CONSTRUCTION OF p-ADIC ADAMS–BARBASCH–VOGAN PACKETS

HIRAKU ATOBE

Abstract. This note is a proceeding of the IMS program "On the Langlands Program: Endoscopy and Beyond" held in National University of Singapore from 17 Dec. 2018 to 18 Jan. 2019. The purpose is to explain p-adic Adams–Barbasch–Vogan packets constructed in [CFMMX].

Contents

1. Overview

In this section, we recall Arthur's results and we explain a motivation for this lecture note.

1.1. L-parameters and A-parameters.

Let F be a non-archimedean local field of characteristic zero. We denote the Weil group of F by W_F. We consider a connected reductive split algebraic group G over F as in the following cases:

- (B_n): $G = \mathrm{SO}_{2n+1}$ is the split special orthogonal group of size $2n+1$.
- (C_n): $G = \mathrm{Sp}_{2n}$ is the split symplectic group of size $2n$.

We denote by $\Pi(G(F))$ the set of equivalence classes of irreducible smooth representations of $G(F)$, and by $\Pi_{\mathrm{temp}}(G(F))$ its subset consisting of equivalence classes of tempered representations.

Let $\widehat{G}$ be the complex dual group of G, i.e., $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ in the case (B_n); and $\widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$ in the case (C_n).

Definition 1.1. *Let G be in the case (B_n) or (C_n).*

1. *An **infinitesimal character** for G is a homomorphism $\lambda\colon W_F \to \widehat{G}$ such that*
   (a) *λ is smooth, i.e., λ has an open kernel;*
   (b) *the image of λ consists of semisimple elements in $\widehat{G}$.*
2. *An **L-parameter** for G is a homomorphism $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ such that*
   (a) *$\phi|_{W_F \times \{\mathbf{1}\}}\colon W_F \to \widehat{G}$ is an infinitesimal character for G;*

(b) $\phi|_{\{\mathbf{1}\}\times\mathrm{SL}_2(\mathbb{C})}\colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ *is algebraic.*

(3) *An* ***A-parameter*** *for G is a homomorphism $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ such that*
   (a) $\psi|_{W_F\times\{\mathbf{1}\}\times\{\mathbf{1}\}}\colon W_F \to \widehat{G}$ *is an infinitesimal character for G;*
   (b) $\psi(W_F)$ *is bounded;*
   (c) $\psi|_{\{\mathbf{1}\}\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})}\colon \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ *is algebraic.*

Two infinitesimal characters (resp. L-parameters, A-parameters) for G are equivalent if they are conjugate under $\widehat{G}$. The set of equivalence classes of infinitesimal characters (resp. L-parameters, A-parameters) for G is denoted by $\Lambda(G/F)$ (resp. $\Phi(G/F)$, $\Psi(G/F)$).

We call an L-parameter ϕ (resp. an A-parameter ψ) tempered if $\phi(W_F) \subset \widehat{G}$ is bounded (resp. $\psi|_{\{\mathbf{1}\}\times\{\mathbf{1}\}\times\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$). Namely, the notion of tempered L-parameters are the same as the one of tempered A-parameters. We denote by $\Phi_{\mathrm{temp}}(G/F)$ (resp. $\Psi_{\mathrm{temp}}(G/F)$) the subset of $\Phi(G/F)$ (resp. $\Psi(G/F)$) consisting of tempered L-parameters (resp. tempered A-parameters) for G. Namely,

$$\Phi(G/F) \supset \Phi_{\mathrm{temp}}(G/F) = \Psi_{\mathrm{temp}}(G/F) \subset \Psi(G/F).$$

For $\lambda \in \Lambda(G/F)$, $\phi \in \Phi(G/F)$, and $\psi \in \Psi(G/F)$, we define their component groups by

$$A_\lambda = Z_{\widehat{G}}(\mathrm{Im}(\lambda))/Z_{\widehat{G}}(\mathrm{Im}(\lambda))^0,$$
$$A_\phi = Z_{\widehat{G}}(\mathrm{Im}(\phi))/Z_{\widehat{G}}(\mathrm{Im}(\phi))^0,$$
$$A_\psi = Z_{\widehat{G}}(\mathrm{Im}(\psi))/Z_{\widehat{G}}(\mathrm{Im}(\psi))^0,$$

respectively. They are elementary 2-groups. When $G = \mathrm{SO}_{2n+1}$, i,e, in the case (B_n), the image of $-\mathbf{1} \in \widehat{G}$ in A_λ (resp. A_ϕ, A_ψ) is also denoted by $-\mathbf{1}$.

Let I_F be the inertia subgroup of W_F, and $\mathrm{Frob}_F \in W_F$ be an arithmetic Frobenius element, i.e., a lift of $[x \mapsto x^q] \in \mathrm{Gal}(\overline{k}_F/k_F)$, where k_F is the residue field of F and q is its cardinality. We normalize the norm map $|\cdot|\colon W_F \to \mathbb{R}^\times$ so that $|\mathrm{Frob}_F| = q$. For $w \in W_F$, we put

$$d_w = \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

For $\psi \in \Psi(G/F)$ and $\phi \in \Phi(G/F)$, we define $\phi_\psi \in \Phi(G/F)$, $\lambda_\psi \in \Lambda(G/F)$ and $\lambda_\phi \in \Lambda(G/F)$ by

$$\phi_\psi(w,a) = \psi(w,a,d_w), \quad \lambda_\psi(w) = \psi(w,d_w,d_w), \quad \lambda_\phi(w) = \phi(w,d_w).$$

We call ϕ_ψ the L-parameter for ψ, and λ_ψ (resp. λ_ϕ) the infinitesimal character for ψ (resp. ϕ).

1.2. ***L*-packets and *A*-packets.** Set $G = \mathrm{SO}_{2n+1}$ in the case (B_n), and $G = \mathrm{Sp}_{2n}$ in the case (C_n). The local main theorem in Arthur's book is stated as follows.

Theorem 1.2 ([Ar13, Theorem 2.2.1])**.** (1) *For each $\psi \in \Psi(G/F)$, there is a finite multiset Π_ψ over $\Pi(G(F))$ with a map*

$$\Pi_\psi \to \mathrm{Irr}(A_\psi),\ \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

enjoying certain endoscopic character identities. When $G = \mathrm{SO}_{2n+1}$, i,e, in the case (B_n), the pairing satisfies $\langle -\mathbf{1}, \pi \rangle_\psi = 1$. We call Π_ψ the ***A-packet*** *of $G(F)$ associated with ψ.*

(2) *When $\psi = \phi \in \Phi_{\mathrm{temp}}(G/F)$, the A-packet Π_ϕ is in fact a subset of $\Pi_{\mathrm{temp}}(G(F))$. Moreover the map $\pi \mapsto \langle \cdot, \pi \rangle_\phi$ is injective, and the image is*

$$\begin{cases} \mathrm{Irr}(A_\psi/\{\pm \mathbf{1}\}) & \text{in the case } (B_n), \\ \mathrm{Irr}(A_\psi) & \text{in the case } (C_n). \end{cases}$$

In addition, $\Pi_\phi \cap \Pi_{\phi'} = \emptyset$ for $\phi \neq \phi'$, and

$$\Pi_{\mathrm{temp}}(G(F)) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G/F)} \Pi_\phi.$$

By the Langlands classification, one can extend Theorem 1.2 (2) to any L-parameter $\phi \in \Phi(G/F)$. Namely, there is a finite subset Π_ϕ of $\Pi(G(F))$ with an injection

$$\Pi_\phi \to \mathrm{Irr}(A_\phi),\ \pi \mapsto \langle \cdot, \pi \rangle_\phi$$

such that $\Pi_\phi \cap \Pi_{\phi'} = \emptyset$ for $\phi \neq \phi'$, and

$$\Pi(G(F)) = \bigsqcup_{\phi \in \Phi(G/F)} \Pi_\phi.$$

We call Π_ϕ the **L-packet** of $G(F)$ associated with ϕ. When $\pi \in \Pi_\phi$, we say that ϕ is the **L-parameter of** π. The map $\pi \mapsto (\phi, \eta)$, where $\phi \in \Phi(G/F)$ is the L-parameter of π and $\eta = \langle \cdot, \pi \rangle_\phi \in \mathrm{Irr}(A_\phi)$, is called the **local Langlands correspondence**.

The A-packets Π_ψ were constructed explicitly by Mœglin. See also Xu's paper [X17] and its references. As a consequence, Mœglin showed the following deep result.

Theorem 1.3 (Mœglin [Mœ11], Xu [X17, Theorem 8.12])**.** *The A-packet Π_ψ is multiplicity-free, i.e., Π_ψ is a subset of $\Pi(G(F))$.*

1.3. **The main conjecture.** The L-packets give a classification of $\Pi(G(F))$ but the endoscopic character identities fail for non-tempered L-packets in general. On the other hand, the A-packets are "local factors" of global A-packets which classify the discrete spectrum of automorphic representations, so that the local meaning of the A-packets is unclear. Furthermore, Mœglin's construction of L-packets might not be so hard (up to constructing supercuspidal representations), but the one of A-packets should be quite difficult (see also [X17]).

One may desire to understand A-packets more directly. Since a classification of $\Pi(G(F))$ is given by L-packets, it may be desirable that A-packets are described in terms of L-packets.

After Arthur formulated conjectures about A-packets in 1980's, in [ABV92], Adams, Barbasch and Vogan constructed A-packets Π_ψ^{ABV} for real reductive groups purely locally. These packets are called the **Adams–Barbasch–Vogan packets** (or shortly, the **ABV packets**). Vogan [V93] had also given a similar definition of A-packets Π_ψ^{ABV} for p-adic reductive groups using the microlocal Euler characteristic derived from the microlocalisation functor.

In [CFMMX], the authors constructed p-adic ABV packets Π_ψ^{ABV} using vanishing cycles functors (instead of the microlocal analysis). It is conjectured (and might be known to experts) that the ABV packets coincide with Vogan's ones, but the definition using vanishing cycles functors is more amenable than Vogan's one. For the detail for the comparison of two definitions, see [CFMMX, §7.6].

The main conjecture in [CFMMX] is roughly stated as follows:

Conjecture 1.4. *For $\psi \in \Psi(G/F)$,*

$$\Pi_\psi = \Pi_\psi^{\mathrm{ABV}}.$$

1.4. **Outline of the definition of p-adic ABV packets.** The p-adic ABV packets Π_ψ^{ABV} will be defined using the vanishing cycles functors of equivariant perverse sheaves on Vogan varieties. Now we explain these terminologies roughly.

(1) The **Vogan variety** is a moduli space of L-parameters. Let $\lambda \in \Lambda(G/F)$ be an infinitesimal character. This gives a complex variety V_λ (Definition 2.1) with an action of a complex reductive group $H_\lambda = Z_{\widehat{G}}(\mathrm{Im}(\lambda))$. There is a canonical bijection

$$\Phi_\lambda(G/F) \coloneqq \{\phi \in \Phi(G/F) \mid \lambda_\phi = \lambda\} \to \{H_\lambda\text{-orbits in } V_\lambda\},\ \phi \mapsto C_\phi.$$

See Proposition 2.2.

(2) When there is a group action $H \times V \to V$ in the category of algebraic varieties (over $\mathbb{C}$), one can define a category $\mathsf{Per}_H(V)$ of H-equivariant perverse sheaves on V. Every simple object in $\mathsf{Per}_H(V)$ is (isomorphic to) an intersection complex $\mathcal{IC}(C, \mathcal{L})$, where C is an H-orbit in V and $\mathcal{L}$ is a simple H-equivariant local system on C. Hence the simple objects in $\mathsf{Per}_H(V)$ are parametrized by pairs (C, ρ), where C is an H-orbit in V and ρ is an isomorphism class of irreducible representations of equivariant fundamental group A_C of C. Moreover, if $(V, H) = (V_\lambda, H_\lambda)$ and $C = C_\phi$, then $A_C \cong A_\phi$ (Lemma 2.3).

(3) The local Langalnds correspondence together with the bijection

$$\begin{array}{ccc} \{(\phi, \eta) \mid \phi \in \Phi_\lambda(G/F),\, \eta \in \mathrm{Irr}(A_\phi)\} & & (\phi, \eta) \\ \downarrow & & \downarrow \\ \{(C, \rho) \mid H_\lambda\text{-orbit } C \subset V_\lambda,\, \rho \in \mathrm{Irr}(A_C)\} & & (C_\phi, \eta) \end{array}$$

gives an injection

$$\bigsqcup_{\phi \in \Phi_\lambda(G/F)} \Pi_\phi \to \mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}},\ \pi \mapsto \mathcal{P}(\pi).$$

See Proposition 4.1.

(4) For an H_λ-orbit C in V_λ, define a subvariety $T^*_C(V_\lambda)_{\mathrm{reg}}$ of the conormal bundle $T^*_C(V_\lambda)$ by

$$T^*_C(V_\lambda)_{\mathrm{reg}} = T^*_C(V_\lambda) \setminus \bigcup_{C \subsetneq \overline{C}_1} T^*_{C_1}(V_\lambda),$$

where C_1 runs over all H_λ-orbits in V_λ such that $C \subsetneq \overline{C}_1$. Then the vanishing cycles functor gives an exact functor

$${}^p\mathsf{Ev}_C \colon \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}}).$$

See Theorem 4.2.

(5) We say that $(x, \xi) \in T^*_C(V_\lambda)$ is **strongly regular** if its H_λ-orbit is open and dense in $T^*_C(V_\lambda)$. We denote by $T^*_C(V_\lambda)_{\mathrm{sreg}}$ the strongly regular part of $T^*_C(V_\lambda)$. Then $T^*_C(V_\lambda)_{\mathrm{sreg}} \subset T^*_C(V_\lambda)_{\mathrm{reg}}$ is an H_λ-orbit if it is non-empty (Proposition 3.4). One can define a functor

$$\mathsf{Evs}_C \colon \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}}),$$

where $\mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}})$ is the category of H_λ-equivariant local systems on $T^*_C(V_\lambda)_{\mathrm{sreg}}$, by

$$\mathsf{Evs}_C\mathcal{P} = {}^p\mathsf{Ev}_C\mathcal{P}[-\dim V_\lambda]|_{T^*_C(V_\lambda)_{\mathrm{sreg}}}.$$

Finally, we define a normalization $\mathsf{NEvs}_C\colon \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}})$ of Evs_C. Its properties are stated in Theorem 4.4.

