Kazhdan-Lusztig representations and Whittaker space of some genuine representations

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Main Result:

$$\dim \mathrm{Wh}_\psi(\pi_S) = \langle \sigma_{\mathscr{X}}, \sigma_S \rangle_W.$$

Questions:

$$\dim \mathrm{Wh}_\psi(\pi_S)_{\mathcal{O}_y} \quad = \langle \sigma_{\mathscr{X}}^{\mathcal{O}_y}, \sigma_S \rangle_W?$$

$$\dim \mathrm{Wh}_\psi(\pi_\sigma) \qquad = \langle \sigma_{\mathscr{X}}, \sigma \rangle_{R_\chi}?$$

Covering groups

The dimension of $\mathrm{Wh}_\psi(\pi_S)$

Several questions

Notation

- $\mathbf{G}$: a pinned *split* connected reductive group over a non-archimedean field F with fixed $(\mathbf{B} = \mathbf{T}\mathbf{U}, \{e_\alpha\})$.

- $(X, \Phi; Y, \Phi^\vee)$: root datum of $\mathbf{G}$, where

$$Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}) \text{ and } X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m).$$

  Also, Φ (resp. $\Phi^\vee$) is the set of roots (resp. coroots), and $\Delta \subseteq \Phi$ the set of simple roots associated with $\mathbf{B}$.

$\mathbf{K}_2$-torsors $\widetilde{\mathbf{G}}$

Starting with a *Weyl-invariant integral* quadratic form

$$Q : Y \to \mathbf{Z},$$

the work of Brylinski and Deligne (2001) gives a central extension

$$\mathbf{K}_2 \hookrightarrow \widetilde{\mathbf{G}} \twoheadrightarrow \mathbf{G}.$$

The structure of $\widetilde{\mathbf{G}}$ is governed essentially by three facts:

- $\mathbf{K}_2$ lies in the center of $\widetilde{\mathbf{G}}$;
- $\widetilde{\mathbf{G}}$ splits $\mathbf{G}$-equivariantly and uniquely over unipotent elements in $\mathbf{G}$.
- the group law on $\widetilde{\mathbf{T}}$ can be described by using Q.

Example

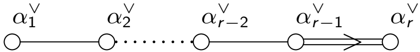
For $\mathbf{G}$ almost-simple and simply-connected, one has a bijection

$$\begin{aligned} \mathrm{Quad}_{\mathbf{Z}}(Y)^W &\longleftrightarrow \quad \mathbf{Z} \\ Q &\longmapsto \quad Q(\alpha^\vee), \end{aligned}$$

where $\alpha^\vee \in Y$ is any *short* coroot.

Example

Dynkin diagram for simple coroots of Sp_{2r}:



Let Q be such that $Q(\alpha_r^\vee) = -1$. It gives the desired $\widetilde{\mathrm{Sp}}_{2r}$.

Topological covers

- Assume $\mu_n \subseteq F^\times$. Then:

$$\begin{array}{ccccc} \mathbf{K}_2(F) & \hookrightarrow & \widetilde{\mathbf{G}}(F) & \twoheadrightarrow & \mathbf{G}(F) \\ \downarrow {\scriptstyle (-,-)_n} & & \downarrow & & \| \\ \mu_n & \hookrightarrow & \widetilde{G} & \twoheadrightarrow & G, \end{array}$$

where $(-,-)_n : \mathbf{K}_2(F) \to \mu_n$ is the n-th Hilbert symbol.

- Let $B_Q(y,z) := Q(y+z) - Q(y) - Q(z)$. Then the commutator on $\widetilde{T}$ is given by

$$[y(a), z(b)] = (a,b)_n^{B_Q(y,z)}.$$

Definition

A representation of $\widetilde{G}$ is called *genuine* if the central subgroup μ_n acts by a fixed embedding $\mu_n \hookrightarrow \mathbf{C}^\times$.

$\mathrm{Irr}_{\mathrm{gen}}(\widetilde{T})$

In general, a covering torus $\widetilde{T}$ is not abelian, but always a Heisenberg type group! Construction of elements in $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{T})$:

- Let $\chi : Z(\widetilde{T}) \to \mathbf{C}^\times$ be a genuine character.
- Choose a maximal abelian subgroup $\widetilde{A} \subseteq \widetilde{T}$ and an extension $\chi' : \widetilde{A} \to \mathbf{C}^\times$.
- Let $i(\chi) := \mathrm{Ind}_{\widetilde{A}'}^{\widetilde{T}} \chi'$ be the induced representation. Then $i(\chi) \in \mathrm{Irr}_{\mathrm{gen}}(\widetilde{T})$ and every element in $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{T})$ arises in this way.

