Generalizing Gödel's Constructible Universe:

The $\mathrm{Ultimate}$-L Conjecture

W. Hugh Woodin

Harvard University

IMS Graduate Summer School in Logic
June 2018

Generalizing L

Relativizing L to an arbitrary predicate P

Suppose P is a set. Define $L_\alpha[P]$ by induction on α by:

1. $L_0[P] = \emptyset$,
2. (Successor case) $L_{\alpha+1}[P] = \mathcal{P}_{\mathrm{Def}}(L_\alpha[P]) \cup \{P \cap L_\alpha[P]\}$,
3. (Limit case) $L_\alpha[P] = \bigcup_{\beta<\alpha} L_\beta[P]$.

- $L[P]$ is the class of all sets X such that $X \in L_\alpha[P]$ for some ordinal α.

- If $P \cap L \in L$ then $L[P] = L$.
- $L[\mathbb{R}] = L$ versus $L(\mathbb{R})$ which is **not** L unless $\mathbb{R} \subset L$.

Lemma

For every set X, there exists a set P such that $X \in L[P]$.

- This is equivalent to the Axiom of Choice.

Normal ultrafilters and $L[U]$

Definition

Suppose that U is a uniform ultrafilter on δ. Then U is a **normal ultrafilter** if for all functions, $f : \delta \to \delta$, if

- $\{\alpha < \delta \mid f(\alpha) < \alpha\} \in U$,

then for some $\beta < \delta$,

- $\{\alpha < \delta \mid f(\alpha) = \beta\} \in U$.

- A normal ultrafilter on δ is necessarily δ-complete.

Theorem (Kunen)

Suppose that $\delta_1 \leq \delta_2$, U_1 is a normal ultrafilter on δ_1, and U_2 is a normal ultrafilter on δ_2. Then:

- $L[U_2] \subseteq L[U_1]$
- *If $\delta_1 = \delta_2$ then*
  - $L[U_1] = L[U_2]$ *and* $U_1 \cap L[U_1] = U_2 \cap L[U_2]$.
- *If $\delta_1 < \delta_2$ there is an elementary embedding $j : L[U_1] \to L[U_2]$.*

$L[U]$ is a generalization of L

Theorem (Silver)

Suppose that U is a normal ultrafilter on δ. Then in $L[U]$:

- $2^\lambda = \lambda^+$ *for infinite cardinals λ.*
- *There is a projective wellordering of the reals.*

Theorem (Kunen)

Suppose that U is a normal ultrafilter on δ.

- *Then δ is the only measurable cardinal in $L[U]$.*

- This generalizes Scott's Theorem to $L[U]$ and so:
  - $V \neq L[U]$.

Weak Extender Models

Theorem

Suppose N is a transitive class, N contains the ordinals, and that N is a model of ZFC. Then for each cardinal δ the following are equivalent.

- *N is a weak extender model of δ is supercompact.*
- *For every $\gamma > \delta$ there exists a δ-complete normal fine ultrafilter U on $\mathcal{P}_\delta(\gamma)$ such that*
  - *$N \cap \mathcal{P}_\delta(\gamma) \in U$,*
  - *$U \cap N \in N$.*

- If δ is a supercompact cardinal then V is a weak extender model of δ is supercompact.

Why weak extender models?

The Basic Thesis

If there is a generalization of L at the level of a supercompact cardinal then it should exist in a version which is a weak extender model of δ is supercompact for some δ.

- Suppose U is δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\gamma)$, such that $\delta^+ \leq \gamma$, and such that γ is a regular cardinal. Then:
  - $L[U] = L$.
- Let W be the induced uniform ultrafilter on γ by restricting U to a set Z on which the "sup function" is 1-to-1. Then:
  - $L[W]$ is a Kunen inner model for 1 measurable cardinal.

Theorem

Suppose N is a weak extender model of δ is supercompact.