(6) For an A-parameter $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$, set

$$\psi_1 = \psi|_{\{\mathbf{1}\}\times\mathrm{SL}_2(\mathbb{C})\times\{\mathbf{1}\}}\colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G},$$
$$\psi_2 = \psi|_{\{\mathbf{1}\}\times\{\mathbf{1}\}\times\mathrm{SL}_2(\mathbb{C})}\colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G},$$

and put

$$x_\psi = d\psi_1\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ \xi_\psi = d\psi_2\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \in \widehat{\mathfrak{g}} = \mathrm{Lie}(\widehat{G}).$$

Then

$$(x_\psi, \xi_\psi) \in T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$$

where $\lambda = \lambda_\psi$ is the infinitesimal character for ψ, and $C_\psi = C_{\phi_\psi}$ is the H_λ-orbit corresponding to the L-parameter ϕ_ψ for ψ. Moreover, the equivariant fundamental group $A_{T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}}$ of $T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$ is isomorphic to A_ψ (Proposition 3.5). In particular, $T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$ is non-empty when $\lambda = \lambda_\psi$, and there is an equivalence map

$$\mathsf{Loc}_{H_\lambda}(T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}) \to \mathrm{Rep}(A_{T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}}) \cong \mathrm{Rep}(A_\psi).$$

(7) Consequently, $\psi \in \Psi(G/F)$ with $\lambda_\psi = \lambda$ determines a functor

$$\mathsf{Ev}_\psi\colon \mathsf{Per}_{H_\lambda}(V_\lambda) \xrightarrow{\mathsf{Evs}_{C_\psi}} \mathsf{Loc}_{H_\lambda}(T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}) \to \mathrm{Rep}(A_\psi),$$

and its normalization NEv_ψ. See §5.1.

(8) Note that there is the zero representation in $\mathrm{Rep}(A_\psi)$. For $\psi \in \Psi(G/F)$, the ABV packet Π^{ABV}_ψ is defined in Definition 5.1 by

$$\Pi^{\mathrm{ABV}}_\psi = \left\{ \pi \in \bigsqcup_{\phi\in\Phi_{\lambda_\psi}(G/F)} \Pi_\phi \;\middle|\; \mathsf{Ev}_\psi(\mathcal{P}(\pi)) \neq 0 \right\}.$$

In the rest of this paper, we explain the terminologies appearing in the definition of the ABV packets. The proofs of several facts are devolved to the original paper [CFMMX]. The main conjecture is stated in Conjecture 5.2.

In Appendix A, we summarize supplements: Algebraic representations of $\mathrm{SL}_2(\mathbb{C})$; equivariant fundamental groups and local systems; and the Kazhdan–Lusztig conjecture. In Appendix B, we recall relations between representations of semisimple Lie groups, $\mathcal{D}$-modules and perverse sheaves in the archimedean case.

2. Vogan varieties and several groups associated to infinitesimal characters

In this section, we define Vogan varieties V_λ together with several reductive groups $H_\lambda \subset J_\lambda \subset K_\lambda \subset \widehat{G}$ associated to infinitesimal characters $\lambda \in \Lambda(G/F)$, and give some examples.

2.1. **Infinitesimal characters for A-parameters.** Let $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ be an A-parameter for G. Recall that $\widehat{G}$ is $\mathrm{Sp}_{2n}(\mathbb{C})$ or $\mathrm{SO}_{2n+1}(\mathbb{C})$ so that one can regard ψ as a self-dual representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. We decompose ψ into a direct sum of representations as follows:

$$\psi = \left(\bigoplus_{i \in I} \rho_i \boxtimes V_i\right) \oplus \left(\bigoplus_{j \in J} \rho_j \boxtimes W_j\right) \oplus \left(\bigoplus_{k \in K} (\rho_k \oplus \rho_k^\vee) \boxtimes U_i\right),$$

where

- for $i \in I \cup J \cup K$, ρ_i is irreducible bounded representation of W_F such that $\rho_i \ncong \rho_{i'}$ and $\rho_i \ncong \rho_{i'}^\vee$ for $i \neq i'$;
- if $i \in I$, then ρ_i is self-dual of the same type as ψ;
- if $j \in J$, then ρ_j is self-dual of the opposite type to ψ;
- if $k \in K$, then ρ_k is not self-dual;
- V_i, W_j, and U_k are representations of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$.

For $i \in I$ (resp. $j \in J$), the structure of $\widehat{G}$ gives a non-degenerate symmetric (resp. alternating) bilinear form on V_i (resp. W_j). The centralizer $Z_{\widehat{G}}(\psi(W_F))$ of $\psi(W_F)$ is the kernel of the map

$$\begin{aligned}&\prod_{i \in I} \mathrm{O}(V_i) \times \prod_{j \in J} \mathrm{Sp}(W_j) \times \prod_{k \in K} \mathrm{GL}(U_k) \to \{\pm 1\},\\ &((\alpha_i)_i, (\beta_j)_j, (\gamma_k)_k) \mapsto \prod_{i \in I} \det(\alpha_i)^{\dim \rho_i}.\end{aligned}$$

2.2. **Several groups associated to infinitesimal characters.** Let $\lambda\colon W_F \to \widehat{G}$ be an infinitesimal character for G. Recall that we fix a Frobenius element $\mathrm{Frob}_F \in W_F$, and we normalize the norm map $|\cdot|\colon W_F \to \mathbb{R}^\times$ so that $|\mathrm{Frob}_F| = q$. We may assume that $\lambda(\mathrm{Frob}_F)$ is diagonal, say $\mathrm{diag}(\alpha_1, \dots, \alpha_N)$ for some $\alpha_1, \dots, \alpha_N \in \mathbb{C}^\times$. Write $\alpha_i = r_i u_i$ with $r_i > 0$ and $|u_i| = 1$. We define $f_\lambda, s_\lambda, t_\lambda \in \widehat{G}$ by

$$\begin{aligned}f_\lambda &= \lambda(\mathrm{Frob}_F) = \mathrm{diag}(\alpha_1, \dots, \alpha_N),\\ s_\lambda &= \mathrm{diag}(r_1, \dots, r_N),\\ t_\lambda &= \mathrm{diag}(u_1, \dots, u_N).\end{aligned}$$

Note that $s_\lambda t_\lambda = t_\lambda s_\lambda = f_\lambda$. We call s_λ (resp. t_λ) the **hyperbolic part** (resp. **elliptic part**) of f_λ. Since $\alpha_i = \alpha_j \implies u_i = u_j$, we have $Z_{\widehat{G}}(f_\lambda) \subset Z_{\widehat{G}}(t_\lambda)$.

Now we define $H_\lambda \subset J_\lambda \subset K_\lambda \subset \widehat{G}$ by

$$\begin{aligned}H_\lambda &= Z_{\widehat{G}}(\lambda(W_F)),\\ J_\lambda &= Z_{\widehat{G}}(\lambda(I_F)) \cap Z_{\widehat{G}}(t_\lambda),\\ K_\lambda &= Z_{\widehat{G}}(\lambda(I_F)).\end{aligned}$$

Remark that $H_\lambda = Z_{\widehat{G}}(\lambda(I_F)) \cap Z_{\widehat{G}}(f_\lambda)$. These are complex reductive groups, but not necessarily connected. We note that f_λ stabilizes K_λ and that $s_\lambda \in J_\lambda$.

2.3. **Vogan varieties.** Let $\mathfrak{h}_\lambda \subset \mathfrak{j}_\lambda \subset \mathfrak{k}_\lambda \subset \widehat{\mathfrak{g}}$ be the Lie algebras of $H_\lambda \subset J_\lambda \subset K_\lambda \subset \widehat{G}$, respectively. The adjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}$ is denoted by Ad.

Definition 2.1. *For $\lambda \in \Lambda(G/F)$, we define*

$$V_\lambda = \{x \in \mathfrak{k}_\lambda \mid \mathrm{Ad}(f_\lambda)x = qx\},$$
$${}^tV_\lambda = \{x \in \mathfrak{k}_\lambda \mid \mathrm{Ad}(f_\lambda)x = q^{-1}x\}.$$

Then H_λ acts on V_λ and ${}^tV_\lambda$ by Ad. *We call V_λ the* ***Vogan variety*** *for λ.*

We note that

$$V_\lambda = \{x \in \mathfrak{j}_\lambda \mid \mathrm{Ad}(s_\lambda)x = qx\},$$
$${}^tV_\lambda = \{x \in \mathfrak{j}_\lambda \mid \mathrm{Ad}(s_\lambda)x = q^{-1}x\}$$

since $f_\lambda = s_\lambda t_\lambda = t_\lambda s_\lambda$ and any eigenvalue of t_λ has complex norm 1.

There is a concrete description of V_λ ([CFMMX, §4.6]). Notice that $s_\lambda \in J_\lambda$ by definition. Let S be a maximal torus of J_λ such that $s_\lambda \in S$. We denote the set of roots of S in J_λ by $R(S, J_\lambda)$. Then

$$V_\lambda \cong \mathbb{A}^d, \quad \text{for } d = |\{\alpha \in R(S, J_\lambda) \mid \alpha(s_\lambda) = q\}|.$$

The Lie algebra $\mathfrak{j}_\lambda$ has a decomposition

$$\mathfrak{j}_\lambda = Z(\mathfrak{j}_\lambda) \oplus [\mathfrak{j}_\lambda, \mathfrak{j}_\lambda],$$

where $Z(\mathfrak{j}_\lambda)$ is the center of $\mathfrak{j}_\lambda$, and $[\mathfrak{j}_\lambda, \mathfrak{j}_\lambda]$ is a semisimple Lie algebra. They are given explicitly by

$$[\mathfrak{j}_\lambda, \mathfrak{j}_\lambda] = \bigoplus_{i \in I} \mathfrak{so}(V_i) \oplus \bigoplus_{j \in J} \mathfrak{sp}(W_j) \oplus \bigoplus_{k \in K} \mathfrak{sl}(U_k), \quad Z(\mathfrak{j}_\lambda) \cong \mathbb{C}^{\oplus |K|}.$$

We choose any non-degenerate symmetric bilinear form on $Z(\mathfrak{j}_\lambda)$. Using it, we extend the Killing form on $[\mathfrak{j}_\lambda, \mathfrak{j}_\lambda]$ to $\mathfrak{j}_\lambda$ so that $[\mathfrak{j}_\lambda, \mathfrak{j}_\lambda]$ and $Z(\mathfrak{j}_\lambda)$ are orthogonal to each other. Then we obtain a non-degenerate symmetric J_λ-invariant bilinear pairing

$$(\ |\ ) \colon \mathfrak{j}_\lambda \times \mathfrak{j}_\lambda \to \mathbb{C}.$$

This pairing allows us to identify ${}^tV_\lambda$ with the linear dual of V_λ. It is known that the Killing forms on $\mathfrak{so}(V_i)$, $\mathfrak{sp}(W_j)$, and $\mathfrak{sl}(U_k)$ are given as follows.

$$\mathfrak{so}(V_i) \times \mathfrak{so}(V_i) \to \mathbb{C},\ (X, Y) \mapsto (\dim V_i - 2) \cdot \mathrm{tr}(XY),$$
$$\mathfrak{sp}(W_j) \times \mathfrak{sp}(W_j) \to \mathbb{C},\ (X, Y) \mapsto (\dim W_j + 2) \cdot \mathrm{tr}(XY),$$
$$\mathfrak{sl}(U_k) \times \mathfrak{sl}(U_k) \to \mathbb{C},\ (X, Y) \mapsto 2 \dim U_k \cdot \mathrm{tr}(XY).$$

2.4. **L-parameters and H_λ-orbits in V_λ.** Let $\Phi_\lambda(G/F)$ be the subset of $\Phi(G/F)$ consisting of L-parameters $\phi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ such that $\lambda_\phi = \lambda$. For $\phi \in \Phi_\lambda(G/F)$, set

$$\varphi = \phi|_{\{\mathbf{1}\} \times \mathrm{SL}_2(\mathbb{C})} \colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G},$$

and put

$$x_\phi = d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}}.$$

Then $x_\phi \in V_\lambda$. The H_λ-orbit of x_ϕ is denoted by C_ϕ.

Proposition 2.2 ([CFMMX, Proposition 3.2.2])**.** *The map $\phi \mapsto C_\phi$ gives a bijection*

$$\Phi_\lambda(G/F) \to H_\lambda \backslash V_\lambda.$$

For an H_λ-orbit C in V_λ, let $A_C = \pi_1(C)_{H_\lambda}$ be the H_λ-equivariant fundamental group of C. For equivariant fundamental groups, see Appendix A.2.

Lemma 2.3 ([CFMMX, Lemma 3.6.1])**.** *For any $\phi \in \Phi_\lambda(G/F)$, we have*

$$A_{C_\phi} \cong A_\phi.$$

Now we set

$$\Pi_\lambda = \bigsqcup_{\phi \in \Phi_\lambda(G/F)} \Pi_\phi$$

to be the union of the L-packets associated with $\phi \in \Phi_\lambda(G/F)$. The LLC together with Proposition 2.2 and Lemma 2.3 gives the following:

Proposition 2.4. *There is a canonical injection*

$$\Pi_\lambda \to \{(C, \rho) \mid H_\lambda\text{-orbit } C \subset V_\lambda,\ \rho \in \mathrm{Irr}(A_C)\}.$$

2.5. **Examples of $\Phi_\lambda(G/F)$ and V_λ.** In this subsection, we describe V_λ, ${}^tV_\lambda$ and $\Phi_\lambda(G/F)$ more explicitly.

