Jacquet-Mao's fundamental lemma

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Outline

Metaplectic correspondence

- Let Ω be a global field (number field or function field) and $\mathbb{A}$ its adele ring.
- The global metaplectic group $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ is a certain extension of $\mathrm{GL}_r(\mathbb{A})$ by $\{\pm 1\}$. We shall write an element in $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ as (g, z) with $g \in \mathrm{GL}_r(\mathbb{A})$ and $z \in \{\pm 1\}$.
- A function f on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ is *genuine* if it satisfies $f(g, z) = f(g, 1).z$
- Denote by $\widetilde{L}^2$ the subspace of $L^2(\mathrm{GL}_r(\Omega) \setminus \widetilde{\mathrm{GL}}_r(\mathbb{A}))$ consisting of genuine functions. A constituent of the $\widetilde{\mathrm{GL}}_r$-module $\widetilde{L}^2$ is called *genuine* automorphic representation.
- The **metaplectic correspondence** is a lifting of the genuine automorphic representations of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ to the automorphic representations of $\mathrm{GL}_r(\mathbb{A})$.

Jacquet's conjecture

Conjecture (Jacquet)

A cuspidal automorphic representation π *of* $\mathrm{GL}_r(\mathbb{A})$ *with trivial central character is a lifting from a genuine cuspidal automorphic representation of* $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ *if and only if* π *is* $(\mathrm{GO}_\epsilon, \chi)$*-distinguished, where* ϵ *is a symmetric matrix,* χ *is a quadratic character of* $\mathbb{A}^\times/\Omega^\times$ *and where* GO_ϵ *is the similitude orthogonal group:*

$$\{g \in \mathrm{GL}_r | {}^t g \epsilon g = \lambda(g)\epsilon,\ \lambda(g) \text{ is a scalar}\}.$$

Recall that a cuspidal automorphic representation π *is* $(\mathrm{GO}_\epsilon, \chi)$*-distinguished if for some* ϕ *lying in the space of* π*, we have (*Z *being the center of* GL_r*)*

$$\int_{Z \cap \mathrm{GO}_\epsilon(\mathbb{A})\mathrm{GO}_\epsilon(\Omega)\backslash \mathrm{GO}_\epsilon(\mathbb{A})} \phi(hg)\chi(\lambda(h))dh \not\equiv 0.$$

Outline

Local metaplectic cover

- Let K be a non-archimedean local field, $\mathcal{O}_K$ be its ring of integers and k_K be its residue field of odd characteristic p.
- The metaplectic group is a certain extension GL_r by $\{\pm 1\}$:

$$1 \to \{\pm 1\} \to \widetilde{\mathrm{GL}}_r(K) \to \mathrm{GL}_r(K) \to 1.$$

  - As a set, we realize $\widetilde{\mathrm{GL}}_r(K)$ as

$$\widetilde{\mathrm{GL}}_r(K) = \mathrm{GL}_r(K) \times \{\pm 1\} = \{(g,z) | g \in \mathrm{GL}_r(K),\ z \in \{\pm 1\}\}.$$

  - The group law is defined in terms of 2-cocycle as follows,

$$(g,z)(g',z') = (gg', zz'\chi(g,g')).$$

2-cocycle χ

- Let $[,] : K^* \times K^* \to \{\pm 1\}$ be the Hilbert symbol.
- $\chi : \mathrm{GL}_r(K) \times \mathrm{GL}_r(K) \to \{\pm 1\}$ is the unique 2-cocycle satisfying:
  - $\chi(t, t') = \prod_{i<j} [t_i, t_j]$, where $t = \mathrm{diag}(t_i)$ and $t' = \mathrm{diag}(t'_i)$.
  - $\chi(t, w) = \chi(w, w') = 1$, where $t \in T_r(K)$ and $w, w' \in W_r$.
  - $\chi(\alpha, t) = [t_\ell, t_{\ell+1}][-1, t_\ell / t_{\ell+1}][-1, \det(t)]$, where α is a matrix of the transposition $(\ell, \ell+1)$.
  - $\chi(ng, g'n') = \chi(g, g')$, where $n, n' \in N_r(K)$.
  - $\chi(t, g) = \chi(t, B(g))$, where $t \in T_r(F)$ and $B(g) = m$ if $g = n_1 m n_2$ with $n_1, n_2 \in N_r(K)$ and $m \in T_r(K) \times W_r$ (note that $B(g)$ is determined uniquely).
  - $\chi(\alpha, g) = \chi(B(\alpha g) B(g)^{-1}, B(g))$.

