What are A-Packets?

Wee Teck Gan
National University of Singapore

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What are A(rthur)-Packets?

- F local field of characteristic 0, e.g. $\mathbb{Q}_p$ or $\mathbb{R}$.
- G a (connected, split) reductive group over F; $G = G(F)$.
- $\mathrm{Irr}(G)$ = the set of equivalence classes of irreducible smooth complex representations of G.
- $\mathrm{Irr}_{unit}(G)$ = the subset of unitarizable representations.

$\boxed{\text{An A-packet of } G \text{ is a finite (multi-)set of elements of } \mathrm{Irr}_{unit}(G)}$

Questions:

- What are these finite sets of unitary representations for?
- How are they defined or characterised?
- How are they constructed?

Start with an easier question: What are L-packets?

Local Langlands Correspondence (LLC)

The LLC is the key local problem in the Langlands program: it gives a classification of $\mathrm{Irr}(G)$.

$\boxed{\text{LLC}}$ (i) There is a surjective finite-to-one map

$$\mathcal{L} : \mathrm{Irr}(G) \longrightarrow \Phi(G),$$

where $\Phi(G)$ is the set of equivalence class of L-parameters for G. This map should satisfy a number of properties which characterize it uniquely.

The fibres of this reciprocity map $\mathcal{L}$ are called L-packets. For $\phi \in \Phi(G)$, denote the associated fiber by Π_ϕ. So

$$\mathrm{Irr}(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi$$

Question: What is $\Phi(G)$?

Weil Group and Weil-Deligne Group

Let $\overline{F}$ be an algebraic closure of F. A variant of the absolute Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$ is the Weil group of F:

- if F is p-adic, W_F is a dense subgroup of $\mathrm{Gal}(\overline{F}/F)$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_F & \longrightarrow & W_F & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_F & \longrightarrow & \Gamma_F & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1 \end{array}$$

- if $F = \mathbb{C}$, $W_F \cong \mathbb{C}^\times$;
- if $F = \mathbb{R}$, $W_F = \mathbb{C}^\times \cup \mathbb{C}^\times \cdot j$ with

$$j^2 = -1 \text{ and } jzj^{-1} = \overline{z} \text{ for } z \in \mathbb{C}^\times.$$

The Weil-Deligne group of F is:

$$WD_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}), & \text{if } F \text{ p-adic;} \\ W_F, & \text{if } F \text{ archimedean.} \end{cases}$$

Dual Group and L-group

Given the connected quasi-split reductive group G over F, Langlands associated to it a connected complex Lie group $\widehat{G}$ called the dual group of G.

Let $T \subset G$ be a maximal torus contained in a Borel subgroup over F. Have associated root datum

$$(X(T), \Phi(G,T), Y(T), \Phi^\vee(G,T)).$$

The root datum of $\widehat{G}$ is

$$(Y(T), \Phi^\vee(G,T), X(T), \Phi(G,T)).$$

The Galois group Γ_F acts on both root data above, giving an action as automorphisms of $\widehat{G}$ and allowing one to define the L-group of G:

$$^L G = \widehat{G} \rtimes \Gamma_F$$

When G is split, this is a direct product.

G	GL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}	G_2
$G^\vee$	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{SO}_{2n}(\mathbb{C})$	$G_2(\mathbb{C})$

L-parameters

An *L-parameter* is an equivalence class of homomorphism

$$\phi : WD_F \longrightarrow {}^L G$$

with some properties and where equivalence is defined by $G^\vee$-conjugacy.

- the composite

$$W_F \longrightarrow WD_F \xrightarrow{\phi} {}^L G \longrightarrow \Gamma_F$$

  is the natural map.
- $\phi|_{W_F}$ is smooth and $\phi(W_F)$ consists of semisimple elements;
- $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.

In short, an L-parameter is more or less a $\widehat{G}$-conjugacy class of local Galois representations valued in the complex Lie group $G^\vee$.

So have:

$$\Phi(G) = \widetilde{\Phi}(G)/\widehat{G}\text{-conjugacy}$$

where $\tilde{\Phi}(G)$ is the space of all maps ϕ as above.

Parametrization of L-packets

$\boxed{\text{LLC}}$ (ii) To refine the reciprocity map

$$\mathcal{L} : \mathrm{Irr}(G) \longrightarrow \Phi(G)$$

to a bijection, we need to parametrize the fibres Π_ϕ of $\mathcal{L}$.

Given $\phi \in \tilde{\Phi}(G)$, have associated stabilizer in $\widehat{G}$:

$$S_\phi = Z_{\widehat{G}}(\phi).$$

This is the centralizer of the image of ϕ in $\widehat{G}$. Consider its (finite) component group:

$$A_\phi = \pi_0(S_\phi).$$

There is a natural map

$$Z(\widehat{G})^{\Gamma_F} \longrightarrow S_\phi \longrightarrow A_\phi.$$

Then one postulates the existence of a bijection

$$\Pi_\phi \longleftrightarrow \mathrm{Irr}(A_\phi / Z(\widehat{G}^{\Gamma_F})).$$

characterised by some properties.

