Counting ℓ-adic local systems using Arthur's trace formula

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Arithmetic $\leftrightarrow$ Geometry

- X_0 projective, smooth, and geometrically connected curve, defined over $\mathbb{F}_q$.
- $F = \mathcal{O}_{X_0,\eta}$: global function field, where η is the generic point of X_0.
- S_0 a finite set of closed points of X_0 / places of F.
- $\pi_1(X_0 - S_0, \overline{\eta}) \cong \mathrm{Gal}(\overline{F}^{S_0-ur}|F)$
- An ℓ-adic Galois representation ($\overline{\mathbb{Q}}_\ell$-coefficients) unramified outside S_0 corresponds to an ℓ-adic local system over $X_0 - S_0$.

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- $X = X_0 \otimes \overline{\mathbb{F}_q}$; $S = S_0 \otimes \overline{\mathbb{F}_q}$; $\mathbf{F}_X$ the Frobenius endomorphism of X.
- A point $s_0 \in S_0$ splits into $[\kappa(s_0) : \mathbb{F}_q]$-points after base change, and $\mathbf{F}_X$ permutes cyclically on the set of this points.
- Let $s \in S$ lying over s_0. $X_{(s)}^* := X_{(s)} - \{s\}$ for $X_{(s)}^*$ the henselisation (or completion) of X at s.
- Let $W_{s_0}, I_{s_0}, P_{s_0}$ be respectively, the local Weil group, inertia subgroup and wild inertia subgroup (to the choice of $\overline{\eta}$).
- $\pi_1(X_{(s)}^*, \overline{\eta}) = I_s \cong I_{s_0}$.

Geometry $\leftrightarrow$ Arithmetic

- $\mathcal{L}$ an ℓ-adic local system on $X - S$, then for each $s \in S$, it induces a I_s representation, called local monodromy at s, denote its isomorphic class by $\mathcal{R}(s)$.
- If $\mathbf{F}_X^* \mathcal{L} \cong \mathcal{L}$. Then we must have $\mathbf{F}_X^* \mathcal{R}(\mathbf{F}_X(s)) \sim \mathcal{R}(s)$. $\Rightarrow$ $\mathcal{R}(s)$ as a representation of I_{s_0} extends to a representation of W_{s_0}.

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- Grothendieck : If ρ is an ℓ-adic representation of W_{s_0}, then $\rho|_{I_{s_0}}$ is quasi-unipotent.
- Let $\widehat{\mathbb{Z}}^{p'}(1) = \lim_{p \nmid n} \mu_n(\overline{\mathbb{F}_q})$.
- We have a short exact sequence : $1 \to P_{s_0} \to I_{s_0} \to \widehat{\mathbb{Z}}^{p'}(1) \to 1$.
- Example : quasi-unipotent representation of $\widehat{\mathbb{Z}}^{p'}(1)$ is of the form :

$$\bigoplus (\chi \otimes \nu_\rho(\chi))$$

where $\nu_\rho(\chi)$ is a unipotent representation of the quotient $\mathbb{Z}_\ell(1)$ of $\widehat{\mathbb{Z}}^{p'}(1)$, χ is a character of $\widehat{\mathbb{Z}}^{p'}(1)$ of finite order.

Global Langlands Correspondence

After Drinfeld and Lafforgue, we have a bijection

$(\mathcal{L}, j)$ where $\mathcal{L}$ is an irreducible ℓ-adic local system on $X - S$ of rank n and

$$j : \mathbf{F}_X^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$$

$$\Updownarrow$$

absolutely cuspidal automorphic representation of $GL_n(\mathbb{A})$

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- The bijection depends on an auxiliary isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.
- The ramified local components correspond to each other in the sense of local Langlands correspondence too.

Example

$\mathcal{L}$ is an irreducible ℓ-adic local system on $X - S$, so that there exist a $j : \mathbf{F}_X^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$.

- Trivial local monodromy : I_s act as identity. Automorphic side : unramified principal series representation.
- The local monodromy at a point $s \in S$ is principal unipotent :

$$I_s \to \mathbb{Z}_\ell(1) \xrightarrow{a \mapsto \exp(aN)} GL_n(\overline{\mathbb{Q}}_\ell)$$

  where N is nilpotent with one Jordan block. Automorphic side : Inertially equivalent to Steinberg representation.