- *Then:*
  - *N has the δ-approximation property.*
  - *N has the δ-covering property.*

Corollary

Suppose N is a weak extender model of δ is supercompact and let $A = N \cap H(\delta^+)$. Then:

- *$N \cap H(\gamma)$ is (uniformly) definable in $H(\gamma)$ from A, for all strong limit cardinals $\gamma > \delta$.*
- *N is Σ_2-definable from A.*

- The theory of weak extender models for supercompactness is part of the first order theory of V.
  - There is no need to work in a theory with classes.

Weak extender models of δ is supercompact are close to V above δ

Theorem

Suppose N is a weak extender model of δ is supercompact and that $\gamma > \delta$ is a singular cardinal. Then:

- γ *is a singular cardinal in* N.
- $\gamma^+ = (\gamma^+)^N$.

This theorem strongly suggests:

- There can be no generalization of Scott's Theorem to **any** axiom which holds in **some** weak extender model of δ is supercompact, for any δ.
  - Since a weak extender model of δ is supercompact cannot be *far* from V.

The Universality Theorem

- The following theorem is a special case of the Universality Theorem for weak extender models.

Theorem

Suppose that N is a weak extender model of δ is supercompact, $\alpha > \delta$ is an ordinal, and that

$$j : N \cap V_{\alpha+1} \to N \cap V_{j(\alpha)+1}$$

is an elementary embedding such that $\delta \leq \mathrm{CRT}(j)$.

- *Then $j \in N$.*

- **Conclusion:** There can be **no generalization** of Scott's Theorem to **any** axiom which holds in **some** weak extender model of δ is supercompact, for any δ.

Large cardinals above δ are downward absolute to weak extender models of δ is supercompact

Theorem

Suppose that N is a weak extender model of δ is supercompact.

$$\kappa > \delta,$$

and that κ is an extendible cardinal.

- *Then κ is an extendible cardinal in N.*

(sketch) Let $A = N \cap H(\delta^+)$ and fix an elementary embedding

$$j : V_{\alpha+\omega} \to V_{j(\alpha)+\omega}$$

such that $\kappa < \alpha$ and such that $\mathrm{CRT}(j) = \kappa > \delta$.

- $N \cap H(\gamma)$ is uniformly definable in $H(\gamma)$ from A for all strong limit cardinals $\gamma > \delta^+$.
  - This implies that $j(N \cap V_{\alpha+\omega}) = N \cap V_{j(\alpha)+\omega}$ since $j(A) = A$.
- Therefore by the Universality Theorem, $j|(N \cap V_{\alpha+1}) \in N$.

Magidor's characterization of supercompactness

Lemma (Magidor)

Suppose that δ is strongly inaccessible. Then the following are equivalent.

(1) *δ is supercompact.*

(2) *For all $\lambda > \delta$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that $\mathrm{CRT}(\pi) = \bar{\delta}$ and such that $\pi(\bar{\delta}) = \delta$.

Theorem

Suppose that N is a weak extender model of δ is supercompact, $\kappa > \delta$, and that κ is supercompact.

- *Then N is a weak extender model of κ is supercompact.*

Too close to be useful?

- Are weak extender models for supercompactness simply **too** close to V to be of any use in the search for generalizations of L?

Theorem (Kunen)

There is no nontrivial elementary embedding

$$\pi : V_{\lambda+2} \to V_{\lambda+2}.$$

Theorem

Suppose that N is a weak extender model of δ is supercompact and $\lambda > \delta$.

- *Then there is no nontrivial elementary embedding*

  $$\pi : N \cap V_{\lambda+2} \to N \cap V_{\lambda+2}$$

  such that $\mathrm{CRT}(\pi) \geq \delta$.

Perhaps not

- Weak extender models for supercompactness can be nontrivially far from V in one key sense.

Theorem (Kunen)

The following are equivalent.

1. *L is far from V (as in the Jensen Dichotomy Theorem).*
2. *There is a nontrivial elementary embedding $j : L \to L$.*

Theorem

Suppose that δ is a supercompact cardinal.