Let $\lambda = \lambda_\psi$ be the infinitesimal character for the A-parameter

$$\psi = \left(\bigoplus_{i \in I} \rho_i \boxtimes V_i\right) \oplus \left(\bigoplus_{j \in J} \rho_j \boxtimes W_j\right)$$

as in §2.1 (with $K = \emptyset$). Suppose that ψ is of good parity, i.e., each irreducible constituent of ψ is self-dual of the same type as ψ. Then for each $i \in I$ and $j \in J$, there are decompositions

$$V_i = (V_{i,t_i} \oplus \cdots \oplus V_{i,1}) \oplus V_{i,0} \oplus (V_{i,1}^* \oplus \cdots \oplus V_{i,t_i}^*),$$
$$W_j = (W_{j,t_j} \oplus \cdots \oplus W_{j,1}) \oplus (W_{j,1}^* \oplus \cdots \oplus W_{j,t_j}^*),$$

where

- $V_{i,k}$ (resp. $W_{j,k}$) is a totally isotropic subspace of V_i (resp. W_j) for $i = 1, \dots, t_i$ (resp. $j = 1, \dots, t_j$);
- $V_{i,k} \oplus V_{i,k}^*$ (resp. $W_{j,k} \oplus W_{j,k}^*$) is non-degenerate for $i = 1, \dots, t_i$ (resp. $j = 1, \dots, t_j$);
- $V_{i,k'} \oplus V_{i,k'}^*$ (resp. $W_{j,k'} \oplus W_{j,k'}^*$) is orthogonal to $V_{i,k} \oplus V_{i,k}^*$ (resp. $W_{j,k} \oplus W_{j,k}^*$) for $k' \neq k$;
- V_0 is the orthogonal complement of $(V_{i,t_i} \oplus \cdots \oplus V_{i,1}) \oplus (V_{i,1}^* \oplus \cdots \oplus V_{i,t_i}^*)$

such that s_λ stabilizes $V_{i,k}$ (resp. $W_{j,k}$) and acts on it by the scalar q^k (resp. $q^{k-\frac{1}{2}}$). Moreover, $\dim(V_{i,k+1}) \leq \dim(V_{i,k})$ and $\dim(W_{j,k+1}) \leq \dim(W_{j,k})$ for $k \geq 1$.

We have

$$J_\lambda \subset \prod_{i \in I} \mathrm{O}(V_i) \times \prod_{j \in J} \mathrm{Sp}(W_j), \quad \mathfrak{j}_\lambda = \bigoplus_{i \in I} \mathfrak{so}(V_i) \oplus \bigoplus_{j \in J} \mathfrak{sp}(W_j),$$

and

$$H_\lambda \subset \prod_{i \in I} \left(\mathrm{O}(V_{i,0}) \times \prod_{k=1}^{t_i} \mathrm{GL}(V_{i,k})\right) \times \prod_{j \in J} \prod_{k=1}^{t_j} \mathrm{GL}(W_{j,k}).$$

More explicitly, J_λ is the kernel of

$$\prod_{i\in I}\mathrm{O}(V_i)\times\prod_{j\in J}\mathrm{Sp}(W_j)\to\{\pm1\},\ ((\alpha_i)_i,(\beta_j)_j)\mapsto\prod_{i\in I}\det(\alpha_i)^{\dim\rho_i},$$

and H_λ is the kernel of

$$\prod_{i\in I}\left(\mathrm{O}(V_{i,0})\times\prod_{k=1}^{t_i}\mathrm{GL}(V_{i,k})\right)\times\prod_{j\in J}\prod_{k=1}^{t_j}\mathrm{GL}(W_{j,k})\to\{\pm1\},$$
$$((\alpha_i,(a_{i,k})_k)_i,(b_{j,k})_{j,k})\mapsto\prod_{i\in I}\det(\alpha_i)^{\dim\rho_i}.$$

The Vogan variety V_λ is the subspace of $\mathfrak{j}_\lambda$ given by

$$\bigoplus_{i\in I}\bigoplus_{k=1}^{t_i}\mathrm{Hom}(V_{i,k-1},V_{i,k})\oplus\bigoplus_{j\in J}\left(\mathrm{Sym}(W_{j,1}^*,W_{j,1})\oplus\bigoplus_{k=2}^{t_j}\mathrm{Hom}(W_{j,k-1},W_{j,k})\right),$$

where $\mathrm{Sym}(W_{j,1}^*,W_{j,1})=\{c\in\mathrm{Hom}(W_{j,1}^*,W_{j,1})\mid c^*=-c\}$ with $c^*\in\mathrm{Hom}(W_{j,1}^*,W_{j,1})$ being defined so that

$$\langle cw_1,w_2\rangle_{W_j}=\langle w_1,c^*w_2\rangle_{W_j}$$

for $w_1,w_2\in W_{j,1}^*$. More precisely, $\oplus_{k=1}^{t_i}\mathrm{Hom}(V_{i,k-1},V_{i,k})$ and $\mathrm{Sym}(W_{j,1}^*,W_{j,1})\oplus(\oplus_{k=2}^{t_j}\mathrm{Hom}(W_{j,k-1},W_{j,k}))$ are regarded as subspaces in $\mathfrak{so}(V_i)$ and $\mathfrak{sp}(W_j)$ by

$$(A_{i,k})_k\mapsto\left(\begin{array}{ccc|c|ccc}
\mathbf{0}_{V_{i,t_i}} & A_{i,t_i} & & & & & \\
 & \ddots & \ddots & & & & \\
 & & \mathbf{0}_{V_{i,1}} & A_{i,1} & & & \\
\hline
 & & & \mathbf{0}_{V_{i,0}} & A_{i,1}^* & & \\
\hline
 & & & & \mathbf{0}_{V_{i,1}^*} & \ddots & \\
 & & & & & \ddots & A_{i,t_i}^* \\
 & & & & & & \mathbf{0}_{V_{i,t_i}^*}
\end{array}\right)\in\mathfrak{so}(V_i),$$

$$(B_{j,k})_k\mapsto\left(\begin{array}{ccccc|ccc}
\mathbf{0}_{W_{j,t_j}} & B_{j,t_j} & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & \mathbf{0}_{W_{j,2}} & B_{j,2} & & & & \\
 & & & \mathbf{0}_{W_{j,1}} & B_{j,1} & & & \\
\hline
 & & & & \mathbf{0}_{W_{j,1}^*} & B_{j,2}^* & & \\
 & & & & & \mathbf{0}_{W_{j,2}^*} & \ddots & \\
 & & & & & & \ddots & B_{j,t_j}^* \\
 & & & & & & & \mathbf{0}_{W_{j,t_j}^*}
\end{array}\right)\in\mathfrak{sp}(W_j),$$

where $A^*_{i,k} \in \mathrm{Hom}(V^*_{i,k}, V^*_{i,k-1})$ (with $V^*_{i,0} \coloneqq V_{i,0}$) and $B^*_{j,k} \in \mathrm{Hom}(W^*_{j,k}, W^*_{j,k-1})$ are defined so that

$$\langle A_{i,k}v, v^*\rangle_{V_i} + \langle v, A^*_{i,k}v^*\rangle_{V_i} = 0,$$
$$\langle B_{j,k}w, w^*\rangle_{W_j} + \langle w, B^*_{j,k}w^*\rangle_{W_j} = 0$$

for $v \in V_{i,k-1}$, $v^* \in V^*_{i,k}$ and $w \in W_{j,k-1}$, $w^* \in W^*_{j,k}$, respectively. Choosing bases of $V_{i,k}$ and $W_{j,k}$, we have an identification

$$V_\lambda = \bigoplus_{i\in I}\bigoplus_{k=1}^{t_i} \mathrm{Mat}_{d_{i,k},d_{i,k-1}}(\mathbb{C}) \oplus \bigoplus_{j\in J}\left(\mathrm{Sym}_{d_{j,1}}(\mathbb{C}) \oplus \bigoplus_{k=2}^{t_j} \mathrm{Mat}_{d_{j,k},d_{j,k-1}}(\mathbb{C})\right),$$

where we set $d_{i,k} = \dim(V_{i,k})$, $d_{j,k} = \dim(W_{j,k})$, and $\mathrm{Sym}_{d_{j,1}}(\mathbb{C})$ is the subspace of $\mathrm{Mat}_{d_{j,1}}(\mathbb{C})$ consisting of symmetric matrices. Therefore, as an algebraic variety, V_λ is the affine space $\mathbb{A}^d$ with

$$d = \sum_{i\in I}\sum_{k=1}^{t_i} d_{i,k}d_{i,k-1} + \sum_{j\in J}\left(\frac{d_{j,1}(d_{j,1}+1)}{2} + \sum_{k=2}^{t_j} d_{j,k}d_{j,k-1}\right).$$

Similarly, ${}^tV_\lambda$ is the subspace of $\mathfrak{j}_\lambda$ given by

$$\bigoplus_{i\in I}\bigoplus_{k=1}^{t_i} \mathrm{Hom}(V_{i,k}, V_{i,k-1}) \oplus \bigoplus_{j\in J}\left(\mathrm{Sym}(W_{j,1}, W^*_{j,1}) \oplus \bigoplus_{k=2}^{t_j} \mathrm{Hom}(W_{j,k}, W_{j,k-1})\right),$$

which is isomorphic to

$$\bigoplus_{i\in I}\bigoplus_{k=1}^{t_i} \mathrm{Mat}_{d_{i,k-1},d_{i,k}}(\mathbb{C}) \oplus \bigoplus_{j\in J}\left(\mathrm{Sym}_{d_{j,1}}(\mathbb{C}) \oplus \bigoplus_{k=2}^{t_j} \mathrm{Mat}_{d_{j,k-1},d_{j,k}}(\mathbb{C})\right).$$

Since J_λ is semisimple, the bilinear pairing $(\ |\ ) \colon \mathfrak{j}_\lambda \times \mathfrak{j}_\lambda \to \mathbb{C}$ is defined by the Killing form. The restriction $(\ |\ ) \colon V_\lambda \times {}^tV_\lambda \to \mathbb{C}$ is given explicitly by

$$\Big(((A_{i,k})_{i,k}, (B_{j,k})_{j,k}) \mid ((A'_{i,k})_{i,k}, (B'_{j,k})_{j,k})\Big)$$
$$= \sum_{i\in I} 2(\dim V_i - 2)\sum_{k=1}^{t_i} \mathrm{tr}(A_{i,k}A'_{i,k}) + \sum_{j\in J}(\dim W_j + 2)\left(\mathrm{tr}(B_{j,1}B'_{j,1}) + 2\sum_{k=2}^{t_j}\mathrm{tr}(B_{j,k}B'_{j,k})\right).$$

The sets $\Phi_\lambda(G/F)$ and V_λ/H_λ can be complicated. For a positive integer k, we denote by S_k the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension k. For its realization, see Appendix A.1 below.

Example 2.5. *Suppose that $\lambda = \lambda_\phi$ with*

$$\phi = \rho \boxtimes (S_3 \oplus S_5),$$

where ρ is irreducible and self-dual of the same type as ϕ. Then $d_0 = 2$, $d_1 = 2$ and $d_2 = 1$ so that

$$H_\lambda = \begin{cases} \mathrm{O}(2,\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) & \text{if } \dim(\rho) \equiv 0 \bmod 2, \\ \mathrm{SO}(2,\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) & \text{if } \dim(\rho) \equiv 1 \bmod 2, \end{cases}$$

and

$$V_\lambda = \mathrm{Mat}_{2,2}(\mathbb{C}) \oplus \mathrm{Mat}_{1,2}(\mathbb{C}).$$

The action of H_λ *on* V_λ *is given by*

$$(\alpha, \gamma_1, \gamma_2) \cdot (X_1, X_2) = (\gamma_1 X_1 \alpha^{-1}, \gamma_2 X_2 \gamma_1^{-1})$$

for $(\alpha, \gamma_1, \gamma_2) \in H_\lambda$ *and* $(X_1, X_2) \in V_\lambda$.

