Low degree cohomologies of congruence groups

Binyong Sun (joint with Jian-Shu Li)

Academy of Mathematics and Systems Science,

Chinese Academy of Sciences

Institute for Mathematical Sciences
National University of Singapore
2019.1.2

Contents

1. Cohomologies of arithmetic manifolds

- G: real reductive group;
- $K \subset G$: a maximal compact subgroup;
- $\Gamma \subset G$: a lattice.

Problem :

$$\mathrm{H}^i(\Gamma;\mathbb{C}) = \mathrm{H}^i(\Gamma\backslash G/K;\underline{\mathbb{C}}) =?$$

Here

$$\underline{\mathbb{C}}(U) := \{\Gamma\text{-invariant locally constant function } f : \tilde{U} \to \mathbb{C}\},$$

and $\tilde{U}$ is the pre-image of U under the map $G/K \to \Gamma\backslash G/K$.

- F: an irreducible finite dimensional representation of G;

More generally,

$$\mathrm{H}^i(\Gamma; F) = \mathrm{H}^i(\Gamma\backslash G/K; \underline{F}) =?$$

where

$$\underline{F}(U) := \{\Gamma\text{-equivariant locally constant function } f : \tilde{U} \to F\},$$

and $\tilde{U}$ is the pre-image of U under the map $G/K \to \Gamma\backslash G/K$.

- Interesting in topology.
- Important in arithmetic.

There is a linear map

$$F^\Gamma \otimes \mathrm{H}^i_{\mathrm{ct}}(G;\mathbb{C}) = \mathrm{H}^i_{\mathrm{ct}}(G;F^\Gamma) \to \mathrm{H}^i_{\mathrm{ct}}(\Gamma;F^\Gamma) \to \mathrm{H}^i(\Gamma;F).$$

Remark: We have

$$\mathrm{H}^i_{\mathrm{ct}}(G;\mathbb{C}) = \mathrm{Hom}_K(\wedge^i(\mathfrak{g}_\mathbb{C}/\mathfrak{k}_\mathbb{C}),\mathbb{C}).$$

Problem

Is this an isomorphism?

Some earlier results:

$\Gamma\backslash G$ is compact: Matsushima, Raghunathan, Venkataramana, Millson-Raghunathan, $\cdots$

$\Gamma\backslash G$ is noncompact : Borel, Franke, Yang, $\cdots$

- **G**: connected reductive linear algebraic group over $\mathbb{Q}$;
- $\Gamma \subset \mathbf{G}(\mathbb{Q})$: a congruence subgroup;
- $$G = \bigcap_{\chi:\mathbf{G}\to \mathrm{GL}(1)/_{\mathbb{Q}}} \ker(|\chi| : \mathbf{G}(\mathbb{R}) \to \mathbb{R}_+^\times).$$

Global Theorem (Li-Sun, 2018, preprint): The above linear map is an isomorphism if $i < r_G$.

Here r_G is the smallest integer, if it exists, such that the continuous group cohomology

$$\mathrm{H}_{\mathrm{ct}}^{r_G}(G; F \otimes \pi) \neq \{0\}$$

for some finite dimensional irreducible representations F of G, and some infinite-dimensional irreducible unitary representations π of G. Otherwise $r_G = \infty$.

Remark:

- In many cases, r_G is the optimal bound.
- The quantity r_G is calculated in all cases (Enright, Kumaresan, Vogan-Zuckerman, Li-Schwermer).

Example 1

- $G = \mathrm{SU}(m, n)$.
  Matsushima: Suppose $\Gamma\backslash G$ is compact, and $i < \frac{\min\{m,n\}}{2}$. Then
  $$\mathrm{H}^i(\Gamma; \mathbb{C}) \cong \mathrm{H}^i_{\mathrm{ct}}(G; \mathbb{C}).$$
  Our theorem: Suppose Γ is a congruence subgroup and $i < \min\{m, n\}$. Then
  $$\mathrm{H}^i(\Gamma; F) \cong F^\Gamma \otimes \mathrm{H}^i_{\mathrm{ct}}(G; \mathbb{C}).$$

Example 2

- $G = \mathrm{SL}_n(\mathbb{R})$.
  Borel : Suppose Γ is a congruence subgroup, $n \geq 4$ and $i \leq \frac{n+2}{4}$. Then
  $$\mathrm{H}^i(\Gamma; \mathbb{C}) \cong \mathrm{H}^i_{\mathrm{ct}}(G; \mathbb{C}).$$
  Our theorem: Suppose Γ is a congruence subgroup and $i \leq n-2$. Then
  $$\mathrm{H}^i(\Gamma; F) \cong F^\Gamma \otimes \mathrm{H}^i_{\mathrm{ct}}(G; \mathbb{C}).$$

- G : a locally compact Hausdorff topological group.