More Geometric Description

Think of the space $\tilde{\Phi}(G)$ as a variety with an action of $\widehat{G}$ by conjugation. If WD_F had been a finitely generated group, this is what people call a character variety.

An L-parameter ϕ is thus a $\widehat{G}$-orbit $\mathcal{O}_\phi$ on $\tilde{\Phi}(G)$ and

$$\mathcal{O}_\phi \cong \widehat{G}/S_\phi.$$

Elements (ρ, V) of $\mathrm{Irr}(A_\phi)$ correspond to $\widehat{G}$-equivairant vector bundles $\mathcal{V}_{\phi,\rho}$ on $\mathcal{O}_\phi$:

$$\mathcal{V}_{\phi,\rho} \cong \widehat{G} \times_{S_{\phi,\rho}} V$$

This more geometric viewpoint will be relevant in the last two lectures of this series.

Known Results

- When $F = \mathbb{R}$ or $\mathbb{C}$, LLC was shown through the work of Harish-Chandra, Langlands, Knapp-Zuckermann,.....
- Assume F p-adic. LLC is known for:
  - $G = \mathrm{GL}(n)$: Harris-Taylor, Henniart, Scholze. In this case, $\mathcal{L}$ is bijective, so L-packets are singletons.
  - $G = \mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{O}(2n)$, $\mathrm{U}(n)$: Arthur, Moeglin, Mok, Kaletha-Minguez-Shin-White,.......
  - $G = \mathrm{Mp}(2n)$: Gan-Savin
  - $G = \mathrm{GSp}(4)$: Gan-Takeda

Recent ongoing work of Fargues and Scholze provides a general framework and strategy for establishing the LLC through more geometric means.

Properties

Have the following chain of special subsets of representations:

$$\boxed{\mathrm{Irr}_{ur}(G) \subset \mathrm{Irr}(G) \supset \mathrm{Irr}_{unit}(G) \supset \mathrm{Irr}_{temp}(G) \supset \mathrm{Irr}_{ds}(G) \supset \mathrm{Irr}_{sc}(G)}$$

$$\downarrow \mathcal{L}$$

$$\boxed{\Phi_{ur}(G) \subset \Phi(G) \supset \Phi_{unit}(G) \supset \Phi_{temp}(G) \supset \Phi_{ds}(G) \supset \Phi_{sc}(G)}$$

Question: Can one characterize $\Phi_{\bullet}(G)$?

- (Unramified) $\phi \in \Phi_{ur}(G)$ iff ϕ is trivial on I_F and $\mathrm{SL}_2(\mathbb{C})$.
- (tempered) $\phi \in \Phi_{temp}(G)$ iff the image of ϕ is bounded.
- (discrete series) $\phi \in \Phi_{ds}(G)$ iff the image of ϕ is not contained in a proper (relevant) parabolic.
- (supercuspidal) $\phi \in \Phi_{sc}(G)$ iff .....an answer will be given in Atobe's lecture tomorrow.
- (unitary) $\Phi_{unit} =$? (this is the problem of the unitary dual)

Characterisation of $\mathcal{L}$

Ideally, one would like to characterise the reciprocity map $\mathcal{L}$ by saying that it respects certain invariants known as L-factors and ϵ-factors that one attaches to elements of $\mathrm{Irr}(G)$ and $\Phi(G)$.

- For $\phi \in \Phi(G)$: given any algebraic finite-dim. representation $r : {}^L G \longrightarrow \mathrm{GL}(V)$, the composite $r \circ \phi : WD_F \longrightarrow \mathrm{GL}(V)$ is a representation of WD_F and one can associate an L-factor:

$$\boxed{L(s, r \circ \phi) = \frac{1}{\det \left(1 - r \circ \rho \left(\mathrm{Frob}, \begin{pmatrix} q_F^{1/2} & 0 \\ 0 & q_F^{-1/2} \end{pmatrix} \right) \right) \mid V_N^{I_F} \Big)}}$$

- For $\pi \in \mathrm{Irr}(G)$: would like to define $L(s, \pi, r)$, but only special cases have been done (Langlands-Shahidi, Rankin-Selberg). In the course of Lei Zhang and Zhilin Luo, a conjectural general approach due to Braverman-Kazhdan will be introduced.

One would like:

$$\boxed{L(s, \pi, r) = L(s, r \circ \phi) \quad \text{if } \mathcal{L}(\pi) = \phi.}$$

Endoscopic Character Identities

There is another characterization of the partition of $\mathrm{Irr}_{temp}(G)$ into L-packets (but without the reciprocity map). This is the theory of endoscopy, which provides a characterization of L-packets through character identities.

Namely, to $\pi \in \mathrm{Irr}(G)$, Harish-Chandra defined its character Θ_π: a G-conjugacy-invariant distribution on G which uniquely determines π.

- given a tempered L-packet Π, there is (up to scaling) a unique linear combination
$$\sum_{\pi \in \Pi} c_\pi \cdot \Theta_\pi$$
which is a stably-invariant distribution.
- some other linear combinations of the Θ_π's are equal to the transfers of stable distributions from endoscopic groups.

I will leave it to the course of Parab and Kaletha on the stable trace formula to elaborate on this slide.

