A BASE CHANGE FUNDAMENTAL LEMMA VIA THE GEOMETRY OF SHTUKAS

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ABSTRACT. These are notes for a talk on the paper [Feng]. We will explain a geometric approach to the base change fundamental lemma over function fields, which is based on ideas of Ngô Bao Châu. The approach works by comparing the cohomology groups of moduli spaces of shtukas, which we will introduce.

CONTENTS

1. AUTOMORPHIC FORMS OVER FUNCTION FIELDS

To orient ourselves, we begin with a brief survey on the evolution of the Langlands program in different contexts.

1.1. **The evolution of automorphic forms.** The subject of automorphic forms is concerned with functions on spaces such as

$$G(\mathbf{Q})\backslash G(\mathbf{A_Q})/\prod_p G(\mathbf{Z}_p). \tag{1.1.1}$$

These spaces are, approximately, smooth manifolds, and we can study them using the techniques of topology, differential geometry, and functional analysis.

For certain G, (1.1.1) can be realized as the complex points of an algebraic variety, which would then be called a *Shimura variety*. This interpretation allows us to additionally harness the tools of algebraic geometry, which significantly enriches the story.

1.1.1. *Function fields.* There are two types of "global fields": the number fields, which are finite extensions of $\mathbf{Q}$, and the *global function fields*, which are the field of rational functions on a smooth projective curve X over $\mathbf{F}_q$. There is a strong analogy between these two types of fields, and it is natural to consider the analogue of (1.1.1) in the function field case.

Fix a smooth projective curve X over $\mathbf{F}_q$ and let F be its function field. The analogue of (1.1.1) is

$$G(F)\backslash G(\mathbf{A}_F)/\prod_{x\in|X|} G(\mathcal{O}_x) \tag{1.1.2}$$

where $\mathcal{O}_x$ is the completed local ring of X at x. Let $F_x = \mathrm{Frac}(\mathcal{O}_x)$; this is called a *local function field*, and it is (non-canonically) isomorphic to $\mathbf{F}_{q^d}((t))$.

The theory of automorphic forms on (1.1.2) has many parallels to the characteristic 0 case. Note, however, that the geometry of (1.1.2) is uninteresting: it is a discrete set. (Since there are no archimedean places, it is the quotient of a space by an open subspace.)

1.1.2. *Geometric Langlands.* In the case where G is split reductive over $\mathbf{F}_q$, Weil observed that (1.1.2) could be interpreted as the set of rational points of a certain algebra-geometric object Bun_G, the *moduli stack of G-bundles* on X.[1] This puts interesting geometry back into the picture. Moreover, we are motivated by Grothendieck's "function-sheaf dictionary" to look for sheaves on Bun_G which corresponds to automorphic functions on (1.1.2). This line of thought leads to the *Geometric Langlands program*, pioneered by Drinfeld and Laumon.

1.2. **Moduli of shtukas.** Using Weil's observation, we can rewrite

$$G(F)\backslash G(\mathbf{A}_F)/\prod_{x\in|X|} G(\mathcal{O}_x) = \left\{ \begin{array}{c} \mathscr{E} = G\text{-bundle on } X \\ \varphi: \mathrm{Frob}^*\mathscr{E} \xrightarrow{\sim} \mathscr{E} \end{array} \right\}. \tag{1.2.1}$$

We will denote the right hand side by Sht_G^0. It is a discrete groupoid, which we will consider as an algebraic stack. We can interpret the space of $\overline{\mathbf{Q}}_\ell$-valued functions on (1.2.1) as $H^0_{\text{ét}}(\mathrm{Sht}_G^0; \overline{\mathbf{Q}}_\ell)$. This seems a little silly, but we can now expand the picture to get something interesting.