For simplicity, we assume that $\dim \rho$ *is even. Then the elements* ϕ *in* $\Phi_\lambda(G/F)$ *and the associated groups* A_ϕ*, the vectors* $x_\phi \in V_\lambda$*, the* H_λ*-orbits* C_ϕ *are listed as follows:*

- $\phi_1 = \rho \otimes (|\cdot|^2 \oplus |\cdot|^1 \oplus |\cdot|^1 \oplus \mathbf{1} \oplus \mathbf{1} \oplus |\cdot|^{-1} \oplus |\cdot|^{-1} \oplus |\cdot|^{-2})$*,* $A_{\phi_1} \cong \mathbb{Z}/2\mathbb{Z}$*, and*
$$x_{\phi_1} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,0) \right) \in C_{\phi_1} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \times \{(0,0)\};$$
- $\phi_2 = \rho \otimes (|\cdot|^2 \oplus |\cdot|^1 \oplus |\cdot|^{\frac{1}{2}} S_2 \oplus |\cdot|^{-\frac{1}{2}} S_2 \oplus |\cdot|^{-1} \oplus |\cdot|^{-2})$*,* $A_{\phi_2} \cong \mathbf{1}$*, and*
$$x_{\phi_2} = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{-1} \\ 0 & 0 \end{pmatrix}, (0,0) \right)$$
$$\in C_{\phi_2} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \;\middle|\; \begin{matrix} (x,y,z,w) \neq (0,0,0,0), \\ xw - yz = x^2 + y^2 = z^2 + w^2 = 0 \end{matrix} \right\} \times \{(0,0)\};$$
- $\phi_3 = \rho \otimes (|\cdot|^2 \oplus |\cdot|^1 \oplus \mathbf{1} \oplus S_3 \oplus |\cdot|^{-1} \oplus |\cdot|^{-2})$*,* $A_{\phi_3} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$*, and*
$$x_{\phi_3} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, (0,0) \right)$$
$$\in C_{\phi_3} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \;\middle|\; \begin{matrix} xw - yz = 0, \\ x^2 + y^2 \neq 0 \text{ or } z^2 + w^2 \neq 0 \end{matrix} \right\} \times \{(0,0)\};$$
- $\phi_4 = \rho \otimes (|\cdot|^2 \oplus S_3 \oplus S_3 \oplus |\cdot|^{-2})$*,* $A_{\phi_4} \cong \mathbb{Z}/2\mathbb{Z}$*, and*
$$x_{\phi_4} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (0,0) \right) \in C_{\phi_4} = \mathrm{GL}_2(\mathbb{C}) \times \{(0,0)\};$$
- $\phi_5 = \rho \otimes (|\cdot|^{\frac{3}{2}} S_2 \oplus |\cdot|^1 \oplus \mathbf{1} \oplus \mathbf{1} \oplus |\cdot|^{-1} \oplus |\cdot|^{-\frac{3}{2}} S_2)$*,* $A_{\phi_5} \cong \mathbb{Z}/2\mathbb{Z}$*, and*
$$x_{\phi_5} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right) \in C_{\phi_5} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \times \{(a,b) \mid (a,b) \neq (0,0)\};$$
- $\phi_6 = \rho \otimes (|\cdot|^{\frac{3}{2}} S_2 \oplus |\cdot|^{\frac{1}{2}} S_2 \oplus |\cdot|^{-\frac{1}{2}} S_2 \oplus |\cdot|^{-\frac{3}{2}} S_2)$*,* $A_{\phi_6} \cong \mathbf{1}$*, and*
$$x_{\phi_6} = \left(\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{pmatrix}, \frac{1}{\sqrt{2}} (1, -\sqrt{-1}) \right)$$
$$\in C_{\phi_6} = \left\{ \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, (a,b) \right) \;\middle|\; \begin{matrix} (x,y,z,w) \neq (0,0,0,0),\ (a,b) \neq (0,0),\ xw - yz = 0, \\ (ax + bz, ay + bw) = (0,0),\ (\overline{a}z - \overline{b}x)^2 + (\overline{a}w - \overline{b}y)^2 = 0 \end{matrix} \right\};$$
- $\phi_7 = \rho \otimes (|\cdot|^{\frac{3}{2}} S_2 \oplus \mathbf{1} \oplus S_3 \oplus |\cdot|^{-\frac{3}{2}} S_2)$*,* $A_{\phi_7} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$*, and*
$$x_{\phi_7} = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, (1,0) \right)$$
$$\in C_{\phi_7} = \left\{ \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, (a,b) \right) \;\middle|\; \begin{matrix} (a,b) \neq (0,0),\ xw - yz = 0, \\ (ax + bz, ay + bw) = (0,0),\ (\overline{a}z - \overline{b}x)^2 + (\overline{a}w - \overline{b}y)^2 \neq 0 \end{matrix} \right\};$$

- $\phi_8 = \rho \otimes (|\cdot|^1 S_3 \oplus |\cdot|^1 \oplus |\cdot|^{-1} \oplus |\cdot|^{-1} S_3)$, $A_{\phi_8} \cong \mathbf{1}$, *and*
$$x_{\phi_8} = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{-1} \\ 0 & 0 \end{pmatrix}, (1,0) \right)$$
$$\in C_{\phi_8} = \left\{ (\begin{pmatrix} x & y \\ z & w \end{pmatrix}, (a,b)) \;\middle|\; \begin{array}{c} (a,b)\neq(0,0),\ xw-yz=0, \\ ax+bz\neq 0,\ (ax+bz)^2+(ay+bw)^2=0 \end{array} \right\};$$
- $\phi_9 = \rho \otimes (|\cdot|^1 \oplus \mathbf{1} \oplus S_5 \oplus |\cdot|^{-1})$, $A_{\phi_9} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, *and*
$$x_{\phi_9} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right)$$
$$\in C_{\phi_9} = \left\{ (\begin{pmatrix} x & y \\ z & w \end{pmatrix}, (a,b)) \;\middle|\; \begin{array}{c} (a,b)\neq(0,0),\ xw-yz=0, \\ (ax+bz)^2+(ay+bw)^2\neq 0 \text{ or } (\overline{a}z-\overline{b}x)^2+(\overline{a}w-\overline{b}y)^2\neq 0 \end{array} \right\};$$
- $\phi_{10} = \rho \otimes (|\cdot|^{\frac{1}{2}} S_4 \oplus |\cdot|^{-\frac{1}{2}} S_4)$, $A_{\phi_{10}} \cong \mathbf{1}$, *and*
$$x_{\phi_{10}} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}(1, \sqrt{-1}) \right)$$
$$\in C_{\phi_{10}} = \left\{ (\begin{pmatrix} x & y \\ z & w \end{pmatrix}, (a,b)) \;\middle|\; \begin{array}{c} (a,b)\neq(0,0),\ xw-yz\neq 0, \\ (ax+bz)^2+(ay+bw)^2=0 \end{array} \right\};$$
- $\phi_{11} = \rho \otimes (S_3 \oplus S_5)$, $A_{\phi_{11}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, *and*
$$x_{\phi_{11}} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (1,0) \right)$$
$$\in C_{\phi_{11}} = \left\{ (\begin{pmatrix} x & y \\ z & w \end{pmatrix}, (a,b)) \;\middle|\; \begin{array}{c} (a,b)\neq(0,0),\ xw-yz\neq 0, \\ (ax+bz)^2+(ay+bw)^2\neq 0 \end{array} \right\}.$$

Note that C_{ϕ_1} is a unique closed orbit and $C_{\phi_{11}}$ is a unique open dense orbit. The closure relations of H_λ-orbits are given as follows:

$$\begin{array}{ccccccccccc} C_{\phi_1} & = & \overline{C_{\phi_1}} & \subset & \overline{C_{\phi_2}} & \subset & \overline{C_{\phi_3}} & \subset & \overline{C_{\phi_4}} & \subset & \overline{C_{\phi_{10}}} & \subset & \overline{C_{\phi_{11}}} \\ & & \cap & & & & \cap & & & & \cup & & \cup \\ & & \overline{C_{\phi_5}} & & \subset & & \overline{C_{\phi_6}} & \subset & \overline{C_{\phi_7}} & \subset & \overline{C_{\phi_8}} & \subset & \overline{C_{\phi_9}}. \end{array}$$

3. A-parameters and strongly regular elements in the conormal bundle

In this section, we define the regular part of the conormal bundle $T^*_C(V_\lambda)$, and we introduce the notion of strongly regular elements.

3.1. **Cotangent space and conormal bundle to the Vogan variety.** Let $T^*(V_\lambda)$ be the cotangent space to the Vogan variety V_λ. For an H_λ-orbit C in V_λ, we denote the conormal bundle by $T^*_C(V_\lambda)$. Recall that V_λ is an affine space.

Proposition 3.1 ([CFMMX, Propositions 5.2.1, 5.3.1])**.** *There is an H_λ-equivariant isomorphism*
$$T^*(V_\lambda) \cong V_\lambda \times {}^tV_\lambda.$$
Similarly,
$$T^*_C(V_\lambda) = \{(x,\xi) \in T^*(V_\lambda) \mid x \in C,\ [x,\xi] = 0\},$$

where $[\cdot,\cdot]$ denotes the Lie bracket on $\mathfrak{j}_\lambda$, and we identity V_λ and ${}^tV_\lambda$ with subspaces of $\mathfrak{j}_\lambda$.

Since $V_\lambda^* \cong {}^tV_\lambda$, we can define an H_λ-equivariant isomorphism

$$T^*(V_\lambda) \to T^*({}^tV_\lambda),\ (x,\xi) \mapsto (\xi,x),$$

which we call **transposition**. As in Proposition 3.1, for an H_λ-orbit B in ${}^tV_\lambda$, we have

$$T^*_B({}^tV_\lambda) = \{(\xi,x) \in T^*({}^tV_\lambda) \mid \xi \in B,\ [\xi,x]=0\}.$$

Lemma 3.2. *For any H_λ-orbit C in V_λ, there is a unique H_λ-orbit C^* in ${}^tV_\lambda$ such that the restriction of the transposition gives an isomorphism*

$$\overline{T^*_C(V_\lambda)} \cong \overline{T^*_{C^*}({}^tV_\lambda)}.$$

The map $C \mapsto C^*$ is a bijection

$$H_\lambda\backslash V_\lambda \to H_\lambda\backslash {}^tV_\lambda.$$

Define $T^*_C(V_\lambda)_{\mathrm{reg}} \subset T^*_C(V_\lambda)$ by

$$T^*_C(V_\lambda)_{\mathrm{reg}} = T^*_C(V_\lambda) \setminus \bigcup_{\substack{C_1 \\ C_1 \neq C \subsetneq \overline{C}_1}} \overline{T^*_{C_1}(V_\lambda)},$$

where C_1 runs over all H_λ-orbits in V_λ such that $C \subsetneq \overline{C}_1$ and $C_1 \neq C$.

Proposition 3.3 ([CFMMX, Proposition 5.4.3])**.** *If $(x,\xi) \in T^*_C(V_\lambda)_{\mathrm{reg}}$, then $(x,\xi) \in C \times C^*$, $[x,\xi]=0$, and $(x \mid \xi)=0$.*

We say that $(x,\xi) \in T^*_C(V_\lambda)$ is **strongly regular** if its H_λ-orbit is open and dense in $T^*_C(V_\lambda)$. We write $T^*_C(V_\lambda)_{\mathrm{sreg}}$ for the strongly regular part of $T^*_C(V_\lambda)$.

Proposition 3.4 ([CFMMX, Proposition 5.5.1])**.** *We have*

$$T^*_C(V_\lambda)_{\mathrm{sreg}} \subset T^*_C(V_\lambda)_{\mathrm{reg}}.$$

Moreover, $T^*_C(V_\lambda)_{\mathrm{sreg}}$ is an H_λ-orbit if it is non-empty.

Let ψ be an A-parameter for G such that $\lambda = \lambda_\psi$. We set

$$\psi_1 = \psi|_{\{\mathbf{1}\}\times \mathrm{SL}_2(\mathbb{C})\times\{\mathbf{1}\}} \colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G},$$
$$\psi_2 = \psi|_{\{\mathbf{1}\}\times\{\mathbf{1}\}\times \mathrm{SL}_2(\mathbb{C})} \colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G},$$

and we put

$$x_\psi = d\psi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \xi_\psi = d\psi_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}} = \mathrm{Lie}(\widehat{G}).$$

It is easy to see that

$$(x_\psi, \xi_\psi) \in T^*_{C_\psi}(V_\lambda),$$

where $C_\psi = C_{\phi_\psi}$ is the H_λ-orbit corresponding to the L-parameter ϕ_ψ for ψ.

Proposition 3.5 ([CFMMX, Propositions 5.6.1, 5.7.1])**.** *We have*

$$(x_\psi, \xi_\psi) \in T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}.$$

Moreover, the equivariant fundamental group $A_{T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}}$ of $T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$ is isomorphic to A_ψ.

Example 3.6. *Consider*

$$\lambda = \rho \otimes (|\cdot|^2 \oplus |\cdot|^1 \oplus |\cdot|^1 \oplus \mathbf{1} \oplus \mathbf{1} \oplus |\cdot|^{-1} \oplus |\cdot|^{-1} \oplus |\cdot|^{-2}),$$

which is the same as in Example 2.5. Then the A-parameters $\psi \in \Psi(G/F)$ such that $\lambda = \lambda_\psi$ and the associated vectors (x_ψ, ξ_ψ) are listed as follows:

- $\psi_1 = \rho \boxtimes \mathbf{1} \boxtimes (S_3 \oplus S_5)$ *and*

$$x_{\psi_1} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,0) \right) \in C_{\phi_1}, \quad \xi_{\psi_1} = \left(\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right) \in {}^tV_\lambda;$$

- $\psi_3 = \rho \boxtimes (S_3 \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes S_5)$ *and*

$$x_{\psi_3} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, (0,0) \right) \in C_{\phi_3}, \quad \xi_{\psi_3} = \left(\begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right) \in {}^tV_\lambda;$$

- $\psi_6 = \rho \boxtimes S_2 \boxtimes S_4$ *and*

$$x_{\psi_6} = \left(\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{pmatrix}, \frac{1}{\sqrt{2}}(1, -\sqrt{-1}) \right) \in C_{\phi_6},$$
$$\xi_{\psi_6} = \left(\frac{1}{2} \begin{pmatrix} 7 & \sqrt{-1} \\ -\sqrt{-1} & 7 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ -3\sqrt{-1} \end{pmatrix} \right) \in {}^tV_\lambda;$$

- $\psi_9 = \rho \boxtimes (S_5 \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes S_3)$ *and*

$$x_{\psi_9} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right) \in C_{\phi_9}, \quad \xi_{\psi_9} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \in {}^tV_\lambda;$$

- $\psi_{10} = \rho \boxtimes S_4 \boxtimes S_2$ *and*

$$x_{\psi_{10}} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}(1, \sqrt{-1}) \right) \in C_{\phi_{10}},$$
$$\xi_{\psi_{10}} = \left(\frac{1}{2} \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix} \right) \in {}^tV_\lambda;$$

- $\psi_{11} = \rho \boxtimes (S_3 \oplus S_5) \boxtimes \mathbf{1}$ *and*

$$x_{\psi_{11}} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (1,0) \right) \in C_{\phi_{11}}, \quad \xi_{\psi_{11}} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \in {}^tV_\lambda.$$

For a realization of S_k, see Appendix A.1.

Let $\mathsf{Loc}_{H_\lambda}(T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}})$ be the category of H_λ-equivariant local systems on $T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$. Then $(x_\psi, \xi_\psi) \in T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$ determines an equivalence

$$\mathsf{Loc}_{H_\lambda}(T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}) \to \mathrm{Rep}(A_{T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}}) \cong \mathrm{Rep}(A_\psi).$$

4. Vanishing cycles functor of perverse sheaves on Vogan varieties

In this section, we define microlocal vanishing cycles functors, which we will use to define the ABV packets.