- *Then there exists a weak extender model N for δ is supercompact such that*
  - *$N^\omega \subset N$.*
  - *There is a nontrivial elementary embedding $j : N \to N$.*

- This theorem shows that the restriction in the Universality Theorem on $\mathrm{CRT}(j)$ is necessary.

The HOD Dichotomy (full version)

Theorem (HOD Dichotomy Theorem)

Suppose that δ is an extendible cardinal. Then one of the following holds.

(1) *No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

Further:

- HOD *is a weak extender model of δ is supercompact.*

(2) *Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

Further:

- HOD *is not a weak extender model of λ is supercompact, for any λ.*
- *There is no weak extender model N of λ is supercompact such that $N \subseteq \mathrm{HOD}$, for any λ.*

A unconditional corollary

Theorem

Suppose that δ is an extendible cardinal, $\kappa \geq \delta$, and that κ is a measurable cardinal.

- *Then κ is a measurable cardinal in HOD.*

Two cases by appealing to the HOD Dichotomy Theorem:

- **Case 1:** HOD is **close** to V. Then HOD is a weak extender model of δ is supercompact.
  - Apply (a simpler variation of) the Universality Theorem.
- **Case 2:** HOD is **far** from V. Then **every** regular cardinal $\kappa \geq \delta$ is a measurable cardinal in HOD;
  - since κ is ω-strongly measurable in HOD.

The axiom $V = \text{Ultimate-}L$

The axiom for $V = \text{Ultimate-}L$

- There is a proper class of Woodin cardinals.
- For each Σ_2-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that
$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi.$$

Scott's Theorem and the rejection of $V = L$

Theorem (Scott)

Assume $V = L$. Then there are no measurable cardinals.

The key question

Is there a generalization of Scott's theorem to the axiom $V = \mathrm{Ultimate}\text{-}L$?

- If so then we must reject the axiom $V = \mathrm{Ultimate}\text{-}L$.

$V = \mathrm{Ultimate}\text{-}L$ and the structure of Γ^∞

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

For each $x \in \mathbb{R}$, there exists a universally Baire set $A \subseteq \mathbb{R}$ such that

$$x \in \mathrm{HOD}^{L(A,\mathbb{R})}.$$

- Assume there is a proper class of Woodin cardinals and that for each $x \in \mathbb{R}$ there exists a universally Baire set $A \subseteq \mathbb{R}$ such that $x \in \mathrm{HOD}^{L(A,\mathbb{R})}$.
  - This is in general yields the simplest possible wellordering of the reals.
    - It implies $\mathbb{R} \subset \mathrm{HOD}$.

Question

Does some large cardinal hypothesis imply that there must exist $x \in \mathbb{R}$ such that

$$x \notin \mathrm{HOD}^{L(A,\mathbb{R})}$$

for any universally Baire set?

$V = \text{Ultimate-}L$ and the structure of Γ^∞

Lemma

Suppose that there is a proper class of Woodin cardinals and that $A, B \in \mathcal{P}(\mathbb{R})$ are each universally Baire. Then the following are equivalent.

(1) $L(A, \mathbb{R}) \subseteq L(B, \mathbb{R})$.

(2) $\Theta^{L(A,\mathbb{R})} \leq \Theta^{L(B,\mathbb{R})}$.

Corollary

Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire. Then

$$\mathrm{HOD}^{L(A,\mathbb{R})} \subset \mathrm{HOD}.$$

Corollary ($V = \text{Ultimate-}L$)

Let Γ^∞ be the set of all universally Baire sets $A \subseteq \mathbb{R}$.

- *Then $\Gamma^\infty \neq \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R})$.*

Projective Sealing Theorems

Theorem (Unconditional Projective Sealing)

Suppose that there is a proper class of Woodin cardinals and that $V[G]$ is a generic extension of V.

- *Then* $V_{\omega+1} \prec V[G]_{\omega+1}$.

- Suppose $V_{\omega+1} \prec V[G]_{\omega+1}$ for generic extensions of V. Then there is no projective wellordering of the reals.

