On unipotent representations of real classical groups (joint with Binyong Sun and Chengbo Zhu)

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Outline

1. Motivation
2. Unipotent representations of real classical groups
3. Key ingredients of the proof

Unipotent Arthur packet

- Arthur parameter: $\psi\colon W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{G}^\vee \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.
  $W_{\mathbb{R}} = \mathbb{C}^\times \cup \mathbf{j}\mathbb{C}^\times$.
- Arthur packet $\Pi_\psi(G)$ whose size is controled by the component group of $Z_{G^\vee}(\psi)$.
- Unipotent Arthur parameter: $\psi|_{\mathbb{C}^\times}$ is trivial
- Question: How to construct elements in $\Pi_\psi(G)$ for a unipotent Arthur parameter?
- Unipotent Arthur parameter ψ
  $\rightsquigarrow$ nilpotent orbits ${}^L\mathcal{O}$ in $\mathfrak{g}^\vee := \mathrm{Lie}(\mathbf{G}^\vee)$ (Jacobson-Morozov).
  $\rightsquigarrow$ special nilpotent orbits $\mathcal{O}$ in $\mathfrak{g} := \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ (Lusztig-Spaltenstein-Barbasch-Vogan)
- $\rightsquigarrow$ Barbasch-Vogan's definition of *special unipotent reprensentation*.

Orbit method

Let G be a Lie group, with Lie algebra $\mathfrak{g}$.

- Philosophical of the orbit method (Kirillov theory):

$$\widehat{G}_u \leftrightsquigarrow \mathfrak{g}^*/G.$$

- Analogy between
  - classical mechanical systems $\rightsquigarrow$ quantum mechanical systems
  - coadjoint orbits $\rightsquigarrow$ unitary representations
- Both processes are called quantization.

Vogan: The orbit method should also serve as a unifying principle for the unitary dual problem of reductive Lie groups.

For different types of coadjoint orbits.

- Hyperbolic orbits: parabolic induction
  - Method of real analysis (Mackey, Gelfand-Naimark, Bruhat, ..., beginning from the 1950's)
- Elliptic orbits: cohomological induction
  - Method of complex analysis (Harish-Chandra, Schmid, Zuckerman, Vogan, beginning from the 1970's)
- Nilpotent orbits: ???
  - Terminology: unipotent representations

Special unipotent representations

- $\mathcal{O}$: a special nilpotent orbit in $\mathfrak{g}$
  $\rightsquigarrow$ infinitesimal character $\chi_{\mathcal{O}}$.

Special unipotent representation with integral infinitesimal character

Assume $\chi_{\mathcal{O}}$ is integral. An irreducible representation π of G is called a *special unipotent* representation attached to $\mathcal{O}$, if

- π has infinitesimal character $\chi_{\mathcal{O}}$ and
- $\mathrm{AV}(\pi) \subset \overline{\mathcal{O}}$.

Let $\mathrm{Unip}_{\mathcal{O}}(G)$ be the set of $\mathcal{O}$-unipotent representations.

- Expectation : $\mathrm{Unip}_{\mathcal{O}}(G)$ and $\Pi_{\psi}(G)$ coincide
- Conjecture : $\mathrm{Unip}_{\mathcal{O}}(G)$ consists of unitary representations. (Proved by Barbasch-Vogan for complex groups ...)

Classical groups and its dual

G_V^J	G_V	$G_{\check{V}}$
real form of	$\mathrm{O}(2n, \mathbb{C})$	$\mathrm{O}(2n, \mathbb{C})$
real form of	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$
real form of	$\mathrm{O}(2n+1, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$
$\mathrm{Mp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$

$\epsilon = 1$ (resp. -1) if G_V is an orthogonal group (resp. a symplectic group)
$\check{\epsilon} = 1$ (resp. -1) if $G_{\check{V}}$ is an orthogonal group (resp. a symplectic group)
Here V and $\check{V}$ are the formed spaces over $\mathbb{C}$.
J is a conjugate linear automorphism of V such that $J^2 = \pm 1$.

Infinitesimal characters

- $\check{\mathcal{O}} \in \mathrm{Nil}(\mathfrak{g}_{\check{V}})$
- Young diagram of $\check{\mathcal{O}}$ has row $\{ a_1 \geq a_2 \geq \cdots \geq a_s > 0 \}$
- $\rightsquigarrow$ a character $\chi_{\check{\mathcal{O}}} : \mathcal{U}(\mathfrak{g}_V)^{G_V} \to \mathbb{C}$:

$$\chi_{\check{\mathcal{O}}} := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0)$$

where

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd;} \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even;} \end{cases}$$

Nilpotent orbits in $\mathfrak{g}_{\check{V}}$

Definition

$\check{\mathcal{O}} = \{ a_1 \geq a_2 \geq \cdots \geq a_k > 0 \} \in \mathrm{Nil}(\mathfrak{g}_{\check{V}})$ is called *purely even* if all a_i are

- odd when $\check{\epsilon} = 1$;
- even when $\check{\epsilon} = -1$.