Thus, every genuine irreducible representation of $\widetilde{T}$ is of the same dimension equal to

$$\sqrt{[\widetilde{T} : Z(\widetilde{T})]} = [\widetilde{T} : \widetilde{A}].$$

$Z(\widetilde{T})$ and $\widetilde{A}$

- Let

$$Y_{Q,n} := \{y \in Y : B_Q(y,z) \in n\mathbf{Z} \text{ for all } z \in Y\}.$$

Then $Z(\widetilde{T}) \subseteq \widetilde{T}$ is the preimage of $i_{Q,n}(Y_{Q,n} \otimes F^\times) \subseteq T$, where

$$i_{Q,n} : Y_{Q,n} \otimes F^\times \to T = Y \otimes F^\times$$

is the natural isogeny induced from the inclusion $Y_{Q,n} \subseteq Y$.

- In the tame case $(p \nmid n)$, one can choose $\widetilde{A} = Z(\widetilde{T}) \cdot \mathbf{T}(O_F)$ and thus

$$\dim i(\chi) = \left|\widetilde{T}/\widetilde{A}\right| = |Y/Y_{Q,n}|.$$

Whittaker space $\mathrm{Wh}_\psi(\pi)$

Let $\widetilde{B} = \widetilde{T}U$ be the Borel subgroup of $\widetilde{G}$. Fix a non-degenerate character

$$\psi : U \to \mathbf{C}^\times.$$

Definition

A Whittaker functional of a genuine representation (π, V_π) of $\widetilde{G}$ is a functional $\ell : V_\pi \to \mathbf{C}$ such that

$$\ell(\pi(u)v) = \psi(u) \cdot \ell(v)$$

for all $u \in U$ and $v \in V_\pi$.

$\mathrm{Wh}_\psi(\pi)$: the vector space of Whittaker functionals on π.

Problem. For fixed $\widetilde{G}$:

(i) describe the function $\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G}) \to \mathbf{N}$ given by

$$\pi \mapsto \dim \mathrm{Wh}_\psi(\pi);$$

equivalently, describe the induced group morphism

$$\dim \mathrm{Wh}_\psi(-) : \ \mathrm{Groth}(\mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})) \longrightarrow \mathbf{Z}.$$

(ii) single out the class π such that $\dim \mathrm{Wh}_\psi(\pi) = 1$.

Question. What do we know?

Selective review

If $\widetilde{G} = G$ (i.e. non-covering), then

- by Gelfand-Kazhdan 1971, Rodier 1972, Shalika 1974, one has

$$\dim \mathrm{Wh}_\psi(\pi) \le 1$$

  for every $\pi \in \mathrm{Irr}(G)$;

- by Rodier 1975, Moeglin-Waldspurger 1987,

$$\dim \mathrm{Wh}_\psi(\pi) = c_{\mathcal{O}_{reg}}(\pi),$$

  the $\mathcal{O}_{reg}$-coefficient in the Harish-Chandra character expansion.

Remark

The multiplicity-one property of Whittaker functionals lies at the heart of the Langlands-Shahidi method of L-functions, and also crucial for some Rankin-Selberg method of L-functions. (See the work of Cai, Friedberg, Ginzburg and Kaplan for recent progress on the R-S method for covering groups.)

For covering groups $\widetilde{G}$:

- Principal series representation $I(\chi) := \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} i(\chi) \otimes \mathbf{1}_U$ of $\widetilde{G}$: $\dim \mathrm{Wh}_\psi(I(\chi)) = \dim i(\chi) = \sqrt{[\widetilde{T} : Z(\widetilde{T})]}$.
- Finiteness of $\dim \mathrm{Wh}_\psi(\pi)$: Kazhdan-Patterson (1984) for $\widetilde{\mathrm{GL}}_r$, Patel (2015) for general $\widetilde{G}$.
- Theta representation: Kazhdan-Patterson (1984) for $\widetilde{\mathrm{GL}}_r$; Gao (2016) for general $\widetilde{G}$.
- Depth-zero supercuspidal representation: Blondel (1992) for $\widetilde{\mathrm{GL}}_r$; Gao-Weissman (2017) for general $\widetilde{G}$.
- Multiplicity-one property: Gao-Shahidi-Szpruch (2017) shows $\dim \mathrm{Wh}_\psi(\pi) \leq 1$ for all $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\widetilde{G})$ if and only if $\widetilde{T} = Z(\widetilde{T})$.

