightarrow \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \times |\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\
&\cong |\cdot|^{-\frac{1}{2}} \times \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}.
\end{aligned}$$

Let σ *be the unique irreducible subrepresentation of* $\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$. *Since* $\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \times |\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ *has a unique irreducible subrepresentation,* $\pi(\psi,(1,\pm1))$ *must be the unique irreducible subrepresentation of* $|\cdot|^{-\frac{1}{2}} \rtimes \sigma$. *Since* $\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ *is reducible and*

$$\mathrm{Jac}_{|\cdot|^{\frac{1}{2}}}\left(\left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}\right) = |\cdot|^{-\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$$

is irreducible, we see that $\mathrm{Jac}_{|\cdot|^{\frac{1}{2}}}(\sigma) = 0$. *Also, we note that* $\mathrm{Jac}_{|\cdot|^{-\frac{1}{2}}}(\sigma) = 0$. *Hence* $\mathrm{Jac}_{|\cdot|^{-\frac{1}{2}},|\cdot|^{\frac{1}{2}}}(\pi(\psi,(1,\pm1))) = 0$. *This contradicts that* $\pi(\psi,(1,\pm1)) \hookrightarrow \left\langle -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$. *Therefore we conclude that*

$$\pi(\psi,(1,\pm1)) \hookrightarrow \left\langle -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \pi(S_2) = |\cdot|^{-1}\mathrm{St}_2 \times \pi(S_2).$$

Hence $\pi(\psi,(1,\pm1))$ *is non-tempered, and belongs to* Π_{ϕ_ψ} *with* $\phi_\psi = |\cdot|^1 S_2 \oplus S_2 \oplus |\cdot|^{-1} S_2$.

5. The case of good parity and the general case

In this section, we construct A-packets Π_ψ for general A-parameters ψ. See [X17b, §8] for more precision. First, we consider the case where ψ is of good parity. Before the construction, we introduce the notion of admissible orders and explain their roles in the case of general linear groups GL_N.

5.1. **Admissible order.** Recall that $\psi \in \Psi(G_n)$ is of good parity if ψ is a sum of irreducible self-dual representations of the same type as ψ. When $\psi = \oplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$, we set $I_\rho = \{i \in I \mid \rho_i \cong \rho\}$, and

$$\psi_\rho = \bigoplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \subset \psi.$$

As in the previous section, we set

$$d_i = \min\{a_i, b_i\}, \quad \zeta_i = \begin{cases} 1 & \text{if } a_i > b_i, \\ -1 & \text{if } a_i < b_i, \end{cases}$$

and we choose $\zeta_i \in \{\pm 1\}$ arbitrarily when $a_i = b_i$. We fix a total order $>_\psi$ on I_ρ satisfying the following condition:

($\mathcal{P}$)**:** For $i, j \in I_\rho$, if $a_i + b_i > a_j + b_j$, $|a_i - b_i| > |a_j - b_j|$, and $\zeta_i = \zeta_j$, then $i >_\psi j$.

We call such an order $>_\psi$ an **admissible order** on I_ρ. Taking $>_\psi$ on I_ρ for each ρ, we extend $>_\psi$ to a partial order on $I = \sqcup_\rho I_\rho$. Note that there are many such orders and there is no canonical choice of them in general. When ψ has a DDR, we can always choose such an order satisfying that for $i, j \in I_\rho$,

$$i >_\psi j \iff a_i + b_i > a_j + b_j.$$

We call such orders the **natural orders** for parameters with DDR.

We say that $\psi_\gg \in \Psi(G_{n'})$ **dominates** ψ **with respect to** $>_\psi$ if

$$\psi_\gg = \bigoplus_{i \in I} \rho_i \boxtimes S_{a'_i} \boxtimes S_{b'_i}$$

such that for each $i \in I$,

$$(a'_i, b'_i) = \begin{cases} (a_i + 2T_i, b_i) & \text{if } \zeta_i = +1, \\ (a_i, b_i + 2T_i) & \text{if } \zeta_i = -1 \end{cases}$$

for some integer $T_i \geq 0$. When $a'_i = b'_i$, we set $\zeta'_i = \zeta_i$. Note that for any $\psi \in \Psi_{\mathrm{gp}}(G_n)$ with admissible order $>_\psi$, one can take $\psi_\gg \in \Psi_{\mathrm{DDR}}(G_{n'})$ such that $\psi_\gg$ dominates ψ with respect to $>_\psi$, and such that the order $>_\psi$ is a natural order for $\psi_\gg$.

5.2. **The case of** GL_N**.** Let $\psi \in \Psi_{\mathrm{gp}}(G_n)$. Fix an admissible order $>_\psi$, and an A-parameter $\psi_\gg \in \Psi_{\mathrm{DDR}}(G_{n'})$ which dominates ψ with respect to $>_\psi$. In this subsection, we construct τ_ψ using $\tau_{\psi_\gg}$.

Let τ be an irreducible representation of $\mathrm{GL}_N(F)$. For $d \leq N/2$, we write

$$\mathrm{s.s.Jac}_{(d, N-2d, d)}(\tau) = \bigoplus_{i \in I} \tau_i^{(1)} \boxtimes \tau_i^{(0)} \boxtimes \tau_i^{(2)},$$

where $\tau_i^{(j)}$ is an irreducible representation of $\mathrm{GL}_d(F)$ if $j = 1, 2$, and of $\mathrm{GL}_{N-2d}(F)$ if $j = 0$. For a fixed irreducible supercuspidal unitary representation ρ of $\mathrm{GL}_d(F)$ and for a real number x, we set

$$\mathrm{Jac}^\theta_{\rho|\cdot|^x}(\tau) = \bigoplus_{\substack{i \in I \\ \tau_i^{(1)} \cong \rho|\cdot|^x, \tau_i^{(2)} \cong \rho^\vee|\cdot|^{-x}}} \tau_i^{(0)}.$$

For example, $\mathrm{Jac}^\theta_{|\cdot|^{\frac{d-1}{2}}}(\mathrm{St}_d) = \mathrm{St}_{d-2}$.

Lemma 5.1. *Let $\psi = \oplus_{i\in I}\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\mathrm{gp}}(G_n)$. Fix an admissible order $>_\psi$, and an A-parameter $\psi_\gg = \oplus_{i\in I}\rho_i \boxtimes S_{a'_i} \boxtimes S_{b'_i} \in \Psi_{\mathrm{DDR}}(G_{n'})$ which dominates ψ with respect to $>_\psi$. Then*

$$\tau_\psi \cong \circ_{i\in I}\Big(\mathrm{Jac}^\theta_{\rho_i|\cdot|^{\frac{a_i-b_i}{2}+\zeta_i},\dots,\rho_i|\cdot|^{\zeta_i\frac{a_i+b_i}{2}}} \circ \mathrm{Jac}^\theta_{\rho_i|\cdot|^{\frac{a_i-b_i}{2}+2\zeta_i},\dots,\rho_i|\cdot|^{\zeta_i(\frac{a_i+b_i}{2}+1)}} \circ\cdots\circ \mathrm{Jac}^\theta_{\rho_i|\cdot|^{\frac{a'_i-b'_i}{2}},\dots,\rho_i|\cdot|^{\zeta_i(\frac{a'_i+b'_i}{2}-1)}}\Big)(\tau_{\psi_\gg}),$$

where $\circ_{i\in I}$ is taken in the decreasing order with respect to $>_\psi$.

For the meaning of the phrase "$\circ_{i\in I}$ is taken in the decreasing order with respect to $>_\psi$", see the following examples.

Example 5.2. (1) *Consider $\psi = (S_d \boxtimes \mathbf{1})^{\oplus n}$ with $d > 1$ so that $\tau_\psi = \mathrm{St}_d \times \cdots \times \mathrm{St}_d$ (n-times). Take $I = \{1,\dots,n\}$ with the order $>_\psi$ given by $i >_\psi i-1$. Then any A-parameter $\psi_\gg$ which has a DDR and dominates ψ, and such that $>_\psi$ is a natural order is of the form*

$$\psi_\gg = \psi_{(t_1,\dots,t_n)} = S_{d+t_1} \boxtimes \mathbf{1} \oplus \cdots \oplus S_{d+t_n} \boxtimes \mathbf{1}$$

with $t_1,\dots,t_n \in 2\mathbb{Z}$ and $t_n > t_{n-1} > \cdots > t_1 \geq 0$. The representation $\tau_{\psi_\gg}$ is given by

$$\tau_{\psi_{(t_1,\dots,t_n)}} = \mathrm{St}_{d+t_1} \times \cdots \times \mathrm{St}_{d+t_n}.$$

When $t_1 > 0$, we have

$$\mathrm{Jac}^\theta_{|\cdot|^{\frac{d+t_1-1}{2}}}(\tau_{\psi_{(t_1,\dots,t_n)}}) = \tau_{\psi_{(t_1-2,\dots,t_n)}}.$$

Hence

$$\mathrm{Jac}^\theta_{|\cdot|^{\frac{d+t_1-1}{2}},\dots,|\cdot|^{\frac{d+1}{2}}}(\tau_{\psi_{(t_1,\dots,t_n)}}) = \tau_{\psi_{(0,t_2,\dots,t_n)}}.$$

Therefore we have

$$\mathrm{Jac}^\theta_{|\cdot|^{\frac{d+t_n-1}{2}},\dots,|\cdot|^{\frac{d+1}{2}}} \circ\cdots\circ \mathrm{Jac}^\theta_{|\cdot|^{\frac{d+t_1-1}{2}},\dots,|\cdot|^{\frac{d+1}{2}}}(\tau_{\psi_{(t_1,\dots,t_n)}}) = \tau_\psi.$$

(2) *Consider $\psi = S_2 \boxtimes S_3 \oplus S_5 \boxtimes S_2$. If we put $(a_1,b_1) = (2,3)$ and $(a_2,b_2) = (5,2)$ with $I = \{1,2\}$, then by the condition $(\mathcal{P})$, the admissible order $>_\psi$ must satisfy $2 >_\psi 1$. An A-parameter $\psi_\gg$ which dominants ψ is given by $\psi_\gg = S_2 \boxtimes S_3 \oplus S_7 \boxtimes S_2$. Then*

$$\tau_\psi = \mathrm{Sp}(\mathrm{St}_2, 3) \times \mathrm{Sp}(\mathrm{St}_5, 2),$$
$$\tau_{\psi_\gg} = \mathrm{Sp}(\mathrm{St}_2, 3) \times \mathrm{Sp}(\mathrm{St}_7, 2).$$

Since

$$\mathrm{Sp}(\mathrm{St}_7, 2) \hookrightarrow \left\langle \frac{5}{2}, \frac{7}{2} \right\rangle \times \mathrm{Sp}(\mathrm{St}_5, 2) \times \left\langle -\frac{7}{2}, -\frac{5}{2} \right\rangle,$$

we have

$$\mathrm{Jac}^\theta_{|\cdot|^{\frac{5}{2}},|\cdot|^{\frac{7}{2}}}(\tau_{\psi_\gg}) = \tau_\psi.$$

5.3. **The case of SO_{2n+1} and Sp_{2n}.** For $\psi = \oplus_{i\in I}\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\mathrm{gp}}(G_n)$, we define d_i, ζ_i, Σ_ψ, $\sim_\psi$, and $\varepsilon_{\underline{l},\underline{\eta}}$ as in §4.2. We fix an admissible order $>_\psi$ on I_ρ for each ρ, and an A-parameter $\psi_\gg \in \Psi_{\mathrm{DDR}}(G_{n'})$ such that $\psi_\gg$ dominates ψ with respect to $>_\psi$, and such that the order $>_\psi$ is a natural order for $\psi_\gg$. For $(\underline{l},\underline{\eta}) \in \Sigma_\psi$, we define $\pi_{>_\psi}(\psi,\underline{l},\underline{\eta})$ by

$$\pi_{>_\psi}(\psi,\underline{l},\underline{\eta}) = \circ_{i\in I}\Big(\mathrm{Jac}_{\rho_i|\cdot|^{\frac{a_i-b_i}{2}+\zeta_i},\dots,\rho_i|\cdot|^{\zeta_i\frac{a_i+b_i}{2}}} \circ \mathrm{Jac}_{\rho_i|\cdot|^{\frac{a_i-b_i}{2}+2\zeta_i},\dots,\rho_i|\cdot|^{\zeta_i(\frac{a_i+b_i}{2}+1)}} \circ \cdots \circ \mathrm{Jac}_{\rho_i|\cdot|^{\frac{a_i'-b_i'}{2}},\dots,\rho_i|\cdot|^{\zeta_i(\frac{a_i'+b_i'}{2}-1)}}\Big)(\pi(\psi_\gg,\underline{l},\underline{\eta})),$$

where $\circ_{i\in I}$ is taken in the decreasing order with respect to $>_\psi$.

Theorem 5.3. *Let $\psi \in \Psi_{\mathrm{gp}}(G_n)$ with an admissible order $>_\psi$. Then, for $(\underline{l},\underline{\eta}) \in \Sigma_\psi$, the representation $\pi_{>_\psi}(\psi,\underline{l},\underline{\eta})$ does not depend on the choice of $\psi_\gg$. Moreover, it is either zero or irreducible. If $\pi_{>_\psi}(\psi,\underline{l},\underline{\eta}) \cong \pi_{>_\psi}(\psi,\underline{l'},\underline{\eta'}) \neq 0$, then $(\underline{l},\underline{\eta}) \sim_\psi (\underline{l'},\underline{\eta'})$. The A-packet Π_ψ is given by*

$$\Pi_\psi = \{\pi_{>_\psi}(\psi,\underline{l},\underline{\eta}) \mid (\underline{l},\underline{\eta}) \in \Sigma_\psi/\sim_\psi\} \setminus \{0\}.$$

In particular, $\#\Pi_\psi \le \#(\Sigma_\psi/\sim_\psi)$.

Example 5.4. *Suppose that $\psi \in \Psi_{\mathrm{gp}}(\mathrm{SO}_7)$ such that $\psi_d = S_2^{\oplus 3}$. Then*

$$\Sigma_\psi = \{(\underline{0},\underline{\eta}) \mid \underline{\eta} = (\eta_1,\eta_2,\eta_3) \in \{\pm1\}^3,\ \eta_1\eta_2\eta_3 = 1\}$$

with trivial equivalence relation $\sim_\psi$, where $\underline{0} = (0,0,0)$. We always take the admissible order $>_\psi$ given by $3 >_\psi 2 >_\psi 1$. If we write $\psi = \oplus_{i=1}^3 S_{a_i} \boxtimes S_{b_i}$ with $(a_i,b_i) \in \{(2,1),(1,2)\}$, the sign $\zeta_i \in \{\pm1\}$ is determined by $\zeta_i = \mathrm{sgn}(a_i-b_i)$.