Let $\mathbf{Nil}^{pe}(\mathfrak{g}_{\check{V}})$ be the set of purely even orbits.

$\check{\mathcal{O}} \in \mathbf{Nil}^{pe}(\mathfrak{g}_{\check{V}})$ is called *quasi-distinguished* if

$$\begin{cases} a_{2i} > a_{2i+1} & \text{whenever } 1 \leq i \text{ and } 2i+1 \leq k, \qquad \text{if } \epsilon = 1; \\ a_{2i-1} > a_{2i} & \text{whenever } 1 \leq i \text{ and } 2i \leq k, \qquad \text{if } \epsilon = -1. \end{cases}$$

Let $\mathbf{Nil}^{qd}(\mathfrak{g}_{\check{V}})$ be the set of quasi-distinguished orbits.

{ purely even orbit } $\supset$ { qusi-distinguished orbit } $\supset$ { distinguished orbit }
$d_{BV}(\mathbf{Nil}^{qd}(\mathfrak{g}_{\check{V}})) \supset$ { rigid special nilpotent orbits }

Special Unipotent representation

Definition

An irreducible representation π of G_V^J is called a *(weakly) special unipotent* representation attached to $\check{\mathcal{O}} \in \mathbf{Nil}^{pe}(\mathfrak{g}_{\check{V}})$, if

- π has infinitesimal character $\chi_{\check{\mathcal{O}}}$ and
- $\mathrm{AV}(\pi) \subset \overline{\mathcal{O}}$ where $\mathcal{O} := d_{BV}(\check{\mathcal{O}})$.
- π is genuine if G_V^J is a metaplectic group.

Let $\mathrm{Unip}_{\check{\mathcal{O}}}(G_V^J)$ be the set of $\check{\mathcal{O}}$-unipotent representations.

The definition is equivalent to $\mathrm{Ann}\,(\pi)$ = the maximal primitive ideal in $\mathcal{U}(\mathfrak{g})$ with infinitesimal character $\chi_{\check{\mathcal{O}}}$.

Conjecture

$\mathrm{Unip}_{\check{\mathcal{O}}}(G_V^J)$ *consists of unitary representations.*

Descent of nilpotent orbits

Take $\check{\mathcal{O}} \in \mathbf{Nil}^{pe}(\mathfrak{g}_{\check{V}})$. Descent sequence in the dual side is:

$$(\check{V}_0, \check{\mathcal{O}}_0), (\check{V}_1, \check{\mathcal{O}}_1), \cdots, (\check{V}_k, \check{\mathcal{O}}_k)$$

such that $\check{V}_i$ is $\check{\epsilon}$-symmetric, and $\check{\mathcal{O}}_i =$ removing the first i-rows of $\check{\mathcal{O}}$.

Definition

A descent sequence for $(\mathcal{O}, G_V^J)$ is a sequence

$$\mathcal{J} : (\epsilon_0, V_0, J_0), (\epsilon_1, V_1, J_1), \cdots, (\epsilon_k, V_k, J_k)$$

- $(\epsilon_0, V_0, J_0) = (\epsilon, V, J)$;
- $\epsilon_i = (-1)^i \epsilon$, and $G_{\check{V}_i}$ is the dual of G_{V_i};
- $J_i^2 = J^2$, i.e. $G_{\check{V}_i}^J$ is real or quaternion, if G_V^J is respectively real or quaternionic.

Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G_V^J)$

Definition

Let $\chi = \chi_0 \otimes \chi_1 \otimes \cdots \otimes \chi_k$ be a finite order character of $G_{V_0}^{J_0} \times G_{V_1}^{J_1} \times \cdots \times G_{V_k}^{J_k}$, Define a smooth representation of G_V^J.

$$\pi_{\mathcal{J},\chi} := \left(\omega_{V_0,V_1} \widehat{\otimes} \omega_{V_1,V_2} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{V_{k-1},V_k} \otimes \chi\right)_{G_{V_1}^{J_1} \times G_{V_2}^{J_2} \times \cdots \times G_{V_k}^{J_k}}$$

Theorem

If $\pi_{\mathcal{J},\chi}$ is nonzero, then it is irreducible, unitarizable, and $\check{\mathcal{O}}$-unipotent.

Expectation: $\mathrm{Unip}_{\check{\mathcal{O}}}(G_V^J) = \{ \pi_{\mathcal{J},\chi} \neq 0 \}$.