In (1.2.1) we have demanded φ, which is a map of G-bundles, to be an isomorphism. We could broaden our scope by allowing it to have "zeros" and "poles" over a point $x \in X$. These are indexed by the set

$$G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x) = X_*^+(G).$$

Thus we have found that the space Sht_G^0, which was not very interesting (just a discrete groupoid), fits naturally into a collection of spaces Sht_G^μ indexed by $\mu \in X_*^+(G)$, which is the moduli space parametrizing

$$\left\{ \begin{array}{c} x \in X \\ \mathscr{E} = G\text{-bundle on } X \\ \varphi: \mathrm{Frob}^*\mathscr{E} \xrightarrow[x]{\leq\mu} \mathscr{E} \end{array} \right\}.$$

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[1]One direction of the bijection can be described as follows. Given a G-bundle $\mathscr{E}$, choose trivializations of $\mathscr{E}$ in a formal neighborhood of each $x \in X$, as well as the generic point (the possibility of trivializing at the generic point is why we need G to be split reductive over k). The transition functions between these trivializations gives an element of $G(\mathbf{A}_F)$, but because of the ambiguity in the choice of trivialization we really only get a well-defined element of $G(F)\backslash G(\mathbf{A}_F)/\prod_{x\in|X|} G(\mathcal{O}_x)$.

We have an obvious map

$$
\begin{array}{c}
\mathrm{Sht}_G^\mu \\
\big\downarrow{\scriptstyle\pi} \\
X
\end{array}
$$

sending $(x, \mathcal{E}, \varphi) \mapsto x$, which tracks "where the pole of φ occurs". The spaces Sht_G^μ are roughly analogous to the Shimura varieties mentioned earlier.

We can now broaden our picture again. Why force φ to have a pole over only one $x \in X$? For each finite set I, we can define a space $\mathrm{Sht}_{G,I}^\mu$ that parametrizes

$$
\left\{
\begin{array}{c}
\{x_i \in X\}_{i \in I} \\
\mathcal{E} = G\text{-bundle on } X \\
\varphi : \mathrm{Frob}^* \mathcal{E} \xrightarrow[\{x_i\}]{\leq \mu} \mathcal{E}
\end{array}
\right\} .
$$

We have similarly a map

$$
\begin{array}{c}
\mathrm{Sht}_{G,I}^\mu \\
\big\downarrow{\scriptstyle\pi} \\
X^I
\end{array}
$$

but for $\#I > 1$, this picture has no analogue for Shimura varieties. Indeed, the analogy matches X with $\mathrm{Spec}\,\mathbf{Z}$, but one cannot form $(\mathrm{Spec}\,\mathbf{Z})^I$ in an interesting way.

1.3. **Higher automorphic forms.** As explained above, traditional automorphic forms could be thought of as elements of $H^0(\mathrm{Sht}_{G,\emptyset}^0; \overline{\mathbf{Q}}_\ell)$. This is naturally generalized by $H^*(\mathrm{Sht}_{G,I}^\mu)$ or $R\pi_*(\mathrm{Sht}_{G,I}^\mu)$. (We are ignoring some technical points here – we should take coefficients in perverse sheaves, and use compactly supported cohomology instead.) It is reasonable to think of elements of these spaces *higher automorphic forms,* and they have played crucial roles in several recent breakthroughs in the study of automorphic forms over function fields.

- The work of Vincent Lafforgue [Laff12], constructing the "automorphic-to-Galois" direction of the Global Langlands correspondence for *arbitrary* reductive G over function fields.

    To say something very brief (and woefully inadequate), Lafforgue's proof works by embedding the space of (traditional) automorphic forms into various spaces of higher automorphic forms, and the using the geometry of shtukas to performing interesting geometric constructions on the latter.
- There are various situations where period integrals of automorphic forms are related to special values of L-functions. Zhiwei Yun and Wei Zhang have shown in [YZ17] that in the function field setting, analogous "period integrals" of *higher* automorphic forms are related to special values of *higher derivatives* of L-functions.

To say something very brief (and woefully inadequate), their proof upgrades Jacquet's relative trace formula proof of Waldspurger's formula to a "higher version" of the relative trace formula, comparing traces on spaces of *higher* automorphic forms.

