On the Ramanujan conjecture over function fields

Will Sawin (Columbia University and Clay Institute)
Nicolas Templier (Cornell University)

X is a smooth projective curve, $F = k(X)$ function field.

The Ramanujan conjecture that every cuspidal automorphic representation π of $\mathrm{GL}(n)$ is tempered is established by Drinfeld (1980's) for $n = 2$ and L.Lafforgue (2000's) for general n.

For GSp_4, there are cuspidal non-tempered automorphic representations as shown by Saito-Kurokawa and Howe–Piatetskii-Shapiro (1970's).

Conjecture (Arthur, 1988)

If π_u belongs to a supercuspidal L-packet for at least one place u, then π is tempered.

Assume in this talk that G is split and $\mathrm{char}(F) \neq 2$.

(The method also works more generally assuming that some transfer conjectures hold.)

Sarnak-Shin-T. (2014): We formulated the Katz-Sarnak heuristics, and the Sato-Tate conjecture for geometric families, resp. harmonic families. Family versions of the individual Sato-Tate conjecture, resp. the Langlands conjecture. Individual and family versions don't imply each other.

Observation (simplified version of Sawin-T. 2018)

Imagine Arthur conjectures as analogs of the Weil conjectures

Why?

- Arthur already established the relation between his conjectures and Lefschetz $\mathrm{SL}(2)$ acting on the cohomology of Shimura varieties.
- Neither Weil's nor Arthur's conjectures tell you what the weights are.

Hard Lefschetz is proved over $\mathbb{C}$ by transcendental methods by Lefschetz (1924), and over any field by algebraic methods by Deligne as consequence of the Weil conjectures, see Katz-Messing (1973).

A new definition in the representation theory of reductive groups over non-archimedean fields (Sawin-T. 2018)

mgs = monomial geometric supercuspidal

Monomial means quotient of **compactly induced representation from a character**.
Geometric means that the character and the subgroup are **preserved under unramified extensions**. More details in the coming slides (think of a character in a graded piece of the Moy-Prasad filtration).

mgs are non-archimedean analogs of holomorphic discrete series

Conjecture (Arthur, 1988)

If π_u belongs to a supercuspidal L-packet for at least one place u, then π is tempered.

Theorem (Sawin-T., in preparation)

If π_u belongs to an mgs packet for at least one place u, then π is tempered at every unramified place.

Part I of our work is posted:

Theorem (Sawin-T., arXiv:1805.12231)

Suppose that for at least one place u, the representation π_u is mgs, and that π is base-changeable from $\mathbb{F}_q(X)$ to $\mathbb{F}_{q^n}(X)$ for infinitely many n. Then π is tempered.

Part II is in progress:

Theorem (Sawin-T., in preparation)

Suppose that for at least one place u, the representation π_u belongs to an mgs packet. Then π is base-changeable from $\mathbb{F}_q(X)$ to $\mathbb{F}_{q^n}(X)$ for every $n \geqslant 1$.

Monomial geometric supercuspidal (mgs)

The definition is motivated by features of the problem and our method to attack it via a geometric study of the trace formula kernel and their families:

- Our method requires a way to check the local condition. Monomial $=$ the condition that a representation contains a vector which transforms according to a one-dimensional character $\chi : J \to \mathbb{C}^\times$ of a subgroup J.
- At minimum we should avoid the residual spectrum, and a supercuspidal condition is the easiest to achieve this. Thus we want that $\operatorname{c-ind}_J^G \chi$ has vanishing Jacquet modules. It is equivalent that the restriction of χ to $J \cap N$ for every proper parabolic $P = MN$ be non-trivial.

Monomial geometric supercuspidal, formal definition

- Geometric objects behave similarly over extension fields. In our case, if we use a geometric property to prove temperedness, this property will be maintained over constant field extensions. Again the easiest way to ensure cuspidality is that our character (J, χ) still prescribes a supercuspidal representation after a constant field extension.

For an algebraic group H over a field κ, a character sheaf $\mathcal{L}$ is a rank one lisse sheaf, with an isomorphism $\mathcal{L} \boxtimes \mathcal{L} \simeq m^*\mathcal{L}$, where $m : H \times H \to H$ is multiplication. For every field extension κ'/κ, the trace function of $\mathcal{L}$ defines a character $\chi_{\kappa'} : H(\kappa') \to \overline{\mathbb{Q}}_\ell^\times$.