Remark

For theta reptn $\Theta(\widetilde{G})$ (generic or not), one has the more general work on the unipotent support $\mathcal{O}(\Theta(\widetilde{G}))$:

- Y.-Q. Cai 2016: for Kazhdan-Patterson covering $\widetilde{\mathrm{GL}}_r^{(n)}$, one has
$$\mathcal{O}(\Theta(\widetilde{\mathrm{GL}}_r^{(n)})) = (n^a b),$$
where $r = an + b$ with $a \geq 0, 0 \leq b < n$.
- Friedberg-Ginzburg 2016 (Conjecture): Let n be odd. If $n > 2r$, then $\mathcal{O}(\Theta(\widetilde{\mathrm{Sp}}_{2r}^{(n)})) = \mathcal{O}_{reg}$. If $n < 2r$, then
$$\mathcal{O}(\Theta(\widetilde{\mathrm{Sp}}_{2r}^{(n)})) = \text{symplectic collapse of } (n^a b),$$
where $2r = an + b$ with $a \geq 0$ and $0 \leq b < n$.
- S. Leslie 2017:
$$\mathcal{O}(\Theta(\widetilde{\mathrm{Sp}}_{2r}^{(4)})) = (2^r).$$

Unramified $I(\chi)$

Assume $p \nmid n$. Let $K = \mathbf{G}(O_F)$. Assume $\widetilde{G}$ splits over K and we fix a splitting and thus identify $K \subseteq \widetilde{G}$. Every K-unramified rptn of $\widetilde{G}$ is a subquotient of $I(\chi) = \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} i(\chi)$, where χ is unramified, i.e.

$$\chi|_{Z(\widetilde{T}) \cap K} = \mathbf{1}.$$

Denote

$$\mathscr{X}_{Q,n} := Y / Y_{Q,n}.$$

For unramified χ, one has

$$\dim \mathrm{Wh}_\psi(I(\chi)) = |\mathscr{X}_{Q,n}|.$$

Question. Let $\mathrm{JH}(I(\chi)) = \{\pi_i : i \in I\}$ be the Jordan-Holder set of $I(\chi)$. What is the map $i \mapsto \dim \mathrm{Wh}_\psi(\pi_i)$? (Note:

$$\sum_{i \in I} \dim \mathrm{Wh}_\psi(\pi_i) = |\mathscr{X}_{Q,n}|.)$$

Regular unramified $I(\chi)$

The group W acts on χ by ${}^w\chi(t) := \chi(w^{-1}tw)$. We assume χ is regular, i.e.

$$\{w \in W : {}^w\chi = \chi\} = \{1\}.$$

For every root α, one has an unramified linear character

$$\chi_\alpha : F^\times \to \mathbf{C}^\times \text{ given by } \chi_\alpha(x) = \chi(\underbrace{\widetilde{\alpha^\vee(x)}^{n_\alpha}}_{\in Z(\widetilde{T})}),$$

where

$$n_\alpha := \frac{n}{\gcd(Q(\alpha^\vee), n)}.$$

Denote

$$\Phi(\chi) := \{\alpha \in \Phi : \chi_\alpha = |\cdot|\}.$$

Theorem (Rodier 1981)

Let $I(\chi)$ be a regular unramified p.s. reptn. Then $\mathrm{JH}(I(\chi))$ is multiplicity-free. Moreover, there is a bijection

$$\mathscr{P}(\Phi(\chi)) \to \mathrm{JH}(I(\chi)) \text{ denoted by } S \mapsto \pi_S$$

satisfying the following properties:

(i) *The representation π_S is characterized by*

$$(\pi_S)_U = \bigoplus_{w \in W_S} \delta_B^{1/2} \cdot i({}^{w^{-1}}\chi),$$

where

$$W_S := \left\{ w \in W : \Phi(\chi) \cap w(\Phi^-) = S \right\}.$$

(ii) *The representation π_S is square integrable modulo the center of $\widetilde{G}$ if and only if $|\Phi(\chi)| = |\Delta|$ and $S = \emptyset$.*

Theorem (Rodier 1981)