The function κ

- We fix the section $\mathbf{s} : \mathrm{GL}_r(K) \to \widetilde{\mathrm{GL}}_r(K)$ given by $\mathbf{s}(g) = (g, 1)$.
- The group $\mathrm{GL}_r(\mathcal{O}_K)$ splits in $\widetilde{\mathrm{GL}}_r(K)$.
- There is a unique splitting $\kappa^* : \mathrm{GL}_r(\mathcal{O}_K) \to \widetilde{\mathrm{GL}}_r(K)$ satisfies

$$\begin{aligned} \kappa^*_{|T_r(K)\cap \mathrm{GL}_r(\mathcal{O}_K)} &= \mathbf{s}_{|T_r(K)\cap \mathrm{GL}_r(\mathcal{O}_K)}, \\ \kappa^*_{|N_r(K)\cap \mathrm{GL}_r(\mathcal{O}_K)} &= \mathbf{s}_{|N_r(K)\cap \mathrm{GL}_r(\mathcal{O}_K)}, \\ \text{and } \kappa^*_{|W_r} &= \mathbf{s}_{|W_r}. \end{aligned}$$

- We then obtain a unique map $\kappa : \mathrm{GL}_r(\mathcal{O}_K) \to \{\pm 1\}$ such that

$$\kappa^*(g) = (g, \kappa(g)) = (\mathrm{Id}_r, \kappa(g))\mathbf{s}(g).$$

Spherical Hecke algebras

- Let $\mathcal{H}_r$ be the spherical Hecke algebra of $\mathrm{GL}_r(K)$.
  - The set of the smooth complex valued functions with compact support on $\mathrm{GL}_r(K)$ who are bi-$\mathrm{GL}_r(\mathcal{O}_K)$-invariant.
  - Equipped a structure of associative unital algebra by the convolution product: $f * \phi(x) = \int_{\mathrm{GL}_r(K)} f(g)\phi(g^{-1}x)dg$.
  - Its unit element is the function defined by $\phi_0(g) = \begin{cases} 1 & \text{if } g \in \mathrm{GL}_r(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$
- Let $\widetilde{\mathcal{H}}_r$ be the spherical Hecke algebra of $\widetilde{\mathrm{GL}}_r(K)$.
  - The set of the genuine smooth complex valued functions with compact support on $\widetilde{\mathrm{GL}}_r(K)$ who are bi-$\mathrm{GL}_r(\mathcal{O}_K)$-invariant.
  - Equipped a structure of associative unital algebra by the convolution product: $f * \phi(x) = \int_{\mathrm{GL}_r(K)} f((g,1))\phi((g,1)^{-1}x)dg$.
  - Its unit element is the function defined by

$$f_0((g,z)) = \begin{cases} \kappa(g).z & \text{if } g \in \mathrm{GL}_r(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$$