Definition A representation of G is a quasi-complete Hausdorff locally convex topological vector space V over $\mathbb{C}$, together with a continuous linear action

$$G \times V \to V.$$

All representations of G form a category : Rep_G.

- Want: $\mathrm{H}^i_{\mathrm{ct}}(G; V)$.
- Defined by : Hochschild-Mostow.

Definition A homomorphism $\phi : V_1 \to V_2$ in Rep_G is strong if, as topological vector spaces,

$$\ker \phi \subset V_1 \text{ is a direct summand}$$

and

$$\mathrm{Im}\phi \subset V_2 \text{ is a direct summand.}$$

Definition A representation V of G is relatively injective if for all injective strong homomorphisms

$$V_1 \hookrightarrow V_2$$

in Rep_G, every homomorphism in $\mathrm{Hom}_G(V_1, V)$ extends to a homomorphism in $\mathrm{Hom}_G(V_2, V)$.

Definition A strong injective resolution of a representation V of G is an exact sequence

$$0 \to V \to J_0 \to J_1 \to J_2 \to \cdots$$

in Rep_G such that all J_i's are relatively injective, all the arrows are strong homomorphisms.

Definition

$$\mathrm{H}^i_{\mathrm{ct}}(G; V) := \mathrm{H}^i(J_\bullet).$$

This is a locally convex topological vector space.

- G : a Lie group.

Definition A representation V of G is said to be smooth if for every $X \in \mathfrak{g}$, the map

$$V \to V, \quad v \mapsto X.v := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(tX).v$$

is well-defined and continuous.

All smooth representations of G form a category : Rep_G^∞.

All smooth representations of G are representations of $\mathfrak{g}_{\mathbb{C}}$.

Definition A homomorphism $\phi : V_1 \to V_2$ in Rep_G^∞ is strong if, as topological vector spaces,

$$\ker \phi \subset V_1 \text{ is a direct summand}$$

and

$$\mathrm{Im}\phi \subset V_2 \text{ is a direct summand.}$$

Definition A representation V in Rep_G^∞ is relatively injective if for all injective strong homomorphisms

$$V_1 \hookrightarrow V_2$$

in Rep_G^∞, every homomorphism in $\mathrm{Hom}_G(V_1, V)$ extends to a homomorphism in $\mathrm{Hom}_G(V_2, V)$.

Definition A strong injective resolution of a representation V in Rep_G^∞ is an exact sequence

$$0 \to V \to J_0 \to J_1 \to J_2 \to \cdots$$

in Rep_G^∞ such that all J_i's are relatively injective, all the arrows are strong homomorphisms.

Definition

$$\mathrm{H}^i_{\mathrm{sm}}(G; V) := \mathrm{H}^i(J_\bullet).$$

This is a locally convex topological vector space.

Facts

- (Hochschild-Mostow) For every V in Rep_G^∞,

$$\mathrm{H}^i_{\mathrm{ct}}(G; V) = \mathrm{H}^i_{\mathrm{sm}}(G; V).$$

- (Hochschild-Mostow) Suppose G has only finitely many connected components. Then for every V in Rep_G^∞,

$$\mathrm{H}^i_{\mathrm{sm}}(G; V) = \mathrm{H}^i(\mathfrak{g}_{\mathbb{C}}, K; V).$$

- (Blanc) For every representation V of G,

$$\mathrm{H}^i_{\mathrm{ct}}(G; V) = \mathrm{H}^i_{\mathrm{sm}}(G; V^\infty)$$

at least when V is Fréchet.

Remarks.