2. THE BASE CHANGE FUNDAMENTAL LEMMA

Today I will explain another result, namely the *base change fundamental lemma*, whose proof also depends on "higher version" of certain classical constructions.

2.1. **Formulation of the base change fundamental lemma.**

2.1.1. *Orbital integrals.* Let F_v be a local field and E_v/F_v be an unramified field extension of degree r.

- Let $f \in C_c^\infty(G(F_v); \mathbf{Q}_\ell)$.
- Let $\gamma \in G(F_v)$ and $G_\gamma(F_v)$ be its centralizer.

Definition 2.1.1. The *orbital integral* $O_\gamma(f)$ is "the integral of f over the orbit of γ":

$$O_\gamma(f) := \int_{G(F_v)/G_\gamma(F_v)} f(g^{-1}\gamma g)\, dg.$$

2.1.2. *Twisted orbital integrals.* Let E_v/F_v be an unramified extension, $\sigma \in \mathrm{Gal}(E_v/F_v)$ a generator. Consider $G(E_v)$ acting on itself by twisted conjugation :

$$g \cdot \delta = g^{-1}\delta\sigma(g).$$

- Let $f \in C_c^\infty(G(E_v); \mathbf{Q}_\ell)$.
- Let $\delta \in G(E_v)$ and $G_{\delta\sigma}(E_v)$ be its twisted centralizer.

Definition 2.1.2. The *twisted orbital integral* $\mathrm{TO}_{\delta\sigma}(f)$ is "the integral of f over the twisted orbit of δ":

$$\mathrm{TO}_{\delta\sigma}(f) := \int_{G(E_v)/G_{\delta\sigma}(E_v)} f(g^{-1}\delta\sigma(g))\, dg.$$

2.1.3. *The base change fundamental lemma.* Roughly speaking, the base change fundamental lemma predicts that:

$$O_{\gamma_1}(f_1) = \mathrm{TO}_{\gamma_2}(f_2).$$

However, to make this meaningful we need to clarify (i) how γ_1 and γ_2 are related, and (ii) how f_1 and f_2 are related? We address (i) first.

2.1.4. *Base change of a conjugacy class.*

Definition 2.1.3 (Norm of a (stable) conjugacy class)**.** Given $\gamma_2 \in G(E_v)$, define

$$\mathrm{Nm}_{E_v/F_v}(\gamma_2) := \gamma_2 \cdot \sigma(\gamma_2) \cdot \ldots \cdot \sigma^{r-1}(\gamma_2).$$

This is well-defined as a *stable* conjugacy class.

2.1.5. *The base change homomorphism.* By the Satake isomorphism, we can view

$$\mathrm{MaxSpec}\,\mathscr{H}(G(F_v))\text{``}=\text{''}\left\{\begin{array}{c}\text{unramified Galois representations}\\ \mathrm{Gal}(F_v)\to\widehat{G}(\overline{\mathbf{Q}}_\ell)\end{array}\right\}.$$

Definition 2.1.4. The restriction map from representations of $\mathrm{Gal}(\overline{F}_v/F_v)$ to $\mathrm{Gal}(\overline{E}_v/E_v)$ induces a *base change homomorphism*

$$b_{E_v/F_v}:\mathscr{H}(G(E_v))\to\mathscr{H}(G(F_v)).$$

Generalizing this, we also have a version of the base change homomorphism for the *center* of a parahoric Hecke algebra

$$\mathscr{H}_{\mathrm{I}}(G(F_v)):=C_c(\mathrm{I}\backslash G(F_v)/\mathrm{I};\overline{\mathbf{Q}}_\ell)$$

where I is a parahoric subgroup, by using the Bernstein isomorphism:

$$Z(\mathscr{H}_{\mathrm{I}}(G(F_v)))\xrightarrow{\sim}\mathscr{H}(G(F_v)),$$

which is given by convolution with $\mathbf{I}_K$.