Global Problem

- Let k be a number field with ring of adeles $\mathbb{A} = \prod'_v (k_v, \mathcal{O}_v)$.
- G connected reductive group over k, so have $G(k) \subset G(\mathbb{A})$.

Is the global problem

$$\boxed{\text{Classify the irreducible representations of } G(\mathbb{A})\text{?}}$$

Recall $G(\mathbb{A}) = \prod'_v (G(k_v), K_v)$ for a family of maximal compact subgroups (K_v). Then if $\pi \in \mathrm{Irr}(G(\mathbb{A}))$, one has

$$\pi \cong \otimes'_v \pi_v \quad \text{(restricted tensor product)}$$

with $\pi_v \in \mathrm{Irr}(G(k_v))$ and $\pi_v \in \mathrm{Irr}_{K_v-ur}(G(k_v))$ for almost all v. So

$$\mathrm{Irr}(G(\mathbb{A})) = \prod_v{}' (\mathrm{Irr}(G(k_v)), \mathrm{Irr}_{K_v-ur}(G(k_v)))$$

Thus the above problem has no global content. The actual global problem is:

$$\boxed{\text{Classify the irreducible } \textit{automorphic} \text{ representations of } G(\mathbb{A}).}$$

$L^2(G(k)\backslash G(\mathbb{A}))$

Assuming G semisimple, it is natural to consider the unitary representation of $G(\mathbb{A})$ on $L^2(G(k)\backslash G(\mathbb{A}))$. By abstract results of functional analysis, one has a decomposition

$$\boxed{L^2(G(k)\backslash G(\mathbb{A})) = L^2_{disc}(G) \oplus L^2_{cont}(G)}$$

into the discrete spectrum and the continuous spectrum. So

$$\boxed{L^2_{disc}(G) = \widehat{\bigoplus_{\pi}} m_{disc}(\pi) \cdot \pi \quad \text{with } m_{\mathrm{disc}}(\pi) \text{ finite.}}$$

This is the most fundamental part: $L^2_{cont}(G)$ can be described in terms of L^2_{disc} of Levi subgroups of G using Eisenstein series.

The discrete spectrum further decomposes:

$$\boxed{L^2_{disc}(G) = L^2_{cusp}(G) \oplus L^2_{res}(G)}$$

into the cuspidal spectrum and residual spectrum (spanned by residues of Eisenstein series).

Automorphic Forms

- A function f on $G(k)\backslash G(\mathbb{A})$ is called an automorphic form if
  - f is smooth of moderate growth
  - f is $Z(\mathfrak{g})$-finite.
- An automorphic form f on G is called a cusp form if, for any parabolic $P = MN$ of G, the constant term

$$f_N(g) = \int_{N(k)\backslash N(\mathbb{A})} f(ng)\, dn$$

  is zero as a function on $G(\mathbb{A})$.
- Let $\mathcal{A}(G) \supset \mathcal{A}_{cusp}(G)$ be the vector space of (smooth) automorphic forms containing the subspace of cusp forms. The group $G(\mathbb{A})$ acts on both by right translation.

An irreducible representation is automorphic if it is a subquotient of $\mathcal{A}(G)$. It is cuspidal automorphic if it is a summand of $\mathcal{A}_{cusp}(G)$. A theorem of Langlands says that $\mathcal{A}(G)$ can be built up from $\mathcal{A}_{cusp}$ of Levi subgroups of G using Eisenstein series.

Automoprhic Discrete Spectrum

The intersection

$$\mathcal{A}_{disc}(G) := \mathcal{A}(G) \cap L^2_{disc}(G)$$

is the space of square-integrable automorphic forms. It is a dense subspace of $L^2_{disc}(G)$ (automorphic discrete spectrum). The main global problem in the Langlands program is:

$$\boxed{\text{Describe how } \mathcal{A}_{disc}(G) \text{ or equivalently } L^2_{disc}(G) \text{ decomposes.}}$$

This contains the problem of classifying cuspidal automorphic representations. Motivated by the LLC, and the spirit of the local-global principle, one might suspect that cuspidal automorphic representations are (conjecturally) parametrized by global Galois representations.

The Global Langlands Correspondence for $\mathrm{GL}(n)$

There is a group L_k called the global Langlands group with the following properties:

- there is a surjective map $L_k \longrightarrow W_k$ (the Weil group of k);
- for each place v of k, there is a distinguished conjugacy classes of embeddings $i_v : L_{k_v} := WD_{k_v} \longrightarrow L_k$;

  So L_k is a variant of $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$.
- there is a bijection (GLC)

$$\{\text{irreducible cuspidal representations } \pi \text{ of } \mathrm{GL}(n)\}$$

$$\updownarrow$$

$$\{\text{irreducible } n\text{-dim. representations } \phi \text{ of } L_k\}$$

  characterised by some properties such as local-global compatibility: if $\pi = \otimes_v' \pi_v$ corresponds to ϕ, then π_v corresponds to $\phi \circ i_v$ under LLC for all v.

Near Equivalence Classes

To decompose $\mathcal{A}_{disc}(G)$, we define an equivalence relation for representations of $G(\mathbb{A})$ weaker than isomorphisms.