(iii) *The representation π_S is tempered if and only if $S = \emptyset$ and the restriction of χ to $\varphi^{-1}\left(\bigcap_{\alpha\in\Phi(\chi)} \mathrm{Ker}(\alpha)\right)$ is unitary, where $\varphi : Z(\widetilde{T}) \twoheadrightarrow Y_{Q,n}$ is the natural quotient.*

(iv) *The representation $\pi_{\Phi(\chi)}$ is the unique irreducible unramified subquotient of $I(\chi)$.*

Remark

Let $\mathscr{C}(X \otimes \mathbf{R}, \chi)$ be the set of connected component of $X \otimes \mathbf{R} - \bigcup_{\alpha\in\Phi(\chi)} \mathrm{Ker}(\alpha^\vee)$. Then one has a bijection

$$\mathscr{P}(\Phi(\chi)) \to \mathscr{C}(X \otimes \mathbf{R}, \chi), \quad S \mapsto \Gamma_S,$$

where

$$\Gamma_S := \left\{ v \in X \otimes \mathbf{R} : \langle \alpha^\vee, v\rangle < 0 \text{ iff } \alpha \in S \right\}.$$

The representation $\sigma_{\mathscr{X}}$

Question. For a regular unramified $I(\chi)$, what is $\dim \mathrm{Wh}_\psi(\pi_S)$?

To answer it, we need to introduce two finite-dimensional representations of W. The first (universal) one:

- Let $\rho \in Y \otimes \mathbf{Z}[1/2]$ be the half sum of positive coroots. Consider the twisted Weyl action on Y:

$$w[y] := w(y - \rho) + \rho \text{ for every } y \in Y.$$

The action $w[\cdot]$ decends to give an action of W on $\mathscr{X}_{Q,n} = Y/Y_{Q,n}$. Denote by

$$\sigma_{\mathscr{X}} : W \to \mathrm{Perm}(\mathscr{X}_{Q,n})$$

the arising permutation representation.

Why $\sigma_{\mathscr{X}}$?

Reasons that $\sigma_{\mathscr{X}}$ appears:

- the action $w[\cdot]$ is an incarnation (the skeleton) of the Chinta-Gunnells action (2009) used to construct the Weyl-group multiple Dirichlet series.
- $w[\cdot]$ is involved in the description of entries of induced map

$$T(w,\chi)^* : \mathrm{Wh}_\psi(I({}^w\chi)) \to \mathrm{Wh}_\psi(I(\chi)).$$

Kazhdan-Lusztig representations of W

- Kazhdan-Lusztig (1979) describes a partition of a Coxeter group (in particular W) into right cells, each of which bears a representation of W. The description involves the famous Kazhdan-Lusztig polynomials.
- Many applications of the Kazhdan-Lusztig theory: decomposition of Verma modules, reptn of Hecke-algebras, proof of Deligne-Langlands, etc.
- Pertaining more to our interest, K-L polynomials are related to description of the transition matrix for the Casselman basis: work of Rogawski, Reeder, Bump-Nakasuji etc.

The second reptn of W (associated to each S):

Proposition

Assume $\Phi(\chi) \subseteq \Delta$. Then for every $S \subseteq \Phi(\chi)$:

- *the set $W_S \subseteq W$ is a union of Kazhdan-Lusztig right cells and thus bears a representation σ_S of W;*
- *in fact,*

$$\sigma_S = \sum_{S' : S \subseteq S' \subseteq \Phi(\chi)} (-1)^{|S'-S|} \cdot \mathrm{Ind}_{W(S')}^{W} \varepsilon_{W(S')} \in \mathscr{R}(W),$$

where $W(S') \subseteq W$ is the parabolic subgroup generated by S'.

Example

If $\Phi(\chi) = \Delta$, then $W_\emptyset = \{1\}, W_\Delta = \{w_G\}$; in this case,

$$\sigma_\emptyset = \mathbf{1}_W, \quad \sigma_\Delta = \varepsilon_W.$$

Definition

A covering group $\widetilde{G}$ associated to Q is called *saturated* if $\mathbf{Z} \cdot n_\alpha \alpha^\vee = (\mathbf{Z} \cdot \alpha^\vee) \cap Y_{Q,n}$ for every simple root α. (Note that $\subseteq$ always holds.)