Monomial geometric supercuspidal, continued

Let G be a quasi-split reductive group over a field κ. A monomial geometric datum is a pair $(H, \mathcal{L})$, of a pro-algebraic group $H \subset G[\![t]\!]$ of finite codimension, and of a character sheaf $\mathcal{L}$ on H. Here, the arc group $G[\![t]\!]$ is an affine group scheme (of infinite dimension), inside the loop group $G(\!(t)\!)$ which is a group ind-scheme.

Definition

We say that $(H, \mathcal{L})$ is **geometric supercuspidal** if for every unipotent radical N of a proper parabolic subgroup of $G(\!(t)\!)$, the restriction of $\mathcal{L}$ to $N \cap H$ is geometrically non-trivial.

If $\kappa = \mathbb{F}_q$, $J_n = H(\mathbb{F}_{q^n})$, viewed as a finite index subgroup of the hyperspecial maximal compact $G(\mathbb{F}_{q^n}[\![t]\!])$, and $\chi_n : H(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ is the character associated to $\mathcal{L}$, the definition is equivalent to

For every $n \geqslant 1$, the induced representation $\mathrm{c\text{-}ind}_{J_n}^{G(\mathbb{F}_{q^n}(\!(t)\!))} \chi_n$ has vanishing Jacquet module for every proper parabolic subgroup of $G(\mathbb{F}_{q^n}(\!(t)\!))$. Typically a direct sum of irreducible supercuspidals.

Unipotent representations are <u>not</u> mgs

The representation considered by Howe-Piatetskii-Shapiro, for $G = \mathrm{Sp}_4$, has local component

$$\pi_u = \text{c-ind}_{G(\mathbb{F}_q[\![t]\!])}^{G(\mathbb{F}_q(\!(t)\!))} \theta_{10}$$

where θ_{10} is the simplest of the unipotent representations of $G(\mathbb{F}_q)$ constructed by Lusztig.

It is **not** geometric supercuspidal. Because applying the Deligne-Lusztig construction with the same inducing data to $G(\mathbb{F}_{q^n})$, the induced representation becomes a principal series for n large enough. Indeed every elliptic $\mathbb{F}_q$-torus in G splits after base change $\otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ for n large enough.

Which supercuspidal representations are mgs?

- All wildly ramified supercuspidals of GL_n are mgs. This follows from Bushnell-Kutzko's construction of cuspidal types.
- Simple supercuspidal representations, and more generally epipelagic representations are mgs. Indeed the construction of Gross, Reeder and Yu, is by nature geometric, their criterion for supercuspidality is semi-stability in the sense of GIT.
- Another source of mgs representations is Adler's construction (1998) of toral supercuspidal representations, starting from a character on a totally ramified maximal torus.
- In the tame case, Kaletha's regular supercuspidals are mgs when Yu's datum does not involve depth zero nor Weil representation (because we have required inducing from characters). Fintzen arXiv:1810.04198 shows that it is sufficient that p does not divide the order of the Weyl group for (Kim-)Yu's construction to be exhaustive.

Theorem (Sawin-T., arXiv:1805.12231)

Suppose that for at least one place u, the representation π_u is mgs, and that π is base-changeable from $\mathbb{F}_q(X)$ to $\mathbb{F}_{q^n}(X)$ for every $n \geqslant 1$. Then π is tempered.

Elliptic modules vs Shtukas vs G-bundles

For the general linear group, the Ramanujan bound is the statement that a cuspidal automorphic representation of $\mathrm{GL}(n)$ over a function field F is tempered at every place. One can distinguish two proofs:

- Laumon (1995) under a cohomological condition at one place, extending Drinfeld's first proof (1977) for $\mathrm{GL}(2)$, using elliptic modules.
- L. Lafforgue (2002) in general, extending Drinfeld's second proof (1978, 1983) for $\mathrm{GL}(2)$, using shtukas.
- Our approach (Sawin-T. 2018), under the mgs condition at one place, is yet different, even in the case of $G = \mathrm{GL}(n)$. We are going to use Bun_G, the moduli of vector bundles, as in the geometric Langlands program. Functions on these moduli spaces give rise to families of automorphic forms satisfying certain local conditions.