- Quasi-distinguished nilpotent orbit $\check{\mathcal{O}} \in \mathbf{Nil}^{qd}(G_{\check{V}})$:

  J.-J. Ma, B. Sun and C.-B. Zhu, "Unipotent representations of real classical group", arXiv:1712.05552.
- General purely even orbit $\check{\mathcal{O}} \in \mathbf{Nil}^{pe}(G_{\check{V}})$:
  work in progress with Barbasch, Sun and Zhu.
- A lot of people have contributed to the problem

  Barbasch, Vogan, Mœglin, He, Adams, Przebinda, Trapa,,....

Associated characters

- Let $\mathcal{O}$ be an nilpotent coadjoint orbit in $\mathfrak{g}^*$. (Here $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$).
- Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
- $\mathfrak{g}_{\mathbb{R}}^* \cap \mathcal{O}/G \leftrightarrow \mathfrak{p}^* \cap \mathcal{O}/\mathbf{K}$ (Kostant-Sekiguchi)
- $\{ G$-equiv. sheaves on $\mathfrak{g}_{\mathbb{R}}^* \cap \mathcal{O} \}$
  $\leftrightarrow \{ \mathbf{K}$-equiv. sheaves on $\mathfrak{p}^* \cap \mathcal{O} \}$.
- $\mathcal{K}_{\mathcal{O}}$: Grothendieck group of $\mathbf{K}$-equiv. coherent sheaves supp. on $\mathfrak{p}^* \cap \mathcal{O}$.
- $\mathcal{K}_{\mathscr{O}}^{\mathrm{aod}} \subset \mathcal{K}_{\mathcal{O}}$: the set of admissible orbit data on a $\mathbf{K}$-orbit $\mathscr{O} \subset \mathfrak{p}^* \cap \mathcal{O}$
- Associated character map:

$$\mathrm{Ch}\colon \{ \pi \in \mathrm{Irr}(G) \mid \mathrm{AV}(\pi) \subset \overline{\mathcal{O}} \} \longrightarrow \mathcal{K}_{\mathcal{O}}.$$

Main theorem

Theorem

For $\check{\mathcal{O}} \in \mathbf{Nil}^{qd}$,

- *the following is a bijection:*

$$\mathrm{Ch}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \longrightarrow \bigsqcup_{\mathscr{O} \subset \mathfrak{p}^* \cap \mathcal{O}} \mathcal{K}_{\mathscr{O}}^{\mathrm{aod}}$$

- $\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{ \pi_{\mathcal{J},\chi} \neq 0 \}$ *consists of unitary representations*

Conjecture (Vogan):

$\pi|_{\mathbf{K}} \cong$ Global section of the vector bundle given by $\mathrm{Ch}_{\mathcal{O}}(\pi)$.

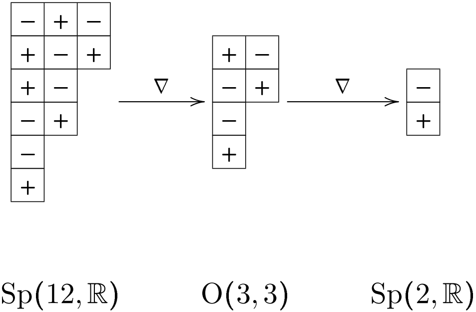
The conjecture is verified in many cases ...,[Barbasch], [Loke-Ma], ...

Local Theta correspondence

- $(G, G') = (\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}))$ is a dual pair in $\mathrm{Sp}(2(p+q)n,\mathbb{R})$.
- *Theta correspondence.* $\mathcal{R}(\widetilde{G},\omega) \xleftrightarrow{\theta} \mathcal{R}(\widetilde{G}',\omega)$
- 1. ω: the Weil repn. of the metaplectic group $\mathrm{Mp}(2(p+q)n,\mathbb{R})$
  2. Fix $\pi \in \mathrm{Irr}_{\mathsf{gen}}(\widetilde{G})$, $\omega_\pi \cong \pi \otimes \Theta(\pi)$.
  3. $\Theta(\pi) \neq 0 \Leftrightarrow \pi$ in the domain $\mathcal{R}(\widetilde{G},\omega)$ of theta correspondence.
     $\Rightarrow \Theta(\pi)$ has a unique irr. quotient $\theta(\pi) \leftarrow$ the *theta lift* of π.
     [Howe 89] [Waldspurger 90] [Gan-Takeda 14] [Gan-Sun 15]
  4. π unitary $\longrightarrow \theta(\pi)$ unitary,
     in stable range [Li 89], semi-stable range [He]
  5. Describe the correspondence in terms of parameters:
     Archimedean place (mainly eq. rank): Adams, Barbasch, Mœglin, Paul, ...
     p-adic place:..., Gan, Ichino, Atobe (tempered), Loke-Ma (supercuspidal repn.), $\cdots$