4.1. **L-packets and equivariant perverse sheaves on V_λ.** For an algebraic variety V, one can consider the category $\mathsf{Per}(V)$ of perverse sheaves, which is a subcategory of the derived category $\mathsf{D}(V) = \mathsf{D}^b_c(V)$ of bounded (constructible) $\overline{\mathbb{Q}}_\ell$-sheaves on V. See [BBD82].

When $H \times V \to V$ is a group action in the category of algebraic varieties, one can define the category $\mathsf{Per}_H(V)$ of H-equivariant perverse sheaves on V, equipped with a forgetful functor

$$\mathsf{Per}_H(V) \to \mathsf{Per}(V).$$

For more precision, see [CFMMX, §3.4–3.6]. Every simple object in $\mathsf{Per}_H(V)$ is the intersection complex $\mathcal{IC}(C, \mathcal{L})$, where C is an H-orbit in V, and $\mathcal{L}$ is a simple H-equivariant local system of C. It is defined by

$$\mathcal{IC}(C, \mathcal{L}) = \mathrm{Im}\left({}^pH^0(j_!\mathcal{L}) \to {}^pH^0(Rj_*\mathcal{L})\right),$$

where j is the embedding $C \hookrightarrow V$. Let $\mathsf{Loc}_H(C)$ denote the category of H-equivariant local systems of C. Hence there exists a canonical bijection:

$$\begin{aligned}\mathsf{Per}_H(V)^{\mathrm{simple}}_{/\mathrm{iso}} &\leftrightarrow \{(C, \mathcal{L}) \mid H\text{-orbit } C \subset V,\ \mathcal{L} \in \mathsf{Loc}_H(C)^{\mathrm{simple}}_{/\mathrm{iso}}\}\\ &\leftrightarrow \{(C, \eta) \mid H\text{-orbit } C \subset V,\ \eta \in \mathrm{Irr}(A_C)\},\end{aligned}$$

where A_C is the H-equivariant fundamental group of C.

Now we consider the Vogan variety $V = V_\lambda$ equipped with the action of the reductive group $H = H_\lambda$. Proposition 2.4 gives the following:

Proposition 4.1 ([CFMMX, Proposition 3.6.2])**.** *There is a canonical injection*

$$\Pi_\lambda \to \mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}},\ \pi \mapsto \mathcal{P}(\pi).$$

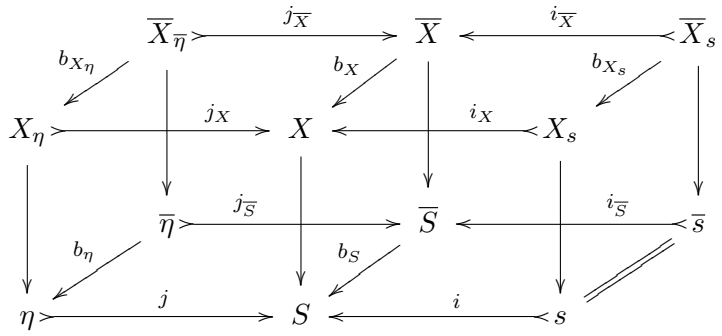
4.2. **Vanishing cycles functors.** In this subsection, we fix the notation for vanishing cycles functors. Let

- $R = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$;
- $\overline{K}$ be a separable closure of K, and $\overline{R}$ be the integral closure of R in $\overline{K}$;
- $S = \mathrm{Spec}(R)$, $\eta = \mathrm{Spec}(K)$, $s = \mathrm{Spec}(\mathbb{C})$, and $\overline{S} = \mathrm{Spec}(\overline{R})$.

Note that S is a trait, i.e., the spectrum of a henselian discrete valuation ring with generic point η and special point s. The morphism $s \to S$ has a canonical section corresponding to $\mathbb{C} \to \mathbb{C}[[t]]$. Hence we have

$$\eta \xrightarrow{j} S \underset{i}{\overset{\curvearrowright}{\leftarrow}} s.$$

Let $\overline{\eta}$ be a geometric point of S localized at η, i.e., $\overline{\eta}\colon \mathrm{Spec}(\overline{K}) \to \eta \to S$. Note that $\mathrm{Gal}(\overline{\eta}/\eta) \cong \widehat{\mathbb{Z}}$. For any morphism $X \to S$, we have the cartesian diagram:



where $\overline{X} = X \times_S \overline{S}$, $\overline{X}_{\overline{\eta}} = \overline{X} \times_{\overline{S}} \overline{\eta}$, and $\overline{X}_s = \overline{X} \times_{\overline{S}} \overline{s}$.

Recall that

- "a sheaf on $X_s \times_s \eta$" means a sheaf on X_s equipped with a $\mathrm{Gal}(\overline{\eta}/\eta)$-action;
- $\mathsf{D}(X_s \times_s \eta) = \mathsf{D}^b_c(X_s \times_s \eta)$ is the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on $X_s \times_s \eta$;
- "a sheaf on $X_s \times_s S$" means a triple $(\mathcal{F}_\eta, \mathcal{F}_s, \mathcal{F}_s \to \mathcal{F}_\eta)$, where $\mathcal{F}_\eta$ and $\mathcal{F}_s$ are sheaves on $X_s \times_s \eta$ and on X_s, respectively, and $\mathcal{F}_s \to \mathcal{F}_\eta$ is a $\mathrm{Gal}(\overline{\eta}/\eta)$-equivariant morphism, with the action on $\mathcal{F}_s$ being trivial;
- $\mathsf{D}(X_s \times_s S) = \mathsf{D}^b_c(X_s \times_s S)$ is the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on $X_s \times_s S$.

There exists a functor

$$\mathsf{R}\Psi_{X_\eta} \colon \mathsf{D}(X_\eta) \to \mathsf{D}(X_s \times_s \eta)$$

called the **nearby cycles functor** [DK73]. This is defined by

$$\mathsf{R}\Psi_{X_\eta}\mathcal{F} = (i_{X_{\overline{s}}})^* (j_{X_{\overline{\eta}}})_* (b_{X_\eta})^* \mathcal{F}$$

for $\mathcal{F} \in \mathsf{D}(X_\eta)$, which is a sheaf on X_s equipped with an action of $\mathrm{Gal}(\overline{\eta}/\eta)$ obtained from the canonical action on $(b_{X_\eta})^*\mathcal{F}$. Finally, there exists a functor

$$\mathsf{R}\Phi_X \colon \mathsf{D}(X) \to \mathsf{D}(X_s \times_s S)$$

called the **vanishing cycles functor**. The triangle

$$\begin{array}{ccc} & \mathsf{R}\Phi_X & \\ {\scriptstyle (1)}\swarrow & & \nwarrow \\ i_{\overline{X}}^* b_X^* & \longrightarrow & \mathsf{R}\Psi_{X_\eta} j_{X_\eta}^* \end{array}$$

is a distinguished triangle in $\mathsf{D}(X_s \times_s S)$. Thus, for $\mathcal{F} \in \mathsf{D}(X)$, we obtain a long exact sequence

$$\cdots \longrightarrow H^q(i_{\overline{X}}^* b_X^* \mathcal{F}) \longrightarrow H^q(\mathsf{R}\Psi_{X_\eta} j_{X_\eta}^* \mathcal{F}) \longrightarrow H^q(\mathsf{R}\Phi_X \mathcal{F}) \longrightarrow H^{q+1}(i_{\overline{X}}^* b_X^* \mathcal{F}) \longrightarrow \cdots .$$

4.3. **Definition of functors.** Now fix an infinitesimal character $\lambda \in \Lambda(G/F)$, and consider the Vogan variety V_λ with an action of a reductive group H_λ. Recall in §2.3 that we fixed a non-degenerate symmetric J_λ-invariant bilinear pairing $(\ |\ ) \colon \mathfrak{j}_\lambda \times \mathfrak{j}_\lambda \to \mathbb{C}$. By restriction, we obtain an s-morphism

$$(\ |\ ) \colon T^*(V_\lambda) \to \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[t]).$$

For an H_λ-orbit $B \subset {}^tV_\lambda$, we denote by

$$X_B = (V_\lambda \times B) \times_{\mathbb{A}^1_{\mathbb{C}}} S \xrightarrow{f_B} S$$

the base change of the restriction of $(\ |\ )$ to $V_\lambda \times B$ along the canonical map $g \colon S \to \mathbb{A}^1_C$. Namely, we have a cartesian diagram

$$\begin{array}{ccc} X_B & \longrightarrow & V_\lambda \times B \\ {\scriptstyle f_B}\downarrow & & \downarrow{\scriptstyle (\ |\ )} \\ S & \xrightarrow{g} & \mathbb{A}^1_{\mathbb{C}}. \end{array}$$

The structure sheaf of X_B is

$$\mathcal{O}_{X_B} = R \otimes_{\mathbb{C}[t]} (\mathcal{O}_{V_\lambda} \otimes_{\mathbb{C}} \mathcal{O}_B).$$

The special fibre of X_B is the s-scheme

$$X_{B,s} = f_B^{-1}(s) = f_B^{-1}(0) = \{(x,\xi) \in V_\lambda \times B \mid (x \mid \xi) = 0\}.$$

We obtain the vanishing cycles functor

$$\mathsf{R}\Phi_{X_B} \colon \mathsf{D}(X_B) \to \mathsf{D}(f_B^{-1}(0) \times_s S).$$

As an s-scheme, H_λ acts on $V_\lambda \times B$. Its base change along $S \to s$ gives an action of $H_\lambda \times_s S$ on $(V_\lambda \times B) \times_s S$. Since $(\ \mid\ )$ is H_λ-invariant, this action preserves

$$\{(x,\xi,t) \in (V_\lambda \times B) \times_s S \mid (x|\xi) = g(t)\}.$$

This is precisely $X_B = (V_\lambda \times B) \times_{\mathbb{A}^1_{\mathbb{C}}} S$. Hence $H_\lambda \times_s S$ acts on X_B in the category of S-schemes.

We denote the H_λ-equivariant derived category on V_λ by $\mathsf{D}_{H_\lambda}(V_\lambda)$. For any H_λ-orbit $C \subset V_\lambda$, we define a functor

$$\mathsf{Ev}_C \colon \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s S)$$

by the diagram

$$\begin{array}{ccccc}
\mathsf{D}_{H_\lambda}(V_\lambda) & & \xrightarrow{\quad\mathsf{Ev}_C\quad} & & \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s S) \\
{\scriptstyle \cdot\boxtimes \mathbf{1}_{C^*}}\downarrow & & & & \uparrow{\scriptstyle \mathrm{Res}} \\
\mathsf{D}_{H_\lambda}(V_\lambda \times C^*) & \xrightarrow{\mathrm{BC}} & \mathsf{D}_{H_\lambda \times_s S}(X_{C^*}) & \xrightarrow{\mathsf{R}\Phi_{X_{C^*}}} & \mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S),
\end{array}$$

where

(1) $\cdot \boxtimes \mathbf{1}_{C^*} \colon \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{H_\lambda}(V_\lambda \times C^*)$ is the pullback along the projection $V_\lambda \times C^* \to V_\lambda$;
(2) $\mathrm{BC} \colon \mathsf{D}_{H_\lambda}(V_\lambda \times C^*) \to \mathsf{D}_{H_\lambda \times_s S}(X_{C^*})$ is the base change functor, which is an exact functor;
(3) $\mathsf{R}\Phi_{X_{C^*}} \colon \mathsf{D}_{H_\lambda \times_s S}(X_{C^*}) \to \mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S)$ is the vanishing cycles functor;
(4) $\mathrm{Res} \colon \mathsf{D}_{H_\lambda}(f_{C^*}^{-1}(0) \times_s S) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s S)$ is given by the pullback along the inclusion $T^*_C(V_\lambda)_{\mathrm{reg}} \hookrightarrow f_{C^*}^{-1}(0)$ induced by Proposition 3.3.

For $\xi_0 \in {}^tV_\lambda$, define $f_{\xi_0} \colon X_{\xi_0} \to S$ by the base change of $V_\lambda \ni x \mapsto (x|\xi_0) \in \mathbb{A}^1_{\mathbb{C}}$, i.e., we have a cartesian diagram

$$\begin{array}{ccc}
X_{\xi_0} & \longrightarrow & V_\lambda \\
{\scriptstyle f_{\xi_0}}\downarrow & & \downarrow{\scriptstyle (\ |\xi_0)} \\
S & \xrightarrow{g} & \mathbb{A}^1_{\mathbb{C}}.
\end{array}$$

The structure sheaf of X_{ξ_0} is

$$\mathcal{O}_{X_{\xi_0}} = R \otimes_{\mathbb{C}[t]} \mathcal{O}_{V_\lambda},$$

where $\mathcal{O}_{V_\lambda}(A) = A[x]$ for any $\mathbb{C}$-algebra A, on which $A[t]$ acts by $t \mapsto (x|\xi_0)$. The special fibre of X_{ξ_0} is the s-scheme

$$X_{\xi_0,s} = f_{\xi_0}^{-1}(s) = f_{\xi_0}^{-1}(0) = \{x \in V_\lambda \mid (x|\xi_0) = 0\}.$$

Define

$$\mathsf{R}\Phi_{f_{\xi_0}} \colon \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{Z_{H_\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0) \times_s S)$$

by

$$\begin{array}{ccc} \mathsf{D}_{H_\lambda}(V_\lambda) & \xrightarrow{\mathsf{R}\Phi_{f_{\xi_0}}} & \mathsf{D}_{Z_{H_\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0) \times_s S) \\ {\scriptstyle \text{forget}}\downarrow & & \uparrow{\scriptstyle \mathsf{R}\Phi_{X_{\xi_0}}} \\ \mathsf{D}_{Z_{H_\lambda}(\xi_0)}(V_\lambda) & \xrightarrow{\text{BC}} & \mathsf{D}_{Z_{H_\lambda}(\xi_0)\times_s S}(X_{\xi_0}) \end{array}$$

where $\mathrm{BC}\colon \mathsf{D}_{Z_{H_\lambda}(\xi_0)}(V_\lambda) \to \mathsf{D}_{Z_{H_\lambda}(\xi_0)\times_s S}(X_{\xi_0})$ is the pullback along $X_{\xi_0} \to V_\lambda$.