Say that two irreducible representations

$$\pi = \otimes'_v \pi_v \text{ and } \pi' = \otimes'_v \pi'_v$$

are nearly equivalent if $\pi_v \cong \pi'_v$ for almost all v.

We have a decomposition of $\mathcal{A}_{disc}(G)$ into direct sums of near equivalence classes:

$$\mathcal{A}_{disc}(G) = \bigoplus_{\Psi} \mathcal{A}_{\Psi}(G).$$

So the problem becomes:

- Find a meaningful indexing set $\{\Psi\}$ for near equivalence classes.
- For each Ψ, describe how $\mathcal{A}_{\Psi}(G)$ decomposes.

Arthur's conjecture formulates an answer to the above problems by introducing the notion of A-parameters.

A-parameters

A global A-parameter for G is a homomorphism

$$\Psi : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G \quad \text{(with some conditions).}$$

Note the extra $\mathrm{SL}_2(\mathbb{C})$, which we shall call the Arthur SL_2.

- Two A-parameters are equivalent if they are conjugate by $\widehat{G}$.
- Two A-parameters Ψ and Ψ' are nearly equivalent if for all v, $\Psi \circ i_v$ and $\Psi' \circ i_v$ are conjugate by $\widehat{G}$, where

  $$i_v : L_{k_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow L_k \times \mathrm{SL}_2(\mathbb{C}).$$

  In other words, Ψ and Ψ' are locally conjugate.
- The A-parameter Ψ is elliptic if its image is not contained in a (relevant) parabolic. Denote the set of equivalence classes of such by $\Psi_{ell}(G)$.
- Say Ψ is tempered or generic if $\Psi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial.

Given an A-parameter Ψ, the local component

$$\Psi_v := \Psi \circ i_v : L_{k_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G$$

is a local A-parameter.

A-parameters and Near Equivalence Classes

An A-parameter Ψ gives rise to a near equivalence class of representations of $G(\mathbb{A})$ as follows:

- For each v, consider the local A-parameter $\Psi_v = \Psi \circ i_v$.
- For nonarchimedean v, one has a surjection

$$|-| : L_{k_v} = WD_{k_v} \twoheadrightarrow W_{k_v} \twoheadrightarrow \mathbb{Z} \cong q_{k_v}^{\mathbb{Z}} \subset \mathbb{R}^{\times}.$$

- define the inclusion $j_v : L_{k_v} \longrightarrow L_{k_v} \times \mathrm{SL}_2(\mathbb{C})$ by

$$w \mapsto (w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})).$$

- Define the local L-parameter associated to Ψ_v by setting:

$$\boxed{\phi_{\Psi_v} = \Psi_v \circ j_v}$$

- for almost all v, ϕ_{Ψ_v} is unramified and so determines $\pi_v^0 \in \mathrm{Irr}_{K_v-ur}(G(k_v))$.
- the collection $(\pi_v^0)_{v \notin S}$ determines a near equivalence class of $\mathrm{Irr}(G(\mathbb{A}))$.

Arthur Conjecture: Parametrisation of N.E. Classes

The above construction gives a bijection

$$\boxed{\Psi_{ell}(G)/\sim_{NE} \quad \longleftrightarrow \{\text{N.E. classes in } \mathcal{A}_{disc}(G)\}}$$

Actually, Arthur conjectures a refinement of the above:

$$\boxed{\Psi_{ell}(G) \longleftrightarrow \{\text{Certain Canonical submodules in } \mathcal{A}_{disc}(G)\}.}$$

The stronger statement says that equivalence classes of elliptic A-parameters give a decomposition of $\mathcal{A}_{disc}(G)$ which is finer than the decomposition into N.E. classes.

For GL(n) and classical groups (Sp($2n$), SO($2n+1$), O($2n$), $\mathrm{U}(n)$), there is no difference between N.E. classes and equivalence classes in $\Psi_{ell}(G)$. For these groups, one expects:

$$\boxed{\Psi_{ell}(G) \longleftrightarrow \{\text{N.E. classes in } \mathcal{A}_{disc}(G)\}}$$

Arthur Conjecture: Description of $\mathcal{A}_\Psi$

Fix a global A-parameter Ψ. Arthur describes the N.E. class (or the canonical submodule) $\mathcal{A}_\Psi$ using:

- a local-global principle: building global objects from local ones.
- a reciprocity law in the form of a multiplicity formula.

Global Component Group. Set $\boxed{A_\Psi = \pi_0\left(Z_{\widehat{G}}(\Psi)\right)}$. Has natural map $A_\Psi \longrightarrow A_{\Psi_v}$ for each v. If one sets:

$$A_{\Psi,\mathbb{A}} = \prod_v A_{\Psi_v},$$

have natural diagonal map $\boxed{\Delta : A_\Psi \longrightarrow A_{\Psi,\mathbb{A}}}$.

A Quadratic Character. Arthur associates to Ψ a quadratic character

$$\epsilon_\Psi : A_\Psi \longrightarrow \{\pm 1\}.$$

We won't give the definition here, but simply note that when Ψ is tempered/generic, $\epsilon_\Psi = 1$.