Two actions, *Relevant* orbits and their classifications

- The actions :
  - N_r acts on $S_r \cap \mathrm{GL}_r$ by $s \mapsto {}^t nsn$,
  - $N_r \times N_r$ acts on GL_r by $g \mapsto n_1^{-1} g n_2$.
- Let $\Psi : K \to \mathbb{C}^*$ be a non-trivial additive character of order 0.
- Let $\theta : N_r(K) \to \mathbb{C}^* \quad n \mapsto \Psi(\sum_{i=1}^{n-1} n_{i,i+1}/2)$.
- The *relevant* orbits :
  - The orbit $N_r s$ is *relevant* if θ^2 is trivial on the stabilizer $(N_r)_s$ of s in N_r.
  - The orbit $(N_r \times N_r)g$ is relevant if θ is trivial on the stabilizer $(N_r \times N_r)_g$ of g in $N_r \times N_r$.
- The classification of the relevant orbits :

Theorem (Friedberg,Goldfeld,Jacquet,..., Mao)

The relevant N_r-orbits in $S_r \cap \mathrm{GL}_r$ have representatives of the form wt where w is the longest Weyl element of a standard parabolic subgroup in GL_r and t lies in the center of the corresponding Levi subgroup. The relevant $N_r \times N_r$-orbits in GL_r have the representatives of the form $w_0 wt$ with w, t being as above and w_0 being the longest Weyl element of GL_r.

Fundamental lemma of Jacquet-Mao

For each wt we have two orbital integrals:

- $I(wt, \phi) = \int_{N_r/(N_r)_{wt}} \phi({}^t nwtn)\theta^2(n)dn,$
- $J(wt, f) = \int_{N_r \times N_r/(N_r \times N_r)_{w_0 wt}} f(\mathbf{s}(n)^{-1}\mathbf{s}(w_0 wt)\mathbf{s}(n'))\theta(n^{-1}n')dndn',$

where $\phi \in \mathcal{H}_r$, $f \in \widetilde{\mathcal{H}}_r$. Jacquet-Mao's fundamental lemma is the following conjecture:

Conjecture (Jacquet-Mao)

There exists a homomorphism $b : \widetilde{\mathcal{H}}_r \to \mathcal{H}_r$ such that

$$J(wt, f) = \Delta(wt)I(wt, b(f)),$$

where $\Delta(wt)$ is an explicit transfer factor and $f \in \widetilde{\mathcal{H}}_r$.

Outline

Reduces to the largest orbits

Theorem

Suppose that a function $\phi \in \mathcal{H}_r$ *and a function* $f \in \widetilde{\mathcal{H}}_r$ *satisfy the following identity*

$$J(\mathrm{Id}_r t, f) = \Delta_r(t) I(\mathrm{Id}_r t, \phi)$$

with $\Delta_r(t)$ *is an explicit function for all* $t \in T_r(F)$*. Then for other representative element* wt *there exists* $\Delta_w(t)$ *(an explicit function can be expressed as a function of* $\Delta_i(t)$ *for* $1 \leq i \leq r$*) such that*

$$J(wt, f) = \Delta_w(t) I(wt, \phi).$$

Sketch of proof I

- We prove this theorem by induction on r.
- For $w \neq w_0$ the transfer factors can be calculated via the transfer factors of smaller ranks.
- To calculate the transfer factor for $w = w_0$, we used the concept of Shalika germs due à Jacquet.
- Let M be the standard subgroup of type $(r-1,1)$.
- Let $\alpha = \mathrm{diag}(a, \ldots, a, a^{1-r}\det(w_M w_0))$, and w_M be the longest Weyl element of M.
- Let $\mathbf{t} \in T_{w_0}$, then $\alpha\mathbf{t} \in T_{w_M}$.
- We have

$$J(w_M\alpha\mathbf{t}, f) = \omega_f(\alpha\mathbf{t}) + \sum_{z|z^r=1} K_{w_M}^{w_0}(z\alpha)J(w_0 z^{-1}\mathbf{t}, f)$$

and

$$I(w_M\alpha\mathbf{t}, \phi) = \omega_\phi(\alpha\mathbf{t}) + \sum_{z|z^r=1} L_{w_M}^{w_0}(z\alpha)I(w_0 z^{-1}\mathbf{t}, \phi),$$

where w_f, w_ϕ are smooth functions of compact support on T_{w_M}.