The following is a summary of properties of Ev_C ([CFMMX, Propositions 6.4.1, 6.5.1, 6.6.2, 6.8.1, 6.9.1]).

Theorem 4.2. *Let $C \subset V_\lambda$ be an H_λ-orbit.*

(a) *The functor*
$$\mathsf{Ev}_C \colon \mathsf{D}_{H_\lambda}(V_\lambda) \to \mathsf{D}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}} \times_s S)$$
is exact.

(b) *For every $\mathcal{F} \in \mathsf{D}_{H_\lambda}(V_\lambda)$ and every $(x_0, \xi_0) \in T^*_C(V_\lambda)_{\mathrm{reg}}$, there is a canonical isomorphism*
$$(\mathsf{Ev}_C \mathcal{F})_{(x_0,\xi_0)} \cong (\mathsf{R}\Phi_{f_{\xi_0}} \mathcal{F})_{x_0}.$$

(c) *If $\mathcal{F} \in \mathsf{D}_{H_\lambda}(V_\lambda)$, then*
$$\mathsf{Ev}_C \mathcal{F} = 0 \quad \textit{unless} \quad C \subset \operatorname{supp} \mathcal{F} = \bigcup_i \operatorname{supp} H^i(\mathcal{F}).$$

(d) *For any H-equivariant local system $\mathcal{L}$ on C,*
$$\mathsf{Ev}_C \mathcal{IC}(C, \mathcal{L}) = \mathsf{Ev}_C \mathcal{IC}(C, \mathbf{1}_C) \otimes (\mathcal{L} \boxtimes \mathbf{1}_{C^*})|_{T^*_C(V_\lambda)_{\mathrm{reg}}},$$
where $\mathbf{1}_C$ is the constant sheaf on C.

(e) *If $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$, then*
$$\mathsf{Ev}_C \mathcal{P}[\dim C^* - 1] \in \mathsf{Per}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}}).$$
We set
$${}^p\mathsf{Ev}_C \coloneqq \mathsf{Ev}_C[\dim C^* - 1] \colon \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Per}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}}).$$

(f) *Suppose that $T^*_C(V_\lambda)_{\mathrm{sreg}}$ is non-empty. If $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$, then the restriction of ${}^p\mathsf{Ev}_C \mathcal{P}$ to $T^*_C(V_\lambda)_{\mathrm{sreg}}$ is a local system concentrated in degree $\dim V_\lambda$. We set*
$$\mathsf{Evs}_C \coloneqq {}^p\mathsf{Ev}_C[-\dim V_\lambda] \colon \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}}).$$

For a description of the stalk of $\mathsf{Ev}_C \mathcal{IC}(C, \mathbf{1}_C)$ at $(x, \xi) \in T^*_C(V_\lambda)_{\mathrm{reg}}$, see [CFMMX, Theorem 6.7.5]. In particular, we note that

$$\operatorname{rank} \mathsf{Ev}_C \mathcal{IC}(C, \mathbf{1}_C) = 1.$$

If we put $\mathcal{T}_C = \mathsf{Evs}_C \mathcal{IC}(C, \mathbf{1}_C) \in \mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}})$, by Theorem 4.2 (d), we have $\mathsf{Evs}_C \mathcal{IC}(C, \mathcal{L}) = \mathcal{T}_C \otimes (\mathcal{L} \boxtimes \mathbf{1}_{C^*})|_{T^*_C(V_\lambda)_{\mathrm{sreg}}}$ for every $\mathcal{L} \in \mathsf{Loc}_{H_\lambda}(C)$.

Now we normalize Ev_C as follows.

Definition 4.3. *We define a functor* $\mathsf{NEv}_C\colon D_{H_\lambda}(V_\lambda) \to D_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{reg}})$ *by*

$$\mathsf{NEv}_C = (\mathsf{Ev}_C\mathcal{IC}(C, \mathbf{1}_C))^\vee \otimes \mathsf{Ev}_C,$$

where we put $(\mathsf{Ev}_C\mathcal{IC}(C, \mathbf{1}_C))^\vee = \mathcal{H}\mathrm{om}(\mathsf{Ev}_C\mathcal{IC}(C, \mathbf{1}_C), \mathbf{1}_{T^*_C(V_\lambda)_{\mathrm{reg}}})$*, and we use the left derived tensor product. We refer to* NEv_C *as the* ***normalized microlocal vanishing cycles functor****.*

When $T^*_C(V_\lambda)_{\mathrm{sreg}}$ is non-empty, we set

$$\mathsf{NEvs}_C\mathcal{F} = (\mathsf{NEv}_C\mathcal{F}[\dim C^* - 1 - \dim V_\lambda])|_{T^*_C(V_\lambda)_{\mathrm{sreg}}}$$

for $\mathcal{F} \in D_{H_\lambda}(V_\lambda)$. Then

$$\mathsf{NEvs}_C = \mathcal{T}^\vee_C \otimes \mathsf{Evs}_C,$$

where $\mathcal{T}^\vee_C$ is the dual local system of $\mathcal{T}_C$.

Theorem 4.4 ([CFMMX, Theorem 6.10.1])**.** *Suppose that* $T^*_C(V_\lambda)_{\mathrm{sreg}}$ *is non-empty.*

(a) *The functor* $\mathsf{NEvs}_C\colon \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathsf{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\mathrm{sreg}})$ *is exact.*
(b) *If* $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$*, then* $\mathsf{NEvs}_C\mathcal{P} = 0$ *unless* $C \subset \operatorname{supp}\mathcal{P}$*.*
(c) *If* $\mathcal{P} \in \mathsf{Per}_{H_\lambda}(V_\lambda)$*, then*

$$\operatorname{rank}(\mathsf{NEvs}_C\mathcal{P}) = \operatorname{rank}(\mathsf{R}\Phi_{f_\xi}\mathcal{P})_x$$

for every $(x, \xi) \in T^*_C(V_\lambda)_{\mathrm{sreg}}$*.*
(d) *For every* $\mathcal{L} \in \mathsf{Loc}_{H_\lambda}(C)$*,*

$$\mathsf{NEvs}_C\mathcal{IC}(C, \mathcal{L}) = (\mathcal{L} \boxtimes \mathbf{1}_{C^*})|_{T^*_C(V_\lambda)_{\mathrm{sreg}}}.$$

In particular,

$$\operatorname{rank}(\mathsf{NEvs}_C\mathcal{IC}(C, \mathcal{L})) = \operatorname{rank}\mathcal{L}.$$

5. A-packets v.s. ABV packets

In this section, we define the ABV packets, and state a conjecture.

5.1. **Definition of ABV packets and the main conjecture.** Now we can define the Adams–Barbasch–Vogan packets. Let $\psi \in \Psi(G/F)$ and set $\lambda = \lambda_\psi$. Consider the H_λ-orbit $C_\psi = C_{\phi_\psi}$ in V_λ corresponding to the L-parameter ϕ_ψ for ψ via Proposition 2.2. Recall in Proposition 4.1 that there is a canonical injection

$$\Pi_\lambda = \bigsqcup_{\phi \in \Phi_\lambda(G/F)} \Pi_\phi \to \mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}},\ \pi \mapsto \mathcal{P}(\pi).$$

On the other hand, ψ gives a strongly regular point $(x_\psi, \xi_\psi) \in T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}$ (Proposition 3.5). It determines an equivalence

$$\mathsf{Loc}_{H_\lambda}(T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}) \to \mathrm{Rep}(A_{T^*_{C_\psi}(V_\lambda)_{\mathrm{sreg}}}) \cong \mathrm{Rep}(A_\psi).$$

Define

$$\mathsf{Ev}_\psi\colon \mathsf{Per}_{H_\lambda}(V_\lambda) \to \mathrm{Rep}(A_\psi)$$

by the composition of Evs_{C_ψ} and this equivalence, and set

$$\mathsf{NEv}_\psi = \mathcal{T}^\vee_\psi \otimes \mathsf{Ev}_\psi,$$

where $\mathcal{T}_\psi = \mathsf{Ev}_\psi\mathcal{IC}(C_\psi, \mathbf{1}_{C_\psi})$ is the representation of A_ψ corresponding to $\mathcal{T}_{C_\psi}$.

Definition 5.1. *For $\psi \in \Psi(G/F)$, we define the **Adams–Barbasch–Vogan packet** (shortly, the **ABV packet**) Π_ψ^{ABV} of $G(F)$ associated with ψ by*

$$\Pi_\psi^{\mathrm{ABV}} = \left\{ \pi \in \Pi_{\lambda_\psi} \;\middle|\; \mathsf{Ev}_{C_\psi} \mathcal{P}(\pi) \neq 0 \right\}.$$

Recall that there is Arthur's A-packet $\Pi_\psi \subset \Pi(G(F))$. The following is the main conjecture in [CFMMX].

Conjecture 5.2. *Let G be split SO_{2n+1} or Sp_{2n}. For an A-parameter $\psi \in \Psi(G/F)$, we would have*

$$\Pi_\psi = \Pi_\psi^{\mathrm{ABV}}.$$

Moreover, the map

$$\Pi_\psi \to \mathrm{Irr}(A_\psi),\ \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

would be given by

$$\langle a_\psi a_s, \pi \rangle_\psi = (-1)^{\dim C_\psi - \dim \mathrm{supp}\, \mathcal{P}(\pi)} \mathrm{tr}(\mathsf{NEv}_\psi \mathcal{P}(\pi))(a_s)$$

for any $a_s \in A_\psi$, where a_ψ is the image of $\psi(\mathbf{1}, \mathbf{1}_2, -\mathbf{1}_2)$ in A_ψ.

Let $\mathsf{K}\Pi_\lambda$ and $\mathsf{KPer}_{H_\lambda}(V_\lambda)$ be the Grothendieck groups of Π_λ and $\mathsf{Per}_{H_\lambda}(V_\lambda)$, i.e., the free abelian groups for which Π_λ and $\mathsf{Per}_{H_\lambda}(V_\lambda)^{\mathrm{simple}}_{/\mathrm{iso}}$ are bases, respectively. Define a bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathsf{K}\Pi_\lambda \times \mathsf{KPer}_{H_\lambda}(V_\lambda) \to \mathbb{Z}$$

by

$$\langle \pi, \mathcal{P} \rangle = \begin{cases} (-1)^{\dim \mathrm{supp}\,(\mathcal{P})} & \text{if } \mathcal{P} = \mathcal{P}(\pi), \\ 0 & \text{otherwise.} \end{cases}$$

If we set

$$\eta_{\psi,s} = \sum_{\pi \in \Pi_\psi} \langle a_\psi a_s, \pi \rangle_\psi \, \pi,$$

then Conjecture 5.2 is equivalent that

$$\langle \eta_{\psi,s}, \mathcal{P} \rangle = (-1)^{\dim C_\psi} \mathrm{tr}(\mathsf{NEv}_\psi \mathcal{P})(a_s)$$

for any $a_s \in A_\psi$ and any $\mathcal{P} \in \mathsf{KPer}_{H_\lambda}(V_\lambda)$.

In [CFMMX, PART II], several examples were given.

Proposition 5.3. *Conjecture 5.2 holds for the following cases:*

1. *q is odd, $G = \mathrm{SL}_2 = \mathrm{Sp}_2$ and $\psi \in \Psi(G/F)$ with $\lambda_\psi = \chi_1 \oplus \chi_2 \oplus \chi_3$, where χ_1, χ_2, χ_3 are the three distinct non-trivial quadratic characters of $F^\times$;*
2. *$G = \mathrm{SO}_3$ and $\psi \in \Psi(G/F)$ with $\lambda_\psi = |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}}$;*
3. *$G = \mathrm{SO}_5$ and $\psi \in \Psi(G/F)$ with $\lambda_\psi = |\cdot|^{\frac{3}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}}$;*
4. *$G = \mathrm{SO}_5$ and $\psi \in \Psi(G/F)$ with $\lambda_\psi = |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}}$;*
5. *$G = \mathrm{SO}_7$ and $\psi \in \Psi(G/F)$ with $\lambda_\psi = |\cdot|^{\frac{3}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}}$.*

Appendix A. Supplements

In this appendix, we explain several supplementary topics.

A.1. **Algebraic representations of** $\mathrm{SL}_2(\mathbb{C})$**.** For each positive integer k, there is a unique (up to isomorphism) irreducible algebraic representation S_k of $\mathrm{SL}_2(\mathbb{C})$ of dimension k. Note that the representation $w \mapsto S_k(d_w)$ of W_F is isomorphic to

$$|\cdot|^{\frac{k-1}{2}} \oplus |\cdot|^{\frac{k-3}{2}} \oplus \cdots \oplus |\cdot|^{-\frac{k-1}{2}}.$$

The representation S_k is realized as the $(k-1)$-th symmetric power $\mathrm{Sym}^{k-1}\mathbb{C}^2$ of the standard representation $S_2 = \mathbb{C}^2$. It has a non-degenerate $\mathrm{SL}_2(\mathbb{C})$-invariant bilinear form

$$[\cdot,\cdot]\colon \mathrm{Sym}^{k-1}\mathbb{C}^2 \times \mathrm{Sym}^{k-1}\mathbb{C}^2 \to \mathbb{C},$$
$$\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}, \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \cdots \begin{pmatrix} c_{k-1} \\ d_{k-1} \end{pmatrix}\right) \mapsto \prod_{\sigma \in \mathfrak{S}_{k-1}} \det \begin{pmatrix} a_i & c_{\sigma(i)} \\ b_i & d_{\sigma(i)} \end{pmatrix}.$$

In particular, S_k is self-dual of sign $(-1)^{k-1}$.