Local A-packets

$\boxed{\text{The Local A-packets}}$

- For each v, associated to Ψ_v is a finite multi-set Π_{Ψ_v} over $\mathrm{Irr}_{unit}(G(k_v))$: this is the local A-packet!
- Let $A_{\Psi_v} = \pi_0(Z_{\widehat{G}}(\Psi_v))$. There is a map

$$\boxed{\Pi_{\Psi_v} \longrightarrow \mathrm{Irr}\left(A_{\Psi_v}/Z(\widehat{G}^{\Gamma_{k_v}})\right).}$$

In other words, one can write:

$$\boxed{\Pi_{\Psi_v} = \{\pi_{\eta_v} : \eta_v \in \mathrm{Irr}\left(A_{\Psi_v}/Z(\widehat{G}^{\Gamma_{k_v}})\right)\}}$$

where π_{η_v} is a finite length unitary representation (possibly 0, possibly reducible).

Properties of Local A-packets

The local A-packets need to satisfy some properties:

- Π_{Ψ_v} contains the L-packet $\Pi_{\phi_{\Psi_v}}$.
- There is a commutative diagram

$$\begin{tikzcd}
\Pi_{\phi_{\Psi_v}} \arrow[r] \arrow[d] & \mathrm{Irr}\left(A_{\phi_{\Psi_v}}/Z(\widehat{G}^{\Gamma_{k_v}})\right) \arrow[d] \\
\Pi_{\Psi_v} \arrow[r] & \mathrm{Irr}\left(A_{\Psi_v}/Z(\widehat{G}^{\Gamma_{k_v}})\right).
\end{tikzcd}$$

- Stability and Endoscopic Character Identities.

One can view the local A-packet Π_{ψ_v} as an enlargement of the local L-packet $\Pi_{\phi_{\Psi_v}}$ so as to achieve (stability) and (endoscopic character identities)

These properties should more or less give a purely local characterization of the local A-packets.

Global A-packets

With the local A-packets Π_{Ψ_v} at hand, we can form the global A-packet:

$$\Pi_\Psi = \otimes'_v \Pi_{\Psi_v}$$

so that elements of Π_Ψ are restricted tensor product of elements of the local A-packets, with the local component being the unique K_v-unramified representation in $\Pi_{\phi_{\Psi_v}}$. If one sets:

$$\overline{A}_{\Psi,\mathbb{A}} = \prod_v (A_{\Psi_v}/Z(\widehat{G}^{\Gamma_{k_v}})) \quad \text{(a compact group)},$$

then

$$\mathrm{Irr}(\overline{A}_{\Psi,\mathbb{A}}) = \prod_v \mathrm{Irr}(A_{\Psi_v}/Z(\widehat{G}^{\Gamma_{k_v}})) \quad \text{(via tensor product)}$$

One can write:

$$\Pi_\Psi = \{\pi_\eta : \eta \in \mathrm{Irr}(\overline{A}_{\Psi,\mathbb{A}})\}.$$

Thus, Π_Ψ is a (possibly infinite) set of unitary representations of $G(\mathbb{A})$.

Multiplicity Formula

The submodule $\mathcal{A}_\Psi \subset \mathcal{A}_{disc}(G)$ will be a linear combination of elements in Π_Ψ:

$$\mathcal{A}_\Psi = \oplus_{\eta \in \mathrm{Irr}(\overline{A}_{\Psi,\mathbb{A}})} m_\eta \pi_\eta$$

for some multiplicities m_η.

$\boxed{\text{Multiplicity Formula}}$ Recall that we have a diagonal map

$$\Delta : A_\Psi \longrightarrow A_{\Psi,\mathbb{A}} \twoheadrightarrow \overline{A}_{\Psi,\mathbb{A}}.$$

Then we have:

$$\boxed{m_\eta = \langle \Delta^*(\eta), \epsilon_\Psi \rangle_{A_\Psi}}$$

Note that

$$L^2((\Delta(A_\Psi), \epsilon_\Psi)\backslash \overline{A}_{\Psi,\mathbb{A}}) \cong \bigoplus_\eta m_\eta \cdot \eta.$$

Summary

Returning to the questions raised at the beginning:

Questions:

- What are these finite sets of unitary representations for?
- How are they defined or characterised?
- How are they constructed?

We have addressed the first question and parts of the second:

- While local L-parameters and L-packets are designed to address the natural local question of classifying $\mathrm{Irr}(G(F))$, A-parameters and A-packets are designed as local ingredients in the solution of the natural global problem of classifying the constituents of $\mathcal{A}_{disc}(G)$.
- Like local L-packets, local A-packets may be characterized by some local properties and endoscopic character identities. But the value of their utility lies in the role they play in solving the above global problem.

The main part of this course is to address the 3rd. We will discuss some examples next.

The Trivial Representation

For G (split) semisimple, the constant functions are contained in $\mathcal{A}_{disc}(G)$. The weak approximation theorem implies that they form a N.E. class. What is its associated A-parameter Ψ_0?