Starting point

- $E_n^{(\ell)}(\mathcal{R})$ the set of irreducible ℓ-adic local systems of rank n with local monodromy at s belongs to $\mathcal{R}(s)$.
- If $\mathcal{R}$ is fixed by $\mathbf{F}_X^*$ then $\mathbf{F}_X^*$ act on $E_n^{(\ell)}(\mathcal{R})$.
- Let
$$C_n(k) = |E_n^{(\ell)}(\mathcal{R})^{\mathbf{F}_X^{*k}}|$$

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Theorem (Drinfeld, 1981)

$S_0 = \emptyset$. If the genus g of the curve is greater than or equal to 2, then there are q-Weil numbers $(\mu_i)_{i=1,\dots,N}$ which are monomials of eigenvalues of Frobenius endomorphism on $H^1_{ét}(X, \mathbb{Q}_\ell)$ and the integers $(m_i)_{i=1,\dots,N}$, such that

$$C_2(k) = q^{(4g-3)k} + \sum_{i=1}^{N} m_i \mu_i^k$$

The weights of μ_i are strictly smaller than $8g-6$.

Other results

There are other results for calculating $C_n(k)$. All of this shows that $C_n(k)$ looks like the number of $\mathbb{F}_{q^k}$-points of a variety.

- Deligne-Flicker(2012) : $|S_0| \geqslant 2$. Principal unipotent local monodromy at every point.
- Flicker(2015) : $n = 2$ $|S_0| = 1$ with principal unipotent monodromy at the point.
- Arinkin : $S_0 \neq \emptyset$, local monodromies are sum of characters with some additional requirements.
- Savin-Templier(2018) : $S_0 \neq \emptyset$, with *mgs* local monodromy at at least one point.

Main theorem

Theorem (Y.)

$S_0 = \varnothing$. If the genus of the curve is greater than or equal to 2, then there are q-Weil numbers $(\mu_i)_{i=1,\dots,N}$ which are monomials of eigenvalues of Frobenius endomorphism on $H^1_{\acute{e}t}(X, \mathbb{Q}_\ell)$ and the integers $(m_i)_{i=1,\dots,N}$, such that

$$C_n(k) = q^{(n^2(g-1)+1)k} + \sum_{i=1}^{N} m_i \mu_i^k$$

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- m_i depend only on the genus of the curve.
- $(n^2(g-1)+1)$ is the half dimension of the moduli space of rank n stable Higgs bundles on X_0.
- μ_i are essentially q-Weil numbers of the the moduli space of stable Higgs bundles on X_0

Counting cuspidal automorphic representations

Two automorphic representations π_1, π_2 are called inertially equivalent if $\exists \chi : \mathbb{Z} \to \mathbb{C}$ such that $\pi_1 \cong \pi_2 \otimes \chi \circ \deg \circ \det$.

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After Langlands correspondence, we need to count the number of inertial equivalent classes of absolutely irreducible cuspidal representations of $GL_n(\mathbb{A})$.

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We use Arthur's non-invariant trace formula.

Automorphic spectrum

- Automorphic cuspidal representations lie in the L^2-automorphic spectrum :

$$L^2_{cusp}(G(F)\backslash G(\mathbb{A})) \subset L^2(G(F)\backslash G(\mathbb{A})) \circlearrowleft G(\mathbb{A})$$

  Test functions acting by convolution is compatible with spectral decompositions. The characteristic function $\mathbb{1}_{G(\mathcal{O})}$ will act on it as a projection to $G(\mathcal{O})$-invariant subspace. So its trace on the cuspidal spectrum gives number of unramified cuspidal representations.
- Difficulties :
  - $L^2(G(F)\backslash G(\mathbb{A}))$ is huge. Besides cuspidal spectrum, there are continuous spectrum and residue spectrum. These two spectrums contribute in different ways.
  - Not all cuspidal representations are absolutely cuspidal.
- Good news : all these can be reduced to absolutely cuspidal automorphic representations. (After Moeglin-Waldspurger and L. Lafforgue).