Theorem (Martin-Steel)

Suppose there are infinitely many Woodin cardinals. Then for each $n < \omega$ there exists a model M such that:

(1) $M \models \mathrm{ZFC} +$ *"There exist n-many Woodin cardinals".*

(2) $M \models \mathrm{ZFC} +$ *"There is a projective wellordering of the reals".*

Strong cardinals and conditional projective sealing

Suppose δ is a Woodin cardinal. Then:

- $V_\delta \models \mathrm{ZFC} +$ "There is a proper class of strong cardinals"

Thus:

- $\mathrm{ZFC} +$ "There is a proper class of strong cardinals" cannot prove projective sealing.

Theorem (Conditional Projective Sealing)

Suppose that δ is a limit of strong cardinals and $V[G]$ is a generic extension of V in which δ is countable.

Suppose $V[H]$ is a generic extension of $V[G]$.

- *Then $V[G]_{\omega+1} \prec V[H]_{\omega+1}$.*

- Thus after collapsing a limit of strong cardinals to be countable, one obtains projective sealing.

- Can Γ^∞ be sealed?

A Sealing Theorem for Γ^∞

Notation

Suppose $V[H]$ is a generic extension of V. Then

- $\Gamma^\infty_H = (\Gamma^\infty)^{V[H]}$
- $\mathbb{R}_H = (\mathbb{R})^{V[H]}$.

Theorem (Conditional Γ^∞ Sealing)

Suppose that δ is a supercompact cardinal and that there is a proper class of Woodin cardinals.
Suppose that $V[G]$ is a generic extension of V in which $(2^\delta)^V$ is countable.
Suppose that $V[H]$ is a generic extension of $V[G]$.

- *Then:*
  - $\Gamma^\infty_G = \mathcal{P}(\mathbb{R}_G) \cap L(\Gamma^\infty_G, \mathbb{R}_G)$.
  - *There is an elementary embedding*

$$j : L(\Gamma^\infty_G, \mathbb{R}_G) \to L(\Gamma^\infty_H, \mathbb{R}_H).$$

What about an Unconditional Γ^∞ Sealing Theorem?

A natural conjecture

By analogy with the Projective Sealing Theorems, there should be some large cardinal hypothesis which suffices to prove:

- Unconditional Γ^∞ Sealing.

But:

If some large cardinal hypothesis proves that

- $\Gamma^\infty = \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R})$

then the axiom $V = \text{Ultimate-}L$ is false.

- So there are potential paths to generalizing Scott's Theorem to the axiom $V = \text{Ultimate-}L$.
- Is there a potential path to showing that there is no generalization of Scott's Theorem to the axiom $V = \text{Ultimate-}L$?

The Ultimate-L Conjecture

Ultimate-L Conjecture

(ZFC) *Suppose that δ is an extendible cardinal. Then (provably) there is a transitive class N such that:*

1. *N is a weak extender model of δ is supercompact.*
2. *$N \models$ "$V = \mathrm{Ultimate}\text{-}L$".*

- The Ultimate-L Conjecture implies there is no generalization of Scott's Theorem to the case of $V = \mathrm{Ultimate}\text{-}L$.
  - By the Universality Theorem.
- The Ultimate-L Conjecture is a number theoretic statement
  - It is an existential statement, so if it is undecidable it must be false. Therefore:
    - It must be either true or false (it cannot be meaningless).
    - Just like the HOD Conjecture.
- The Ultimate-L Conjecture implies a slightly weaker version of the HOD Conjecture.

The summary from Tuesday's lecture

There is a progression of theorems from large cardinal hypotheses that suggest:

- Some version of $V = L$ is true.

Further:

- The theorems become much stronger as the large cardinal hypothesis is increased.

Large cardinals are amplifiers of the structure of V.

A natural conjecture building on this theme

One should be able to augment large cardinal axioms with some simple consequences of $V = \mathrm{Ultimate}\text{-}L$ and actually

- **recover** that $V = \mathrm{Ultimate}\text{-}L$,
  - laying the foundation for an argument that the axiom $V = \mathrm{Ultimate}\text{-}L$ is **true**.