Consider

$$\Psi_0 : L_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G$$

such that $\Psi_0|_{L_k} = 1$ and $\Psi_0|_{\mathrm{SL}_2}$ is the principal SL_2, corresponding to the regular unipotent conjugacy class of $\widehat{G}$ (Jacobson-Morozov).

- The centralizer of $\Psi_0(\mathrm{SL}_2(\mathbb{C}))$ is simply $Z(\widehat{G}) = A_{\Psi_0}$. Moreover, $\epsilon_{\Psi_0} = 1$.
- For each v, the local A-parameter $\Psi_{0,v}$ has same image as Ψ_0. So the local component groups $\overline{A}_{\Psi_{0,v}}$ is trivial.
- Moreover, $\phi_{\Psi_{0,v}}$ is the L-parameter of the trivial representation.

Conclusion: $\Pi_{\Psi_{0,v}} = \Pi_{\phi_{\Psi_{0,v}}} = \{\mathbb{C}\}$, $\Pi_{\Psi_0} = \otimes'_v \Pi_{\Psi_{0,v}} = \{\mathbb{C}\}$ and $\mathcal{A}_{\Psi_0} = \mathbb{C}$.

Example of GL(n)

$\Psi \in \Psi_{ell}(\mathrm{GL}(n))$ is an irreducible n-dim. representation. These look like:

$$\Psi = \Psi_a \boxtimes S_b, \quad \text{with } a \cdot b = n$$

where

- Ψ_a is a generic elliptic L-parameter for GL_a, thus corresponding to a cuspidal representation σ_a of GL_a by GLC.
- S_b is the irred. b-dim. representation of $\mathrm{SL}_2(\mathbb{C})$.

For such Ψ, $A_\Psi = 1$. For each v, we also have $\overline{A}_{\Psi_v} = 1$. Take $\Pi_{\Psi_v} = \Pi_{\phi_{\Psi_v}}$.

Recall that

$$\mathcal{A}_{disc} = \mathcal{A}_{cusp} \oplus \mathcal{A}_{res}.$$

For $G = \mathrm{GL}(n)$, the cuspidal spectrum $\mathcal{A}_{cusp}$ is multiplicity-free and its irreducible summands are indexed by the elliptic generic Ψ (with $b = 1$). So the residual spectrum should be described by the non-generic elliptic Ψ (with $b > 1$).

Results of Moeglin-Waldspurger on Residual Spectrum

Moeglin and Waldspurger gave a decomposition of $\mathcal{A}_{res}(\mathrm{GL}(n))$:

- With $n = a \cdot b$ and $\sigma_a \subset \mathcal{A}_{cusp}\mathrm{GL}(a)$, one considers the representation
$$I(\sigma_a, b) := \sigma_a|\det|^{\frac{b-1}{2}} \times \sigma_a|\det|^{\frac{b-3}{2}} \times .... \times \sigma_a|\det|^{-\frac{b-1}{2}},$$
parabolically induced from the Levi subgroup $\mathrm{GL}(a)^b$.
- $I(\sigma_a, b)$ has a unique irreducible quotient $J(\sigma_a, b)$ with the expected L-parameter (Speh reps).
- Using iterated residues of Eisenstein series, [MW] showed
$$J(\sigma_a, b) \hookrightarrow \mathcal{A}_{res}(\mathrm{GL}(n))$$
and
$$\mathcal{A}_{res}(\mathrm{GL}(n)) \cong \bigoplus_{ab=n} \bigoplus_{\sigma_a \subset \mathcal{A}_{cusp}(\mathrm{GL}(a))} J(\sigma_a, b).$$

This verifies Arthur's conjecture for $\mathrm{GL}(n)$: it describes $\mathcal{A}_{disc}(\mathrm{GL}(n))$ in terms of $\mathcal{A}_{cusp}(\mathrm{GL}(a))$ for all $a < n$ dividing n, in exactly the precise form predicted by Arthur.

PGL(2) = SO(3) versus SL(2) = Sp(2)

The case of SO(3) = PGL(2) is a special case of the discussion on GL(n). So we focus on $G = \mathrm{SL}(2)$ with $\widehat{G} = \mathrm{SO}_3(\mathbb{C})$.

- $\Psi \in \Psi_{ell}(\mathrm{SL}(2))$ can be regarded as a 3-dimensional representation of $L_k \times \mathrm{SL}_2(\mathbb{C})$ which is of orthogonal type. Two such Ψ's are equivalent as A-parameters iff they are isomorphic as 3-dim. representations.
- If $\Psi|_{\mathrm{SL}_2} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{C})$ is the natural surjective map, then $\Psi = \Psi_0$ (giving the trivial representation).
- If $\Psi|_{\mathrm{SL}_2} = 1$, then Ψ is generic and is a multiplicity-free sum of irreducible reps of orthogonal type. There are 3 possibilities.
  - If Ψ irreducible, then $A_\Psi = 1$.
  - If $\Psi = (2 - dim) + (1 - dim)$, $A_\psi \cong \mathbb{Z}/2\mathbb{Z}$.
  - If $\Psi =$ sum of 3 quadratic characters, then $A_\Psi \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example of SL(2)

For each place v, Ψ_v need not be elliptic (even if Ψ is) and A_{Ψ_v} can be any one of the above 3 possibilities. So in this case,

- one has non-singleton local A-packets;
- not every member of a global A-packet occurs in $\mathcal{A}_{disc}$: the multiplicity formula provides a constraint.