If we put

$$\mathbf{e}_p = \frac{1}{p!} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{k-p-1} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{p} \in \mathrm{Sym}^{k-1}\mathbb{C}^2$$

for $0 \le p \le k-1$, then we have

$$[\mathbf{e}_p, \mathbf{e}_q] = \begin{cases} (-1)^p & \text{if } q = k-p-1, \\ 0 & \text{otherwise.} \end{cases}$$

We identify $\mathrm{Sym}^{k-1}\mathbb{C}^2$ with $\mathbb{C}^k$ by $\mathbf{e}_0, \dots, \mathbf{e}_{k-1}$. Since

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mathbf{e}_p = \sum_{l=0}^{p} \frac{t^{p-l}}{(p-l)!} \mathbf{e}_l, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mathbf{e}_p = \sum_{l=p}^{k-1} \frac{l!(k-p-1)!}{p!(l-p)!(k-l-1)!} t^{l-p} \mathbf{e}_l,$$

the actions of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$ on $\mathrm{Sym}^{k-1}\mathbb{C}^2 \cong \mathbb{C}^k$ are given by

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & & & \\ 1(k-1) & 0 & & & \\ & 2(k-2) & \ddots & & \\ & & \ddots & 0 & \\ & & & (k-1)1 & 0 \end{pmatrix},$$

respectively, where they are in $\mathfrak{so}(k,\mathbb{C})$ or $\mathfrak{sp}(k,\mathbb{C})$ with respect to

$$\begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{k-1} & & & \end{pmatrix}$$

according to $k \equiv 1 \bmod 2$ or $k \equiv 0 \bmod 2$.

A.2. **Equivariant fundamental groups and local systems.** Here, we define equivariant fundamental groups and equivariant local systems.

Let C and X be topological spaces equipped with continuous actions of a topological group H. A subset U of X is **stable** if $h \cdot u \in U$ for any $u \in U$ and $h \in H$. We call X **H-equivariantly connected** if only $\emptyset$ and X are the open and closed stable subsets of X. A map $f\colon X \to C$ is said to be **H-equivariant** if $f(h \cdot x) = h \cdot f(x)$ for any $x \in X$ and $h \in H$. Moreover, it is called an **H-equivariant covering of** C if f is a covering in the usual sense, i.e., f is surjective and for any $p \in C$, there exists an open neighborhood U of p in C such that $f^{-1}(U)$ is a disjoint union of open subsets V_i of X satisfying that the restriction $f|V_i$ is a homeomorphism $V_i \xrightarrow{\sim} U$. An **H-equivariant base point** of X is an H-equivariant map $b\colon H \to X$. Fix an H-equivariant base point $c\colon H \to C$ of C. We say that an H-equivariant map $f\colon X \to C$ is **base point-preserving** if $f(b(h)) = c(h)$ for any $h \in H$. Such a map is denoted by $f\colon (X, b) \to (C, c)$. A **universal H-equivariant covering of** C is an H-equivariant covering map $\pi\colon (\widetilde{C}, \widetilde{c}) \to (C, c)$ with $\widetilde{C}$ being H-equivariantly connected such that for any H-equivariant covering $f\colon (X, b) \to (C, c)$ with X being H-equivariantly connected, there exists a unique H-equivariant continuous map $\hat{f}\colon (\widetilde{C}, \widetilde{c}) \to (X, b)$ such that the diagram

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\hat{f}} & X \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle f} & \\ & C & \end{array}$$

commutes. We define the **H-equivariant fundamental group** $\pi_1(C)_H$ by the group of H-equivariant homeomorphisms $\hat{f}\colon \widetilde{C} \to \widetilde{C}$ such that the diagram

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\hat{f}} & \widetilde{C} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & C & \end{array}$$

commutes. By the universality, $\pi_1(C)_H$ is uniquely determined by C, up to a unique isomorphism.

Example A.1. *Let $C = \{0\}$ be the set of a point, equipped with the trivial action of the orthogonal group $H = \mathrm{O}(m, \mathbb{C})$. Then the covering map is a map from a discrete topological space X equipped with a continuous action of $\mathrm{O}(m, \mathbb{C})$. Hence this action factors through the quotient $\mathrm{O}(m, \mathbb{C})/\mathrm{SO}(m, \mathbb{C}) \cong \{\pm 1\}$. The discrete space X is $\mathrm{O}(m, \mathbb{C})$-equivariantly connected if and only if the action of $\mathrm{O}(m, \mathbb{C})$ is transitive. In particular, an $\mathrm{O}(m, \mathbb{C})$-equivariant universal cover of C is given by the set $\widetilde{C} = \{p_+, p_-\}$ of two points with the action $\alpha \cdot p_\epsilon = p_{\epsilon \det \alpha}$ for $\alpha \in \mathrm{O}(m, \mathbb{C})$ and $\epsilon \in \{\pm 1\}$. Therefore,*

$$\pi_1(\{0\})_{\mathrm{O}(m,\mathbb{C})} = \{\pm 1\}.$$

More generally, when C is a quotient H/Z for some subgroup Z of H with the canonical base point $c\colon H \to C$, for any H-equivariant covering $f\colon (X, x) \to (C, c)$ of C, we have $hx = x$ for $h \in Z^0$. In particular, a universal H-equivariant covering of C is given by $\widetilde{C} = H/Z^0$, and

the H-equivariant fundamental group is given by

$$\pi_1(X)_H \cong \pi_0(Z) = Z/Z^0.$$

Let $m\colon H \times X \to X$ be an group action in the category of algebraic varieties (over $\mathbb{C}$). We also consider the projection $p\colon H \times X \to X$. An **H-equivariant sheaf** on X is a sheaf $\mathcal{F}$ on X equipped with an isomorphism of sheaves on $H \times X$

$$\varphi\colon m^{-1}\mathcal{F} \xrightarrow{\sim} p^{-1}\mathcal{F}$$

such that φ satisfies the usual cocycle condition on $H \times H \times X$. On the stalk level, φ implies an isomorphism $\mathcal{F}_{hx} \cong \mathcal{F}_x$ for $h \in H$ and $x \in X$. The cocycle condition says that the isomorphism $\mathcal{F}_{ghx} \cong \mathcal{F}_x$ is the same as the composition $\mathcal{F}_{ghx} \cong \mathcal{F}_{hx} \cong \mathcal{F}_x$ for $g, h \in H$ and $x \in X$.

Recall that a local system on X is a locally constant sheaf on X. We denote the category of H-equivariant local systems on X by $\mathsf{Loc}_H(X)$. There is a relation between $\mathsf{Loc}_H(X)$ and representations of $\pi_1(X)_H$. Recall that $\pi_1(X)_H$ is the group of H-equivariant homeomorphisms on the universal H-equivariant cover $\widetilde{X}$ of X which commute with the projection $\pi\colon \widetilde{X} \to X$. Given a representation $\rho\colon \pi_1(X)_H \to \mathrm{GL}(V)$, we consider the sections of the bundle $(\widetilde{X} \times V)/\pi_1(X)_H \to X$. More precisely, for an open set U of X, we set $\mathcal{L}_\rho(U)$ to be the space of locally constant functions $f\colon \pi^{-1}(U) \to V$ satisfying

$$f(\gamma x) = \rho(\gamma) f(x)$$

for $\gamma \in \pi_1(X)_H$. Then $\mathcal{L}_\rho$ is an H-equivariant local system on X. The map $\rho \mapsto \mathcal{L}_\rho$ gives an identification $\mathrm{Rep}(\pi_1(X)_H) \to \mathsf{Loc}_H(X)$.

A.3. **Kazhdan–Lusztig conjecture.** Let $\lambda \in \Lambda(G/F)$. Recall that $\mathsf{K}\Pi_\lambda$ and $\mathsf{KPer}_{H_\lambda}(V_\lambda)$ are the free abelian groups with the canonical bases

$$\{\pi(\phi,\eta) \mid \phi \in \Phi_\lambda(G/F),\ \eta \in \widehat{A_\phi},\ \eta(-\mathbf{1}) = 1 \text{ if } G = \mathrm{SO}_{2n+1}\},$$
$$\left\{\mathcal{IC}(C,\mathcal{L}) \;\middle|\; H\text{-orbit } C \subset V_\lambda,\ \mathcal{L} \in \mathsf{Loc}_{H_\lambda}(C)^{\mathrm{simple}}_{/\mathrm{iso}}\right\},$$

respectively. They also have other bases, consisting of standard modules and standard sheaves.

For given $\phi \in \Phi_\lambda(G/F)$, one can obtain a parabolic subgroup $P = MN$ of G such that ϕ factors through the embedding $\widehat{M} \hookrightarrow \widehat{G}$ so that ϕ can be regarded as an L-parameter for M, which is essentially tempered. Then $\eta \in \widehat{A_\phi}$ (with $\eta(-\mathbf{1}) = 1$) gives an irreducible essentially tempered representation $\pi_M(\phi,\eta)$ of $M(F)$. Moreover, one can assume that the exponents of $\pi_M(\phi,\eta)$ are in the positive Weyl chamber with respect to P. In this case, the parabolic induction

$$M(\phi,\eta) = \mathrm{Ind}_{P(F)}^{G(F)}(\pi_M(\phi,\eta))$$

is called a **standard module**. The representation $\pi(\phi,\eta)$ of $G(F)$ is the unique irreducible quotient of the standard module $M(\phi,\eta)$. Moreover,

$$\{M(\phi,\eta) \mid \phi \in \Phi_\lambda(G/F),\ \eta \in \widehat{A_\phi},\ \eta(-\mathbf{1}) = 1 \text{ if } G = \mathrm{SO}_{2n+1}\}$$

forms a basis of $\mathsf{K}\Pi_\lambda$. In particular, for two pairs (ϕ,η) and (ϕ',η'), there exists a non-negative integer $m_{\mathrm{rep}}((\phi,\eta),(\phi',\eta'))$ such that

$$M(\phi',\eta') = \sum_{(\phi,\eta)} m_{\mathrm{rep}}((\phi,\eta),(\phi',\eta')) \cdot \pi(\phi,\eta)$$

in $\mathsf{K}\Pi_\lambda$. We call $(m_{\mathrm{rep}}((\phi,\eta),(\phi',\eta')))_{(\phi,\eta),(\phi',\eta')}$ the **multiplicity matrix**. It is known that this matrix is an "upper-triangular unipotent matrix" in a certain sense (see e.g., [Ar13, §2.2]).

Similarly, for an H_λ-orbit $C \subset V_\lambda$ and a simple equivariant local system $\mathcal{L} \in \mathsf{Loc}_{H_\lambda}(C)^{\mathrm{simple}}_{/\mathrm{iso}}$, we consider the **shifted standard sheaf**

$$\mathcal{S}(C,\mathcal{L}) = j_{C!}\mathcal{L}[\dim C],$$

where $j_C \colon C \hookrightarrow V_\lambda$ is the inclusion. Then

$$\left\{ \mathcal{S}(C,\mathcal{L}) \;\middle|\; H\text{-orbit } C \subset V_\lambda,\ \mathcal{L} \in \mathsf{Loc}_{H_\lambda}(C)^{\mathrm{simple}}_{/\mathrm{iso}} \right\}$$

forms a basis of $\mathsf{KPer}_{H_\lambda}(V_\lambda)$. In particular, for two pairs $(C,\mathcal{L})$ and $(C',\mathcal{L}')$, there exists an integer $m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L}))$ such that

$$\mathcal{IC}(C,\mathcal{L}) = \sum_{(C',\mathcal{L}')} m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})) \cdot \mathcal{S}(C',\mathcal{L}')$$

in $\mathsf{KPer}_{H_\lambda}(V_\lambda)$. We call $(m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})))_{(C',\mathcal{L}'),(C,\mathcal{L})}$ the **geometric multiplicity matrix**. It is known that $m_{\mathrm{geo}}((C,\mathcal{L}),(C,\mathcal{L})) = 1$ and $m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})) = 0$ unless $C' \subset \overline{C}$. Moreover, if we set

$$m'_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})) = (-1)^{\dim C - \dim C'} m_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})),$$

then it is a non-negative integer. We call $(m'_{\mathrm{geo}}((C',\mathcal{L}'),(C,\mathcal{L})))_{(C',\mathcal{L}'),(C,\mathcal{L})}$ the **normalized geometric multiplicity matrix**.

Recall that a pair (ϕ,η) gives a pair $(C_\phi,\mathcal{L}_\rho)$, and there exists a bilinear form

$$\langle \cdot,\cdot \rangle : \mathsf{K}\Pi_\lambda \times \mathsf{KPer}_{H_\lambda}(V_\lambda) \to \mathbb{Z}$$

given by

$$\langle \pi(\phi,\eta), \mathcal{IC}(C,\mathcal{L}) \rangle = \begin{cases} (-1)^{\dim C} & \text{if } (C,\mathcal{L}) = (C_\phi,\mathcal{L}_\rho), \\ 0 & \text{otherwise.} \end{cases}$$

The **Kazhdan–Lusztig conjecture** predicts that

$$\langle M(\phi,\eta), \mathcal{S}(C,\mathcal{L}) \rangle = \begin{cases} (-1)^{\dim C} & \text{if } (C,\mathcal{L}) = (C_\phi,\mathcal{L}_\rho), \\ 0 & \text{otherwise.} \end{cases}$$

When G and $\lambda = \lambda_\psi$ are in the cases in Proposition 5.3, this conjecture is proven by showing

$$m_{\mathrm{rep}}((\phi,\eta),(\phi',\eta')) = m'_{\mathrm{geo}}((C_\phi,\mathcal{L}_\phi),(C_{\phi'},\mathcal{L}_{\eta'}))$$

for every pairs (ϕ,η) and (ϕ',η').

Appendix B. Representations, $\mathcal{D}$-modules, and perverse sheaves

To define p-adic ABV packets, one uses a relation between irreducible representations of $G(F)$ with simple objects of equivariant perverse sheaves (Proposition 2.2). For real reductive groups, this relation is a conclusion of a deep story. In this appendix, we try to explain this story. Note that it is hard to say that this appendix is mathematically accurate. The readers should refer to relevant references for details.