Other results via different methods

- Lomeli (2015): Suppose that π is generic cuspidal, and that G is a quasi-split classical group. Then π is tempered at every unramified place. This confirms for quasi-split classical group a conjecture of Shahidi.
- Consequence of V. Lafforgue (2012): Suppose that for at least one place u, the parameter of π_u constructed by Genestier-Lafforgue is elliptic. Then π is tempered at every unramified place. See also Lafforgue-Zhu (November 2018).
- When Arthur's endoscopic classification will be established over function fields, then the conjecture will be established for classical groups.

Arthur's SL(2)

Global Weil group W_F, Local Weil group W_{F_u}, and inertia group I_{F_u}.

$$I_{F_u} \subset W_{F_u} \subset W_F \hookrightarrow W_F \times \mathrm{SL}(2)$$

Arthur parameter $\psi : W_F \times \mathrm{SL}(2) \to {}^L G$. Trivial on $\mathrm{SL}(2)$ = Tempered.

$\psi|_{I_{F_u}}$ elliptic $\Rightarrow$ $\psi|_{W_{F_u}}$ elliptic $\Rightarrow$ $\psi|_{W_F}$ elliptic $\Rightarrow$ ψ tempered

elliptic = image not contained in a proper parabolic.

π_u mgs $\overset{?}{\Rightarrow}$ $\psi|_{I_{F_u}}$ elliptic $\Rightarrow$ ψ tempered $\overset{?}{\Leftrightarrow}$ π tempered

At this time we have no information about $\psi|_{I_{F_u}}$ because it has been constructed very indirectly in V. Lafforgue's method.

We now proceed to explain the main ideas of the proof.

Informally: if cyclic base change holds, then Ramanujan holds.

Theorem (Sawin-T., arXiv:1805.12231)

Suppose that for at least one place u, the representation π_u is mgs, and that π is base-changeable from $\mathbb{F}_q(X)$ to $\mathbb{F}_{q^n}(X)$ for every $n \geqslant 1$. Then π is tempered.

Let π be an automorphic representation of G over X which is mgs at the place u.

- We embed the representation π in a family $\mathcal{V}$ in the q-aspect with prescribed local behavior, with mgs datum at the place u.

More precisely: Let D be the set of ramified places. Choose the integer $m_v \geqslant 1$ such that at places $v \in D$, π_v has a non-zero $K(m_v)$-fixed vector (principal congruence subgroup).

Definition

The finite set $\mathcal{V}(\mathbb{F}_q)$ consists of automorphic representations Π such that Π_u has a non-zero (J_u, χ_u)-invariant vector, Π_v has a non-zero $K(m_v)$-fixed vector for $v \in D$, $v \neq u$, and Π_v is unramified for $v \notin D$.

By construction $\pi \in \mathcal{V}(\mathbb{F}_q)$.
Because the inducing data (J_u, χ_u) comes from a monomial geometric datum, we can do the same construction to define $\mathcal{V}(\mathbb{F}_{q^n})$ for every $n \geqslant 1$.

Let π be an automorphic representation of G over X which is mgs at the place u. Fix an unramified point $x \in |X|$.

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- We construct a Hecke complex on $\mathrm{Bun}_G \times \mathrm{Bun}_G$, and show that it is perverse pure.
- The trace function of the Hecke complex is a classical automorphic kernel. For every coweight λ, the spectral side of the trace formula yields, **uniformly on** $n \geqslant 1$:

$$q^{-dn} \sum_{\Pi \in \mathcal{V}(\mathbb{F}_{q^n})} |\mathrm{tr}_\lambda(\Pi_x)|^2 \leqslant C_\lambda = \text{dim of some étale cohomology.}$$

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- We establish/assume cyclic base change for the automorphic representation π for $\mathbb{F}_q(X)$ to a packet $\{\Pi\}$ for $\mathbb{F}_{q^n}(X)$.
- The tensor power trick enables us to conclude that $|\,\mathrm{tr}_\lambda(\pi_x)| \leqslant \dim(V_\lambda) q^d$, and therefore π_x is tempered.

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We take here the unitary normalization of the Satake isomorphism, i.e. a tempered Satake parameter is a compact semisimple element. So π_x is tempered iff $|\,\mathrm{tr}_\lambda(\pi_x)| \leqslant \dim(V_\lambda)$, for all λ.