- $\mathrm{H}^i_{\mathrm{ct}}(\mathfrak{g}_{\mathbb{C}}, K; V)$ is calculated by the cochain complex

$$\mathrm{Hom}_K(\wedge^\bullet \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}, V).$$

- The smoothing V^∞ of V is defined by the Cartesian diagram

$$\begin{array}{ccc} V^\infty & \longrightarrow & C^\infty(G;V) \\ \downarrow & & \downarrow \subset \\ V & \xrightarrow{v \mapsto (g \mapsto g.v)} & C(G;V)\,. \end{array}$$

4. A key local result

- G: real reductive group;
- F: an irreducible finite dimensional representation of G;
- $P \subset G$: a proper parabolic subgroup of G;
- N : the unipotent radical of P;
- $L := P/N$.

Local Theorem (Li-Sun, 2018, preprint) For every irreducible unitarizable Casselman-Wallach representation σ of L, and every dominant positive character $\nu : L \to \mathbb{R}_+^\times$,

$$\mathrm{H}^i_{\mathrm{ct}}(G; F \otimes \mathrm{Ind}_P^G(\nu \otimes \sigma)) = \{0\} \qquad \text{for all } i < r_G.$$

- Take a splitting of $P \to L$ and write $P = LN$.
- $\mathfrak{h} \subset \mathfrak{l}$: a maximally split Cartan subalgebra of $\mathfrak{l}$.

Definition A positive character $\nu : L \to \mathbb{R}_+^\times$ is said to be dominant if

$$\langle \nu|_{\mathfrak{h}_\mathbb{C}}, \alpha^\vee \rangle \geq 0 \quad \text{for all } \alpha \in \Delta(\mathfrak{h}_\mathbb{C}, \mathfrak{n}_\mathbb{C}).$$

Here the compleixfied differential of $\nu : L \to \mathbb{R}_+^\times$ is denoted by $\nu : \mathfrak{l}_\mathbb{C} \to \mathbb{C}$.

Proof: Case by case analysis of the root systems.

5. A result of Franke

- G: real reductive group;
- $K \subset G$: a maximal compact subgroup;
- $\Gamma \subset G$: a lattice;
- F : an irreducible finite dimensional representation of G.

Shapiro's Lemma :

$$\mathrm{H}^i(\Gamma; F) = \mathrm{H}^i_{\mathrm{sm}}(G; \mathrm{Ind}_\Gamma^G F) = \mathrm{H}^i_{\mathrm{sm}}(G; C^\infty(\Gamma\backslash G) \otimes F)$$

Suppose

- $\mathbf{G}$: connected reductive linear algebraic group over $\mathbb{Q}$;
- $\Gamma \subset \mathbf{G}(\mathbb{Q})$: a congruence subgroup;
- $$G = \bigcap_{\chi:\mathbf{G}\to\mathrm{GL}(1)/_{\mathbb{Q}}} \ker(|\chi| : \mathbf{G}(\mathbb{R}) \to \mathbb{R}_+^\times).$$
- $\mathcal{A}(\Gamma\backslash G)$: the space of smooth automorphic forms.

  Then

$$\mathcal{A}(\Gamma\backslash G) = \varinjlim_{I \text{ is a finite codimensional ideal of } \mathrm{Z}(\mathfrak{g}_{\mathbb{C}})} \mathcal{A}(\Gamma\backslash G)^I,$$

where

$$\mathcal{A}(\Gamma\backslash G)^I := \{f \in \mathcal{A}(\Gamma\backslash G) : f \text{ is annihilated by } I\}$$

is a Casselman-Wallach representation of G.

Theorem: (Franke)

$$\mathrm{H}^i_{\mathrm{sm}}(G; C^\infty(\Gamma\backslash G) \otimes F) = \mathrm{H}^i_{\mathrm{sm}}(G; \mathcal{A}(\Gamma\backslash G) \otimes F).$$

Remark.