Define $\psi_\gg = \oplus_{i=1}^3 S_{a_i'} \boxtimes S_{b_i'}$ so that $(a_i',b_i') \in \{(2i,1),(1,2i)\}$ and $(a_i'-b_i')(a_i-b_i) > 0$. Then $\psi_\gg$ dominates ψ with respect to $>_\psi$, and has a DDR with $>_\psi$ being a natural order. In fact, $\psi_\gg$ is elementary. Given $\underline{\eta} = (\eta_1,\eta_2,\eta_3) \in \{\pm1\}$ such that $\eta_1\eta_2\eta_3 = 1$, define $\varepsilon \in \widehat{A_{\psi_\gg}}$ by $\varepsilon(\mathbf{1},2i) = \eta_i$. Then

$$\pi_{>_\psi}(\psi,\underline{0},\underline{\eta}) = \mathrm{Jac}_{|\cdot|^{\zeta_3\frac{3}{2}}} \circ \mathrm{Jac}_{|\cdot|^{\zeta_3\frac{5}{2}}} \circ \mathrm{Jac}_{|\cdot|^{\zeta_2\frac{3}{2}}}(\pi(\psi_\gg,\varepsilon)).$$

(1) *Suppose that $\zeta_1 = \zeta_2 = \zeta_3 = \zeta$. By Lemma 3.5 (together with using the Aubert involution if necessary), $\mathrm{Jac}_{|\cdot|^{\zeta\frac{3}{2}}}(\pi(\psi_\gg,\varepsilon))$ is nonzero if and only if $\eta_1 = \eta_2$. In this case, it is irreducible and we have*

$$\mathrm{Jac}_{|\cdot|^{\zeta\frac{3}{2}}}(\pi(\psi_\gg,(+,+,+))) \oplus \mathrm{Jac}_{|\cdot|^{\zeta\frac{3}{2}}}(\pi(\psi_\gg,(-,-,+))) \cong \begin{cases} \mathrm{St}_2 \rtimes \pi(S_6) & \text{if } \zeta = +1, \\ \det_2 \rtimes \mathbf{1}_{\mathrm{SO}_5(F)} & \text{if } \zeta = -1. \end{cases}$$

Here, $\pi(S_6) \in \Pi_{S_6}$ is a discrete series representation of $\mathrm{SO}_7(F)$, and $\det_2$ is the determinant character of $\mathrm{GL}_2(F)$. Hence

$$\pi_{>_\psi}(\psi,\underline{0},(+,+,+)) \oplus \pi_{>_\psi}(\psi,\underline{0},(-,-,+)) \cong \begin{cases} \mathrm{St}_2 \rtimes \pi(S_2) & \text{if } \zeta = +1, \\ \det_2 \rtimes \mathbf{1}_{\mathrm{SO}_3(F)} & \text{if } \zeta = -1. \end{cases}$$

Since the right hand side is irreducible, exactly one of $\pi_{>_\psi}(\psi,\underline{0},(+,+,+))$ *or* $\pi_{>_\psi}(\psi,\underline{0},(-,-,+))$ *is nonzero. Therefore, we have*

$$\Pi_\psi=\begin{cases}\{\mathrm{St}_2\rtimes\pi(S_2)\} & \text{if } \zeta=+1,\\ \{\det{}_2\rtimes\mathbf{1}_{\mathrm{SO}_3(F)}\} & \text{if } \zeta=-1.\end{cases}$$

In particular, $\#\Pi_\psi=1$ *while* $\#(\Sigma_\psi/\sim_\psi)=4$. *Note that* Π_ψ *is equal to* Π_{ϕ_ψ} *with*

$$\phi_\psi=\begin{cases}S_2^{\oplus 3} & \text{if } \zeta=+1,\\ \left(|\cdot|^{\frac12}\oplus|\cdot|^{-\frac12}\right)^{\oplus 3} & \text{if } \zeta=-1.\end{cases}$$

In fact, one can show that $\pi_{>_\psi}(\psi,\underline{0},(+,+,+))\neq 0$ *and* $\pi_{>_\psi}(\psi,\underline{0},(-,-,+))=0$.

(2) *Suppose that* $\{\zeta_1,\zeta_2,\zeta_3\}=\{+1,-1\}$. *we can assume without loss of generality that* $\zeta_2=\zeta_3=\zeta\neq\zeta_1$.

- *If* $\underline{\eta}=(+,+,+)$, *since* $\pi(\psi_\gg,\varepsilon)$ *is a subrepresentation of*

$$\begin{aligned}&|\cdot|^{-\zeta\frac12}\times|\cdot|^{\zeta\frac32}\times|\cdot|^{\zeta\frac12}\times\left\langle\zeta\frac52,\zeta\frac32,\zeta\frac12\right\rangle\rtimes\mathbf{1}_{\mathrm{SO}_1(F)}\\ &\hookrightarrow|\cdot|^{\zeta\frac32}\times\left\langle\zeta\frac52,\zeta\frac32\right\rangle\times|\cdot|^{-\zeta\frac12}\times|\cdot|^{\zeta\frac12}\times|\cdot|^{\zeta\frac12}\rtimes\mathbf{1}_{\mathrm{SO}_1(F)},\end{aligned}$$

  we have $\mathrm{Jac}_{|\cdot|^{-\zeta\frac12}}(\pi_{>_\psi}(\psi,\underline{0},\underline{\eta}))=\mathrm{Jac}_{|\cdot|^{\zeta\frac32},|\cdot|^{\zeta\frac52},|\cdot|^{\zeta\frac32},|\cdot|^{-\zeta\frac12}}(\pi(\psi_\gg,\varepsilon))\neq0$. *Note that*

$$\mathrm{Jac}_{|\cdot|^{-\zeta\frac12}}(\pi(\psi_\gg,\varepsilon))=\pi(\psi',\varepsilon')$$

  with $(\psi')_d=(\psi_\gg)_d-S_2$ *and* $\varepsilon'=\varepsilon|A_{\psi'}$. *Using* [X17a, Lemma 5.6], *Lemma 3.5 and the Aubert involution if necessary, we see that*

$$\begin{aligned}\mathrm{Jac}_{|\cdot|^{-\zeta\frac12}}(\pi(\psi_\gg,\underline{\eta}))&\cong\mathrm{Jac}_{|\cdot|^{-\zeta\frac12},|\cdot|^{\zeta\frac32},|\cdot|^{\zeta\frac52},|\cdot|^{\zeta\frac32}}(\pi(\psi_\gg,\varepsilon))\\ &\hookrightarrow\begin{cases}\mathrm{St}_2\rtimes\mathbf{1}_{\mathrm{SO}_1(F)} & \text{if } \zeta=+1,\\ \det{}_2\rtimes\mathbf{1}_{\mathrm{SO}_1(F)} & \text{if } \zeta=-1.\end{cases}\end{aligned}$$

  Therefore, $\pi_{>_\psi}(\psi,\underline{0},\underline{\eta})$ *is nonzero and is an irreducible subrepresentation of*

$$\begin{cases}|\cdot|^{-\frac12}\times\mathrm{St}_2\rtimes\mathbf{1}_{\mathrm{SO}_1(F)} & \text{if } \zeta=+1,\\ |\cdot|^{\frac12}\times\det{}_2\rtimes\mathbf{1}_{\mathrm{SO}_1(F)} & \text{if } \zeta=-1.\end{cases}$$

- *If* $\underline{\eta}=(+,-,-)$, *since* $\pi(\psi_\gg,\varepsilon)$ *is a subrepresentation of*

$$\begin{aligned}&|\cdot|^{-\zeta\frac12}\times|\cdot|^{\zeta\frac32}\times|\cdot|^{\zeta\frac52}\times\left\langle\zeta\frac32,\zeta\frac12\right\rangle\times|\cdot|^{-\zeta\frac12}\rtimes\mathbf{1}_{\mathrm{SO}_1(F)}\\ &\hookrightarrow|\cdot|^{\zeta\frac32}\times|\cdot|^{\zeta\frac52}\times|\cdot|^{\zeta\frac32}\times|\cdot|^{-\zeta\frac12}\times|\cdot|^{\zeta\frac12}\times|\cdot|^{-\zeta\frac12}\rtimes\mathbf{1}_{\mathrm{SO}_1(F)},\end{aligned}$$

  we have $\mathrm{Jac}_{|\cdot|^{-\zeta\frac12}}(\pi_{>_\psi}(\psi,\underline{0},\underline{\eta}))=\mathrm{Jac}_{|\cdot|^{\zeta\frac32},|\cdot|^{\zeta\frac52},|\cdot|^{\zeta\frac32},|\cdot|^{-\zeta\frac12}}(\pi(\psi_\gg,\varepsilon))\neq0$. *By the same argument as above, we see that* $\pi_{>_\psi}(\psi,\underline{0},\underline{\eta})$ *is nonzero and is an irreducible subrepresentation of*

$$\begin{cases}|\cdot|^{-\frac12}\times\mathrm{St}_2\rtimes\mathbf{1}_{\mathrm{SO}_1(F)} & \text{if } \zeta=+1,\\ |\cdot|^{\frac12}\times\det{}_2\rtimes\mathbf{1}_{\mathrm{SO}_1(F)} & \text{if } \zeta=-1.\end{cases}$$

- *If* $\underline{\eta} = (-,+,-)$, *since* $\pi(\psi_{\gg}, \varepsilon)$ *is supercuspidal, we have* $\mathrm{Jac}_{|\cdot|^{\zeta\frac{3}{2}}}(\pi(\psi_{\gg}, \varepsilon)) = 0$ *so that* $\pi_{>_\psi}(\psi, \underline{0}, \underline{\eta}) = 0$.
- *If* $\underline{\eta} = (-,-,+)$, *since* $\pi(\psi_{\gg}, \varepsilon)$ *is the same representation as in the case (1), we have* $\pi_{>_\psi}(\psi, \underline{0}, \underline{\eta}) = 0$.

Therefore, we have

$$\Pi_\psi = \{\pi_{>_\psi}(\psi, \underline{0}, (+,+,+)),\ \pi_{>_\psi}(\psi, \underline{0}, (+,-,-))\}.$$

In particular, $\#\Pi_\psi = 2$ *while* $\#(\Sigma_\psi/\sim_\psi) = 4$. *Moreover,*

$$\bigoplus_{\pi \in \Pi_\psi} \pi$$

is the maximal semisimple submodule of $\langle -\zeta/2 \rangle \times \langle \zeta/2, -\zeta/2 \rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$. *Note that* $\Pi_{\phi_\psi} \subset \Pi_\psi$ *but*

$$\#\Pi_{\phi_\psi} = \begin{cases} 2 & \text{with } \phi_\psi = |\cdot|^{\frac{1}{2}} \oplus S_2 \oplus S_2 \oplus |\cdot|^{-\frac{1}{2}} & \text{if } \zeta = +1, \\ 1 & \text{with } \phi_\psi = |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus S_2 \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} & \text{if } \zeta = -1. \end{cases}$$

5.4. **General case.** Finally, let $\psi \in \Psi(G_n)$ be a general A-parameter for G_n. Then we can decompose

$$\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee,$$

where ψ_0 is an A-parameter for G_{n_0} of good parity, and ψ_1 is a sum of irreducible representations of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ which are not the same type as ψ. Fix an admissible order $>_{\psi_0}$ for ψ_0. For $(\underline{l}, \underline{\eta}) \in \Sigma_{\psi_0}$, we set

$$\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) = \tau_{\psi_1} \rtimes \pi_{>_{\psi_0}}(\psi_0, \underline{l}, \underline{\eta}).$$

Theorem 5.5. *The representation* $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ *is either zero or irreducible, and is independent of the choice of* ψ_1. *The A-packet* Π_ψ *is given by*

$$\Pi_\psi = \{\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \mid (\underline{l}, \underline{\eta}) \in \Sigma_{\psi_0}/\sim_{\psi_0}\} \setminus \{0\}.$$

To summarize, we obtain Mœglin's multiplicity-free result for A-packets.

Theorem 5.6 (Mœglin [Mœ11], Xu [X17b, Theorem 8.12])**.** *For* $\psi \in \Psi(G_n)$, *the A-packet* Π_ψ *is multiplicity-free, i.e.,* Π_ψ *is a subset of* $\mathrm{Irr}_{\mathrm{unit}}(G_n(F))$.

5.5. **Complementary results.** There are useful results of Mœglin.

Proposition 5.7 ([Mœ09b, 4.2 Corollaire])**.** *For* $\psi, \psi' \in \Psi(G_n)$, *if* $\Pi_\psi \cap \Pi_{\psi'} \neq \emptyset$, *then* $\psi_d \cong \psi'_d$.

We call $\psi \in \Psi(G_n)$ unramified if $\psi | I_F \times \mathrm{SL}_2(\mathbb{C}) \times \{\mathbf{1}_2\}$ is trivial, where I_F is the inertia subgroup of W_F.

Proposition 5.8 ([Mœ09b, 4.4 Proposition])**.** *If* $\psi \in \Psi(G_n)$ *is unramified, then* Π_ψ *has a unique unramified representation of* $G_n(F)$.

6. A non-vanishing criterion

In the previous sections, we have constructed Π_ψ explicitly for any $\psi \in \Psi(G_n)$. However, when ψ is of good parity, the representation $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ can be zero for $(\underline{l}, \underline{\eta}) \in \Sigma_\psi$. In this section, we give a procedure to determine whether $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ is zero or not. This is the work in [X].

By the construction in §5.4, we may assume in this section that ψ is of good parity.

6.1. **Definitions and the algorithm.**

As in the previous section, we write

$$\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}.$$

We set

$$d_i = \min\{a_i, b_i\}, \quad \zeta_i = \begin{cases} 1 & \text{if } a_i > b_i, \\ -1 & \text{if } a_i < b_i. \end{cases}$$

When $a_i = b_i$, we choose $\zeta_i \in \{\pm 1\}$ arbitrarily.