This inequality an average Ramanujan bound. Establishing the upper-bound $\leqslant C_\lambda$ independently of $n \geqslant 1$ is of crucial importance. The left-hand side could have a priori exponential growth of the form $q^{O(n|\lambda|)}$, which is the size of the Bruhat cell Gr_λ over $\mathbb{F}_{q^n}$. (Morally this corresponds to the Jacquet–Shalika / Borel–Wallach bound for the unitary dual.)

average Ramanujan: $q^{-dn} |\mathrm{tr}_\lambda(\pi_x)|^{2n} \leqslant C_\lambda, \quad \forall \lambda, n$

$\Longrightarrow |\,\mathrm{tr}_\lambda(\pi_x)| \leqslant \dim(V_\lambda) q^d$ individual Ramanujan.

In this final estimate, $\dim(V_\lambda)$ has polynomial growth in λ, which is the true dependence in λ. The tensor power trick is of crucial help here, in bootstraping the uniformity of the average Ramanujan bound.

Deligne's proof of Weil conjectures (Weil I, 1974)

Consider the case of a smooth hypersurface X in projective space.

- Deligne embeds X into the family $\mathcal{X} \to B$ of all such hypersurfaces of fixed dimension and degree. Taking a gereric Lefschetz pencil $U \subset B$, we restrict to a one-dimensional family $\mathcal{X} \to U$ for an open curve U.
- By Grothendieck's base change theorem, there is a representation of $\pi_1(U)$ on $H^*(X)$. Deligne shows that the family has big monodromy, i.e. $\pi_1(U)$ has open image in the symplectic group.
- By a nonnegativity argument (Chebyshev, Landau, Rankin-Selberg), Deligne deduces an upper bound on the eigenvalues of Frobenius, and also a lower bound by Poincaré duality.
- (tensor power trick) Deligne considers the action of $\pi_1(U)$ on $H^*(X^m)^{\otimes n}$, with $n \to \infty$, then $m \to \infty$ (in this order!). He uses classical invariants of representations of the symplectic group, Künneth formula, and concludes that the eigenvalues of Frobenius on $H^d(X)$ have absolute value $q^{\frac{d}{2}}$.

This establishes purity of cohomology, simultaneously for all hypersurfaces.

Arthur conjectures for automorphic forms are analogs to Weil conjectures for algebraic varieties.

Our strategy was to transport Deligne's proof for algebraic families to harmonic families. This is very different from the previous approaches to Ramanujan, which was to realize the automorphic form in the cohomology of Shimura/Drinfeld moduli spaces, and then to apply Deligne's theorem.

The fine details are much more complicated however because we used Deligne's theorem in the context of Bun_G as a key step of the proof, and the tensor power trick is used in a different way than in Deligne's, closer to Bombieri-Stepanov's proof for curves.

An analogy to understand the harmonic family $\mathcal{V}$

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$\mathcal{V}$ is **not** defined by algebraic equations. Imagine nevertheless the family $\mathcal{V}$ as a "kind" of algebraic variety. Surprisingly:

- For every $n \geqslant 1$, one can define $\mathcal{V}(\mathbb{F}_{q^n})$. It is a multi-set of packets of automorphic forms on $G(\mathbb{A}_F \otimes \mathbb{F}_{q^n})$ with prescribed local behavior.
- $|\mathcal{V}(\mathbb{F}_{q^n})| = \sum_i \pm\alpha_i^n$ for some q-Weil numbers α_i (Drinfeld).
- Base change $\mathcal{V}(\mathbb{F}_q) \hookrightarrow \mathcal{V}(\mathbb{F}_{q^n})$.
- One visualizes Hecke eigenvalues $\mathrm{tr}_\lambda(\pi_x)$ as kind of a sheaf Hk on $\mathcal{V}$. Ramanujan property means that the sheaf is pointwise pure.
- The average Ramanujan property can be viewed as a vanishing of certain cohomology groups.
- Tensor power trick: base change and positivity implies pointwise purity.
- Monodromy = Sato-Tate conjecture for the family.
- Katz-Sarnak heuristics should hold, and we have a strategy to establish it by thinking of L-functions again as kind of sections of $\mathcal{V}$.
- mirror symmetry (future)

For classical algebraic families these statements are established by Deligne.

Thus, our project is, in the setting of harmonic families such as $\mathcal{V}$, to repeat the ℓ-adic story.

Automorphic cyclic base change plays a crucial role, in the same way that base change of the ground field/base scheme is crucial in ℓ-adic theory.

Automorphic base change together with geometry of ℓ-adic sheaves. Each ingredient alone is insufficient to establish Ramanujan. It is the **combination** of the two that succeeds.