This suggests that the automorphic spectrum of SL(2) is more complicated than that of PGL(2). Indeed:

- The multiplicity formula for generic elliptic Ψ was first discovered by Labesse-Langlands for SL(2) around 1970;
- They developed the theory of endoscopy for SL(2) to deal with the cases with nontrivial A_Ψ.
- The fact that $\mathcal{A}_{disc}(\mathrm{SL}(2))$ is multiplicity-free was only shown around 2000 by Ramakrishnan.

Hopefully, all these will be explained in the course of Parab and Kaletha.

Example of $\mathrm{PGSp}(4) = \mathrm{SO}(5)$

For $G = \mathrm{PGSp}(4) = \mathrm{SO}(5)$, $\widehat{G} = \mathrm{Sp}(4, \mathbb{C})$. We enumerate the different families of elliptic A-parameters, according to the unipotent conjugacy classes corresponding to $\Psi|_{\mathrm{SL}_2}$.

- Principal SL_2 (trivial rep.): $\Psi|_{\mathrm{SL}_2}$ is irreducible.
- Long root SL_2 (Saito-Kurokawa): $\Psi|_{\mathrm{SL}_2} \cong S_2 \oplus 2 \cdot \mathbb{C}$
- Short root SL_2 (Soudry and Howe-PS): $\Psi|_{\mathrm{SL}_2} = 2 \cdot S_2$.
- Trivial SL_2: this is the generic case.

We shall consider the two intermediate families in some detail.

Saito-Kurokawa Parameters

The long root SL_2 is given by the embedding

$$\mathrm{SL}_2'(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) = \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_2'(\mathbb{C}) \hookrightarrow \mathrm{Sp}_4(\mathbb{C}).$$

The centralizer of SL_2 is $\mathrm{SL}_2' \times Z(\mathrm{SL}_2)$. So an A-parameter of this type has form:

$$\Psi_{\rho,\chi} : L_k \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2'(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C}),$$

with

$$\rho : L_k \longrightarrow \mathrm{SL}_2'(\mathbb{C}) \quad \text{irreducible}, \quad \chi : L_k \longrightarrow \mu_2 \subset \mathrm{SL}_2(\mathbb{C}).$$

Hence ρ corresponds to a cuspidal rep. σ of PGL(2).

A Saito-Kurokawa A-parameter is associated to a cuspidal rep. σ of PGL(2) and a quadratic character χ of GL(1).

Saito-Kurokawa A-Packets

For such a $\Psi = \Psi_{\pi,\chi}$

- $A_\Psi = \mu_2 \times \mu_2$ and $Z(\widehat{G}) = \mu_2^\Delta$, so $\overline{A}_\Psi \cong \mu_2$.
- For each v,

$$\overline{A}_{\Psi_v} = \begin{cases} \mu_2 \text{ if } \rho_v \text{ irreducible, i.e. } \sigma_v \text{ D.S.} \\ 1 \text{ if } \rho_v \text{ reducible, i.e. } \sigma_v \text{ not D.S.} \end{cases}$$

- In the two respective cases, the local A-packets have the form

$$\boxed{\Pi_{\Psi_v} = \{\pi_v^+, \pi_v^-\} \quad \text{or} \quad \Pi_{\Psi_v} = \{\pi_v^+\}.}$$

- Moreover, $\Pi_{\phi_{\Psi_v}} = \{\pi_v^+\}$ and $\chi_v|-| \rtimes \sigma_v \twoheadrightarrow \pi_v^+$.
- Let S be the finite set of places where σ_v is D.S. Then

$$\boxed{\Pi_\Psi = \{\pi^\epsilon = \otimes_v \pi_v^{\epsilon_v} : \epsilon_v = \pm\}} \quad \text{so } \# A_\Psi = 2^{\# S}.$$

- The character ϵ_Ψ of $\overline{A}_\Psi = \mu_2$ is trivial iff $\epsilon(1/2, \sigma) = 1$.
- The multiplicity formula is:

$$\boxed{m(\pi^\epsilon) = \begin{cases} 1 \text{ if } \prod_v \epsilon_v = \epsilon(1/2, \sigma). \\ 0 \text{ if } \prod_v \epsilon_v = -\epsilon(1/2, \sigma_v). \end{cases}}$$

Soudry-type and Howe-PS type

The short root SL_2 is given by the natural embedding

$$\mathrm{O}_2(\mathbb{C}) \times_{\mu_2} \mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{Sp}_4(\mathbb{C}),$$

with $\mathrm{O}_2(\mathbb{C})$ the centralizer of SL_2. An elliptic A -parameter Ψ of this type is of form

$$\Psi_\rho : L_k \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}_2(\mathbb{C}) \times_{\mu_2} \mathrm{SL}_2(\mathbb{C}) \subset \mathrm{Sp}_4(\mathbb{C})$$

where

$$\rho : L_k \longrightarrow \mathrm{O}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$$

is an elliptic parameter for O_2.