Sketch of proof II

- For $|a|$ small enough:
  - $\omega_f(\alpha\mathbf{t}) = \omega_\phi(\alpha\mathbf{t}) = 0$.
  - We have an explicit formula for $K_{w_M}^{w_0}(\alpha)$ and for $L_{w_M}^{w_0}(\alpha)$.
  - $K_{w_M}^{w_0}(\alpha)/L_{w_M}^{w_0}(\alpha)$ does not depend on a.
- The identity $\sum_{z|z^r=1} K_{w_M}^{w_0}(z\alpha)m(z) = 0$ implies that $m(z) = 0$ for all z.

The FL for the unit element of the Hecke algebra

Geometric method in positive characteristic

We shall use the geometric method due à Ngo :

- Interpret the two orbital integrals as traces of Frobenius over two complex of ℓ-adics sheaves.
- Two above complexes are very complicated, so to "deform" them we shall replace them by the analogue global complexes (over the fraction field $k(t) = k(\mathbb{P}^1)$).
- Over a "good open", we prove the global identity.
- Using the perversity of two global complexes to extend the result obtained on the good open.
- Finally, we reduce the fundamental lemma from the global identity.

Extend result obtained to a general case

- We use the principle of Cluckers and Loeser. It said that: "Assume that the fundamental is true in positive characteristic cases. If all the ingredients of the fundamental lemma are definable in the sense of Cluckers-Loeser then the fundamental lemma is also true in general case with the charactersitic p large enough".
- To extend the result obtained in positive characteristic to a general case, we need to check the definability of all the ingredients of the fundamental lemma.

Geometric extension

- Arbarello, De Concini and Kac associate to each $g \in \mathrm{GL}_r(K)$ a line

$$D_g = (\mathcal{O}_K^r | g\mathcal{O}_K^r) := (\bigwedge \mathcal{O}_K^r / g\mathcal{O}_K^r \cap \mathcal{O}_K^r)^{\otimes(-1)} \otimes (\bigwedge g\mathcal{O}_K^r / g\mathcal{O}_K^r \cap \mathcal{O}_K^r)$$

- This construction provides a central extension $\widetilde{\mathrm{GL}}_r'(K)$ of $\mathrm{GL}_r(K)$ by k_K^*.
  - As a set, we realize $\widetilde{\mathrm{GL}}_r'(K)$ as

$$\widetilde{\mathrm{GL}}_r'(K) = \{(g, v) | g \in \mathrm{GL}_r(K),\ v \in D_g - 0\}.$$

  - The group law is defined by the isomorphism of multiplication

$$D_g \otimes D_{g'} \stackrel{\times g}{\to} (\mathcal{O}_K^r | g\mathcal{O}_K^r) \otimes (g\mathcal{O}_K^r | gg'\mathcal{O}_K^r) \stackrel{can}{\simeq} D_{gg'}.$$

- Let $\widetilde{\mathrm{GL}}_{r,\mathrm{geo}}(K) = \det^*(\widetilde{\mathrm{GL}}_1'(K)) - \widetilde{\mathrm{GL}}_r'(K)$ (the Baer sum is noted additively).
- $\mathrm{GL}_r(\mathcal{O}_K)$ splits canonically in $\widetilde{\mathrm{GL}}_{r,\mathrm{geo}}(K)$. We denote this splitting by triv.

Geometric construction of the function κ

- Let $\zeta : k_K^* \to \{\pm 1\}$ be the non trivial quadratic character.

Theorem

The metaplectic extension is obtained from the extension $\widetilde{\mathrm{GL}}_{r,\mathrm{geo}}(K)$ by pushing forward ζ.

- Let $\mathbf{s}_{\mathrm{geo}}$ be the section of $\widetilde{\mathrm{GL}}_{r,\mathrm{geo}}(K)$ which corresponds to the section $\mathbf{s}$ of the metaplectic extension.
- We have then $\kappa(g) = \zeta(\underline{\kappa}(g))$ with $\underline{\kappa}$ is the quotient $\mathrm{triv}/\mathbf{s}_{\mathrm{geo}}$.
- Using this interpretation, we obtained an explicit formula for the function κ.