Definition 6.1. *Suppose that $\rho \boxtimes S_a \boxtimes S_b \subset \psi$, and $r \in \mathbb{Z}$ with $r > 0$. Let J be a subset of I_ρ, and J^c be its complement in I_ρ.*

(1) *We say that $\rho \boxtimes S_a \boxtimes S_b$ is in **level** r **"far away" from** J if*

$$\frac{|a-b|}{2} > 2^{r|J|} \cdot \left(\sum_{j \in J} \left(\frac{a_j + b_j}{2} - 1 \right) + |J| \sum_{i \in I_\rho} d_i \right).$$

In this case, we write

$$\rho \boxtimes S_a \boxtimes S_b \gg_r \psi_J = \bigoplus_{j \in J} \rho \boxtimes S_{a_j} \boxtimes S_{b_j}.$$

(2) *We say that J is **"separated" from** J^c if the following conditions are satisfied.*

(a) *For any $j \in J$ and $j' \in J_c$, either*

$$\frac{|a_{j'} - b_{j'}|}{2} > \frac{a_j + b_j}{2} - 1 \quad \text{or} \quad \frac{|a_j - b_j|}{2} > \frac{a_{j'} + b_{j'}}{2} - 1.$$

This condition is equivalent that

$$\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{C})}(S_{a_j} \otimes S_{b_j}, S_{a_{j'}} \otimes S_{b_{j'}}) = 0.$$

(b) *For any admissible order $>_J$ on J, there exists a parameter $\psi_{J_\gg} = \oplus_{j \in J} \rho \boxtimes S_{a'_j} \boxtimes S_{b'_j}$ with DDR which dominates $\psi_J = \oplus_{j \in J} \rho \boxtimes S_{a_j} \boxtimes S_{b_j}$ such that for any $j \in J$ and $j' \in J^c$,*

$$\frac{|a_{j'} - b_{j'}|}{2} > \frac{a_j + b_j}{2} - 1 \implies \frac{|a_{j'} - b_{j'}|}{2} > \frac{a'_j + b'_j}{2} - 1.$$

(c) *For any admissible order $>_{J^c}$ on J^c, there exists a parameter $\psi_{J^c_\gg} = \oplus_{j \in J^c} \rho \boxtimes S_{a'_j} \boxtimes S_{b'_j}$ with DDR which dominates $\psi_{J^c} = \oplus_{j \in J^c} \rho \boxtimes S_{a_j} \boxtimes S_{b_j}$ such that for any $j \in J$ and $j' \in J^c$,*

$$\frac{|a_j - b_j|}{2} > \frac{a_{j'} + b_{j'}}{2} - 1 \implies \frac{|a_j - b_j|}{2} > \frac{a'_{j'} + b'_{j'}}{2} - 1.$$

(3) *The index set I_ρ is in **"good shape"** if we can index $I_\rho = \{1, \dots, N_\rho\}$ such that $a_i + b_i \geq a_{i-1} + b_{i-1}$ and $|a_i - b_i| \geq |a_{i-1} - b_{i-1}|$ for any i, and we can divide $I_\rho = \sqcup_j I_\rho^{(j)}$ such that*
 (a) $I_\rho^{(j)} = \{i, i-1\}$ *with* $\zeta_i = \zeta_{i-1}$, *or* $I_\rho^{(j)} = \{i\}$;
 (b) $I_\rho^{(j)}$ *is "separated" from* $I_\rho \setminus I_\rho^{(j)}$.
 Then there is a natural order $>_\psi$ on I_ρ given by $i >_\psi i-1$.

(4) *We say that ψ is in the **generalized basic case** if I_ρ is in "good shape" for any ρ.*

When ψ is in the generalized basic case with natural order $>_\psi$, there is a criterion for the non-vanishing of $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ (Proposition 6.2). We reduce the general case to the generalized basic case changing an admissible order $>_\psi$ (Proposition 6.5), and using three reduction operators "Pull" (Propositions 6.6, 6.7), "Expand" (Proposition 6.8), and "Change sign" (Proposition 6.9).

More precisely, we use the following algorithm.

Algorithm: Let

$$\psi = \bigoplus_\rho \bigoplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\mathrm{gp}}(G_n),$$

and $(\underline{l}, \underline{\eta}) \in \Sigma_\psi / \sim_\psi$. Choose an admissible order $>_\psi$ on I_ρ for each ρ, and we index $I_\rho = \{1, 2, \dots, N_\rho\}$ such that $i >_\psi i-1$.

Step 1: Is ψ in the generalized basic case?

- If yes, use Proposition 6.2 (after using Proposition 6.5 unless $>_\psi$ is a natural order).
- If no, choose ρ and $1 \leq m \leq N_\rho$ such that I_ρ is not in "good shape", and for $i > m$,

$$\rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg_2 \bigoplus_{j=1}^{m} \rho \boxtimes S_{a_j} \boxtimes S_{b_j},$$

 and such that $\{m+1, \dots, N_\rho\}$ are in "good shape". Go to Step 2.

Step 2: Choose $1 \leq i_0 \leq m$ such that

$$\max_{1 \leq i \leq m} (a_i + b_i) = a_{i_0} + b_{i_0}.$$

Consider

$$S = \{i \leq m \mid S_{a_i} \otimes S_{b_i} \subsetneq S_{a_{i_0}} \otimes S_{b_{i_0}},\ \zeta_i = \zeta_{i_0}\},$$
$$\overline{S} = \{i \leq m \mid S_{a_i} \otimes S_{b_i} \subseteq S_{a_{i_0}} \otimes S_{b_{i_0}},\ \zeta_i = \zeta_{i_0}\}.$$

Then there are three possibilities.

- If $S \neq \emptyset$, go to Step 3a.
- If $S = \emptyset$ but $\overline{S} \supsetneq \{i_0\}$, go to Step 3b.
- If $\overline{S} = \{i_0\}$, go to Step 3c.

Step 3a: Take $i_0' \in S$ such that

$$\max_{i \in S}(a_i + b_i) = a_{i_0'} + b_{i_0'}.$$

Using Proposition 6.5, we rearrange the order $>_\psi$ for $i \leq m$ so that

$$\rho \boxtimes S_{a_m} \boxtimes S_{b_m} = \rho \boxtimes S_{a_{i_0}} \boxtimes S_{b_{i_0}},$$

$$\rho \boxtimes S_{a_{m-1}} \boxtimes S_{b_{m-1}} = \rho \boxtimes S_{a_{i_0'}} \boxtimes S_{b_{i_0'}}.$$

Then we can "Pull" the pair $\rho \boxtimes S_{a_m} \boxtimes S_{b_m}$ and $\rho \boxtimes S_{a_{m-1}} \boxtimes S_{b_{m-1}}$ using Proposition 6.6. Consequently, the non-vanishing of $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ is equivalent to those of 3 representations $\{\pi_{>_{\psi^*}}(\psi^*, \underline{l}^*, \underline{\eta}^*)\}$. If we replace ψ with ψ^*, one can replace m with $m-1$ or $m-2$. After such replacements, go back to Step 2 if $m \geq 1$. If $m = 0$, then I_ρ is in "good shape", and go back to Step 1.

Step 3b: Take $i_0' \in \overline{S}$ such that $i_0' \neq i_0$. Using Proposition 6.5, we rearrange the order $>_\psi$ for $i \leq m$ as in Step 3a. Then we can "Pull" the pair $\rho \boxtimes S_{a_m} \boxtimes S_{b_m}$ and $\rho \boxtimes S_{a_{m-1}} \boxtimes S_{b_{m-1}}$ using Proposition 6.7. Consequently, the non-vanishing of $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ is equivalent to those of 2 representations $\{\pi_{>_{\psi^*}}(\psi^*, \underline{l}^*, \underline{\eta}^*)\}$. If we replace ψ with ψ^*, one can replace m with $m-1$ or $m-2$. After such replacements, go back to Step 2 if $m \geq 1$. If $m = 0$, then I_ρ is in "good shape", and go back to Step 1.

Step 3c: Using Proposition 6.5, we rearrange the order $>_\psi$ for $i \leq m$ so that

$$\rho \boxtimes S_{a_m} \boxtimes S_{b_m} = \rho \boxtimes S_{a_{i_0}} \boxtimes S_{b_{i_0}}.$$

Then we can "Expand" $\rho \boxtimes S_{a_m} \boxtimes S_{b_m}$ using Proposition 6.8. Consequently, the non-vanishing of $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ is equivalent to that of a representation $\pi_{>_{\psi^*}}(\psi^*, \underline{l}^*, \underline{\eta}^*)$. Go to Step 4.

Step 4: For the parameter ψ^* obtained in Step 3c, consider the set S^* as in Step 2 with $\rho \boxtimes S_{a_{i_0}} \boxtimes S_{b_{i_0}} := \rho \boxtimes S_{a_m^*} \boxtimes S_{b_m^*}$, which is the "Expansion" of $\rho \boxtimes S_{a_m} \boxtimes S_{b_m}$.

- If $S^* \neq \emptyset$, go back to Step 3a after replacing ψ with ψ^*.
- If $S^* = \emptyset$, it is necessary that $|a_m^* - b_m^*| \leq 1$, and $\zeta_i \neq \zeta_n$ for $i < m$. Go to Step 5

Step 5: Using Proposition 6.5, we rearrange the order $>_\psi$ for $i \leq m$ so that

$$\begin{pmatrix} 1 \\ 2 \\ \vdots \\ m \end{pmatrix} \rightsquigarrow \begin{pmatrix} m \\ 1 \\ \vdots \\ m-1 \end{pmatrix}.$$

Then we can "Change sign" of $\rho \boxtimes S_{a_m^*} \boxtimes S_{b_m^*}$ using Proposition 6.9. Consequently, the non-vanishing of $\pi_{>_{\psi^*}}(\psi^*, \underline{l}^*, \underline{\eta}^*)$ is equivalent to that of a representation $\pi_{>_{\psi^{**}}}(\psi^{**}, \underline{l}^{**}, \underline{\eta}^{**})$. If we replace ψ^* with ψ^{**}, the set S^* becomes non-empty. After such replacements, go back to Step 1.

Step 6: By the above steps, the non-vanishing of $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ is equivalent to those of several representations $\{\pi_{>_{\psi^\star}}(\psi^\star, \underline{l}^\star, \underline{\eta}^\star)\}$, where each $\psi^\star$ is in the generalized basic case. Use Proposition 6.2, we obtain the conditions on $(\underline{l}, \underline{\eta})$ for $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$.

In the rest of this section, we state several propositions appearing in this algorithm.

6.2. **Generalized basic case.** Recall that ψ is in the generalized basic case if for each ρ, the index set I_ρ is in "good shape", i.e., $I_\rho = \{1, \ldots, N_\rho\}$ with $a_i + b_i \geq a_{i-1} + b_{i-1}$ and $|a_i - b_i| \geq |a_{i-1} - b_{i-1}|$ for any i, and we can divide $I_\rho = \sqcup_j I_\rho^{(j)}$ such that

(a) $I_\rho^{(j)} = \{i, i-1\}$ with $\zeta_i = \zeta_{i-1}$, or $I_\rho^{(j)} = \{i\}$;
(b) $I_\rho^{(j)}$ is "separated" from $I_\rho \setminus I_\rho^{(j)}$.

In this case, we use the natural order $>_\psi$ on I_ρ such that $i >_\psi i-1$. In the generalized basic case, there is a non-vanishing criterion for $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$.

Proposition 6.2 ([X, Proposition 4.3]). *When ψ is in the generalized basic case, $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if for any ρ and any $I_\rho^{(j)}$, if $I_\rho^{(j)} = \{i, i-1\}$ has two elements, then*

$$\begin{cases} \eta_i = (-1)^{d_{i-1}}\eta_{i-1} \implies l_i + l_{i-1} > \dfrac{a_{i-1}+b_{i-1}}{2} - \dfrac{|a_i - b_i|}{2} - 1, \\ \eta_i \neq (-1)^{d_{i-1}}\eta_{i-1} \implies -\dfrac{|a_i-b_i| - |a_{i-1}-b_{i-1}|}{2} \leq l_i - l_{i-1} \leq \dfrac{(a_i+b_i)-(a_{i-1}+b_{i-1})}{2}. \end{cases}$$

Remark 6.3. *Let ψ be in the generalized basic case, and $I_\rho = \sqcup_j I_\rho^{(j)}$ be the division as in Definition 6.1 (3).*

(1) *If $\# I_\rho^{(j)} = 1$ for any ρ and any j, then by Definition 6.1 (2)-(a), ψ has a DDR.*
(2) *We say that ψ is in the **basic case** if there exists a unique pair (ρ, j) such that $\# I_\rho^{(j)} = 2$. Mœglin gave a non-vanishing criterion for $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ when ψ is in the basic case such that $I_\rho^{(j)} = \{i, i-1\}$ with $S_{a_i} \otimes S_{b_i} \cong S_{a_{i-1}} \otimes S_{b_{i-1}}$. Proposition 6.2 is a generalization of Mœglin's result.*

Using Proposition 6.2, we obtain the following necessary conditions on non-vanishing of $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta})$ in general.

Lemma 6.4 ([X, Lemmas 4.6, 4.7]). *Fix an admissible order $>_\psi$ on I_ρ, and let $k >_\psi k-1$ be two adjacent elements in I_ρ such that $\zeta_k = \zeta_{k-1}$. Suppose that $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$.*

(1) *If $S_{a_k} \otimes S_{b_k} \supseteq S_{a_{k-1}} \otimes S_{b_{k-1}}$, then*

$$\begin{cases} \eta_k = (-1)^{d_{k-1}}\eta_{k-1} \implies l_k + l_{k-1} > d_{k-1} - 1, \\ \eta_k \neq (-1)^{d_{k-1}}\eta_{k-1} \implies 0 \leq l_k - l_{k-1} \leq d_k - d_{k-1}. \end{cases}$$

(2) *If $S_{a_k} \otimes S_{b_k} \subseteq S_{a_{k-1}} \otimes S_{b_{k-1}}$, then*

$$\begin{cases} \eta_k = (-1)^{d_{k-1}}\eta_{k-1} \implies l_k + l_{k-1} > d_k - 1, \\ \eta_k \neq (-1)^{d_{k-1}}\eta_{k-1} \implies 0 \leq l_{k-1} - l_k \leq d_{k-1} - d_k. \end{cases}$$

We denote by $\Sigma_\psi^{(+)}$ (resp. $\Sigma_\psi^{(-)}$) the subset of Σ_ψ satisfying the conditions in Lemma 6.4 (1) (resp. (2)) for any ρ and any adjacent pair $\{k, k-1\}$ in I_ρ with $\zeta_k = \zeta_{k-1}$. Note that if $(\underline{l}, \underline{\eta}) \sim_\psi (\underline{l}', \underline{\eta}')$ and $(\underline{l}, \underline{\eta}) \in \Sigma_\psi^{(\pm)}$, then $(\underline{l}', \underline{\eta}') \in \Sigma_\psi^{(\pm)}$.

6.3. **Change of admissible orders.** Let $\psi \in \Psi_{\mathrm{gp}}(G_n)$. We choose an admissible order $>_\psi$ on I_ρ, and we index $I_\rho = \{1, \dots, N_\rho\}$ such that $i >_\psi i-1$. Now for fixed $1 < k \leq N_\rho$, we denote by $>'_\psi$ the order on I_ρ obtained so that $i >'_\psi j$ if and only if $i >_\psi j$ and $(i,j) \neq (k, k-1)$, or $(i,j) = (k-1, k)$. Assume that $>'_\psi$ is also an admissible order.

When $\zeta_k = \zeta_{k-1}$ and $S_{a_k} \otimes S_{b_k} \supseteq S_{a_{k-1}} \otimes S_{b_{k-1}}$, we define $(\underline{l}', \underline{\eta}') = S^+(\underline{l}, \underline{\eta})$ for $(\underline{l}, \underline{\eta}) \in \Sigma_\psi^{(+)}$ as follows:

- If $i \neq k, k-1$, then $l'_i = l_i$ and $\eta'_i = \eta_i$.