Close embeddings and finitely generated models

Definition

Suppose that M, N are transitive sets, $M \models \mathrm{ZFC}$, and that

$$\pi : M \to N$$

is an elementary embedding. Then π is **close** to M if for each $X \in M$ and each $a \in \pi(X)$,

$$\{Z \in \mathcal{P}(X) \cap M \mid a \in \pi(Z)\} \in M.$$

Definition

Suppose that N is a transitive set such that

$$N \models \mathrm{ZFC} + \text{“}V = \mathrm{HOD}\text{”}.$$

Then N is **finitely generated** if there exists $a \in N$ such that every element of N is definable from a.

Why close embeddings?

Lemma

Suppose that M, N are transitive sets,

$$M \models \mathrm{ZFC} + \text{“}V = \mathrm{HOD}\text{”},$$

and that M is finitely generated.

- *Suppose that*
  - $\pi_0 : M \to N$
  - $\pi_1 : M \to N$

  are elementary embeddings each of which is close to M.
- *Then $\pi_0 = \pi_1$.*

- Without the requirement of closeness, the conclusion that $\pi_0 = \pi_1$ can fail.

Weak Comparison

Definition

Suppose that $V = \mathrm{HOD}$. Then **Weak Comparison** holds if for all $X, Y \prec_{\Sigma_2} V$ the following hold where M_X is the transitive collapse of X and M_Y is the transitive collapse of Y.

- Suppose that M_X and M_Y are finitely generated models of ZFC, $M_X \neq M_Y$, and
  - $M_X \cap \mathbb{R} = M_Y \cap \mathbb{R}$.
- Then there exist a transitive set M^*, and elementary embeddings
  - $\pi_X : M_X \to M^*$
  - $\pi_Y : M_Y \to M^*$

  such that π_X is close to M_X and π_Y is close to M_Y.

Why weak comparison?

- By Shoenfield's Absoluteness Theorem, the conclusion of Weak Comparison is absolute.
- Weak Comparison holds in the current generation of generalizations of L.
- Weak Comparison looks difficult to force.

Summary:

- Weak Comparison provides a good test question for generalizing L to levels of the large cardinal hierarchy.

Question

Assume there is a supercompact cardinal and that $V = \mathrm{HOD}$.

- Can Weak Comparison hold?

- (conjecture) $V = \mathrm{Ultimate}\text{-}L$ implies Weak Comparison.

Goldberg's Ultrapower Axiom

Notation

Suppose that $N \models \mathrm{ZFC}$ is an inner model of ZFC, $U \in N$ and

$$N \models \text{“}U \text{ is a countably complete ultrafilter”}$$

- N_U denotes the transitive collapse of $\mathrm{Ult}_0(N, U)$
- $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Definition (**The Ultrapower Axiom**)

Suppose that U and W are countably complete ultrafilters. Then there exist $W^* \in V_U$ and $U^* \in V_W$ such that the following hold.

(1) $V_U \models \text{“}W^* \text{ is a countably complete ultrafilter”}$.

(2) $V_W \models \text{“}U^* \text{ is a countably complete ultrafilter”}$.

(3) $(V_U)_{W^*} = (V_W)_{U^*}$.

(4) $j_{W^*}^{V_U} \circ j_U^V = j_{U^*}^{V_W} \circ j_W^V$.

- If $V = \mathrm{HOD}$ then (3) implies (4).

Weak Comparison and the Ultrapower Axiom

- The Ultrapower Axiom simply asserts that amalgamation holds for the ultrapowers of V by countably complete ultrafilters.
- If there are no measurable cardinals then the Ultrapower Axiom holds trivially
  - since every countably complete ultrafilter is principal.

Theorem (Goldberg)

Suppose that $V = \mathrm{HOD}$ *and that there exists*

$$X \prec_{\Sigma_2} V$$

such that $M_X \models \mathrm{ZFC}$ *where* M_X *is the transitive collapse of* X. *Suppose that Weak Comparison holds.*

- *Then the Ultrapower Axiom holds.*

- If X does not exist then Weak Comparison holds vacuously.
- If there is a supercompact cardinal, or even just a strong cardinal, then X must exist.