Conjecture 2.1.5 (Base change FL for GL_n, parahoric version)**.** *If $\delta\in\mathrm{GL}_n(E_v)$ is such that* $\mathrm{Nm}(\delta)$ *is regular semisimple and separable, then we have*

$$\mathrm{TO}_{\delta\sigma}(f)=\mathrm{O}_{\mathrm{Nm}_{E_v/F_v}(\delta)}(b_{E_v/F_v}(f))$$

for all $f\in Z(\mathscr{H}_{\mathrm{I}}(G(E_v)))$.

Remark 2.1.6. The full Base Change Fundamental Lemma (in either its spherical or Iwahori versions) asserts more than just what is contained in Conjecture 2.1.5; see [Hai09]. We also caution that the formulation of base change for general G is more complicated; we have taken advantage of the fact that GL_n does not suffer from (or enjoy?) endoscopy to simplify the formulation.

2.2. **Results.** We now briefly survey known results towards the base change fundamental lemma.

2.2.1. *Results over p-adic fields.* We begin by discussing the case where F_v is a p-adic field, where the results are more complete. The spherical BCFL was proved by Kottwitz for unit the element [Kott86], and independently extended to the whole (spherical) Hecke algebra by Clozel [Clo90] and Labesse [Lab90], even for *general G*. Extending their methods, the parahoric BCFL proved by Haines in [Hai09]. The proofs are *analytic*. They depend on the (twisted) trace formula, which has not been developed in the function-field setting.

2.2.2. *Result over function fields.* We now let F_v be a local function field over $\mathbf{F}_q$. In this case, we do not know the BCFL in general. For GL_n, Ngô proved Conjecture 2.1.5 for the spherical Hecke algebra, almost. In [Feng], still for GL_n, I extended his methods to prove Conjecture 2.1.5, in the same "almost" sense[2]. There is a clear strategy to extend the proof to the whole Hecke algebra, but as of yet we have not undertaken the pain to carry this out. As I will explain, the argument is *geometric.*

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[2]The meaning of "almost" is that the test function f are taken in the Hecke algebra of SL_n, viewed as a subspace of the Hecke algebra of GL_n

2.3. **Applications.** Broadly speaking, the base change fundamental lemma arises in applications as a kind of *exchange rate* between different "currencies". (Specific examples will be given shortly below.) More precisely, one wants to compare two types of trace formulae, arising from different sources, and one does this by comparing "geometric sides", which are expressed in terms of some kind of orbital integral. But it is often the case that one trace formula involves *twisted* orbital integrals, while the other involves just orbital integrals. These are the two different "currencies", and one needs fundamental lemma to convert between them.

2.3.1. *Zeta functions of Shimura varieties.* The motivation for Haines' pursuit of a parahoric version of the BCFL was to calculate the Hasse-Weil zeta function of a Shimura variety with parahoric level structure in terms of automorphic L-functions. The problem is to obtain the local factor at a place of parahoric reduction. Using the *Langlands-Kottwitz method*, one can compute the semi-simple zeta factor in terms of (twisted) orbital integrals, and then one needs to convert the twisted orbital integrals into ordinary orbital integrals in order to relate them to automorphic representations (via the Arthur-Selberg trace formula).

2.3.2. *Base change for automorphic representations.* As already mentioned, in the function-field setting Vincent Lafforgue has constructed Galois representations attached to automorphic forms for all G. Building on his methods and various other results, Böckle-Harris-Khare-Thorne proved *potential* automorphy for Galois representations [BHKT], under some technical hypotheses but still with remarkable generality with respect to G.

This reduces the global Langlands correspondence over function fields to the base change problem. The case of *cyclic base change* with respect to an extension E/F, i.e. comparing automorphic forms for G and $G' = \mathrm{Res}_{E/F} G$, has been studied classically (in the characteristic 0 case) with some success. It proceeds by a comparison of the trace formula for G and the twisted trace formula for G'. The geometric side of the trace formula for G will involve orbital integrals, and the geometric side of the twisted trace formula for G' will involve twisted orbital integrals, so one needs to be able to compare the two.