Geometric setup

Let now k be an arbitrary field, and still $F = k(X)$ be the function field of a curve X.
Fix a divisor $D = m[u]$ for some closed point $u \in |X|$. Let π be an automorphic representation of $G(\mathbb{A}_F)$ such that π_u is mgs.
For simplicity of exposition, assume in this lecture that G is split over k, and that π is unramified away from u. We let $\mathrm{Bun} = \mathrm{Bun}_{G(D)}$ be the stack of G-bundles on X with D-level structure. Weil's parametrization let us write $\mathrm{Bun}(k)$ as the adelic double quotient $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$.

From the mgs datum $(H, \mathcal{L})$, we obtain the congruence subgroup J:

$$\begin{tikzcd} U_m(G(\mathfrak{o}_u)) \arrow[r, hook] & J \arrow[r, hook] \arrow[d, two heads] & G(\mathfrak{o}_u) \arrow[d, two heads] \\ & H(\kappa_u) \arrow[r, hook] & G(\kappa_u[t]/t^m) \end{tikzcd}$$

and the character $\chi : J \twoheadrightarrow H(\kappa_u) \to \overline{\mathbb{Q}}_\ell^\times$. We have a standard matrix coefficient of $\text{c-ind}_J^{G(F_u)} \chi$ defined by

$$f_u(g) := \begin{cases} \chi(g), & \text{if } g \in J, \\ 0, & \text{if } g \notin J. \end{cases}$$

Recall $F_u = \kappa_u(\!(t)\!)$, and $\mathfrak{o}_u = \kappa_u[\![t]\!]$,

$$\text{c-ind}_J^{G(F_u)} \chi \twoheadrightarrow \pi_u$$

Moduli spaces

- Λ^+ – Weyl cone in the cocharacter lattice of G (which is naturally in bijection with a Weyl cone in the character lattice of $\hat{G}$).
- $W : |X| \to \Lambda^+$ – a function that sends all but finitely many points to the trivial cocharacter and sends all the points of D to the trivial cocharacter. We will view the trivial cocharacter as the zero element of Λ^+, so that the support of W is the finite set of points sent to nontrivial characters.
- Hk_W – the moduli space of pairs of G-bundles with an isomorphism away from the support of W, and with a trivialization of the first bundle along D, such that near each point x of the support of W, when the isomorphism is viewed as a point in the formal loop space $G(\!(t)\!)$, it projects to a point in the affine Grassmanian that lies in the closed cell corresponding to W_x.
- We define a map $\Delta^W : \mathrm{Hk}_W \times H \to \mathrm{Bun} \times \mathrm{Bun}$ where the left projection is taking the first G-bundle with trivialization and the right projection is taking the second G-bundle, using the isomorphism to carry over the trivialization, and then twisting the trivialization by the element of H.

Key theorem that implies the purity of the Hecke kernel

Let H be a connected k-subgroup of $G(k[t]/t^m)$, and $\mathcal{L}$ a character sheaf on H.

Theorem (Sawin-T. 2018)

Assume the field k has characteristic $\neq 2$, and that $(H, \mathcal{L})$ is geometrically supercuspidal. Then the natural map

$$\Delta_!^W(IC_{\mathrm{Hk}_W} \boxtimes \mathcal{L}) \to \Delta_*^W(IC_{\mathrm{Hk}_W} \boxtimes \mathcal{L})$$

is an isomorphism for every $W : |X| \to \Lambda^+$, with $u \notin \mathrm{supp}(W)$.

The trace function of the sheaf $K_W = R\Delta_!^W(IC_{Hk_W} \boxtimes \mathcal{L})$ is a Hecke kernel $\mathsf{K}_W(x,y)$ on the adelic quotient $\mathrm{Bun}(k) = G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$. Let $f = f_u f_W f^{u,W}$, where $f^{u,W}$ is the characteristic function of the maximal compact subgroup away from $\{u\} \cup \mathrm{supp}(W)$, f_W is the spherical function corresponding to the Hecke operator W, and f_u is the matrix coefficient defined in a previous slide. The construction of K_W is such that

$$\mathsf{K}_W(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

For example in the case when W is trivial, it is simply the idempotent projector onto a space of automorphic forms with prescribed behavior.