Strongly compact cardinals

Definition

Suppose that κ is an uncountable regular cardinal. Then κ is a **strongly compact cardinal** if for each $\lambda > \kappa$ there exists an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that:

1. U is a κ-complete ultrafilter,
2. U is a fine ultrafilter.

- Every supercompact cardinal is a strongly compact cardinal.

A natural question immediately arises:

Question

Suppose κ is a strongly compact cardinal. Must κ be a supercompact cardinal?

Menas' Theorem

Theorem (Menas)

Suppose κ is a measurable cardinal and that κ is a limit of strongly compact cardinals.

- *Then κ is a strongly compact cardinal.*

Lemma

Suppose κ is a supercompact cardinal and let S be the set of $\gamma < \kappa$ such that γ is a measurable cardinal.

- *Then S is a stationary subset of κ.*

Corollary (Menas)

Suppose that κ is the least measurable cardinal which is a limit of supercompact cardinals.

- *Then κ is a strongly compact cardinal and κ is not a supercompact cardinal.*

The Ultrapower Axiom and strongly compact cardinals

- The Identity Crisis Theorem of Magidor:

Theorem (Magidor)

Suppose κ is a supercompact cardinal. Then there is a (class) generic extension of V in which:

- *κ is a strongly compact cardinal.*
- *κ is the* **only** *measurable cardinal.*

Theorem (Goldberg)

Assume the Ultrapower Axiom and that for some κ:

- *κ is a strongly compact cardinal.*
- *κ is not a supercompact cardinal.*

Then κ is a limit of supercompact cardinals.

- The Ultrapower Axiom resolves the "identity crisis".
  - By Menas' Theorem, this is best possible.

The Ultrapower Axiom and the GCH

Theorem (Goldberg)

Asume the Ultrapower Axiom and that κ is a supercompact cardinal.

- *Then $2^\lambda = \lambda^+$ for all $\lambda \geq \kappa$.*

- The Ultrapower Axiom is absolute between V and $V[G]$ for all generic extensions whose associated Boolean algebra is of cardinality below the least strongly inaccessible cardinal of V.
- Therefore the Ultrapower Axiom even augmented by large cardinal assumptions **cannot** imply either of:
  - The Continuum Hypothesis.
  - $V = \mathrm{HOD}$.

Supercompact cardinals and HOD

Lemma

Suppose κ is a supercompact cardinal and that $V = \mathrm{HOD}$. Then

$$V_\kappa \models \text{“}V = \mathrm{HOD}\text{”}$$

- The converse is not true: if κ is supercompact and

$$V_\kappa \models \text{“}V = \mathrm{HOD}\text{”}$$

  then $V \neq \mathrm{HOD}$ can hold.
  - However, if in addition κ is an extendible cardinal then necessarily

$$V = \mathrm{HOD}.$$

The Ultrapower Axiom and HOD

Theorem (Goldberg)

Assume the Ultrapower Axiom , κ is a supercompact cardinal, and

$$V_\kappa \models \text{``}V = \mathrm{HOD}\text{''}.$$

Then:

- *For all regular cardinals $\gamma \geq \kappa$,*

  $$H(\gamma^{++}) = \mathrm{HOD}^{H(\gamma^{++})}$$

  More precisely,

  - *Every set $x \in H(\gamma^{++})$ is definable in $H(\gamma^{++})$ from some $\alpha < \gamma^{++}$.*

- *$V = \mathrm{HOD}$.*

- Thus in the context of the Ultrapower Axiom, the existence of a supercompact cardinal greatly amplifies the assumption that $V = \mathrm{HOD}$ by giving:
  - A uniform local version which must hold above the supercompact cardinal.
    - Just like with GCH, this is best possible.