Example

If $n = 1$, then every linear $G = \widetilde{G}$ is saturated. If $\mathbf{G}$ is semisimple and simply-connected, then $\widetilde{G}$ is saturated if and only if its dual group (in the sense of Finkelberg-Lysenko, McNamara, Reich and Weissman) is of adjoint type.

Theorem (G.)

Let $\widetilde{G}$ be a saturated covering group and $I(\chi)$ a regular unramified principal series with $\Phi(\chi) \subseteq \Delta$. Then for every $S \subseteq \Phi(\chi)$, one has

$$\dim \mathrm{Wh}_\psi(\pi_S) = \langle \sigma_{\mathscr{X}}, \sigma_S \rangle_W.$$

Proof (a sketch):

(i) Reduction. For every $S \subseteq \Phi(\chi)$, denote by $\widetilde{P}_S = \widetilde{M}_S N_S \subset \widetilde{G}$ the associated parabolic subgroup. Let $\pi_S^{\widetilde{M}_S}$ be the (theta) representation of $\widetilde{M}_S$ associated with S. One has

$$\pi_S = \sum_{S': S \subseteq S' \subseteq \Phi(\chi)} (-1)^{|S'-S|} \cdot \mathrm{Ind}_{\widetilde{P}_{S'}}^{\widetilde{G}} \pi_{S'}^{\widetilde{M}_{S'}} \in \mathscr{R}(\widetilde{G}).$$

Thus, it suffices to show

$$\dim \mathrm{Wh}_\psi(\mathrm{Ind}_{\widetilde{P}_S}^{\widetilde{G}} \pi_S^{\widetilde{M}_S}) = \langle \sigma_{\mathscr{X}}, \mathrm{Ind}_{W(S)}^{W} \varepsilon_{W(S)} \rangle_W \qquad (1)$$

for every $S \subseteq \Phi(\chi)$.

(ii) Rodier's heredity and Frobenius reciprocity applied to (1) gives

$$\dim \mathrm{Wh}_\psi(\pi_S^{\widetilde{M}_S}) = \langle \sigma_{\mathscr{X}}, \varepsilon_{W(S)} \rangle_{W(S)}. \tag{2}$$

(iii) Following analysis of Kazhdan-Patterson, show that (if $\widetilde{G}$ is saturated) the equality (2) holds. Main idea:

$$\begin{aligned} &\mathrm{Wh}_\psi(\pi_S^{\widetilde{M}_S}) \\ =& \bigcap_{\alpha \in S} \mathrm{Ker}\big(T(w_\alpha, {}^{w_\alpha}\chi)^* : \mathrm{Wh}_\psi(I(\chi)) \to \mathrm{Wh}_\psi(I({}^{w_\alpha}\chi)) \big). \end{aligned}$$

Example

If $\Phi(\chi) = \Delta$, then $\pi_\emptyset \hookrightarrow I(\chi)$ is the unique subreptn and is the covering analog of the Steinberg representation. Moreover, π_Δ is the unique Langlands quotient of $I(\chi)$.

In this case one has,

$$\dim \mathrm{Wh}_\psi(\pi_\emptyset) = \langle \sigma_{\mathscr{X}}, \mathbf{1}_W \rangle_W = \# \text{ of } W\text{-orbits in } \mathscr{X}_{Q,n}.$$

On the other hand,

$$\dim \mathrm{Wh}_\psi(\pi_\Delta) = \langle \sigma_{\mathscr{X}}, \varepsilon_W \rangle_W = \# \text{ of free } W\text{-orbits in } \mathscr{X}_{Q,n}.$$

If $n = 1$, then $\pi_\emptyset$ is the unique irreducible generic subquotient of $I(\chi)$.

As a covering analog of the injectivity of generic representations for linear groups, we have

Corollary

Assume $\widetilde{G}$ saturated and $\Phi(\chi) \subseteq \Delta$. Then

$$\dim \mathrm{Wh}_\psi(\pi_S) \geq \dim \mathrm{Wh}_\psi(\pi_{\Phi(\chi)})$$

for every $S \subseteq \Phi(\chi)$.

Remark

It is not true in general that

$$\dim \mathrm{Wh}_\psi(\pi_S) \leq \dim \mathrm{Wh}_\psi(\pi_\emptyset)$$

for every $S \subseteq \Phi(\chi)$. One has

$$\dim \mathrm{Wh}_\psi(\pi_S) \sim \frac{|\mathscr{X}_{Q,n}| \cdot \dim(\sigma_S)}{|W|}$$

as $n \to \infty$.