3. IDEAS OF THE PROOF

We outline the key geometric ideas in Ngô's proof [Ngo06] of the spherical version of Conjecture 2.1.5. Our argument for the parahoric case will follow the same geometric framework, with some additional complications that are discussed in §4.

We will restrict our attention to $G = \mathrm{GL}_n$. Although the ideas clearly have broader scope, the phenomenon of endoscopy complicates matters for general G, and we do not yet have a proof beyond GL_n.

3.1. **Trivial base change.** For a degenerate version of base change, replace the field extension E/F by the totally split étale algebra extension F^r/F. Then $\mathrm{Res}_{F^r/F} G = G^r$. The base of representations from G to G^r is trivial, but let's imagine studying this base change by comparing "trace formulae" for each group. This amounts to the following algebra exercise.

Lemma 3.1.1 (Comparison of trace formulae for trivial base change)**.** *Let $f_1, \ldots, f_r$ be endomorphisms of a vector space V. Let $\tau : V^{\otimes r} \to V^{\otimes r}$ be the automorphism given by cyclic rotation. Then*

$$\mathrm{Tr}((f_1 \otimes \ldots \otimes f_r) \circ \tau, V^{\otimes r}) = \mathrm{Tr}(f_1 \circ \ldots \circ f_r, V).$$

In particular, we would find

$$\mathrm{Tr}((f_1 \circ \ldots f_r) \circ \tau, H_c^*(\mathrm{Sht}_{G^r}^0; \overline{\mathbf{Q}}_\ell)) = \mathrm{Tr}(f_1 \circ \ldots \circ f_r, H_c^*(\mathrm{Sht}_G^0; \overline{\mathbf{Q}}_\ell)). \tag{3.1.1}$$

3.2. **Higher trace formulae.** Equation (3.1.1) is not very interesting. But we can ask for what a "higher version" (in the sense of §1.3) of (3.1.1) would look like, and it turns out to be interesting.

An indication that the comparison becomes non-trivial is that it's somewhat subtle to pinpoint which two things should be compared. We postulate that on the left side, $H_c^*(\mathrm{Sht}_{G^r}^0)$ should be replaced with $H_c^*(\mathrm{Sht}_{G^r}^{(\mu,\ldots,\mu)})$. It turns out that on the right hand side, we then need to consider $H_c^*(\mathrm{Sht}_G^{r,\mu})$ where $\mathrm{Sht}_G^{r,\mu}$ parametrizes

$$\left\{ \begin{array}{c} x \in X \\ \{\mathcal{E}_i = G\text{-bundle on } X\}_{i=1}^r \\ \mathrm{Frob}^* \mathcal{E}_r \xrightarrow[x]{\leq \mu} \mathcal{E}_1 \xrightarrow[x]{\leq \mu} \ldots \xrightarrow[x]{\leq \mu} \mathcal{E}_r \end{array} \right\}.$$

Now, this space has an obvious symmetry by a cyclic rotation τ', sending

$$\left[\mathrm{Frob}^* \mathcal{E}_r \xrightarrow[x]{\leq \mu} \mathcal{E}_1 \xrightarrow[x]{\leq \mu} \ldots \xrightarrow[x]{\leq \mu} \mathcal{E}_r \right] \mapsto \left[\mathrm{Frob}^* \mathcal{E}_{r-1} \xrightarrow[x]{\leq \mu} \mathrm{Frob}^* \mathcal{E}_r \xrightarrow[x]{\leq \mu} \mathcal{E}_1 \xrightarrow[x]{\leq \mu} \ldots \xrightarrow[x]{\leq \mu} \mathcal{E}_{r-1} \right].$$

Remark 3.2.1. It turns out that the canonical map $\mathrm{Sht}_G^{r,\mu} \to \mathrm{Sht}_G^{r\mu}$ induces an isomorphism on cohomology, so we could in principle just phrase our "higher trace" as the cohomology of $\mathrm{Sht}_G^{r\mu}$. However, the extra operator τ' becomes invisible at the level of $\mathrm{Sht}_G^{r\mu}$, so it is better to look at $\mathrm{Sht}_G^{r,\mu}$.