Two possibilities:

- ρ is irreducible 2-dim. rep.: Soudry type.
- $\rho = \chi_1 \oplus \chi_2$, with χ_i quadratic characters: Howe-PS type.

Exercise: Work out the structure of the A-packets for these two cases, as I did for the Saito-Kurokawa parameter.

Example of G_2

When $G = G_2$, $\widehat{G} = G_2(\mathbb{C})$. We consider the most interesting family of A-parameters, associated to the subregular unipotent orbits. (I learned Arthur's conjecture through this example)

FACT: There is a map

$$\iota : SL_2(\mathbb{C}) \twoheadrightarrow \mathrm{SO}_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$$

such that the centralizer of the image is the finite group S_3. So have:

$$\boxed{S_3 \times \mathrm{SO}_3(\mathbb{C}) \subset G_2(\mathbb{C}).}$$

If an A-parameter Ψ satisfies $\Psi|_{\mathrm{SL}_2} = \iota$, then

$$\boxed{\Psi|_{L_k} : L_k \twoheadrightarrow W_k \to S_3.}$$

Such morphisms (modulo conjugacy) correspond to giving a separable cubic algebra over k (e.g. a cubic field extension). So call such Ψ's cubic unipotent A-parameters.

(Cubic) Unipotent A-Parameters

Let's consider the example where $\Psi|_{L_k} = 1$ (corresponding to the split cubic algebra k^3). Then:

- $\mathrm{Im}(\Psi) = \iota(\mathsf{SL}_2(\mathbb{C})) = \mathrm{Im}(\Psi_v)$ for each v.
- So $A_\Psi = S_3 = A_{\Psi_v}$ (non-abelian!)
- Hence $A_{\psi,\mathbb{A}} = S_3(\mathbb{A})$ and the diagonal map $\Delta : A_\Psi \to A_{\Psi,\mathbb{A}}$ is simply the natural embedding

$$\Delta : S_3(k) \hookrightarrow S_3(\mathbb{A}).$$

- S_3 has 3 irreducible characters: 1, $\epsilon = \mathrm{sign}$ and a 2-dim. rep. r. So

$$\boxed{\Pi_{\Psi_v} = \{\pi_{1,v}, \pi_{r,v}, \pi_{\epsilon,v}\}}$$

- One has: $\epsilon_\Psi = 1$.

Unbounded Multiplicity

A member of the global A-packet is determined by two disjoint finite sets S_r and S_ϵ of places of k:

$$\pi_{S_r,S_\epsilon} = \left(\otimes_{v\in S_r}\pi_{r,v}\right)\otimes\left(\otimes_{v\in S_\epsilon}\pi_{\epsilon,v}\right)\otimes\left(\otimes_{v\notin S_r\cup S_\epsilon}\pi_{1,v}\right).$$

What is the multiplicity of this representation in $\mathcal{A}_{disc}$?

Consider the special case $S_\epsilon = \emptyset$. The multiplicity formula gives:

$$m_{S_r} = \langle r^{\otimes|S_r|}, 1\rangle_{S_3} = \frac{1}{6}\cdot(2^{|S_r|} + 2\cdot(-1)^{|S_r|}).$$

Observe that

$$m_{S_r}\to\infty \quad \text{as } |S_r|\to\infty.$$

Hence, Arthur's conjecture predicts that $\mathcal{A}_{disc}(G_2)$ has unbounded multiplicity. This was demonstrated by Gan-Gurevich-Jiang in 2002.

Classical Groups

- In his book, Arthur has established his conjecture for the quasi split groups $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{O}(2n)$.
- Question: How can one prove Arthur's conjecture without knowing the existence of L_k, or equivalently the Global Langlands Correspondence for $\mathrm{GL}(n)$?
- Answer: As illustrated in the examples we looked at, to write down A-parameters of classical groups, one does not need L_k: one only needs irreducible representations of L_k. By GLC, these correspond to cuspidal reps of $\mathrm{GL}(n)$. Hence, one can formulate Arthur's conjecture for classical groups purely in terms of cuspidal reps. of $\mathrm{GL}(n)$, thus suppressing mention of L_k or its representations.

Upshot: Arthur's result is a classification of $\mathcal{A}_{disc}(G)$ in terms of cuspidal reps of $\mathrm{GL}(n)$.

This is the crowning achievement of the theory of endoscopy. The course of Gordon and Altug deals with going beyond endoscopy.

Questions

Arthur showed the existence of the A-packets using the stable trace formula, i.e. by global means. However, if Ψ_v is non-generic, ie. $\Psi_v|_{\mathrm{SL}_2}$ nontrivial, he does not know these local packets very explicitly, beyond the fact that they satisfy some character identities.

- Is $\pi_\eta \in \Pi_{\Psi_v}$ nonzero?
- Is π_η reducible? What is its length?
- Is π_η multiplicity-free?
- Is Π_{Ψ_v} a set rather than a multi-set, i.e. is $\oplus_\eta \pi_\eta$ multiplicity-free?
- Can the constituents of π_η be explicitly described, such as by the LLC?
- Can the A-packet Π_{Ψ_v} be constructed purely by local means?

The purpose of this course is to address some of these questions.