- If $\eta_k = (-1)^{d_{k-1}}\eta_{k-1}$, then

$$\begin{cases} l'_{k-1} = l_{k-1}, \\ l'_k = l_k + 2l_{k-1} - d_{k-1}, \\ \eta'_{k-1} = -(-1)^{d_k}\eta_{k-1}, \\ \eta'_k = (-1)^{d_{k-1}}\eta_k. \end{cases}$$

- If $\eta_k \neq (-1)^{d_{k-1}}\eta_{k-1}$ and $l_k - 2l_{k-1} < (d_k+1)/2 - d_{k-1}$, then

$$\begin{cases} l'_{k-1} = l_{k-1}, \\ l'_k = l_k - 2l_{k-1} + d_{k-1}, \\ \eta'_{k-1} = -(-1)^{d_k}\eta_{k-1}, \\ \eta'_k = (-1)^{d_{k-1}}\eta_k. \end{cases}$$

- If $\eta_k \neq (-1)^{d_{k-1}}\eta_{k-1}$ and $l_k - 2l_{k-1} \geq (d_k+1)/2 - d_{k-1}$, then

$$\begin{cases} l'_{k-1} = l_{k-1}, \\ l'_k = -l_k + 2l_{k-1} + d_k - d_{k-1}, \\ \eta'_{k-1} = -(-1)^{d_k}\eta_{k-1}, \\ \eta'_k = -(-1)^{d_{k-1}}\eta_k. \end{cases}$$

When $\zeta_k = \zeta_{k-1}$ and $S_{a_k} \otimes S_{b_k} \subseteq S_{a_{k-1}} \otimes S_{b_{k-1}}$, we define $(\underline{l}', \underline{\eta}') = S^-(\underline{l}, \underline{\eta})$ for $(\underline{l}, \underline{\eta}) \in \Sigma_\psi^{(-)}$ as follows:

- If $i \neq k, k-1$, then $l'_i = l_i$ and $\eta'_i = \eta_i$.
- If $\eta_k = (-1)^{d_{k-1}}\eta_{k-1}$, then

$$\begin{cases} l'_k = l_k, \\ l'_{k-1} = l_{k-1} + 2l_k - d_k, \\ \eta'_k = -(-1)^{d_{k-1}}\eta_k, \\ \eta'_{k-1} = (-1)^{d_k}\eta_{k-1}. \end{cases}$$

- If $\eta_k \neq (-1)^{d_{k-1}}\eta_{k-1}$ and $l_{k-1} - 2l_k < (d_{k-1}+1)/2 - d_k$, then

$$\begin{cases} l'_k = l_k, \\ l'_{k-1} = l_{k-1} - 2l_k + d_k, \\ \eta'_k = -(-1)^{d_{k-1}}\eta_k, \\ \eta'_{k-1} = (-1)^{d_k}\eta_{k-1}. \end{cases}$$

- If $\eta_k \neq (-1)^{d_{k-1}}\eta_{k-1}$ and $l_{k-1} - 2l_k \geq (d_{k-1}+1)/2 - d_k$, then

$$\begin{cases} l'_k = l_k, \\ l'_{k-1} = -l_{k-1} + 2l_k + d_{k-1} - d_k, \\ \eta'_k = -(-1)^{d_{k-1}}\eta_k, \\ \eta'_{k-1} = -(-1)^{d_k}\eta_{k-1}. \end{cases}$$

Then one can check that $S^{\pm}\colon \Sigma_{\psi}^{(\pm)}/\sim_{\psi} \to \Sigma_{\psi}^{(\mp)}/\sim_{\psi}$ and that they are inverse to each other. Hence S^{+} and S^{-} are bijective.

When $\zeta_k \neq \zeta_{k-1}$, not assuming any extra conditions on $a_k, b_k, a_{k-1}, b_{k-1}$, we define $(\underline{l}', \underline{\eta}') = U(\underline{l}, \underline{\eta})$ for $(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}$ by $l'_i = l_i$ and $\eta'_i = \eta_i$ for $i \neq k, k-1$, and by

$$\begin{cases} l'_{k-1} = l_{k-1}, \\ l'_k = l_k, \\ \eta'_{k-1} = (-1)^{d_k}\eta_{k-1}, \\ \eta'_k = (-1)^{d_{k-1}}\eta_k. \end{cases}$$

Then one can check that $U(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}$ and that $U \circ U = \mathrm{id}$. Hence $U\colon \Sigma_{\psi}/\sim_{\psi} \to \Sigma_{\psi}/\sim_{\psi}$ is bijective.

Proposition 6.5 ([X, Propositions 5.1, 5.3])**.** (1) *Suppose that $\zeta_k = \zeta_{k-1}$ and $S_{a_k} \otimes S_{b_k} \supseteq S_{a_{k-1}} \otimes S_{b_{k-1}}$. Then $\pi_{>_{\psi}}(\psi, \underline{l}, \underline{\eta}) = 0$ unless $(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}^{(+)}/\sim_{\psi}$, in which case,*

$$\pi_{>_{\psi}}(\psi, \underline{l}, \underline{\eta}) \cong \pi_{>'_{\psi}}(\psi, \underline{l}', \underline{\eta}')$$

with $(\underline{l}', \underline{\eta}') = S^{+}(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}^{(-)}/\sim_{\psi}$.

(2) *Suppose that $\zeta_k = \zeta_{k-1}$ and $S_{a_k} \otimes S_{b_k} \subseteq S_{a_{k-1}} \otimes S_{b_{k-1}}$. Then $\pi_{>_{\psi}}(\psi, \underline{l}, \underline{\eta}) = 0$ unless $(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}^{(-)}/\sim_{\psi}$, in which case,*

$$\pi_{>_{\psi}}(\psi, \underline{l}, \underline{\eta}) \cong \pi_{>'_{\psi}}(\psi, \underline{l}', \underline{\eta}')$$

with $(\underline{l}', \underline{\eta}') = S^{-}(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}^{(+)}/\sim_{\psi}$.

(3) *Suppose that $\zeta_k \neq \zeta_{k-1}$. Then*

$$\pi_{>_{\psi}}(\psi, \underline{l}, \underline{\eta}) \cong \pi_{>'_{\psi}}(\psi, \underline{l}', \underline{\eta}')$$

for any $(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}/\sim_{\psi}$ with $(\underline{l}', \underline{\eta}') = U(\underline{l}, \underline{\eta}) \in \Sigma_{\psi}/\sim_{\psi}$.

6.4. **Reduction operator 1: "Pull".** In this subsection, we introduce a reduction operator "Pull". Choose an admissible order $>_{\psi}$. We index $I_{\rho} = \{1, \dots, N_{\rho}\}$ such that $i >_{\psi} i-1$. First, we suppose that there exists m such that

- for $i > m$,
$$\rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg_1 \bigoplus_{j=1}^{m} \rho \boxtimes S_{a_j} \boxtimes S_{b_j};$$
- $S_{a_m} \otimes S_{b_m} \supsetneq S_{a_{m-1}} \otimes S_{b_{m-1}}$ and $\zeta_m = \zeta_{m-1}$.

We denote by $>'_{\psi}$ the order on I_{ρ} obtained so that $i >'_{\psi} j$ if and only if $i >_{\psi} j$ and $(i,j) \neq (m, m-1)$, or $(i,j) = (m-1, m)$. Set $(a_m^{\sharp}, b_m^{\sharp})$, $(a_{m-1}^{\sharp}, b_{m-1}^{\sharp})$, $(a_m^{\flat}, b_m^{\flat})$, and $(a_m^{\natural}, b_m^{\natural})$ so that

- $\max\{a_m^{\sharp}, b_m^{\sharp}\} = \max\{a_m, b_m\} + 2T_m$, $\min\{a_m^{\sharp}, b_m^{\sharp}\} = \min\{a_m, b_m\}$, and so that $\zeta_m^{\sharp} = \zeta_m$, where T_m is an arbitrary positive integer such that
$$\rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg_1 \rho \boxtimes S_{a_m^{\sharp}} \boxtimes S_{b_m^{\sharp}}$$
for $i > m$;

- $\max\{a^\sharp_{m-1}, b^\sharp_{m-1}\} = \max\{a_{m-1}, b_{m-1}\} + 2T_{m-1}$, $\min\{a^\sharp_{m-1}, b^\sharp_{m-1}\} = \min\{a_{m-1}, b_{m-1}\}$, and so that $\zeta^\sharp_{m-1} = \zeta_{m-1}$, where T_{m-1} is the integer given by
$$T_{m-1} = T_m + \frac{|a_m - b_m| - |a_{m-1} - b_{m-1}|}{2};$$
- $\max\{a^\flat_m, b^\flat_m\} = \max\{a_m, b_m\} + 2T$, $\min\{a^\flat_m, b^\flat_m\} = \min\{a_m, b_m\}$, and so that $\zeta^\flat_m = \zeta_m$, where T is an arbitrary positive integer such that
$$T < \frac{|a_i - b_i| - (a_m + b_m)}{2} + 1$$
for $i > m$;
- $\max\{a^\natural_{m-1}, b^\natural_{m-1}\} = \max\{a_{m-1}, b_{m-1}\} + 2T$, $\min\{a^\natural_{m-1}, b^\natural_{m-1}\} = \min\{a_{m-1}, b_{m-1}\}$, and so that $\zeta^\natural_{m-1} = \zeta_{m-1}$, where T is an arbitrary positive integer such that
$$T < \frac{|a_i - b_i| - (a_{m-1} + b_{m-1})}{2} + 1$$
for $i > m$.

Define $\psi^\sharp$, $\psi^\flat$, and $\psi^\natural$ by
$$\psi^\sharp = \psi - \left(\bigoplus_{i=m-1}^{m} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}\right) + \left(\bigoplus_{i=m-1}^{m} \rho \boxtimes S_{a^\sharp_i} \boxtimes S_{b^\sharp_i}\right),$$
$$\psi^\flat = \psi - \rho \boxtimes S_{a_m} \boxtimes S_{b_m} + \rho \boxtimes S_{a^\flat_m} \boxtimes S_{b^\flat_m},$$
$$\psi^\natural = \psi - \rho \boxtimes S_{a_{m-1}} \boxtimes S_{b_{m-1}} + \rho \boxtimes S_{a^\natural_{m-1}} \boxtimes S_{b^\natural_{m-1}}.$$
We may identify Σ_ψ with Σ_{ψ^*} canonically for $* \in \{\sharp, \flat, \natural\}$.

Proposition 6.6 ([X, Proposition 6.1])**.** *Let $(\underline{l}, \underline{\eta}) \in \Sigma_\psi$ and set $(\underline{l}', \underline{\eta}') = S^+(\underline{l}, \underline{\eta})$. Then $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if all of*
$$\pi_{>_\psi}(\psi^\sharp, \underline{l}, \underline{\eta}), \quad \pi_{>_\psi}(\psi^\flat, \underline{l}, \underline{\eta}), \quad \pi_{>'_\psi}(\psi^\natural, \underline{l}', \underline{\eta}')$$
are nonzero.

Next, we suppose that there exists m such that

- for $i > m$,
$$\rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg_1 \bigoplus_{j=1}^{m} \rho \boxtimes S_{a_j} \boxtimes S_{b_j};$$
- $S_{a_m} \otimes S_{b_m} = S_{a_{m-1}} \otimes S_{b_{m-1}}$ and $\zeta_m = \zeta_{m-1}$;
- there is no $i < n$ such that $S_{a_i} \otimes S_{b_i} \subsetneq S_{a_m} \otimes S_{b_m}$ and $\zeta_i = \zeta_m$.

Set $(a^\sharp_m, b^\sharp_m)$, $(a^\sharp_{m-1}, b^\sharp_{m-1})$, and $(a^\flat_m, b^\flat_m)$ so that

- $\max\{a^\sharp_m, b^\sharp_m\} = \max\{a_m, b_m\} + 2T_m$, $\min\{a^\sharp_m, b^\sharp_m\} = \min\{a_m, b_m\}$, and so that $\zeta^*_m = \zeta_m$, where T_m is an arbitrary positive integer such that
$$\rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg_1 \rho \boxtimes S_{a^\sharp_m} \boxtimes S_{b^\sharp_m}$$
for $i > m$.

- $\max\{a^\sharp_{m-1}, b^\sharp_{m-1}\} = \max\{a_{m-1}, b_{m-1}\} + 2T_{m-1}$, $\min\{a^\sharp_{m-1}, b^\sharp_{m-1}\} = \min\{a_{m-1}, b_{m-1}\}$, and so that $\zeta^\sharp_{m-1} = \zeta_{m-1}$, where $T_{m-1} = T_m$;
- $\max\{a^\flat_m, b^\flat_m\} = \max\{a_m, b_m\} + 2T$, $\min\{a^\flat_m, b^\flat_m\} = \min\{a_m, b_m\}$, and so that $\zeta^\flat_m = \zeta_m$, where T is an arbitrary positive integer such that
$$T < \frac{|a_i - b_i| - (a_m + b_m)}{2} + 1$$
for $i > m$.

Define $\psi^\sharp$ and $\psi^\flat$ by

$$\psi^\sharp = \psi - \left(\bigoplus_{i=m-1}^{m} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}\right) + \left(\bigoplus_{i=m-1}^{m} \rho \boxtimes S_{a_i^\sharp} \boxtimes S_{b_i^\sharp}\right),$$
$$\psi^\flat = \psi - \rho \boxtimes S_{a_m} \boxtimes S_{b_m} + \rho \boxtimes S_{a_m^\flat} \boxtimes S_{b_m^\flat}.$$

We may identify Σ_ψ with Σ_{ψ^*} canonically for $* \in \{\sharp, \flat\}$.

Proposition 6.7 ([X, Proposition 6.3])**.** *Let $(\underline{l}, \underline{\eta}) \in \Sigma_\psi$. Then $\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0$ if and only if all of*
$$\pi_{>_\psi}(\psi^\sharp, \underline{l}, \underline{\eta}), \quad \pi_{>_\psi}(\psi^\flat, \underline{l}, \underline{\eta})$$
are nonzero.

6.5. **Reduction operator 2: "Expand".** In this subsection, we introduce a reduction operator "Expand". Choose an admissible order $>_\psi$. We index $I_\rho = \{1, \dots, N_\rho\}$ such that $i >_\psi i-1$. Suppose that there exists m such that

- for $i > m$,
$$\rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg_2 \bigoplus_{j=1}^{m} \rho \boxtimes S_{a_j} \boxtimes S_{b_j};$$
- $a_m + b_m \geq a_i + b_i$ for $i < m$;
- there is no $i < n$ satisfying $S_{a_i} \otimes S_{b_i} \subseteq S_{a_m} \otimes S_{b_m}$ and $\zeta_i = \zeta_m$.