HOD_A and Vopěnka's Theorem

Definition

Suppose A is a set. HOD_A is the class of all sets X such that there exist $\alpha \in \mathrm{Ord}$ and $M \subset V_\alpha$ such that

1. $A \in V_\alpha$.
2. $X \in M$ and M is transitive.
3. Every element of M is definable in V_α from ordinal parameters and A.

Theorem (Vopěnka)

For each set A, HOD_A is a set-generic extension of HOD.

- From the perspective of Set Theoretic Geology:
  - For each set A, HOD is a ground of HOD_A.

The Ultrapower Axiom and the grounds of V

Theorem (Goldberg)

Asume the Ultrapower Axiom and that κ is a supercompact cardinal. Suppose A is a wellordering of V_κ.

- *Then $V = \mathrm{HOD}_A$.*

Corollary (Goldberg)

Asume the Ultrapower Axiom and that there is a supercompact cardinal.

- *Then HOD is a ground of V.*

The HOD of the mantle of V

Putting everything together:

Theorem

Asume the Ultrapower Axiom and that there is an extendible cardinal. Let $\mathbb{M}$ be the mantle of V.

- *Then $\mathbb{M} \models$ "$V = \mathrm{HOD}$".*

(sketch)

- By Goldberg's Theorem, $V = \mathrm{HOD}_A$ for some set A.
- Therefore by Vopěnka's Theorem:
  - If N is a ground of V then HOD^N is a ground of N and so:
    - HOD^N is a ground of V.
- By Usuba's Mantle Theorem, $\mathbb{M}$ is a ground of V.
- Thus $\mathrm{HOD}^{\mathbb{M}}$ is a ground of V.
- Therefore $\mathbb{M} \subseteq \mathrm{HOD}^{\mathbb{M}}$ and so $\mathbb{M} = \mathrm{HOD}^{\mathbb{M}}$.

The mantle, V, HOD, and large cardinals

Theorem (after Hamkins et al)

Suppose $V[G]$ is the **Easton** *extension of V where for each limit cardinal γ, if $V_\gamma \prec_{\Sigma_2} V$ then G adds a fast club at γ^+. Then:*

- *V is not a ground of $V[G]$.*
- *V is the mantle of $V[G]$ and $\mathrm{HOD}^V = \mathrm{HOD}^{V[G]}$.*

- Many large cardinals are preserved, but:
  - There are **no** extendible cardinals in $V[G]$.

Theorem (after Hamkins et al)

Suppose $V[G]$ is the **Backward Easton** *extension of V where for each strong limit cardinal γ, G adds a fast club at γ^+. Then:*

- *$V[G]$ is the mantle of $V[G]$.*
- $\mathrm{HOD}^{V[G]} \subset \mathrm{HOD}^V$.
- *Every extendible cardinal of V is extendible in $V[G]$.*

- By changing G slightly one can arrange $\mathrm{HOD}^{V[G]} = V$.

The mantle of V and HOD when $V = \mathrm{Ultimate}\text{-}L$

Theorem

Assume $V = \mathrm{Ultimate}\text{-}L$. Then:

- *V has no nontrivial grounds.*
- *Suppose $V[G]$ is a set-generic extension of V. Then*
  - *V is the mantle of $V[G]$.*

Theorem

Assume $V = \mathrm{Ultimate}\text{-}L$. Then:

- *$V = \mathrm{HOD}$.*

- An obvious conjecture emerges.

The Mantle Conjecture

Mantle Conjecture

Asume the Ultrapower Axiom and that there is an extendible cardinal. Let $\mathbb{M}$ be the mantle of V.

- Then $\mathbb{M} \models$ "$V = \mathrm{Ultimate}\text{-}L$".

- The conjunction of the $\mathrm{Ultimate}\text{-}L$ Conjecture and the Mantle Conjecture would provide the basis for a powerful argument that the axiom, $V = \mathrm{Ultimate}\text{-}L$, is **true**, by citing as reasons:
  - **convergence** (of different approaches to the same axiom).
  - **recovery** (of axioms from their basic consequences).