Example

Let $\widetilde{\mathrm{SL}}_3$ be the n-fold cover associated with Q such that $Q(\alpha^\vee) = -1$ for every coroot $\alpha^\vee$. Then $\widetilde{\mathrm{SL}}_3$ is saturated if and only if $3 \nmid n$; in this case,

$$\mathscr{X}_{Q,n} \simeq \mathbf{Z}/n\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}.$$

Let χ be such that $\Phi(\chi) = \Delta = \{\alpha_1, \alpha_2\}$. Then

$$W_\emptyset = \{1\}, \quad W_{\{\alpha_1\}} = \{w_1, w_1 w_2\};$$

$$W_{\{\alpha_2\}} = \{w_2, w_2 w_1\}, \quad W_\Delta = \{w_G\}.$$

We obtain:

	$\pi_\emptyset$	$\pi_{\{\alpha_1\}}$	$\pi_{\{\alpha_2\}}$	π_Δ
$\dim \mathrm{Wh}_\psi(-)$	$\frac{n^2+3n+2}{6}$	$\frac{n^2-1}{3}$	$\frac{n^2-1}{3}$	$\frac{n^2-3n+2}{6}$

Example

Let $\widetilde{\mathrm{Sp}}_4$ be the n-fold cover associated with Q such that $Q(\alpha_2^\vee) = -1$, where α_2 is the unique long root in $\Delta = \{\alpha_1, \alpha_2\}$. Then $\widetilde{\mathrm{Sp}}_4$ is saturated if and only if $2 \nmid n$; in this case,

$$\mathscr{X}_{Q,n} \simeq \mathbf{Z}/n\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}.$$

Let χ be such that $\Phi(\chi) = \Delta$. Then

$$W_\emptyset = \{1\}, \quad W_{\{\alpha_1\}} = \{w_1, w_1 w_2, w_1 w_2 w_1\};$$

$$W_{\{\alpha_2\}} = \{w_2, w_2 w_1, w_2 w_1 w_2\}, \quad W_\Delta = \{w_G\}.$$

We obtain:

	$\pi_\emptyset$	$\pi_{\{\alpha_1\}}$	$\pi_{\{\alpha_2\}}$	π_Δ
$\dim \mathrm{Wh}_\psi(-)$	$\frac{n^2+4n+3}{8}$	$\frac{3(n^2-1)}{8}$	$\frac{3(n^2-1)}{8}$	$\frac{n^2-4n+3}{8}$

Questions:

(i) Consider every orbit $\mathcal{O}_y \subseteq \mathscr{X}_{Q,n}$. It is possible to define a subspace $\mathrm{Wh}_\psi(\pi_S)_{\mathcal{O}_y} \subseteq \mathrm{Wh}_\psi(\pi_S)$ (non-intrinsically). We expect:

$$\dim \mathrm{Wh}_\psi(\pi_S)_{\mathcal{O}_y} = \langle \sigma_{\mathscr{X}}^{\mathcal{O}_y}, \sigma_S \rangle_W.$$

(ii) If $\widetilde{G}$ is not saturated (for example the double cover of SL_2), then $\dim \mathrm{Wh}_\psi(\pi_S)$ depends on more precise information of χ, and not just on $\Phi(\chi)$. In this case, what is a formula for $\dim \mathrm{Wh}_\psi(\pi_S)$ in terms of the given data?

(iii) How about *ramified* regular principal series? What is the substitute for $\sigma_{\mathscr{X}}$?

(iv) Let $\widetilde{G}$ be a saturated n-fold cover of a semisimple s.c. group G. Let χ be a *unitary* unramified genuine character of $Z(\widetilde{T})$. Let $R_\chi \subseteq W$ be the R-group associated to $I(\chi)$. Then from the natural correspondence

$$\mathrm{Irr}(R_\chi) \to \mathrm{JH}(I(\chi)), \quad \sigma \mapsto \pi_\sigma,$$

one has

$$\dim \mathrm{Wh}_\psi(\pi_\sigma) = \langle \sigma_{\mathscr{X}}, \sigma \rangle_{R_\chi} ?$$

(The work of D. Szpruch on $\widetilde{\mathrm{GSp}}_{2r}^{(2)}$ shows that it is not enough to assume G_{der} is simply-connected.)

(v) How about general $I(\chi)$, which is irregular and non-unitary?


THANK
YOU