Let
$$t_m = \min \left\{ \left[\frac{|a_m - b_m|}{2}\right] \right\} \cup \left\{ \frac{|a_m - b_m| - |a_i - b_i|}{2} \;\middle|\; i < m,\ \zeta_i = \zeta_m \right\}.$$

Here, $[x]$ denotes the greatest integer which is not larger than x. Set (a^*_m, b^*_m) so that $\max\{a^*_m, b^*_m\} = \max\{a_m, b_m\}$, $\min\{a^*_m, b^*_m\} = \min\{a_m, b_m\} + 2t_m$, and so that $\zeta^*_m = \zeta_m$. Define ψ^* by
$$\psi^* = \psi - \rho \boxtimes S_{a_m} \boxtimes S_{b_m} + \rho \boxtimes S_{a^*_m} \boxtimes S_{b^*_m}.$$

Proposition 6.8 ([X, Proposition 6.4])**.** *For any $(\underline{l}, \underline{\eta}) \in \Sigma_\psi$, we set $(\underline{l}^*, \underline{\eta}^*)$ to be $l^*_i = l_i$ for $i \neq m$, $l^*_m = l_m + t_m$, and $\eta^*_i = \eta_i$ for any i. Then*
$$\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0 \iff \pi_{>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0.$$

6.6. **Reduction operator 3: "Change sign".** In this subsection, we introduce a reduction operator "Change sign". Choose an admissible order $>_\psi$. We index $I_\rho = \{1, \ldots, N_\rho\}$ such that $i >_\psi i-1$. Suppose that there exists m such that

- for $i > m$,
$$\rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg_1 \bigoplus_{j=1}^{m} \rho \boxtimes S_{a_j} \boxtimes S_{b_j};$$
- $a_1 + b_1 \geq a_i + b_i$ and $\zeta_i \neq \zeta_1$ for $1 < i \leq m$;
- $|a_1 - b_1| \leq 1$.

Set $(a_1^*, b_1^*) = (b_1, a_1) + (|a_1 - b_1|, |a_1 - b_1|)$ and $\zeta_1^* = -\zeta_1$. Define ψ^* by
$$\psi^* = \psi - \rho \boxtimes S_{a_1} \boxtimes S_{b_1} + \rho \boxtimes S_{a_1^*} \boxtimes S_{b_1^*}.$$

Proposition 6.9 ([X, Propositions 6.5, 6.6])**.** *Let $(\underline{l}, \underline{\eta}) \in \Sigma_\psi$. When $l_1 = d_1/2$, we assume that $\eta_1 = -1$. We set $(\underline{l}^*, \underline{\eta}^*)$ to be $l_i^* = l_i$, $\eta_i^* = \eta_i$ for $i \neq 1$ and*
$$l_1^* = \begin{cases} l_1 + 1 & \text{if } |a_1 - b_1| = 1,\ \eta_1 = +1, \\ l_1 & \text{otherwise}, \end{cases} \qquad \eta_1^* = \begin{cases} \eta_1 & \text{if } |a_1 - b_1| = 0, \\ -\eta_1 & \text{if } |a_1 - b_1| = 1. \end{cases}$$
Then
$$\pi_{>_\psi}(\psi, \underline{l}, \underline{\eta}) \neq 0 \iff \pi_{>_\psi}(\psi^*, \underline{l}^*, \underline{\eta}^*) \neq 0.$$

6.7. **Example.** We give an example for adapting the algorithm. We again consider $\psi \in \Psi_{\mathrm{gp}}(\mathrm{SO}_7)$ such that $\psi_d = S_2^{\oplus 3}$. As in Example 5.4, $\Sigma_\psi = \{(\underline{0}, \underline{\eta}) \mid \underline{\eta} \in \{\pm 1\}^3,\ \eta_1\eta_2\eta_3 = 1\}$.

When $\zeta_1 = \zeta_2 = \zeta_3$, by Lemma 6.4, $\pi_{>_\psi}(\psi, \underline{0}, \underline{\eta}) = 0$ unless $\eta = (+,+,+)$. Since $\Pi_\psi \neq \emptyset$, we have $\pi_{>_\psi}(\psi, \underline{0}, (+,+,+)) \neq 0$.

When $\zeta_1 = \zeta_2 = \zeta \neq \zeta_3$, by Lemma 6.4, $\pi_{>_\psi}(\psi, \underline{0}, \underline{\eta}) = 0$ unless $\eta_1 = \eta_2$. Assume this condition. Note that $\eta_3 = +1$. We apply Algorithm in §6.1.

Step 1 for ψ: The parameter ψ is not in the generalized basic case. Setting $m = 3$, go to Step 2.

Step 2 for ψ: Take $i_0 = 3$. Then $S = \emptyset$ and $\overline{S} = \{3\}$. Go to Step 3c.

Step 3c for ψ: Proposition 6.8 gives no information. Go to Step 4.

Step 4 for ψ: Since $S = \emptyset$, go to Step 5.

Step 5 for ψ: Let $>'_\psi$ be the new order such that $2 >'_\psi 1 >'_\psi 3$. Then by Proposition 6.5 (3), we have $\pi_{>_\psi}(\psi, \underline{0}, \underline{\eta}) \cong \pi_{>'_\psi}(\psi, \underline{0}, (-\eta_1, -\eta_2, \eta_3))$. By Proposition 6.9, $\pi_{>'_\psi}(\psi, \underline{0}, (-\eta_1, -\eta_2, \eta_3)) \neq 0$ if and only if $\pi_{>'_\psi}(\psi^*, \underline{l^*}, \underline{\eta^*}) \neq 0$, where $\psi^* = \psi - S_{a_3} \boxtimes S_{b_3} \oplus S_{b_3+1} \boxtimes S_{a_3+1}$, $\underline{l^*} = (0,0,1)$, and $\underline{\eta^*} = (-\eta_1, -\eta_2, -\eta_3) = -\underline{\eta}$. Go back to Step 1.

Step 1 for ψ^*: The parameter ψ^* is not in the generalized basic case. Setting $m = 3$, go to Step 2.

Step 2 for ψ^*: Take $i_0 = 3$. Since $\zeta_1^* = \zeta_2^* = \zeta_3^*$, we have $S = \{1, 2\}$. Go to Step 3a.

Step 3a for ψ^*: Take $i_0' = 2$. By Proposition 6.9 (2), $\pi_{>'_\psi}(\psi, \underline{l^*}, \underline{\eta^*}) \neq 0$ if and only if $\pi_{>_\psi}(\psi^*, \underline{l^*}, \underline{\eta^{**}}) \neq 0$, where $\underline{\eta^{**}} = (+,+,-)$ if $\underline{\eta} = (+,+,+)$, and $\underline{\eta^{**}} = (-,-,+)$ if $\underline{\eta} = (-,-,+)$. By Proposition 6.6, $\pi_{>_\psi}(\psi^*, \underline{l^*}, \underline{\eta^{**}}) \neq 0$ if and only if all of
$$\pi_{>_\psi}(\psi^\sharp, \underline{l^*}, \underline{\eta^{**}}), \quad \pi_{>_\psi}(\psi^\flat, \underline{l^*}, \underline{\eta^{**}}), \quad \pi_{>_{\psi^\natural}}(\psi^\natural, \underline{l^\natural}, \underline{\eta^\natural})$$
are nonzero, where
- $\{a_3^\sharp, b_3^\sharp\} = \{5, 2\}$, $\{a_2^\sharp, b_2^\sharp\} = \{4, 1\}$, $\{a_1^\sharp, b_1^\sharp\} = \{2, 1\}$, and $\zeta_1^\sharp = \zeta_2^\sharp = \zeta_3^\sharp = \zeta$;

- $\{a_3^\flat, b_3^\flat\} = \{7,2\}$, $\{a_2^\flat, b_2^\flat\} = \{2,1\}$, $\{a_1^\flat, b_1^\flat\} = \{2,1\}$, and $\zeta_1^\flat = \zeta_2^\flat = \zeta_3^\flat = \zeta$;
- $\{a_3^\natural, b_3^\natural\} = \{3,2\}$, $\{a_2^\natural, b_2^\natural\} = \{8,1\}$, $\{a_1^\natural, b_1^\natural\} = \{2,1\}$, and $\zeta_1^\natural = \zeta_2^\natural = \zeta_3^\natural = \zeta$;
- $2 >_{\psi^\natural} 3 >_{\psi^\natural} 1$, $\underline{l}^\natural = \underline{0}$, and $\underline{\eta}^\natural = (+,-,+)$ if $\underline{\eta} = (+,+,+)$, and $\underline{\eta}^\natural = (-,+,-)$ if $\underline{\eta} = (-,-,+)$. Go back to Step 1.

Step 1 for $\psi^\sharp$, $\psi^\flat$, $\psi^\natural$: All of $\psi^\sharp$, $\psi^\flat$, and $\psi^\natural$ are in the generalized basic case. By Proposition 6.2, we see that all of $\pi_{>_\psi}(\psi^\sharp, \underline{l}^*, \underline{\eta}^{**})$, $\pi_{>_\psi}(\psi^\flat, \underline{l}^*, \underline{\eta}^{**})$, and $\pi_{>_{\psi^\natural}}(\psi^\natural, \underline{l}^\natural, \underline{\eta}^\natural)$ are nonzero. Go to Step 6.

Step 6 for ψ: As a consequence, we see that $\pi_{>_\psi}(\psi, \underline{0}, \underline{\eta}) \neq 0$ for $\underline{\eta} = (+,+,+)$ and $\underline{\eta} = (-,-,+)$. This is compatible with Example 5.4. (Note that in Example 5.4 (2), we assume that $\zeta_2 = \zeta_3 = \zeta \neq \zeta_1$, but now we assume that $\zeta_1 = \zeta_2 = \zeta \neq \zeta_3$.)

Part 2. The Archimedean case

The theory of A-packets are also established in the archimedean case. In Part 2, we review the archimedean case for $G_n = \mathrm{SO}_{2n+1}$ or Sp_{2n}.

7. A-parameters

In this section, we recall the A-parameters in the archimedean case.

7.1. **Weil groups and their representations.** The Weil groups of $\mathbb{C}$ and $\mathbb{R}$ are given by

$$W_{\mathbb{C}} = \mathbb{C}^\times, \quad W_{\mathbb{R}} = \mathbb{C}^\times \sqcup \mathbb{C}^\times j,$$

respectively, where

$$j^2 = -1, \quad jzj^{-1} = \overline{z} \quad \text{for } z \in \mathbb{C}^\times.$$

Then there exists a canonical exact sequence

$$1 \longrightarrow W_{\mathbb{C}} \longrightarrow W_{\mathbb{R}} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1.$$

The norm maps $|\cdot|\colon W_{\mathbb{C}} \to \mathbb{R}^\times$ and $|\cdot|\colon W_{\mathbb{R}} \to \mathbb{R}^\times$ are given by $|z| = z\overline{z}$ for $z \in W_{\mathbb{C}} \subset W_{\mathbb{R}}$, and $|j| = 1$. Note that $|\cdot|$ on $W_{\mathbb{C}} = \mathbb{C}^\times$ is not the absolute value but the modulus character of $\mathbb{C}^\times$ on $\mathbb{C}$.

For each integer k, we define a unitary character χ_k of $W_{\mathbb{C}} = \mathbb{C}^\times$ by

$$\chi_k(z) = \overline{z}^{-k}(z\overline{z})^{\frac{k}{2}}$$

for $z \in \mathbb{C}^\times$. We sometimes write $\chi_k(z)$ as $(z/\overline{z})^{\frac{k}{2}}$, but one has to keep $\chi_k(-1) = (-1)^k$ in mind. Any character of $W_{\mathbb{C}}$ is of the form

$$|\cdot|^\alpha \chi_k$$

for some $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}$.

There are exactly two quadratic characters of $W_{\mathbb{R}}$. One is the trivial character $\mathbf{1}$, and the other is the sign character

$$\mathrm{sgn}\colon W_{\mathbb{R}} \to \{\pm 1\}$$

given by $\mathrm{sgn}(j) = -1$ and $\mathrm{sgn}(z) = 1$ for $z \in \mathbb{C}^\times$. Any character of $W_{\mathbb{R}}$ is of the form

$$|\cdot|^\alpha \mathrm{sgn}^\delta$$

for some $\alpha \in \mathbb{C}$ and $\delta \in \{0, 1\}$. The character $|\cdot|\mathrm{sgn}$ implies an isomorphism

$$W_{\mathbb{R}}^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{R}^\times, \quad j \mapsto -1, \ z \mapsto z\overline{z}.$$

Via this isomorphism, we identify $W_{\mathbb{R}}^{\mathrm{ab}}$ with $\mathbb{R}^\times$. In particular, any character of $W_{\mathbb{R}}$ is regarded as a character of $\mathbb{R}^\times$.

For each integer k, we define a 2-dimensional representation

$$\rho_k\colon W_{\mathbb{R}} \to \mathrm{GL}_2(\mathbb{C})$$

by

$$\rho_k(j) = \begin{pmatrix} 0 & (-1)^k \\ 1 & 0 \end{pmatrix}, \quad \rho_k(z) = \begin{pmatrix} \chi_k(z) & 0 \\ 0 & \chi_k(\overline{z}) \end{pmatrix} \quad \text{for } z \in \mathbb{C}^\times.$$

It is the induced representation from the character χ_k of $W_{\mathbb{C}}$. Note that $\rho_k \cong \rho_{-k}$, $\rho_0 \cong \mathbf{1} \oplus \mathrm{sgn}$, and that ρ_k is irreducible when $k \neq 0$. Moreover, ρ_k is orthogonal (resp. symplectic) if k is

even (resp. k is odd). Any irreducible representation of $W_{\mathbb{R}}$ is a character or a 2-dimensional representation of the form
$$|\cdot|^{\alpha}\rho_k$$
for some $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}$ with $k > 0$.

By abuse of notation, for $k \in \mathbb{Z}$, we denote by ρ_k the irreducible (limit of) discrete series representation of $\mathrm{GL}_2(F)$ with minimal $\mathrm{O}(2)$-type $\pm(|k|+1)$.

7.2. **The case of** GL_N**.** Let $F = \mathbb{R}$ or $F = \mathbb{C}$. A homomorphism
$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$$
is a **representation** of $W_F \times \mathrm{SL}_2(\mathbb{C})$ if

- $\psi|W_{\mathbb{C}}$ is continuous;
- $\psi|\mathrm{SL}_2(\mathbb{C})$ is algebraic.

An A**-parameter** for $\mathrm{GL}_N(F)$ is a representation $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ such that $\psi(W_F)$ is bounded.

For an A-parameter ψ for $\mathrm{GL}_N(F)$, one can associate an irreducible unitary representation τ_ψ of $\mathrm{GL}_N(F)$ as follows: When ψ is irreducible, ψ is of the form $\psi = \chi \boxtimes S_d$ for some a character χ of W_F, or $\psi = \rho_k \boxtimes S_d$ for some $k > 0$ (with $F = \mathbb{R}$). When $\psi = \chi \boxtimes S_d$ so that $N = d$, we set $\tau_\psi = \chi \circ \det_d$. When $F = \mathbb{R}$ and $\psi = \rho_k \boxtimes S_d$ so that $N = 2d$, we set τ_ψ to be the unique irreducible subrepresentation of
$$\rho_k|\cdot|^{\frac{d-1}{2}} \times \rho_k|\cdot|^{\frac{d-3}{2}} \times \cdots \times \rho_k|\cdot|^{-\frac{d-1}{2}}.$$
In general, ψ can be decomposed into a direct sum
$$\psi = \psi_1 \oplus \cdots \oplus \psi_r,$$
where $\psi_1, \dots, \psi_r$ are irreducible representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$. Then we set
$$\tau_\psi = \tau_{\psi_1} \times \cdots \times \tau_{\psi_r},$$
which is irreducible.