- Extend F to a representation of $\mathbf{G}(\mathbb{R})$. Then

$$\mathrm{H}^i_{\mathrm{sm}}(G; \mathcal{A}(\Gamma\backslash G) \otimes F) = \mathrm{H}^i_{\mathrm{sm}}(\mathbf{G}(\mathbb{R}); \mathcal{A}(\Gamma\backslash \mathbf{G}(\mathbb{R})) \otimes F).$$

6. Franke's filtration

- $\mathbf{P}_0$: a minimal split torus in $\mathbf{G}$;
- $\mathbf{A}_0$: a maximal split torus in $\mathbf{P}_0$;
- $\mathbf{P} = \mathbf{L}_\mathbf{P} \ltimes \mathbf{N}_\mathbf{P}$: a standard parabolic;
- $\mathbf{A}_\mathbf{P}$: the largest split central torus in $\mathbf{L}$;
- $\mathfrak{a}_\mathbf{P}$: the Lie algebra of $\mathbf{A}_\mathbf{P}(\mathbb{R})$;
- $\mathfrak{a}_{\mathbf{P},\mathbb{C}}$: the complexificaton;
- $\breve{\mathfrak{a}}_{\mathbf{P},\mathbb{C}}$ the dual space;
- $\mathcal{A}_\mathbf{P}(\mathbf{G})$: the space of smooth automorphic forms on $(\mathbf{L}_\mathbf{P}(\mathbb{Q})\mathbf{N}_\mathbf{P}(\mathbb{A}))\backslash\mathbf{G}(\mathbb{A})$. This is a representation of $\mathbf{G}(\mathbb{A})$. It is smooth as a representation of $\mathbf{G}(\mathbb{R})$.
- $\mathcal{A}(\mathbf{G}) := \mathcal{A}_\mathbf{G}(\mathbf{G})$.

A smooth representation

$$\mathbf{A}_{\mathbf{P}}(\mathbb{R}) \curvearrowright \mathcal{A}_{\mathbf{P}}(\mathbf{G}), \quad (a.\phi)(x) := a^{-\rho_{\mathbf{P}}}\phi(ax),$$

where $\rho_{\mathbf{P}} \in \breve{\mathfrak{a}}_{\mathbf{P}}$ denotes the half sum of the weights (with the multiplicities) associated to $\mathbf{N}_{\mathbf{P}}$.

Differential :

$$\mathfrak{a}_{\mathbf{P},\mathbb{C}} \curvearrowright \mathcal{A}_{\mathbf{P}}(\mathbf{G}).$$

The generalized eigenspace decomposition :

$$\mathcal{A}_{\mathbf{P}}(\mathbf{G}) = \bigoplus_{\lambda \in \breve{\mathfrak{a}}_{\mathbf{P},\mathbb{C}}} \mathcal{A}_{\mathbf{P}}(\mathbf{G})_{\lambda}. \tag{1}$$

A result of Langlands :

$$\breve{\mathfrak{a}}_{\mathbf{P}_0} = \bigsqcup_{\mathbf{P} \text{ is a standard parabolic}} \left(\breve{\mathfrak{a}}_{\mathsf{P}}^{+} - \overline{{}^{+}\breve{\mathfrak{a}}_{\mathbf{P}_0}^{\mathsf{P}}} \right)$$

Since

$$\overline{\mathfrak{a}_{\mathbf{P}_0}^{+}} = \bigsqcup_{\mathsf{P} \text{ is a standard parabolic}} \breve{\mathfrak{a}}_{\mathbf{P}}^{+},$$

we get a map (the dominant part)

$$(\,\cdot\,)_{+} : \breve{\mathfrak{a}}_{\mathbf{P}_0} \to \overline{\breve{\mathfrak{a}}_{\mathbf{P}_0}^{+}}$$

Franke's filtration : for each $t \geq 0$, define

$$\begin{aligned} & \mathcal{A}(\mathbf{G})_{[\leq t]} \\ := & \left\{ \phi \in \mathcal{A}(\mathbf{G}) \,:\, \phi_{\mathbf{P}} \in \bigoplus_{\lambda \in \check{\mathfrak{a}}_{\mathbf{P},\mathbb{C}},\ \langle (\mathrm{Re}(\lambda))_+, \rho_0^\vee \rangle \leq t} \mathcal{A}_{\mathbf{P}}(\mathbf{G})_\lambda, \text{ for all } \mathbf{P} \right\}, \end{aligned}$$

where $\phi_{\mathbf{P}} \in \mathcal{A}_{\mathbf{P}}(\mathbf{G})$ denotes the constant term of ϕ along $\mathbf{P}$, and $\rho_0^\vee \in \check{\mathfrak{a}}_{\mathbf{P}_0}$ is the sum of the fundamental coweights.