Corollary (Sawin-T., Key step implies average Ramanujan bound)

The sheaf K_W on $\mathrm{Bun} \times \mathrm{Bun}$ is perverse pure.

In some sense, there are tremendous cancellations in the corresponding exponential sums appearing in the Hecke kernel. Compare with Sarnak's purity problem for sup-norms.

Remark. * versus !, compactification of the diagonal

In the case D is trivial, i.e. if we remove the mgs condition at u, and consider directly diagonal $\Delta : \mathrm{Bun} \to \mathrm{Bun} \times \mathrm{Bun}$ without any character twist by level conditions, then $R\Delta_!(\overline{\mathbb{Q}}_\ell)$ differs from $R\Delta_*(\overline{\mathbb{Q}}_\ell)$.

In the case where $G = SL_2$, D and W are trivial, the trace function of $R\Delta_*(\overline{\mathbb{Q}}_\ell)$ was calculated by Schieder (2017). Viewing its trace function as a kernel, the induced operator on the space of automorphic forms was calculated recently by Drinfeld–Wang (2015), who found that it acts as the identity on cusp forms, and a similar calculation was done by Wang for general groups.

Ideas of proof of the theorem: partial compactification

Let $\overline{G}$ be the closure of $G \subseteq \operatorname{End} V \subseteq \mathbb{P}(\operatorname{End} V + 1)$, where $\operatorname{End} V + 1$ is a shorthand for $\operatorname{End} V \oplus k$. (The map $G \to GL(V) \to \operatorname{End} V$ is an immersion because V is a faithful representation).
Let $\overline{\mathrm{Hk}}_{H,W,V}$ be the moduli space of two pairs $(\alpha_1, t_1), (\alpha_2, t_2)$ of a G-bundle and a trivialization over D and a section

$$\varphi \in \mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$$

with compatibility conditions.

Unramified base change

We now explain part II of our work, which is in progress. The proof shall follow faithfully the method of comparison of trace formulas for cyclic base change of Shintani, Saito, Langlands, Arthur–Clozel, Kottwitz, Labesse. We return to $k = \mathbb{F}_q$.

Theorem (Sawin-T., in preparation)

Suppose that for at least one place u, the representation π_u belongs to an mgs packet. Then π is base-changeable from $\mathbb{F}_q(X)$ to $\mathbb{F}_{q^n}(X)$ for every $n \geqslant 1$.

Stabilizing functions and mgs packets

A test function is stabilizing if its orbital integrals on regular semisimple elements vanish for non-elliptic elements, and are constant on stable conjugacy classes.

Definition (mgs packet)

A finite set of mgs data $(H_i, \mathcal{L}_i)$ such that

$$\gamma \mapsto \sum_i \int_{G_\gamma(F)\backslash G(F)} f_i(x^{-1}\gamma x)dx$$

is constant on stable conjugacy classes of regular semisimple elements γ.

Recall that f_i is the matrix coefficient of the induced representation

$$f_i(g) := \begin{cases} \chi_i(g), & \text{if } g \in J_i, \\ 0, & \text{if } g \notin J_i. \end{cases}$$

Construction of of mgs packets

Epipelagic representations can be assembled into mgs packets by a combination of the following results:

- Waldspurger (1997) global proof of transfer for Lie algebras;
- DeBacker, Adler-Spice (2009) induced supercuspidal characters;
- Kaletha (2015) results on epipelagic L-packets, confirming conjectures of Gross-Reeder and Reeder-Yu.

Certain regular supercuspidals can be assembled into mgs packets by following Kaletha (2016) and Spice (2017).

Let $n \geqslant 1$, and $E = \mathbb{F}_{q^n}(X)$. Locally, $F_u = \kappa_u(\!(t)\!)$, and $E_u = \prod_{u'|u} \kappa_{u'}(\!(t)\!)$ with $[\kappa_{u'} : \kappa_u] = n_1$, and $n_1 = n/\gcd(n, [\kappa_u : k])$.

Theorem

Suppose that π_u belongs to an mgs packet. There exists a base change representation Π for $G(\mathbb{A}_E)$, of the representation π for $G(\mathbb{A}_F)$, such that

- *for every unramified place $v \neq u$, Π_v and π_v correspond under local base change,*
- *there is a place $u'|u$ such that $\Pi_{u'}$ is monomial geometric supercuspidal, with a non-zero (J_{n_1}, χ_{n_1})-invariant vector. (recall that J_{n_1} is a finite index congruence subgroup of $G(\mathfrak{o}_{u'})$).*

Trace formulas

1. Arthur-Selberg trace formula: any test function.
2. Arthur's cohomological trace formula: Euler-Poincare function. Applications to cohomology of locally symmetric spaces.
3. Simple trace formula of Deligne-Kazhdan: cuspidal at one place, supported on the set of regular semisimple elliptic elements at a second place.