Let G be a quasi-split connected semisimple group over $\mathbb{R}$, and K be a maximal compact subgroup of $G(\mathbb{R})$. Assume that $G(\mathbb{R})$ is connected. Fix a rational Borel subgroup $B = TU$ of

G. We denote the Lie algebras of $G(\mathbb{R})$, $T(\mathbb{R})$ and K by $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{k}$, and their complexifications by $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$, respectively. The Weyl group is denoted by $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

B.1. **Casselman–Wallach globalization.** Let $\mathrm{Rep}(G(\mathbb{R}))$ be the category of smooth admissible Fréchet representations of moderate growth, and $\mathrm{Mod}_{\mathrm{adm}}(\mathfrak{g}_{\mathbb{C}}, K)$ be the category of admissible Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. The isomorphism classes of irreducible objects in these categories are denoted by $\mathrm{Irr}(G(\mathbb{R}))$ and $\mathrm{Irr}(\mathfrak{g}_{\mathbb{C}}, K)$, respectively. The category $\mathrm{Rep}(G(\mathbb{R}))$ seems to be difficult for topological reasons, whereas $\mathrm{Mod}_{\mathrm{adm}}(\mathfrak{g}_{\mathbb{C}}, K)$ seems to be easier because it is purely algebraic.

Taking K-finite vectors, we obtain a functor

$$\mathrm{HC}\colon \mathrm{Rep}(G(\mathbb{R})) \to \mathrm{Mod}_{\mathrm{adm}}(\mathfrak{g}_{\mathbb{C}}, K).$$

Theorem B.1 (Casselman–Wallach [C89], [W92], [BK14])**.** *There exists a quasi-inverse functor* $\mathrm{Mod}_{\mathrm{adm}}(\mathfrak{g}_{\mathbb{C}}, K) \to \mathrm{Rep}(G(\mathbb{R}))$ *of* HC*, called the* ***Casselman–Wallach globalization functor****. Hence the functor* $\mathrm{HC}\colon \mathrm{Rep}(G(\mathbb{R})) \to \mathrm{Mod}_{\mathrm{adm}}(\mathfrak{g}_{\mathbb{C}}, K)$ *is an equivalence of categories.*

By the Casselman–Wallach globalization, one can consider $\mathrm{Irr}(\mathfrak{g}_{\mathbb{C}}, K)$ instead of $\mathrm{Irr}(G(\mathbb{R}))$.

B.2. **Casselman's subrepresentation theorem.** A character χ of $T(\mathbb{R})$ gives a principal series representation $I(\chi) = \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}(\chi)$. Let $I(\chi)_K$ be the K-finite part of $I(\chi)$.

Theorem B.2 (Casselman's subrepresentation theorem [C78])**.** *For any* $\pi \in \mathrm{Mod}_{\mathrm{adm}}(\mathfrak{g}_{\mathbb{C}}, K)$*, there exists a character* χ *of* $T(\mathbb{R})$ *such that*

$$\pi \hookrightarrow I(\chi)_K.$$

There is a p-adic analogue of this theorem, but it asserts that any irreducible smooth representation of $G(F)$, where F is p-adic, is a subrepresentation of the parabolic induction from a supercuspidal representation. One may understand that Casselman's subrepresentation theorem says that real reductive Lie groups have few supercuspidal representations.

B.3. **Beilinson–Bernstein correspondence.** Recall that the principal series representation $I(\chi)$ is a space of sections of the $G(\mathbb{R})$-equivariant vector bundle $G(\mathbb{R}) \times_{B(\mathbb{R})} \chi \to G(\mathbb{R})/B(\mathbb{R})$. Now let us consider the $\mathcal{D}$-modules on the complete flag variety $\mathcal{B} = G(\mathbb{C})/B(\mathbb{C})$. For a reference, see [HTT08, §11].

Fix $\lambda \in \mathfrak{t}_{\mathbb{C}}^*/W$. For a representative $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, the character $\chi_{\lambda+\rho}$ of $T(\mathbb{C})$ corresponding to $\lambda + \rho$ gives a $G(\mathbb{C})$-equivariant line bundle $\mathcal{L}(\lambda + \rho)$ on $\mathcal{B}$. We consider the sheaf D_λ of twisted differential operators acting on $\mathcal{L}(\lambda + \rho)$. Let $\mathrm{Mod}_{qc}(D_\lambda)$ be the abelian category of D_λ-modules which are quasi-coherent over $\mathcal{O}_{\mathcal{B}}$, and $\mathrm{Mod}(\mathfrak{g}_{\mathbb{C}})_\lambda$ be the category of $U(\mathfrak{g}_{\mathbb{C}})$-modules with infinitesimal character λ, where $U(\mathfrak{g}_{\mathbb{C}})$ is the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. The global section functor gives a functor

$$\Gamma(\mathcal{B}, \cdot)\colon \mathrm{Mod}_{qc}(D_\lambda) \to \mathrm{Mod}(\mathfrak{g}_{\mathbb{C}})_\lambda.$$

The set of roots of T and its positive system with respect to B are denoted by Δ and Δ^+, respectively. We put

$$P = \{\lambda \in \mathfrak{t}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\ (\alpha \in \Delta)\}.$$

It is called the **weight lattice in** $\mathfrak{t}^*$.

Theorem B.3 (Beilinson–Bernstein correspondence [HTT08, Corollary 11.2.6])**.** *When $\lambda \in P$ satisfies that*

$$\left\langle \lambda, \alpha^\vee \right\rangle < 0, \quad \alpha \in \Delta^+,$$

then the functor $\Gamma(\mathcal{B}, \cdot)$ induces equivalences

$$\mathrm{Mod}_{qc}(D_\lambda) \cong \mathrm{Mod}(\mathfrak{g}_{\mathbb{C}})_\lambda$$

of abelian categories. The inverse functor is given by $D_\lambda \otimes_{U(\mathfrak{g}_{\mathbb{C}})} (\cdot)$.

However, admissible representations of $G(\mathbb{R})$ are related with not just $\mathfrak{g}_{\mathbb{C}}$-modules, but $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. Let $\mathrm{Rep}(G(\mathbb{R}))_\lambda$ and $\mathrm{Mod}(\mathfrak{g}_{\mathbb{C}}, K)_\lambda$ be the subcategories of $\mathrm{Rep}(G(\mathbb{R}))$ and $\mathrm{Mod}(\mathfrak{g}_{\mathbb{C}}, K)$ consisting of objects with infinitesimal character λ, respectively. To relate the category $\mathrm{Mod}(\mathfrak{g}_{\mathbb{C}}, K)_\lambda$ of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules with $\mathcal{D}$-modules, we need to consider the category of K-equivariant $\mathcal{D}$-modules $\mathrm{Mod}_{qc}(D_\lambda, K)$. Then we have

$$\mathrm{Mod}_{qc}(D_\lambda, K) \cong \mathrm{Mod}(\mathfrak{g}_{\mathbb{C}}, K)_\lambda.$$

For more precision, see, e.g., [HTT08, Theorem 11.5.3, Remark 11.5.4] and its references. We notice that an object in $\mathrm{Mod}(\mathfrak{g}_{\mathbb{C}}, K)_\lambda$ is not necessarily admissible.

B.4. **Riemann–Hilbert correspondence.** Next, we recall Hilbert's twenty-first problem or the Riemann–Hilbert problem. Before stating this problem, let us consider the following example.

Fix a complex number $a \in \mathbb{C}$, and consider the differential equation

$$\frac{df}{dz} = \frac{a}{z} f$$

on $\mathbb{C} \setminus \{0\}$. This equation has regular singularities at 0 and ∞ in the projective line $\mathbb{P}^1_{\mathbb{C}}$. The local solutions of the equation are of the form $f(z) = c \cdot z^a$ for constants $c \in \mathbb{C}$. If $a \notin \mathbb{Z}$, then the function z^a cannot be made well-defined on all of $\mathbb{C} \setminus \{0\}$. This means that the vector bundle

$$E_a = \{c \cdot z^a \mid c \in \mathbb{C}\} \ni c \cdot z^a \mapsto z \in \mathbb{C} \setminus \{0\}$$

is a non-trivial line bundle (local system) on $\mathbb{C} \setminus \{0\}$. In other words, the differential equation has non-trivial monodromy. Explicitly, this monodromy is the 1-dimensional representation of the fundamental group $\pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$ in which a generator (a loop around the origin) acts by multiplication by $e^{2\pi\sqrt{-1}a}$.

This is a typical example that local solutions of regular differential equations give local systems

$$\left(\frac{df}{dz} = \frac{a}{z} f \right) \xmapsto{\text{local solutions}} E_a = \{c \cdot z^a \mid c \in \mathbb{C}\} \in \mathsf{Loc}(\mathbb{C} \setminus \{0\}).$$

The converse of this observation, namely, **the existence of linear differential equations having a prescribed monodromy** is called Hilbert's twenty-first problem, or more commonly, the Riemann–Hilbert problem.

Now, roughly speaking, $\mathcal{D}$-modules are "gluing of differential equations", whereas perverse sheaves are "gluing of local systems". The above observation is generalized to a correspondence between $\mathcal{D}$-modules and perverse sheaves.

Theorem B.4 (Riemann–Hilbert correspondence [HTT08, Theorem 7.2.5])**.** *Let X be a complex manifold or a smooth algebraic variety (over $\mathbb{C}$). We denote by $\mathrm{Mod}_{rh}(D_X)$ the category of "regular holonomic" $\mathcal{D}$-modules on X, and by $\mathsf{Per}(\mathbb{C}_X)$ the category of perverse sheaves on X. Then there exists an equivalence of categories*

$$\mathrm{DR}_X \colon \mathrm{Mod}_{rh}(D_X) \to \mathsf{Per}(\mathbb{C}_X).$$

*The functor DR_X is called the **de Rham functor**.*

In fact, one can define the category of equivariant perverse sheaves on X, and can relate it with the one of equivariant $\mathcal{D}$-modules. See also (the proof of) [HTT08, Theorem 11.6.1].

B.5. **Conclusion.** We conclude that one can relate irreducible representations of $G(\mathbb{R})$ with equivariant perverse sheaves via the following correspondences in the following rough diagram:

Irreducible representations of $G(\mathbb{R})$

$\updownarrow$ K-finite part, Casselman–Wallach globalization

Simple $(\mathfrak{g}_{\mathbb{C}}, K)$-modules

$\updownarrow$ the Beilinson–Bernstein correspondence

Simple K-equivariant $\mathcal{D}$-modules on the flag variety $\mathcal{B}$

$\updownarrow$ the Riemann–Hilbert correspondence

Simple K-equivariant perverse sheaves on the flag variety $\mathcal{B}$.

Acknowledgements. This note was written for the IMS program "On the Langlands Program: Endoscopy and Beyond". The author would like to be grateful to Professor Wee Teck Gan for inviting him to this program, and to IMS-NUS for its generous support. He also be thankful to Dr. Takuma Hayashi, Dr. Hiroki Kato and Dr. Daichi Takeuchi for telling him various invaluable facts about perverse sheaves and vanishing cycles functors.

References

[ABV92] J. Adams, D. Barbasch and D. A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups*. Progress in Mathematics, **104**. *Birkhäuser Boston, Inc., Boston, MA*, 1992. xii+318 pp.

[Ar13] J. Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups*. American Mathematical Society Colloquium Publications, **61**. *American Mathematical Society, Providence, RI*, 2013.

[BBD82] A. A. Beĭlinson, J. Bernstein and P. Deligne, *Faisceaux pervers. Analysis and topology on singular spaces, I (Luminy, 1981)*, 5–171, Astérisque, **100**, *Soc. Math. France, Paris*, 1982.

[BK14] J. Bernstein and B. Krötz, *Smooth Fréchet globalizations of Harish-Chandra modules. Israel J. Math.* **199** (2014), no. 1, 45–111.

[C78] W. Casselman, *Jacquet modules for real reductive groups. Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pp. 557–563, *Acad. Sci. Fennica, Helsinki,* 1980.

[C89] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G. Canad. J. Math.* **41** (1989), no. 3, 385–438.

[CFMMX] C. Cunningham, A. Fiori, J. Mracek, A. Moussaoui and B. Xu, *Arthur packets for p-adic groups by way of microlocal vanishing cycles of perverse sheaves, with examples*. arXiv:1705.01885v3.

[DK73] P. Deligne and N. Katz, *Groupes de monodromie en géométrie algébrique. II*. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). Dirigé par P. Deligne et N. Katz. Lecture Notes in Mathematics, Vol. 340, *Springer-Verlag, Berlin-New York*, 1973. x+438 pp. 14–06.

[HTT08] R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*. Translated from the 1995 Japanese edition by Takeuchi. Progress in Mathematics, **236**. *Birkhäuser Boston, Inc., Boston, MA*, 2008. xii+407 pp.

[Mœ11] C. Mœglin, *Multiplicité* 1 *dans les paquets d'Arthur aux places p-adiques. On certain L-functions*, 333–374, Clay Math. Proc., **13**, *Amer. Math. Soc., Providence, RI*, 2011.

[V93] D. A. Vogan, Jr., *The local Langlands conjecture. Representation theory of groups and algebras*, 305–379, Contemp. Math., **145**, *Amer. Math. Soc., Providence, RI*, 1993.

[W92] N. R. Wallach, *Real reductive groups. II.* Pure and Applied Mathematics, **132-II**. *Academic Press, Inc., Boston, MA,* 1992. xiv+454 pp.

[X17] B. Xu, *On Mœglin's parametrization of Arthur packets for p-adic quasisplit* $\mathrm{Sp}(N)$ *and* $\mathrm{SO}(N)$. *Canad. J. Math.* **69** (2017), no. 4, 890–960.

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan

Email address: atobe@ms.u-tokyo.ac.jp