About the proof

$G = GL_n$. We fix the Borel subgroup B of upper triangular matrices. In the following P always mean a standard parabolic subgroup, M its Levi.

- $\mathfrak{a}_P$, Δ_P, Φ_P, X_M^G, $G(\mathbb{A})^e$, $\mathrm{Fix}(P, \pi)$, $\mathcal{P}(M)$, $k^T(x, y)$, $\mathrm{Higgs}_{n,e}^{st}(X_0)$...

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1. For the test function $\mathbb{1}_{G(\mathcal{O})}$, the trace formula is an identity $J_{geom} = \sum_{(P,\pi)} J_{(P,\pi)}$ where we fix an $a \in \mathbb{A}^\times$ of degree 1 and

$$J_{geom} = \int_{G(F)\backslash G(\mathbb{A})/a^{\mathbb{Z}}} k^{T=0}(x,x)\mathrm{d}x$$

Idea of the proof : calculate geometric side of the trace formula, then calculate spectral side.

- We consider a slightly variant : for any $e \in \mathbb{Z}$:

$$J_{geom}^{T,e} = \int_{G(F)\backslash G(\mathbb{A})^e} k^T(x,x)\mathrm{d}x$$

Hence $J_{geom} = \sum_{e=0}^{n} J_{geom}^{T=0,e}$.

Calculation of the geometric side

We don't calculate orbital integrals but we deal directly with Arthur's truncated integral.

- Essential ingredients :

1. $J^{T,e}_{geom}$ depends quasi-polynomially on $T \in \mathfrak{a}_B$.
2. We have an explicit expression when T is deep enough in the positive Weyl chamber. In fact we have

Theorem (L. Lafforgue)

If T is deep enough in the positive Weyl chamber depending only on the test function, then we have

$$k^T(x,x) = F^G(g,T) \sum_{\gamma \in G(F)} \mathbb{1}_{G(\mathcal{O})}(x^{-1}\gamma x)$$

where $F^G(g,T)$ is a characteristic function defined using Harder-Narasimhan filtration of vector bundles.

Results on the calculation of geometric side

Definition

Let $\mathcal{E}$ be a vector bundle over X_0. We say $\mathcal{E}$ is isocline if for all decomposition non-trivial $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{G}$, we have $\mu(\mathcal{F}) = \mu(\mathcal{E})$.

Let $\mathcal{P}_n^e(X_0)$ be the number of isomorphic classes of isocline vector bundles of rank n of degree e on X_0

Theorem (Y.)

Let $e \in \mathbb{Z}$, one has

$$J_{geom}^{T=0,e} = \mathcal{P}_n^e(X_0)$$

Moreover, when $(n, e) = 1$, we also have

$$J_{geom}^{T=0,e} = q^{-n^2(g_X-1)-1} |\mathrm{Higgs}_{n,e}^{st}(X_0)(\mathbb{F}_q)| \ .$$

Calculation of the spectral side

After L. Lafforgue, we have a decomposition :

$$J_{geom}^{T=0,e} = \sum_{(P,\pi)} J^e_{(P,\pi)}$$

where the sum is over the set of so called discrete pairs.

Spectral side has two part to deal with :

1. Calculate each $J^e_{(P,\pi)}$: we need to calculate weighted characters.
2. Sum up together all these terms : essentially a combinatorial problem.

Weighted Characters :

We consider a special case : (P,π) is a discrete pair so that factors of π are two by two non-inertially-equivalent. In this case, $J^e_{(P,\pi)}$ admits a simpler expression :

$$\frac{1}{|\mathrm{Fix}(P,\pi)|}\sum_{\lambda_\pi\in\mathrm{Fix}(P,\pi)}\int_{\mathrm{Im}X_M^G}\mathrm{Tr}_{\mathcal{A}_{P,\pi}^{G(\mathcal{O})}}\Big(\lim_{\mu\to 1}\sum_{Q\in\mathcal{P}(M)}\widehat{\mathbb{1}}_Q^e(\mu\lambda_\pi)\mathrm{M}_{P|Q}(\lambda)^{-1}\mathrm{M}_{P|Q}(\lambda/\mu\lambda_\pi)\circ\lambda_\pi\Big)\mathrm{d}\lambda,$$

where $\widehat{\mathbb{1}}_Q^e$ are rational functions on X_M^G, they're function field analogy of the functions θ_Q^{-1} defined by Arthur. They have the following properties :