We have a diagram

$$\mathrm{Sht}_{G^r}^{(\mu,\ldots,\mu)} \qquad\qquad\qquad \mathrm{Sht}_G^{r,\mu}$$

$$X$$

Our claim is then:

Claim 3.2.2. For $x_0 \in X$, we have

$$\mathrm{Tr}((f_1, \ldots, f_r) \circ \tau \circ \mathrm{Frob}, H_c^*(\mathrm{Sht}_{G^r}^{(\mu,\ldots,\mu)}|_{x_0})) = \mathrm{Tr}((f_1 \circ \ldots \circ f_r) \circ \tau' \circ \mathrm{Frob}, H_c^*(\mathrm{Sht}_G^{r,\mu}|_{x_0})).$$

3.3. **Comparison of higher trace formulae.** To prove Claim (3.2.2), we could try to compute both sides using the Langlands-Kottwitz method. The output is:

$$\mathrm{Tr}((f_1,\ldots,f_r)\circ\tau\circ\mathrm{Frob}, H_c^*(\mathrm{Sht}_{G^r}^{(\mu,\ldots,\mu)})|_{x_0}) \overset{?}{=\!=\!=} \mathrm{Tr}((f_1\circ\ldots\circ f_r)\circ\tau'\circ\mathrm{Frob}, H_c^*(\mathrm{Sht}_G^{r,\mu})|_{x_0})$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\|$$

$$\sum(\ldots)\underbrace{\mathrm{TO}_{\delta\sigma}(\psi_{\mu,r})}_{\text{factor at } x_0} \qquad\qquad\qquad \sum(\ldots)\underbrace{\mathrm{O}_{b(\gamma)}(b(\psi_{\mu,r}))}_{\text{factor at } x_0}.$$

We haven't precisely defined the objects $\delta\sigma, \psi_{\mu,r}$ appearing above, or explained all the terms of the formula, and we won't do that here. The important observation is that we are confronted by exactly the problem of the base change fundamental lemma – we need to be able to translate $\mathrm{TO}_{\delta\sigma}$'s into O_γ's!

Let's now try to *reverse* this reasoning. Suppose we could prove Claim 3.2.2 by some *other* method, not using the base change fundamental lemma. Then we would have some non-trivial comparison of $\mathrm{TO}_{\delta\sigma}$'s and O_γ's. It is not hard to show that, by advantageous choices of the Hecke operators $f_1,\ldots,f_r$, we could manage to isolate $\mathrm{TO}_{\delta\sigma}$ and O_γ, and obtain precisely the BCFL.

Lemma 3.3.1. *Claim 3.2.2 implies Conjecture 2.1.5, almost*[3].

It then remains to somehow prove Claim 3.2.2. For this, the key idea is to *let the points move*. That is, letting $[r] = \{1,\ldots,r\}$, we consider spaces $\widetilde{\mathrm{Sht}}_{G^r,[r]}^{(\mu,\ldots,\mu)}$ and $\mathrm{Sht}_{G,[r]}^{r,\mu}$ parametrizing

$$\widetilde{\mathrm{Sht}}_{G^r,[r]}^{(\mu,\ldots,\mu)} = \left\{ \begin{array}{c} \{x_i \in X\}_{i\in[r]} \\ \{\mathscr{E}_i = G\text{-bundle on } X\}_{i\in[r]} \\ \varphi_i\colon \mathrm{Frob}^*\mathscr{E}_i \xrightarrow[x_i]{\leq\mu} \mathscr{E}_i \quad i\in[r] \end{array} \right\}.$$

$$\mathrm{Sht}_{G,[r]}^{r,\mu} = \left\{ \begin{array}{c} \{x_i \in X\}_{i=1,\ldots,r} \\ \{\mathscr{E}_i = G\text{-bundle on } X\}_{i=1}^r \\ \mathrm{Frob}^*\mathscr{E}_r \xrightarrow[x_1]{\leq\mu} \mathscr{E}_1 \xrightarrow[x_2]{\leq\mu} \mathscr{E}_2 \xrightarrow[x_3]{\leq\mu} \ldots \xrightarrow[x_r]{\leq\mu} \mathscr{E}_r \end{array} \right\}.$$