7.3. **The case of** SO_{2n+1} **and** Sp_{2n}**.** We denote by SO_{2n+1} the special orthogonal group with respect to a quadratic space over F of dimension $2n+1$. We do not assume here that SO_{2n+1} is (quasi-)split over F. Namely, when $F = \mathbb{R}$, the group $\mathrm{SO}_{2n+1}(\mathbb{R})$ of $\mathbb{R}$-points is isomorphic to $\mathrm{SO}(p,q)$ for some (p,q) with $p+q = 2n+1$.

An A**-parameter** for SO_{2n+1} is a symplectic representation
$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$$
such that $\psi(W_F)$ is bounded. Similarly, an A**-parameter** for Sp_{2n} is an orthogonal representation
$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$$
such that $\psi(W_F)$ is bounded. For $G = \mathrm{SO}_{2n+1}$ or $G = \mathrm{Sp}_{2n}$, we set $\Psi(G)$ to be the set of equivalence classes of A-parameters for G. We say that $\psi \in \Psi(G)$ is **tempered** if $\psi|\{1\} \times \mathrm{SL}_2(\mathbb{C})$ is trivial. We denote by $\Phi_{\mathrm{temp}}(G)$ the subset of $\Psi(G)$ consisting of tempered A-parameters.

One can define the **component group** A_ψ of ψ, and the **central element** $z_\psi \in A_\psi$ as in the p-adic case. Then $\mathcal{S}_\psi = \pi_0(Z_{\widehat{G}}(\mathrm{Im}(\psi))/Z(\widehat{G})^{W_F})$ is canonically isomorphic to $A_\psi/\langle z_\psi\rangle$.

As explained in Theorem 0.1 (at least when G is split), for $\psi \in \Psi(G)$, there is an A-packet Π_ψ, which is a finite multiset over $\mathrm{Irr}_{\mathrm{unit}}(G(F))$, together with a map

$$\Pi_\psi \to \widehat{A_\psi}, \ \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

enjoying certain (twisted and standard) endoscopic character identities such that $\langle z_\psi, \pi \rangle_\psi = 1$ for any $\pi \in \Pi_\psi$. The purpose of Part 2 is to review a construction of A-packets when $F = \mathbb{R}$ or $F = \mathbb{C}$.

8. Complex case

In this section, we consider $F = \mathbb{C}$. Let $G = \mathrm{SO}_{2n+1}$ or Sp_{2n} over $\mathbb{C}$. We identify G with $G(\mathbb{C})$. Fix a Borel subgroup B of G, and a maximal compact subgroup K of G. Define

- $T = B \cap K$, a maximal torus in K;
- $H = Z_G(T)$, a Cartan subgroup of G;
- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{b} = \mathrm{Lie}(B)$, $\mathfrak{k}_0 = \mathrm{Lie}(K)$, $\mathfrak{t}_0 = \mathrm{Lie}(T)$, $\mathfrak{h} = \mathrm{Lie}(H)$, the Lie algebras;
- $W = W(\mathfrak{g}, \mathfrak{h})$ the Weyl group;
- $\mathfrak{a}_0 = \sqrt{-1}\mathfrak{t}_0$, and $A = \exp(\mathfrak{a}_0)$.

Then $H = TA$, and (B, H) is a Borel pair of G. Set $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ when $G = \mathrm{SO}_{2n+1}$, and $\widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$ when $G = \mathrm{Sp}_{2n}$. Fix a Borel pair $(\widehat{B}, \widehat{H})$ of $\widehat{G}$. The Lie algebra of $\widehat{G}$ is denoted by $\widehat{\mathfrak{g}}$.

8.1. **Local Langalnds correspondence over $\mathbb{C}$.** Recall that an **L-parameter** is a continuous homomorphism

$$\phi \colon W_\mathbb{C} = \mathbb{C}^\times \to \widehat{G}$$

such that the image consists of semisimple elements. Let $\Phi(G)$ be the set of conjugacy classes of L-parameters. Taking a conjugation if necessary, we may assume that the image of ϕ is contained in $\widehat{H}$. Since ϕ is semisimple, we can decompose it into a direct sum

$$\phi = |\cdot|^{s_1}\chi_{k_1} \oplus \cdots \oplus |\cdot|^{s_N}\chi_{k_N}$$

with $N = 2n$ or $N = 2n+1$. We define $\lambda_\phi, \mu_\phi \in X_*(\widehat{H}) \otimes_\mathbb{Z} \mathbb{C} = X^*(H) \otimes_\mathbb{Z} \mathbb{C} = \mathfrak{t}^* \cong \mathbb{C}^N$ by

$$\lambda_\phi = \left(s_1 + \frac{k_1}{2}, \dots, s_N + \frac{k_N}{2}\right), \quad \mu_\phi = \left(s_1 - \frac{k_1}{2}, \dots, s_N - \frac{k_N}{2}\right).$$

This satisfies that $\lambda_\phi - \mu_\phi = (k_1, \dots, k_N) \in X^*(H)$. The map $\phi \mapsto (\lambda_\phi, \mu_\phi)$ gives a canonical bijection

$$\Phi(G) \to \{(\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}^* \mid \lambda - \mu \in X^*(H)\}/\Delta W.$$

For $\lambda, \mu \in \mathfrak{h}^*$ such that $\lambda - \mu \in X^*(H)$, take $\phi \colon \mathbb{C}^\times \to \widehat{H}$ such that $\lambda = \lambda_\phi$ and $\mu = \mu_\phi$. Then ϕ can be regarded as a character of H. We set $X(\lambda, \mu)$ to be the K-finite part of the normalized induction $\mathrm{Ind}_B^G(\phi)$. We call $X(\lambda, \mu)$ the **principal series representation with parameter** (λ, μ). Define $\overline{X}(\lambda, \mu)$ by the unique irreducible subquotient of $X(\lambda, \mu)$ containing the K-representation of extremal weight $\lambda - \mu$. We call $\overline{X}(\lambda, \mu)$ the **Langlands subquotient of** $X(\lambda, \mu)$. The $W \times W$-orbit of (λ, μ) is called the **infinitesimal character of** $\overline{X}(\lambda, \mu)$.

Proposition 8.1 (Zhelobenko). *The map* $(\lambda, \mu) \mapsto \overline{X}(\lambda, \mu)$ *gives a bijection*
$$\{(\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}^* \mid \lambda - \mu \in X^*(H)\}/W \to \mathrm{Irr}(G),$$
where $\mathrm{Irr}(G)$ *is the set of equivalence classes of irreducible* $(\mathfrak{g}, K)$*-modules.*

Corollary 8.2 (Local Langlands correspondence over $\mathbb{C}$). *There exists a canonical bijection*
$$\Phi(G) \to \mathrm{Irr}(G), \ \phi \mapsto \overline{X}(\lambda_\phi, \mu_\phi).$$

8.2. **Reduction.** Next, we consider the A-parameters $\psi \in \Psi(G)$. To construct the A-packets Π_ψ, we consider a subset $\Psi_{\mathrm{gp}}(G)$ of $\Psi(G)$. We define that an A-parameter ψ belongs to $\Psi_{\mathrm{gp}}(G)$ if ψ is a sum of irreducible self-dual representations of the same type as ψ. In this case, we say that ψ is **of good parity**.

Since $W_\mathbb{C} = \mathbb{C}^\times$ has no irreducible self-dual representations other than $\mathbf{1}$, any parameter $\psi \in \Psi_{\mathrm{gp}}(G)$ is of the form
$$\psi = \bigoplus_{i=1}^t S_{d_i},$$
where $d_1 \geq \cdots \geq d_t$ are positive even (resp. odd) integers such that $d_1 + \cdots + d_t = 2n$ (resp. $d_1 + \cdots + d_t = 2n+1$) when $G = \mathrm{SO}_{2n+1}$ (resp. when $G = \mathrm{Sp}_{2n}$).

In general, $\psi \in \Psi(G)$ can be decomposed into a direct sum
$$\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee,$$
where ψ_0 is an A-parameter of good parity for a classical group G_0 of the same type as G, and ψ_1 is a sum of irreducible representations of $W_\mathbb{C} \times \mathrm{SL}_2(\mathbb{C})$ which are not the same type as ψ. In this setting, the A-packet Π_ψ can be described by Π_{ψ_0}.

Theorem 8.3 ([MR17, Theorem 6.12]). *Let* $\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee$ *be as above. Then for any* $\pi_0 \in \Pi_{\psi_0}$*, the induced representation*
$$\tau_{\psi_1} \rtimes \pi_0$$
is irreducible, and does not depend on the choice of ψ_1*. The* A*-packet* Π_ψ *is given by*
$$\Pi_\psi = \left\{\tau_{\psi_1} \rtimes \pi_0 \mid \pi_0 \in \Pi_{\psi_0}\right\}.$$

8.3. **The case of good parity.** In this subsection, we assume that $\psi \in \Psi(G)$ is of good parity. In this case, we may regard ψ as an algebraic representation
$$\psi \colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}.$$
By differential, we obtain a map $d\psi \colon \mathfrak{sl}_2(\mathbb{C}) \to \widehat{\mathfrak{g}}$. Taking a conjugation if necessary, we may assume that
$$d\psi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \widehat{\mathfrak{h}} \cong \mathfrak{h}^*.$$
Its W-orbit is denoted by $\lambda_\psi \in \mathfrak{h}^*/W$.

We set $\mathcal{U}_\psi$ to be the $\widehat{G}$-orbit of
$$d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}},$$
which is a special nilpotent orbit. By the duality of Lusztig–Spaltenstein, we obtain a nilpotent orbit $\mathcal{O}_\psi \subset \mathfrak{g}$ from $\mathcal{U}_\psi$. Using the Killing form, we may identify $\mathfrak{g} \cong \mathfrak{g}^*$, and we regard $\mathcal{O}_\psi$ as a nilpotent orbit in $\mathfrak{g}^*$.

Associated to a representation π of G, one can define a subset in the nilpotent cone in $\mathfrak{g}^*$, denoted by $\mathrm{WF}(\pi)$ and called the **wavefront set**. Since we focus on complex groups, when π is irreducible, $\mathrm{WF}(\pi)$ is the closure of a nilpotent orbit.

Definition 8.4 (Barbasch–Vogan [BV85]). *For $\psi \in \Psi_{\mathrm{gp}}(G)$, we define a packet Π_ψ^{BV} by the set of irreducible Harish-Chandra modules π of G with infinitesimal character $(\lambda_\psi, \lambda_\psi)$, and with wavefront set $\mathrm{WF}(\pi) = \overline{\mathcal{O}_\psi}$.*

Let A_ψ be the component group of ψ. There is a quotient $\overline{A}_\psi$ of A_ψ, called **Lusztig's quotient**, such that the packet Π_ψ^{BV} is parametrized by the character group $\widehat{\overline{A}_\psi}$, i.e., there is a bijection

$$\widehat{\overline{A}_\psi} \ni \eta \mapsto \pi_\eta \in \Pi_\psi^{\mathrm{BV}}.$$

Theorem 8.5 ([MR17, Theorem 10.1]). *Let $\psi \in \Psi_{\mathrm{gp}}(G)$. Then we have $\Pi_\psi = \Pi_\psi^{\mathrm{BV}}$. Moreover, the map $\Pi_\psi \ni \pi \to \langle \cdot, \pi \rangle_\psi \in \widehat{A_\psi}$ is given by $\pi_\eta \mapsto \eta$. In particular, this map is injective, and the image of this map is $\widehat{\overline{A}_\psi}$.*

Using a deep result of Barbasch [B89] for his classification of the unitary dual of G, one can describe the A-packet Π_ψ^{BV} for $\psi \in \Psi_{\mathrm{gp}}(G)$ more precisely. In the rest of this section, we explain this description.

8.4. **Barbasch–Vogan packets: Type B_n.** In this subsection, we set $G = \mathrm{SO}_{2n+1}$. Then $\psi \in \Psi(G)$ is of the form $\psi = \oplus_{i=1}^{t'} S_{d_i}$, where d_i is even for any i, and $\sum_{i=1}^{t'} d_i = 2n$. When t' is odd, we set $t = t'$. When t' is even, we set $d_t = 0$ and $t = t' + 1$. We may assume that $d_1 \geq \cdots \geq d_t$.

We define a subalgebra $\mathfrak{m}$ of $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ by

$$\mathfrak{m} = \mathfrak{gl}\left(\frac{d_2 + d_3}{2}, \mathbb{C}\right) \times \cdots \times \mathfrak{gl}\left(\frac{d_{t-1} + d_t}{2}, \mathbb{C}\right) \times \mathfrak{so}_{d_1+1}(\mathbb{C}).$$

Set $k = (t-1)/2$. For each $j \in \{1, \dots, k\}$, let F^j be the irreducible finite dimensional holomorphic representation of $\mathfrak{gl}((d_{2j} + d_{2j+1})/2, \mathbb{C})$ with infinitesimal character

$$\lambda_1^j = \left(\frac{d_{2j} - 1}{2}, \frac{d_{2j} - 3}{2}, \dots, -\frac{d_{2j+1} - 3}{2}, -\frac{d_{2j+1} - 1}{2}\right).$$

Namely, F^j is the 1-dimensional representation given by

$$F^j \colon \mathfrak{gl}\left(\frac{d_{2j} + d_{2j+1}}{2}, \mathbb{C}\right) \ni X \mapsto \frac{d_{2j} - d_{2j+1}}{4} \mathrm{tr}(X) \in \mathbb{C}.$$

When $d_{2j} > d_{2j+1}$, we define $\widetilde{F}^j$ by the irreducible finite dimensional holomorphic representation of $\mathfrak{gl}((d_{2j} + d_{2j+1})/2, \mathbb{C})$ with infinitesimal character

$$\lambda_{-1}^j = \left(\frac{d_{2j} - 1}{2}, \dots, \frac{d_{2j+1} + 3}{2}, \frac{d_{2j+1} - 1}{2}, \dots, -\frac{d_{2j+1} - 1}{2}, -\frac{d_{2j+1} + 1}{2}\right).$$

Namely, λ_{-1}^j is obtained from λ_1^j by the sign change of $(d_{2j+1} + 1)/2$.