The FL for an arbitrary element of the Hecke algebra

Geometric Satake equivalence

- Due to Ginzburg, Mirkovic and Vilonen, we have a natural isomorphism (Satake equivalence) of $\overline{\mathbb{Q}}_\ell$-algebras between the spherical Hecke algebra $\mathcal{H}_r$ and $K_0(\mathrm{Rep}(\mathrm{GL}_r^\vee))$ where $\mathrm{GL}_r^\vee$ is a dual group (viewed as an algebraic group over $\overline{\mathbb{Q}}_\ell$) of GL_r. Ngo give another proof for some results of them in the positive case.
- Due to Lysenko and Finkelberg, Reich, Lysenko, we have also a natural isomorphism (metaplectic Satake equivalence) between the spherical Hecke algebra $\widetilde{\mathcal{H}}_r$ and $K_0(\mathrm{Rep}(\widetilde{\mathrm{GL}}_r^\vee))$.
- An expected homomorphism b can be constructed via the following diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{H}}_r & \overset{b}{\dashrightarrow} & \mathcal{H}_r \\ \downarrow & & \downarrow \\ K_0(\mathrm{Rep}(\widetilde{\mathrm{GL}}_r^\vee)) & \longrightarrow & K_0(\mathrm{Rep}(\mathrm{GL}_r^\vee)) \end{array} .$$

The dual group $\widetilde{\mathrm{GL}}_r^\vee$

- When r is even: $\widetilde{\mathrm{GL}}_r^\vee \simeq \mathrm{GL}_r \simeq \mathrm{GL}_r^\vee$.
- When r is odd: $\widetilde{\mathrm{GL}}_r^\vee = \{(g,a) | g \in \mathrm{GL}_r, \det(g) = a^2\}$.
  There exists an isogeny $\widetilde{\mathrm{GL}}_r^\vee \to \mathrm{GL}_r$ defined by $(g,a) \mapsto g$.
- The FL for an arbitrary element of the Hecke algebra is verified in the case when r is even. The case where r is odd is in progress.

Some bases of $\mathcal{H}_r$

- Noting $\varpi^{\underline{\lambda}} := \mathrm{diag}(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \ldots, \varpi^{\lambda_r})$ for $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.
- Cartan decompostion

$$\mathrm{GL}_r(K) = \coprod_{\underline{\lambda}=(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)} \mathrm{GL}_r(\mathcal{O}_K)\varpi^{\underline{\lambda}}\mathrm{GL}_r(\mathcal{O}_K).$$

- The characteristic functions $c_{\underline{\lambda}}$ of $\mathrm{GL}_r(\mathcal{O}_K)\varpi^{\underline{\lambda}}\mathrm{GL}_r(\mathcal{O}_K)$ form a basis of $\mathcal{H}_r$
- $\mathrm{GL}_r(\mathcal{O}_K)$ has a structure of group scheme, denoted by $G_{\mathcal{O}}$.
- $\mathrm{GL}_r(K)/\mathrm{GL}_r(\mathcal{O}_K)$ has a structure of ind-scheme, denoted by Gr.
- We denote by $\mathrm{Gr}^{\underline{\lambda}}$ the $G_{\mathcal{O}}$-orbits of Gr associated to $\underline{\lambda}$.
- We denote by $\mathcal{A}_{\underline{\lambda}} := \mathrm{IC}(\overline{\mathrm{Gr}^{\underline{\lambda}}}, \overline{\mathbb{Q}_\ell})$.
- Let $a_{\underline{\lambda}} : \mathrm{Gr}(k) \to \overline{\mathbb{Q}_\ell} : x \mapsto \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_{\underline{\lambda}})_x)$.
- The functions $a_{\underline{\lambda}}$ form a geometric basis of $\mathcal{H}_r$.