These spaces naturally live over X^r, and their restrictions to the diagonal $\Delta(X) \hookrightarrow X^r$

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[3]This means, as in §2.2, that one only gets Conjecture 2.1.5 for f coming from the Hecke algebra of SL_n.

obvious recover our old spaces $\mathrm{Sht}_{G^r}^{(\mu,\dots,\mu)}$ and $\mathrm{Sht}_G^{r,\mu}$, i.e. all squares in the following diagram are cartesian.
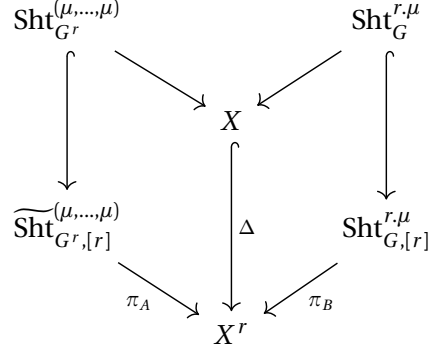
Why should this make things any easier? The point is that we now have a deformation of our original problem, and if we can prove the analogue of Claim 3.2.2 over many points of X^r, away from Δ, then we can deduce by "continuity" (this is quite non-trivial; see §4) that it holds over Δ as well. And indeed, if we look at a point $\underline{x} = (x_1,\dots,x_r) \in X^r$ with the x_i pairwise distinct (and satisfying another condition, which we won't discuss), then the Langlands-Kottwitz method produces *only* (untwisted!) orbital integrals for both $\widetilde{\mathrm{Sht}}_{G^r,[r]}^{(\mu,\dots,\mu)}|_{\underline{x}}$ and $\mathrm{Sht}_{G,[r]}^{r,\mu}|_{\underline{x}}$, allowing the two traces to be equated *without any sort of fundamental lemma*.

3.4. **Summary.** The geometric proofs of fundamental lemmas often have a bewildering flavor of circularity, so we review our steps to clarify the structure of the proof:

(1) We were originally interested in comparing *global* traces, coming from $\mathrm{Res}_{E/F} G$ and G.
(2) We (should be able to) reduce the comparison of these global traces to a comparison of local integrals, $\mathrm{TO}_{\delta\sigma}$ and O_γ.
(3) We used the Langlands-Kottwitz method to reduce the comparison of $\mathrm{TO}_{\delta\sigma}$ and O_γ to the comparison of *higher* global traces, coming from the much more degenerate situation of $\mathrm{Res}_{Fr/F} G = G^r$ and G.
(4) We deformed these global traces to *different* global traces. This deformation uses algebraic geometry, and has no apparent analogue in the arithmetic (characteristic 0) situation.
(5) Finally, we use the Langlands-Kottwiz method *again* to compute these different global traces, reducing to a comparison of local quantities, O_γ and O_γ, which can be equated directly.

4. THE PARAHORIC CASE

We now briefly indicate some of the additional complications that arise in the parahoric situation.

4.1. **Continuity.** At the end of §3.3, we made a vague allusion to a "continuity" principle in order to complete the proof. Making this precise and rigorous is not a trivial matter.