Remark. The spaces of almost square integrable automorphic forms:

$$\begin{aligned} & \mathcal{A}(\mathbf{G})_{[\leq 0]} \\ = & \mathcal{A}^{\bar{2}}(\mathbf{G}) \\ := & \{\phi \in \mathcal{A}^{\bar{2}}(\mathbf{G}) \,:\, \phi(\,\cdot\, g) \in \mathrm{L}^{2+\epsilon}(\mathbf{G}(\mathbb{Q})\backslash \mathbf{G}^1(\mathbb{A})) \text{ for all } g\}. \end{aligned}$$

Define

$$\mathcal{A}(\mathbf{G})_{[<t]} := \left\{ \phi \in \mathcal{A}(\mathbf{G}) \,:\, \phi_{\mathbf{P}} \in \bigoplus_{\lambda \in \check{\mathfrak{a}}_{\mathbf{P},\mathbb{C}},\ \langle (\mathrm{Re}(\lambda))_+, \rho_0^\vee \rangle < t} \mathcal{A}_{\mathbf{P}}(\mathbf{G})_\lambda, \ \text{ for all } \mathbf{P} \right\},$$

and

$$\mathcal{A}(\mathbf{G})_{[t]} := \frac{\mathcal{A}(\mathbf{G})_{[\leq t]}}{\mathcal{A}(\mathbf{G})_{[<t]}}.$$

Theorem (Franke)

$$\mathcal{A}(\mathbf{G})_{[t]} = \bigoplus_{\mathbf{P}} \bigoplus_{\lambda\in\check{\mathfrak{a}}_{\mathbf{P},\mathbb{C}},\, \mathrm{Re}(\lambda)\in\check{\mathfrak{a}}_{\mathbf{P}}^{+},\, \langle \mathrm{Re}(\lambda), \rho_0^{\vee}\rangle = t} \mathcal{A}_{\mathbf{P}}^{\bar{2}}(\mathbf{G})_{\lambda},$$

where

$$\begin{aligned} & \mathcal{A}_{\mathbf{P}}^{\bar{2}}(\mathbf{G}) \\ := \ & \{\phi \in \mathcal{A}_{\mathbf{P}}(\mathbf{G}) \,:\, \phi(\,\cdot\, g) \in \mathsf{L}^{2+\epsilon}(\mathbf{L}_{\mathbf{P}}(\mathbb{Q})\backslash \mathbf{L}_{\mathbf{P}}^{1}(\mathbb{A})) \text{ for all } g\} \\ = \ & \mathrm{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} \mathcal{A}^{\bar{2}}(\mathbf{L}_{\mathbf{P}}). \end{aligned}$$

7. Proof of the global theorem

- Franke's theorem + the local theorem implies that

$$\mathrm{H}^i_{\mathrm{sm}}(\mathbf{G}(\mathbb{R}); \mathcal{A}(\mathbf{G}) \otimes F) = \mathrm{H}^i_{\mathrm{sm}}(\mathbf{G}(\mathbb{R}); \mathcal{A}^{\bar{2}}(\mathbf{G}) \otimes F)$$

  for $i < r_G$.

- The theory of Eisenstein series + the local theorem implies that

$$\mathrm{H}^i_{\mathrm{sm}}(\mathbf{G}(\mathbb{R}); \mathcal{A}^{\bar{2}}(\mathbf{G}) \otimes F) = \mathrm{H}^i_{\mathrm{sm}}(\mathbf{G}(\mathbb{R}); \mathcal{A}^2(\mathbf{G}) \otimes F)$$

  for $i < r_G$.

- This implies that

$$\mathrm{H}^i_{\mathrm{sm}}(G; \mathcal{A}(\Gamma \backslash G) \otimes F) = \mathrm{H}^i_{\mathrm{sm}}(G; \mathcal{A}^2(\Gamma \backslash G) \otimes F)$$

  for $i < r_G$.

- Finally,

$$\mathrm{H}^i_{\mathrm{sm}}(G; \mathcal{A}^2(\Gamma \backslash G) \otimes F) = F^\Gamma \otimes \mathrm{H}^i_{\mathrm{sm}}(G; \mathbb{C}).$$

Thank you!