For each $j \in \{1, \dots, k\}$, we define a representation $\mathcal{F}_1^j$ of $\mathfrak{gl}((d_{2j} + d_{2j+1})/2, \mathbb{C})$ by

$$\mathcal{F}_1^j(X) = F^j(X) \otimes F^j(\overline{X}) \quad \text{for} \quad X \in \mathfrak{gl}((d_{2j} + d_{2j+1})/2, \mathbb{C}).$$

In addition, when $d_{2j} > d_{2j+1}$, we define a representation $\mathcal{F}^j_{-1}$ of $\mathfrak{gl}((d_{2j}+d_{2j+1})/2, \mathbb{C})$ by

$$\mathcal{F}^j_{-1}(X) = F^j(X) \otimes \widetilde{F}^j(\overline{X}) \quad \text{for} \quad X \in \mathfrak{gl}((d_{2j}+d_{2j+1})/2, \mathbb{C}).$$

Note that for $\epsilon_j \in \{\pm 1\}$ such that $\epsilon_j = 1$ if $d_{2j} = d_{2j+1}$, the representation $\mathcal{F}^j_{\epsilon_j}$ of the Lie algebra $\mathfrak{gl}((d_{2j}+d_{2j+1})/2, \mathbb{C})$ can be lifted to an irreducible representation of the Lie group $\mathrm{GL}((d_{2j}+d_{2j+1})/2, \mathbb{C})$, which is denoted by the same notation $\mathcal{F}^j_{\epsilon_j}$. For example,

$$\mathcal{F}^j_1 \colon \mathrm{GL}((d_{2j}+d_{2j+1})/2, \mathbb{C}) \ni g \mapsto \left(\det(g)\overline{\det(g)}\right)^{\frac{d_{2j}-d_{2j+1}}{4}} \in \mathbb{C}^\times.$$

Let $P = MN$ be a parabolic subgroup of G with $\mathrm{Lie}(M) = \mathfrak{m}$. Then for $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \{\pm 1\}^k$ such that $\epsilon_j = 1$ if $d_{2j} = d_{2j+1}$, we define π_ϵ by the irreducible subquotient of the parabolic induction

$$\mathcal{F}^1_{\epsilon_1} \times \cdots \times \mathcal{F}^k_{\epsilon_k} \rtimes \mathbf{1}_{\mathrm{SO}_{d_1+1}(\mathbb{C})}$$

containing its minimal K-type. If we set

$$\lambda = \left(\lambda^1_1, \dots, \lambda^k_1, \frac{d_1-1}{2}, \dots, \frac{1}{2}\right),$$
$$\mu = \left(\lambda^1_{\epsilon_1}, \dots, \lambda^k_{\epsilon_k}, \frac{d_1-1}{2}, \dots, \frac{1}{2}\right),$$

then

$$\pi_\epsilon \cong \overline{X}(\lambda, \mu).$$

The Barbasch–Vogan packet Π^{BV}_ψ is given by

$$\Pi^{\mathrm{BV}}_\psi = \left\{ \pi_\epsilon \;\middle|\; \epsilon \in \{\pm 1\}^k,\ \epsilon_j = 1 \text{ if } d_{2j} = d_{2j+1} \right\}.$$

In particular, $\#\Pi^{\mathrm{BV}}_\psi = 2^m$ with $m = \#\{j \in \{1, \dots, k\} \mid d_{2j} \neq d_{2j+1}\}$.

8.5. **Barbasch–Vogan packets: Type C_n.** In this subsection, we set $G = \mathrm{Sp}_{2n}$. Then $\psi \in \Psi(G)$ is of the form $\psi = \oplus_{i=1}^t S_{d_i}$, where d_i is odd for any i, and $\sum_{i=1}^t d_i = 2n+1$. In particular, we note that t is odd. We may assume that $d_1 \geq \cdots \geq d_t$.

We define a subalgebra $\mathfrak{m}$ of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ by

$$\mathfrak{m} = \mathfrak{gl}\left(\frac{d_1+d_2}{2}, \mathbb{C}\right) \times \cdots \times \mathfrak{gl}\left(\frac{d_{t-2}+d_{t-1}}{2}, \mathbb{C}\right) \times \mathfrak{sp}_{d_t-1}(\mathbb{C}).$$

Set $k = (t-1)/2$. For each $j \in \{1, \dots, k\}$, let F^j be the irreducible finite dimensional holomorphic representation of $\mathfrak{gl}((d_{2j-1}+d_{2j})/2, \mathbb{C})$ with infinitesimal character

$$\lambda^j_1 = \left(\frac{d_{2j-1}-1}{2}, \frac{d_{2j-1}-3}{2}, \dots, -\frac{d_{2j}-3}{2}, -\frac{d_{2j}-1}{2}\right).$$

Namely, F^j is the 1-dimensional representation given by

$$F^j \colon \mathfrak{gl}\left(\frac{d_{2j-1}+d_{2j}}{2}, \mathbb{C}\right) \ni X \mapsto \frac{d_{2j-1}-d_{2j}}{4}\mathrm{tr}(X) \in \mathbb{C}.$$

When $d_{2j-1} > d_{2j}$, we define $\widetilde{F}^j$ by the irreducible finite dimensional holomorphic representation of $\mathfrak{gl}((d_{2j-1}+d_{2j})/2, \mathbb{C})$ with infinitesimal character

$$\lambda^j_{-1} = \left(\frac{d_{2j-1}-1}{2}, \dots, \frac{d_{2j}+3}{2}, \frac{d_{2j}-1}{2}, \dots, -\frac{d_{2j}-1}{2}, -\frac{d_{2j}+1}{2}\right).$$

Namely, λ^j_{-1} is obtained from λ^j_1 by the sign change of $(d_{2j}+1)/2$.

For each $j \in \{1, \dots, k\}$, we define a representation $\mathcal{F}^j_1$ of $\mathfrak{gl}((d_{2j-1}+d_{2j})/2, \mathbb{C})$ by

$$\mathcal{F}^j_1(X) = F^j(X) \otimes F^j(\overline{X}) \quad \text{for} \quad X \in \mathfrak{gl}((d_{2j-1}+d_{2j})/2, \mathbb{C}).$$

In addition, when $d_{2j-1} > d_{2j}$, we define a representation $\mathcal{F}^j_{-1}$ of $\mathfrak{gl}((d_{2j-1}+d_{2j})/2, \mathbb{C})$ by

$$\mathcal{F}^j_{-1}(X) = F^j(X) \otimes \widetilde{F}^j(\overline{X}) \quad \text{for} \quad X \in \mathfrak{gl}((d_{2j-1}+d_{2j})/2, \mathbb{C}).$$

Note that for $\epsilon_j \in \{\pm 1\}$ such that $\epsilon_j = 1$ if $d_{2j-1} = d_{2j}$, the representation $\mathcal{F}^j_{\epsilon_j}$ of the Lie algebra $\mathfrak{gl}((d_{2j-1}+d_{2j})/2, \mathbb{C})$ can be lifted an irreducible representation of the Lie group $\mathrm{GL}((d_{2j-1}+d_{2j})/2, \mathbb{C})$, which is denoted by the same notation $\mathcal{F}^j_{\epsilon_j}$. For example,

$$\mathcal{F}^j_1 \colon \mathrm{GL}((d_{2j-1}+d_{2j})/2, \mathbb{C}) \ni g \mapsto \left(\det(g)\overline{\det(g)}\right)^{\frac{d_{2j-1}-d_{2j}}{4}} \in \mathbb{C}^\times.$$

Let $P = MN$ be a parabolic subgroup of G with $\mathrm{Lie}(M) = \mathfrak{m}$. Then for $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \{\pm 1\}^k$ such that $\epsilon_j = 1$ if $d_{2j-1} = d_{2j}$, we define π_ϵ by the irreducible subquotient of the parabolic induction

$$\mathcal{F}^1_{\epsilon_1} \times \cdots \times \mathcal{F}^k_{\epsilon_k} \rtimes \mathbf{1}_{\mathrm{Sp}_{d_t-1}(\mathbb{C})}$$

containing its minimal K-type. If we set

$$\lambda = \left(\lambda^1_1, \dots, \lambda^k_1, \frac{d_t-1}{2}, \dots, 1\right),$$
$$\mu = \left(\lambda^1_{\epsilon_1}, \dots, \lambda^k_{\epsilon_k}, \frac{d_t-1}{2}, \dots, 1\right),$$

then

$$\pi_\epsilon \cong \overline{X}(\lambda, \mu).$$

The Barbasch–Vogan packet Π^{BV}_ψ is given by

$$\Pi^{\mathrm{BV}}_\psi = \left\{\pi_\epsilon \;\middle|\; \epsilon \in \{\pm 1\}^k,\ \epsilon_j = 1 \text{ if } d_{2j-1} = d_{2j}\right\}.$$

In particular, $\#\Pi^{\mathrm{BV}}_\psi = 2^m$ with $m = \#\{j \in \{1, \dots, k\} \mid d_{2j-1} \neq d_{2j}\}$.

9. Real case

In this section, we consider $F = \mathbb{R}$. Let $G = \mathrm{SO}_{2n+1}$ or Sp_{2n} over $\mathbb{R}$.

9.1. **Filtration of A-parameters.** For $\psi \in \Psi(G)$, we define a unitary representation ψ_d of $W_{\mathbb{C}} = \mathbb{C}^\times$ by

$$\psi_d(z) = \psi\left(z, \begin{pmatrix} (z/\overline{z})^{\frac{1}{2}} & 0 \\ 0 & (z/\overline{z})^{-\frac{1}{2}} \end{pmatrix}\right) \quad \text{for } z \in \mathbb{C}^\times.$$

Definition 9.1. *We define a chain*

$$\Psi(G) \supset \Psi_{\mathrm{gp}}(G) \supset \Psi_{\mathrm{vreg}}(G) \supset \Psi_{\mathrm{unip}}(G)$$

as follows:

(1) $\psi \in \Psi_{\mathrm{gp}}(G)$ *if* ψ *is a sum of irreducible self-dual representations of the same type as* ψ*. In this case, we say that* ψ *is* ***of good parity****.*

(2) $\psi \in \Psi_{\mathrm{vreg}}(G)$ *if* ψ *is of good parity and* ψ *is of the form*

$$\psi = \left(\bigoplus_{i=1}^{r} \rho_{k_i} \boxtimes S_{d_i}\right) \oplus \left(\bigoplus_{j \in J} \mathrm{sgn}^{\delta_j} \boxtimes S_{d'_j}\right)$$

such that $k_i - k_{i+1} \geq d_i + d_{i+1}$ *for* $1 \leq i < r$ *and* $k_r \geq d_r + \max_{j \in J} d'_j$*. In this case, we say that* ψ *is* ***very regular****.*

(3) $\psi \in \Psi_{\mathrm{unip}}(G)$ *if* ψ *is of good parity and* $\psi | W_{\mathbb{C}}$ *is trivial, i.e.,* ψ *is very regular with* $r = 0$*. In this case, we say that* ψ *is* ***unipotent****.*

We also define $\Psi_{\mathrm{AJ}}(G) \subset \Psi_{\mathrm{vreg}}(G)$ *so that:*

(4) $\psi \in \Psi_{\mathrm{AJ}}(G)$ *if* ψ *is of good parity and* ψ_d *is multiplicity-free, i.e.,* ψ *is very regular with* $\#J \leq 1$*. In this case, we say that* ψ *is* ***Adams–Johnson****.*

For $\psi \in \Psi_{\mathrm{AJ}}(G)$, Adams–Johnson [AJ87] constructed a packet Π_ψ^{AJ} using derived functor modules $A_{\mathfrak{q}}(\lambda)$ for λ in the good range. It is called an **Adams–Johnson packet**. Later, Arancibia–Mœglin–Renard [AMR] showed that $\Pi_\psi = \Pi_\psi^{\mathrm{AJ}}$, i.e., Arthur's packets are Adams–Johnson packets when $\psi \in \Psi_{\mathrm{AJ}}(G)$.

On the other hand, Mœglin [Mœ17] constructed a packet Π_ψ for $\psi \in \Psi_{\mathrm{unip}}(G)$ using theta liftings. It is called a **unipotent packet**. After these works, Mœglin–Renard [MRa] constructed a packet Π_ψ for $\psi \in \Psi_{\mathrm{vreg}}(G)$ from unipotent packets using cohomological inductions. They showed that this A-packet is multiplicity-free.

To extend the A-packet to $\psi \in \Psi_{\mathrm{gp}}(G)$, Mœglin–Renard [MRb] used the translation principle. However, since this translation must be used while crossing the walls, the translation functor is difficult to understand, and the multiplicity-free result cannot easily be deduced from the case where ψ is very regular.

Finally, for general ψ, the packet Π_ψ is constructed by irreducible parabolic inductions ([MRa, Proposition 4.3, Théorème 4.4]). In particular, the multiplicity-free result for general ψ is reduced to the case where ψ is of good parity (([MRa, Corollaire 4.5])).

In the next subsection, we explain the construction of Π_ψ only for $\psi \in \Psi_{\mathrm{AJ}}(G)$. For other cases, see the relevant papers.

9.2. **Adams–Johnson packets.** Note that a representation ψ of $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ is in $\Psi_{\mathrm{AJ}}(\mathrm{SO}_{2n+1})$ if and only if

$$\psi = \left(\bigoplus_{i=1}^{r} \rho_{k_i} \boxtimes S_{d_i}\right) \oplus \mathrm{sgn}^{\delta} \boxtimes S_{d_0},$$

where

- $k_i > 0$ and $d_i > 0$ for $1 \leq i \leq r$;
- $k_i + d_i \equiv 0 \bmod 2$ for $1 \leq i \leq r$ and $d_0 \equiv 0 \bmod 2$;
- $2\sum_{i=1}^{r} d_i + d_0 = 2n$;
- $\delta \in \{0, 1\}$;
- $k_i - k_{i+1} \geq d_i + d_{i+1}$ for $1 \leq i < r$ and $k_r \geq d_r + d_0$,

and is in $\Psi_{\mathrm{AJ}}(\mathrm{Sp}_{2n})$ if and only if

$$\psi = \left(\bigoplus_{i=1}^{r} \rho_{k_i} \boxtimes S_{d_i} \right) \oplus \mathrm{sgn}^{\delta} \boxtimes S_{d_0},$$

where

- $k_i > 0$ and $d_i > 0$ for $1 \leq i \leq r$;
- $k_i + d_i \equiv 1 \bmod 2$ for $1 \leq i \leq r$ and $d_0 \equiv 1 \bmod 2$;
- $2\sum_{i=1}^{r} d_i + d_0 = 2n + 1$;
- $\delta \in \{0, 1\}$ such that $\delta \equiv \sum_{i=1}^{r} d_i \bmod 2$;
- $k_i - k_{i+1} \geq d_i + d_{i+1}$ for $1 \leq i < r$, and $k_r \geq d_r + d_0$.

In this subsection, we fix such ψ.