Some bases of $\widetilde{\mathcal{H}}_r$

- Let $\underline{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_r\}$ satisfies the following conditions :
  - $\sum_{j \neq i} \lambda_j$ are even for all $i \in \{1, \ldots, r\}$,
  - $\lambda_i + \lambda_{i+1}$ are even for all $i \in \{1, \ldots, r-1\}$.
- Let $d_{\underline{\lambda}}^{\bullet}$ be the characteristic function of $\mathrm{GL}_r(\mathcal{O}_K)(\varpi^{\underline{\lambda}}, \bullet 1)\mathrm{GL}_r(\mathcal{O}_K)$ where $\bullet \in \{+, -\}$.
- The function $d_{\underline{\lambda}} := d_{\underline{\lambda}}^{+} - d_{\underline{\lambda}}^{-}$ form a basis of $\widetilde{\mathcal{H}}_r$.
- $\widetilde{\mathrm{GL}}_{r,\mathrm{geo}}(K)/\mathrm{GL}_r(\mathcal{O}_K)$ is a $\mathbb{G}_m$-gerbe of Gr, denoted by Gra.
- Let $\mathrm{Gra}^{\underline{\lambda}}$ be the preimage of $\mathrm{Gr}^{\underline{\lambda}}$.
- There exist unique $G_{\mathcal{O}}$-equivariant rank one local system $W^{\underline{\lambda}}$ over $\mathrm{Gra}^{\underline{\lambda}}$ equipped with a $(\mathbb{G}_m, \mathcal{L}^{\zeta})$-equivariant structure. We denote by $\mathcal{B}_{\underline{\lambda}}$ the intermediate extension of $W^{\underline{\lambda}}[\dim(\mathrm{Gra}^{\underline{\lambda}})]$ to Gra.
- Let $b_{\underline{\lambda}} : \mathrm{Gra}(k) \to \overline{\mathbb{Q}_\ell} : x \mapsto \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{B}_{\underline{\lambda}})_x)$.
- The functions $b_{\underline{\lambda}}$ form a geometric basis of $\widetilde{\mathcal{H}}_r$.

FL in the case where r is even and of positive charactersitics

We denote by $\gamma(a,\Psi)$ the Weil constant defined by the formula

$$\int \Phi^{\vee}(x)\Psi\left(\frac{1}{2}ax^2\right)dx = |a|^{-1/2}\gamma(a,\Psi)\int \Phi(x)\Psi\left(-\frac{1}{2}a^{-1}x^2\right)dx,$$

Theorem

Let $t = \mathrm{diag}(t_1,\dots,t_r)$, we note $a_i = \prod_{j=1}^{i} t_j$. Let $\underline{\lambda} = (\lambda_1 \geq \dots \geq \lambda_r)$. We have then

$$J(t, b_{2\underline{\lambda}}) = \Delta(t) I(t, a_{\underline{\lambda}}),$$

where (noting $\zeta : k^* \to \{\pm 1\}$ the non-trivial quadratic character and agreeing that $a_0 = 1$)

$$\Delta(t) = |\prod_{i=1}^{r-1} a_i|^{-1/2}\zeta(-1)^{\sum_{j\not\equiv r(\mathrm{mod} 2)} v(a_j)} \prod_{j\not\equiv r(\mathrm{mod} 2)} \gamma(a_j a_{j-1}^{-1},\Psi).$$

In the case of mix characteristics

- Using the transfer principle of Cluckers and Loeser, Casselman-Cely-Hales proved the Langlands-Shelstad fundamental lemma for the spherical Hecke algebras.
- The point of this work is that they can handle the definability of the homomorphism between two Hecke algebras defined "analogously" our homomorphism b.
- Mimic their work with some care on the "metaplectic" side, I believe that we can extend our result in the case of positive characteristic to the general one.

Thanks you for your attention!