Arthur's cohomological trace formulas are well-suited for reducing the transfer and fundamental lemma to the unit element, using Labesse elementary functions (Clozel, Hales, Waldspurger). Since we already have a supercuspidal condition at one place, it is the best option for our purposes.

Prestabilization of the simple trace formula (Langlands)

$$\sum_{\{\gamma\}_{\text{stable}}} \sum_{\kappa \in \mathfrak{K}} O_\gamma^\kappa(f)$$

$$O_{\gamma_v}^\kappa(f_v) := \sum_{\gamma_v' \sim_{\text{stable}} \gamma_v} e(I_{\gamma_v'}) \langle \kappa, \operatorname{inv}(\gamma_v, \gamma_v') \rangle O_{\gamma_v}(f_v).$$

If $\kappa = 1$, then $O_{\gamma_v}^1(f_v) = SO_{\gamma_v}(f_v)$ is a stable orbital integral.

Matching of κ-orbital integrals (Kottwitz)

Kottwitz (1986) proved the matching of unit elements in the Hecke algebra.

The same paper establishes a more precise matching at the level of conjugacy classes, and for a general class of compact open subgroups.

This is significant because it enables Labesse to establish the matching of elementary functions. And similarly, in our situation we can also establish the matching of the mgs coefficients f_u.

Relation to previous works - Rigidity and Family

Definition of rigidity: there is a single automorphic function f on $\mathrm{Bun}_{G(D)}(X)$ with some prescribed ramification conditions. Gross (2009): $X = \mathbb{P}^1$, G is split, simple supercuspidal at one place $u \in |X|$, Steinberg at a second place, and unramified elsewhere. Heinloth–Ngô–Yun (2010):

1. construct a Hecke eigensheaf whose trace function is f,
2. show temperedness of representation π generated by f,
3. determine the monodromy group (Zariski closure of the attached Galois representation).

In our work, the rigid assumption is no longer necessary.

We use the notion of family: the representation π is still prescribed at u, and can be arbitrary at other places. Each of 1, 2, 3 make sense. Our replacement for 1 will be a geometrization of the Hecke kernels, and using this, a proof of assertion 2 is the topic of this lecture. 3 is the Sato-Tate equidistribution for family.

Relation to previous works - Convolution

Properties of convolution of sheaves on $\mathbb{G}_m^N$, and Tannakian formalism are established by Gabber-Loeser (1996) and Katz's book *Convolution and Equidistribution* (2012). Also more generally elliptic curves and abelian varieties.

Our set-up has more generally sheaves on $\mathrm{Bun}_G(X)$ for a curve X. We want to extend previous results. This means that characters on the torus are replaced by automorphic forms on $\mathrm{Bun}_G(X)$. The Poisson summation formula, Fourier and Mellin transforms are replaced by the trace formula.

Note: The case of $\mathrm{Bun}_{\mathrm{GL}(1)}(X) =$ Jacobian of X, is common to previous work. One can also think of our set-up as a non-abelian generalization.

A close analogue of the argument may be found in the Bombieri-Stepanov proof of the Riemann hypothesis for curves over finite fields. Weil's proof for a curve C of genus g over $\mathbb{F}_q$ immediately proves in one stroke that $|\#C(\mathbb{F}_q) - q - 1| \leq 2g\sqrt{q}$. The proof of Stepanov involves more steps. In the hyperelliptic case, one first deduces an upper bound estimate, then by considering the conjugate curve, $|\#C(\mathbb{F}_q) - q - 1| \leq (2g+1)\sqrt{q}$. To improve the constant from $2g+1$ to the correct value $2g$, it is necessary to use the rationality of the L-function, and bootstraping the estimate for $\#C(\mathbb{F}_{q^n})$ for n large.
Our method closely follows the strategy of the last deduction. The main difference is that, while the bound $(2g+1)\sqrt{q}$ is sufficient for most practical purposes, the constant factor which we amplify away is ineffective, and would render the estimate useless in the λ aspect if not dealt with.