There are natural perverse sheaves, denoted $\mathscr{F}_A$ and $\mathscr{F}_B$, that one takes for the coefficients of cohomology of $\widetilde{\mathrm{Sht}}^{(\mu,\dots,\mu)}_{G^r,[r]}$ and $\mathrm{Sht}^{r,\mu}_{G,[r]}$, respectively. The statement we want is that $R\pi_{A!}\mathscr{F}_A$ is a *local system*, and similarly for $R\pi_{B!}\mathscr{F}_B$. For simplicity let's just discuss π_A; the story for π_B is completely analogous. This sort of property follows if one knows that the geometry of π_A is "nice enough". Typically, this means that π_A should be "smooth" and "proper".

- Typically the moduli stack of shtukas is of infinite type, and will not be proper. However, since we are only interested in a local problem, we can get around this by *changing G* from GL_n to a sufficiently ramified division algebra. This doesn't change what happens locally, except at a few points, but constrains the global problem to be proper.
- The moduli stack of shtukas are not smooth. However, *when G is reductive over all of X*, the singularities will be "locally constant", which one quantifies by exhibiting a well-behaved resolution of singularities, and so if one puts in the appropriate perverse sheaves for coefficients then the situation is okay. In summary, the non-smoothness is not problematic in the case of hyperspecial level structure.

4.2. **Integral models.** If we want to get a base change fundamental lemma for parahoric Hecke algebras, then we must take parahoric level structure. In this case, the second bullet point is not okay. In fact, in this case it is not obvious how to even *define* Sht_G over all of X. This is the problem of constructing "integral models". In the function field case, it is not too hard to identify favorable integral models (see [Feng, §2]), though for Shimura varieties this problem is much more difficult, and has only recently been solved by Kisin-Pappas [KP].

4.3. **Nearby cycles.** Even after one has integral models, they acquire singularities above points where G is not reductive in a "discontinuous" way, so $R\pi_{A!}\mathscr{F}_A$ cannot be locally constant.

To correct this, one modifies the sheaf of coefficients in order to make the cohomology become locally constant. This comes from a process called *nearby cycles*. Given a family $Y \to S$, with S a valuation ring with generic point η and special point s, and a sheaf $\mathscr{F}$ on Y, the "nearby cycles sheaf $R\Psi(\mathscr{F})$" is a sheaf on Y_s which basically constructed to have the property that (if $Y \to S$ is proper),

$$H^*_c(Y_\eta; \mathscr{F}) \xleftarrow{\sim} H^*_c(Y_s; R\Psi(\mathscr{F})).$$

We take $Y = \widetilde{\mathrm{Sht}}^{(\mu,\dots,\mu)}_{G^r,[r]}$ restricted to the henselization of X^r at a diagonal point x_0, and $\mathscr{F}$ to be the restriction of $\mathscr{F}_A$. Taking cohomology of nearby cycles then restores the continuity property of cohomology, but pushes the problem to one of computing.

$$\mathrm{Tr}((f_1,\dots,f_r) \circ \tau \circ \mathrm{Frob}, H^*_c(\widetilde{\mathrm{Sht}}^{(\mu,\dots,\mu)}_{G^r,[r]}|_{x_0}; R\Psi(\mathscr{F}_A))).$$

By the Grothendieck-Lefschetz formula, this amounts to point-counting plus calculating the traces on stalks of $R\Psi(\mathscr{F}_A)$. The latter is not easily accessed from the construction of nearby cycles.

4.4. **Local models.** Describing the stalks $R\Psi(\mathscr{F}_A)$ explicitly amounts to describing explicitly the singularities which appear on the special fiber of the degeneration in §4.3. The key to access this problem is to use a *local model* to compare the situation to one which is both simpler, and has more algebraic structure.

A local model is another family which has the same singularities, but potentially a different (and simpler) global structure. In this case a local model is furnished by a degeneration of so-called "affine flag varieties". Not only are these simpler in that they do not involve "arithmetic", but they have a "convolution" (which one can think of roughly as being like a group structure) which equips their category of sheaves with a multiplicative structure. The key is then to prove a structural statement: "nearby cycles is *central* for the convolution". It turns out that the property of being central is special enough that we can use it to pin down $R\Psi(\mathscr{F}_A)$ in a sufficiently explicit way.

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