Lift/Descent of nilpotent orbits

- $\mathbf{Nil}_{\mathbf{K}}(\mathfrak{p}^*)$ are parameterized by signed young diagrams



- **Kraft-Procesi**, **Ohta**,

  resolution of singularities of nilpotent orbit closure

- Define a sequence $\mathscr{O} = \mathscr{O}_0 \xrightarrow{\nabla} \mathscr{O}_1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathscr{O}_r$ so that $\mathscr{O}_r = \{0\}$.
  $\mathscr{O}_i$ is a K_i-orbit in $\mathfrak{p}_i^*$.
- Let χ_i be a character of G_i.
- Define, $\pi_r = \chi_r$ and $\pi_i = \chi_i \otimes \overline{\Theta}(\pi_{i+1})$.
- $\overline{\Theta}(\_)$ is the "theta lift" given by matrix coefficient integrals (a variation of the theta lift)

  Inductively, WTS:

- $\pi_i \neq 0$,
- $\mathrm{Ch}_{\mathcal{O}_i}(\pi_i)$ is in $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}_i}$.
- $\{\pi_i \mid \chi \text{ vary}\} \xrightarrow{\mathrm{Ch}_{\mathcal{O}_i}} \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}_i}$ is surjective.
- π_i has infinitesimal character $\chi_{\check{\mathcal{O}}_i}$
  (use the correspondence of infinitesimal characters by **Prezbinda**).
- As a consequence, $\pi_i \in \mathrm{Unip}_{\check{\mathcal{O}}_i}(G_{V_i}^{J_i})$.

- *Unitarity*:
  - Estimate of matrix coefficients using the explicit realization of the Weil representations.
    Work of **Li**, **He**, and an idea of **Harris-Li-Sun** showing the nonnegativity of a matrix coefficient integral.
- The non-vanishing, and compute associated character:
  - **Geometry**: moment maps provide the upper bound.
  - **Analysis**: degenerate principal series force the lower bound.
  - Geometry meets Analysis: the equality.

Geometry of the moment maps

- The $(G, G') = (\mathrm{O}(p,q), \mathrm{Sp}(2n, \mathbb{R}))$ setting:

$$
M_{p,q} = \mathfrak{p}^* \xleftarrow{(A,B)\mapsto AB^T} M_{p,n} \oplus M_{q,n} \xrightarrow{(A,B)\mapsto (A^TA, B^TB)} \mathfrak{p}'^* = \mathrm{Sym_n} \oplus \mathrm{Sym_n}
$$

- Upper bound of associated character: Suppose $\nabla(\mathcal{O}) = \mathcal{O}'$, there is a natural map (defined geometrically)

$$
\vartheta : \mathcal{K}_{\mathcal{O}'} \to \mathcal{K}_{\mathcal{O}}.
$$

so that

$$
\mathrm{Ch}_{\mathcal{O}}(\Theta(\pi')) \leq \vartheta(\mathrm{Ch}_{\mathcal{O}'}(\pi')),
$$

for any π' with $\mathrm{AV}(\pi') \subset \overline{\mathcal{O}'}$

Analysis

- We still need a lower bound for $\overline{\Theta}(\pi_i)$.
- **A idea of He**: perform another theta lift!

$$\overline{\Theta}'(\overline{\Theta}(\pi_i))$$

where $\overline{\Theta}'$ is the theta lift for a certain pair.

This is an doubling method!

- **The upshot**: $\overline{\Theta}'(\overline{\Theta}(\pi_i))$ is (close to) a parabolic induction representation (using structure of degenerate principle series by **Lee-Zhu and Yamana**).

A formula for associated characters

Theorem

Let $\mathcal{O} = [c_0, c_1, \cdots, c_k]$ *and* $\mathcal{O}' = \nabla(\mathcal{O})$. *Assume that* $c_0 > c_1$ *when* G_V^J *is a real symplectic group or metaplectic group.*
Then for every $\mathcal{O}'$*-bounded Casselman-Wallach representation* π' *in the convergent range,*

- $\overline{\Theta}_{V',V}(\pi')$ *is* $\mathcal{O}$*-bounded and*
- $$\mathrm{Ch}_{\mathcal{O}}(\overline{\Theta}_{V',V}(\pi')) = \vartheta_{\mathcal{O}',\mathcal{O}}(\mathrm{Ch}_{\mathcal{O}}(\pi')).$$

Exhaustion

Counting method:

- lower bound $= \#\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}$.
- Count # unipotent representations for split classical groups, developed by Barabasch, Vogan, McGovern ...
- Stable range theta lifts (injective, unipotent → unipotent) to split classical group

  $\rightsquigarrow$ upper bound of # unipotent representation.

  The bound is sharp!
- So, we got all!

Thank you!