We use the following coordinates for $\mathrm{SO}(p, q)$ and $\mathrm{Sp}_{2n}(\mathbb{R})$:

$$\mathrm{SO}(p,q) = \left\{ g \in \mathrm{SL}_{p+q}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} g = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \right\},$$
$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \right\}.$$

We choose a maximal torus T of G defined over $\mathbb{R}$ such that $T(\mathbb{R})$ is compact as follows. When $G(\mathbb{R}) = \mathrm{SO}(p, q)$, setting

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

the group $T(\mathbb{R})$ consists of the matrices of the form

$$\left(\begin{array}{cccc|ccc} r_{\theta_1} & & & & & & \\ & \ddots & & & & & \\ & & r_{\theta_{(p-1)/2}} & & & & \\ & & & 1 & & & \\ \hline & & & & r_{\theta'_1} & & \\ & & & & & \ddots & \\ & & & & & & r_{\theta'_{q/2}} \end{array} \right) \quad \text{or} \quad \left(\begin{array}{ccc|cccc} r_{\theta_1} & & & & & & \\ & \ddots & & & & & \\ & & r_{\theta_{p/2}} & & & & \\ \hline & & & r_{\theta'_1} & & & \\ & & & & \ddots & & \\ & & & & & r_{\theta'_{(q-1)/2}} & \\ & & & & & & 1 \end{array} \right)$$

for $\theta_i, \theta'_j \in \mathbb{R}$ according to $p \not\equiv q \equiv 0 \bmod 2$ or $p \not\equiv q \equiv 1 \bmod 2$. When $G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R})$, we set

$$T(\mathbb{R}) = \left\{ \left(\begin{array}{ccc|ccc} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ \hline -b_1 & & & a_1 & & \\ & \ddots & & & \ddots & \\ & & -b_n & & & a_n \end{array} \right) \middle| \begin{array}{c} a_i, b_i \in \mathbb{R}, \\ a_i^2 + b_i^2 = 1 \end{array} \right\}.$$

Let θ be the Cartan involution given by

$$\theta \colon g \mapsto t_0 g t_0^{-1}$$

with

$$t_0 = \begin{cases} \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} & \text{if } G(\mathbb{R}) = \mathrm{SO}(p,q), \\ \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}). \end{cases}$$

Note that $t_0 \in T(\mathbb{R})$ such that $t_0^2 \in Z(G(\mathbb{R}))$. Take a θ-stable Borel subgroup B of G containing T.

Let Σ_ψ be the set of matrices

$$\begin{pmatrix} p_1 & q_1 \\ \vdots & \vdots \\ p_r & q_r \end{pmatrix}$$

such that

- p_i and q_i are non-negative integers for $i = 1, \dots, r$;
- $p_i + q_i = d_i$ for $i = 1, \dots, r$;
- $2\sum_{i=1}^r p_i \leq p$ and $2\sum_{i=1}^r q_i \leq q$ if $G(\mathbb{R}) = \mathrm{SO}(p,q)$ with $p + q = 2n+1$.

Note that there exists a canonical bijection

$$\Sigma_\psi \cong \begin{cases} \mathfrak{S}_{d_0/2} \times \left(\prod_{i=1}^r \mathfrak{S}_{d_i} \right) \Big\backslash \mathfrak{S}_n \Big/ \mathfrak{S}_{[p/2]} \times \mathfrak{S}_{[q/2]} & \text{if } G(\mathbb{R}) = \mathrm{SO}(p,q), \\ \prod_{i=1}^r \mathcal{P}_2(d_i) & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}), \end{cases}$$

where $[x]$ denotes the greatest integer which is not larger than x, and $\mathcal{P}_2(d_i)$ is the set of pairs of integers (p_i, q_i) with $p_i, q_i \geq 0$ such that $p_i + q_i = d_i$.

For $w = (p_i \ q_i)_i \in \Sigma_\psi$, we take a θ-stable parabolic subgroup $Q_w = L_w U_w$ of G containing B such that the Levi L_w is defined over $\mathbb{R}$, and its $\mathbb{R}$-points $L_w(\mathbb{R})$ is given as follows: Define

$\iota\colon \mathrm{M}_{p,q}(\mathbb{C}) \to \mathrm{M}_{2p,2q}(\mathbb{R})$ by

$$\iota\left(\begin{pmatrix} x_{1,1} & \dots & x_{1,q} \\ \vdots & \ddots & \vdots \\ x_{p,1} & \dots & x_{p,q} \end{pmatrix} + \sqrt{-1}\begin{pmatrix} y_{1,1} & \dots & y_{1,q} \\ \vdots & \ddots & \vdots \\ y_{p,1} & \dots & y_{p,q} \end{pmatrix}\right) = \left(\begin{array}{cc|c|cc} x_{1,1} & y_{1,1} & \dots & x_{1,q} & y_{1,q} \\ -y_{1,1} & x_{1,1} & \dots & -y_{1,q} & x_{1,q} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline x_{p,1} & y_{p,1} & \dots & x_{p,q} & y_{p,q} \\ -y_{p,1} & x_{p,1} & \dots & -y_{p,q} & x_{p,q} \end{array}\right)$$

for $x_{i,j}, y_{i,j} \in \mathbb{R}$. We put

$$\eta = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \in \mathrm{O}(2q).$$

Note that $\eta\iota(d)\eta^{-1} = \iota(\overline{d})$ for $d \in \mathrm{M}_q(\mathbb{C})$. When $G(\mathbb{R}) = \mathrm{SO}(p,q)$, setting $p_0 = p - 2\sum_{i=1}^r p_i$ and $q_0 = q - 2\sum_{i=1}^r q_i$ so that $p_0 + q_0 = d_0 + 1$, the group $L_w(\mathbb{R})$ consists of the matrices of the form

$$h = \left(\begin{array}{ccc|c|ccc|c} \iota(a_1) & & & & \iota(b_1)\eta^{-1} & & & \\ & \ddots & & & & \ddots & & \\ & & \iota(a_r) & & & & \iota(b_r)\eta^{-1} & \\ \hline & & & A & & & & B \\ \hline \eta\iota(c_1) & & & & \eta\iota(d_1)\eta^{-1} & & & \\ & \ddots & & & & \ddots & & \\ & & \eta\iota(c_r) & & & & \eta\iota(d_r)\eta^{-1} & \\ \hline & & & C & & & & D \end{array}\right)$$

for $a_i \in \mathrm{M}_{p_i,p_i}(\mathbb{C})$, $b_i \in \mathrm{M}_{p_i,q_i}(\mathbb{C})$, $c_i \in \mathrm{M}_{q_i,p_i}(\mathbb{C})$, $d_i \in \mathrm{M}_{q_i,q_i}(\mathbb{C})$ such that $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{U}(p_i,q_i)$ for $i = 1, \dots, r$, and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(p_0,q_0)$. When $G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R})$, the group $L_w(\mathbb{R})$ consists of the matrices of the form

$$h = \left(\begin{array}{cc|c|cc|c|cc|c|cc|c} a_1 & b_1 & & & & & a_1' & -b_1' & & & & \\ c_1 & d_1 & & & & & c_1' & -d_1' & & & & \\ \hline & & \ddots & & & & & & \ddots & & & \\ \hline & & & a_r & b_r & & & & & a_r' & -b_r' & \\ & & & c_r & d_r & & & & & c_r' & -d_r' & \\ \hline & & & & & A & & & & & & B \\ \hline -a_1' & -b_1' & & & & & a_1 & -b_1 & & & & \\ c_1' & d_1' & & & & & -c_1 & d_1 & & & & \\ \hline & & \ddots & & & & & & \ddots & & & \\ \hline & & & -a_r' & -b_r' & & & & & a_r & -b_r & \\ & & & c_r' & d_r' & & & & & -c_r & d_r & \\ \hline & & & & & C & & & & & & D \end{array}\right),$$

where $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} + \sqrt{-1}\begin{pmatrix} a'_i & b'_i \\ c'_i & d'_i \end{pmatrix} \in \mathrm{U}(p_i, q_i)$ for $i = 1, \dots, r$, and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{d_0-1}(\mathbb{R})$. Note that L_w contains T, and

$$L_w(\mathbb{R}) \cong \mathrm{U}(p_1, q_1) \times \cdots \times \mathrm{U}(p_r, q_r) \times \begin{cases} \mathrm{SO}(p_0, q_0) & \text{if } G(\mathbb{R}) = \mathrm{SO}(p, q), \\ \mathrm{Sp}_{d_0-1}(\mathbb{C}) & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}). \end{cases}$$

Set

$$\lambda^j = \left(\frac{k_j + d_j - 1}{2}, \frac{k_j + d_j - 3}{2}, \dots \frac{k_j - d_j + 1}{2} \right) \in \left(\frac{1}{2}\mathbb{Z}\right)^{d_j}$$

and

$$\lambda_\psi = \begin{cases} \left(\lambda^1, \dots, \lambda^r, \frac{d_0 - 1}{2}, \frac{d_0 - 3}{2}, \dots, \frac{1}{2} \right) & \text{if } G = \mathrm{SO}_{2n+1}, \\ \left(\lambda^1, \dots, \lambda^r, \frac{d_0 - 1}{2}, \frac{d_0 - 3}{2}, \dots, 1 \right) & \text{if } G = \mathrm{Sp}_{2n}. \end{cases}$$

Let ρ be the half sum of positive roots of T with respect to B, so that

$$\rho = \begin{cases} \left(n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2} \right) & \text{if } G = \mathrm{SO}_{2n+1}, \\ (n, n-1, \dots, 1) & \text{if } G = \mathrm{Sp}_{2n}. \end{cases}$$

Fix $w = (p_i\ q_i)_i \in \Sigma_\psi$. Define a unitary character $\chi_{\lambda_\psi} \colon L_w(\mathbb{R}) \to \mathbb{C}^\times$ as follows: If $h \in L_w(\mathbb{R})$ is of the above form, we set

$$\chi_{\lambda_\psi}(h) = \begin{cases} \mathrm{spin}^\delta \begin{pmatrix} A & B \\ C & D \end{pmatrix} \prod_{j=1}^r \det \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}^{\lambda_1^j - n + \frac{1}{2} + (d_1 + \cdots + d_{j-1})} & \text{if } G(\mathbb{R}) = \mathrm{SO}(p, q), \\ \prod_{j=1}^r \det \left(\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} + \sqrt{-1} \begin{pmatrix} a'_j & b'_j \\ c'_j & d'_j \end{pmatrix} \right)^{\lambda_1^j - n + (d_1 + \cdots + d_{j-1})} & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}), \end{cases}$$

where $\mathrm{spin} \colon \mathrm{SO}(p_0, q_0) \to \{\pm 1\}$ is the spinor character. When $p_0 q_0 = 0$, we interpret spin to be the trivial character. The restriction of χ_{λ_ψ} to $T(\mathbb{R})$ is equal to $\lambda_\psi - \rho \in X^*(T) \cong \mathbb{Z}^n$.

For each $w \in \Sigma_\psi$, we define

$$\pi_w = A_{Q_w}(\chi_{\lambda_\psi})$$

to be the derived functor modules. Since the character χ_{λ_ψ} is in the good range, i.e., $\mathrm{Re}\,\langle \lambda_\psi, \alpha \rangle > 0$ for any simple root α of T appearing $\mathrm{Lie}(U_w)$, it is nonzero and irreducible with infinitesimal character λ_ψ. Moreover, $\pi_w \not\cong \pi_{w'}$ if $w \neq w'$.

Definition 9.2 (Adams–Johnson [AJ87])**.** *For $\psi \in \Psi_{\mathrm{AJ}}(G_n)$, we define a packet Π_ψ^{AJ} by*

$$\Pi_\psi^{\mathrm{AJ}} = \{ \pi_w \mid w \in \Sigma_\psi \}.$$

Arancibia–Mœglin–Renard identified Adams–Johnson packets with Arthur's one.

Theorem 9.3 ([AMR])**.** *For $\psi \in \Psi_{\mathrm{AJ}}(G)$, we have*

$$\Pi_\psi = \Pi_\psi^{\mathrm{AJ}}.$$

The map $\Pi_\psi \ni \pi \mapsto \langle\cdot,\pi\rangle_\psi \in \widehat{A_\psi}$ is determined by

$$\langle z_\psi,\pi_w\rangle_\psi = \begin{cases} (-1)^{[\frac{p-q}{2}]} & \text{if } G(\mathbb{R}) = \mathrm{SO}(p,q),\\ 1 & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}) \end{cases}$$

and

$$\langle \alpha_i,\pi_w\rangle_\psi = (-1)^{\frac{p_i-q_i-\delta_i}{2}}$$

for $w = (p_i\ q_i)_i \in \Sigma_\psi$ and $\alpha_i \in A_\psi$ associated to $\rho_{k_i} \boxtimes S_{d_i}$, where

$$\delta_i = \begin{cases} 0 & \text{if } d_i \equiv 0 \bmod 2,\\ (-1)^{\sum_{j=1}^{i-1} d_j} & \text{if } d_i \equiv 1 \bmod 2. \end{cases}$$

Remark 9.4. *We remark that*

$$\sum_{\substack{p+q=2n+1\\ p \text{ is odd}}} \# \left(\mathfrak{S}_{\frac{d_0}{2}} \times \left(\prod_{i=1}^r \mathfrak{S}_{d_i}\right) \backslash \mathfrak{S}_n / \mathfrak{S}_{\frac{p-1}{2}} \times \mathfrak{S}_{\frac{q}{2}}\right) = \left(\frac{d_0}{2}+1\right)\prod_{i=1}^r (d_i+1),$$

$$\# \prod_{i=1}^r \mathcal{P}_2(d_i) = \prod_{i=1}^r (d_i+1).$$

One might regard these computations as an analogue to Lemma 4.2. Hence A-parameters with DDR may be regarded as a p-adic analogue of "Adams–Johnson" parameters.

As an example, we consider $\psi = \phi \in \Psi_{\mathrm{AJ}}(G) \cap \Phi(G)$, i.e., the case where $d_1 = \cdots = d_r = 1$ and $d_0 \in \{0,1\}$ so that $r = n$. In this case, for any $w = (p_i\ q_i)_i \in \Sigma_\psi$, we have $(p_i,q_i) \in \{(1,0),(0,1)\}$. Moreover, the Levi L_w is just the fixed maximal torus T. The irreducible representation π_w is the discrete series representation of Harish-Chandra parameter

$$\begin{cases} (\lambda_{i_1},\dots,\lambda_{i_{p_0}};\lambda_{j_1},\dots,\lambda_{j_{q_0}}) & \text{if } G(\mathbb{R}) = \mathrm{SO}(p,q),\\ ((p_1-q_1)\lambda_1,\dots,(p_n-q_n)\lambda_n) & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}), \end{cases}$$

where we write $\lambda_\psi = (\lambda_1,\dots,\lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$, and we define $i_1 < \cdots < i_{p_0}$ and $j_1 < \cdots < j_{q_0}$ so that $\{i_1,\dots,i_{p_0}\} = \{i \in \{1,\dots,n\} \mid p_i = 1\}$ and $\{j_1,\dots,j_{q_0}\} = \{j \in \{1,\dots,n\} \mid q_j = 1\}$. Hence $p_0 = [p/2]$ and $q_0 = [q/2]$. The pairing $\langle\cdot,\pi_w\rangle_\phi$ is given by

$$\langle \alpha_i,\pi_w\rangle_\phi = (-1)^{i-1}(p_i-q_i)$$

for $i = 1,\dots,n$.

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