- $\widehat{\mathbb{1}}_Q^e(\lambda)\prod_{\alpha\in\Delta_Q}(1-\lambda^{\alpha^\vee})$ is a regular function on X_M^G for any $Q\in\mathcal{P}(M)$.
- The limit above always converge due to functional equations of intertwining operators.

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- The intertwining operators act as scalar multiplications which can are explicit in terms of Rankin-Selberg L-functions.
- The twisting operator "$\circ\lambda_\pi$" will act as identity by a simple calcul global Whittaker model.

Weighted Characters :

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In number field case, Arthur has calculated weighted characters explicitly. We prove that when $(e,n)=1$, the Arthur's result holds in our case :

Proposition

If $\lambda_\pi\notin X_G^G$, then corresponding integral above is zero. If $\lambda_\pi\in X_G^G$, then the corresponding integral equals to

$$(\lambda_\pi)_1\sum_F\prod_{\beta\in F}(\mathrm{N}(n_\beta(\pi,\cdot))-\mathrm{P}(n_\beta(\pi,\cdot)))$$

where the sum runs over subsets F of $-\Phi_P$ such that F is a base of $\mathfrak{a}_M^{G*}$. The functions N et P give expectively the number of zeros and poles of $n_\beta(\pi,z)=q^*\frac{\mathrm{L}(\pi_i\times\pi_j^\vee,z)}{\mathrm{L}(\pi_i\times\pi_j^\vee,q^{-1}z)}$ in the region $|z|<1$.

Weighted Characters :

The sum over $\sum_F$ is still a problem for us. We solve this by graph theory : Consider : $\mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_r$. Let $\Phi = \{e_i - e_j | i < j\}$, F be a subset of Φ of cardinality $r-1$. Consider a graph $\mathcal{F}$ with the set of vertex $\{1, \cdots, r\}$, and the of edges are those (i,j) such that $e_i - e_j \in F$.

Proposition

F is a linearly independent family $\Leftrightarrow$ $\mathcal{F}$ is a tree.

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Theorem (Kirchhoff)

Let $(y_\beta)_{\beta\in\Phi}$ be a family of complex numbers, and A the symmetric matrix such that the element of index (i,j) $i<j$ is $y_{e_i - e_j} \in \mathbb{C}$. The diagonal elements of A are determined so that the sum of elements of every line is zero. Then

$$\sum_F \prod_{\beta\in F} y_\beta = \kappa(A)$$

where the sum over F runs through the subsets of Φ of cardinality $r-1$ which is linearly independent, $\kappa(A)$ is any cofactor of A.

Results on the calculation of spectral side

Let

$$\mathrm{aut}_l(z) = \exp(\sum_{m\geqslant 1}\sum_{k\geqslant 1}\frac{mC_m(kl)}{k}z^{mkl}) \ .$$

For all $v \geqslant 1$, let $[z^v]$ be an operator such that

$$[z^v]\sum_{i\geqslant 0} a_i z^i = a_v.$$

Theorem

Let the genus g of the curve not be 1. Let $e \in \mathbb{Z}$ such that $(e, n) = 1$, we have

$$J_e = \sum_{l|n}\frac{\mu(l)}{n(2g-2)}\sum_{\lambda\vdash n}\frac{1}{\sum_j a_j}\prod_{j\geqslant 1}[z^{a_j}]\mathrm{aut}_{X_l}(z^l)^{\frac{(2g-2)S_j(\lambda)}{l}}$$

where $\sum_{\lambda\vdash n}$ takes over all partitions $\lambda = (1^{a_1}, 2^{a_2}, \dots)$ of n and $S_j(\lambda) := \sum_{\nu\geqslant 1} a_\nu \min\{\nu, j\}$.

End

Thank you for your attention !