Generalizing Gödel's Constructible Universe:

The HOD Dichotomy

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Definition

Suppose λ is an uncountable cardinal.

- λ is a **singular cardinal** if there exists a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.
- λ is a **regular cardinal** if there does not exist a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.

Lemma (Axiom of Choice)

Every (infinite) successor cardinal is a regular cardinal.

Definition

Suppose λ is an uncountable cardinal. Then $\mathrm{cof}(\lambda)$ is the minimum possible $|X|$ where $X \subset \lambda$ is cofinal in λ.

- $\mathrm{cof}(\lambda)$ is always a regular cardinal.
- If λ is regular then $\mathrm{cof}(\lambda) = \lambda$.
- If λ is singular then $\mathrm{cof}(\lambda) < \lambda$.

The Jensen Dichotomy Theorem

Theorem (Jensen)

Exactly one of the following holds.

(1) *For all singular cardinals γ, γ is a singular cardinal in L and*

$$\gamma^+ = (\gamma^+)^L.$$

- *L is* **close** *to V.*

(2) *Every uncountable cardinal is a regular limit cardinal in L.*

- *L is* **far** *from V.*

A strong version of Scott's Theorem:

Theorem (Silver)

Assume that there is a measurable cardinal.

- *Then L is far from V.*

Tarski's Theorem and Gödel's Response

Theorem (Tarski)

Suppose $M \models \mathrm{ZF}$ and let X be the set of all $a \in M$ such that a is definable in M without parameters.

- *Then X is not a definable in M without parameters.*

Theorem (Gödel)

Suppose that $M \models \mathrm{ZF}$ and let X be the set of all $a \in M$ such that a is definable in M from b for some ordinal b of M.

- *Then X is Σ_2-definable in M without parameters.*

Gödel's transitive class HOD

- Recall that a set M is transitive if every element of M is a subset of M.

Definition

HOD is the class of all sets X such that there exist $\alpha \in \mathrm{Ord}$ and $M \subset V_\alpha$ such that

1. $X \in M$ and M is transitive.
2. Every element of M is definable in V_α from ordinal parameters.

- (ZF) The Axiom of Choice holds in HOD.
- $L \subseteq \mathrm{HOD}$.
- HOD is the union of all transitive sets M such that every element of M is definable in V from ordinal parameters.
  - By Gödel's Response.

Stationary sets

Definition

Suppose λ is an uncountable regular cardinal.

1. A set $C \subset \lambda$ is **closed and unbounded** if C is cofinal in λ and C contains all of its limit points below λ:
   - For all limit ordinals $\eta < \lambda$, if $C \cap \eta$ is cofinal in η then $\eta \in C$.
2. A set $S \subset \lambda$ is **stationary** if $S \cap C \neq \emptyset$ for all closed unbounded sets $C \subset \lambda$.

Example:

- Let $S \subset \omega_2$ be the set all ordinals α such that $\mathrm{cof}(\alpha) = \omega$.
  - S is a stationary subset of ω_2,
  - $\omega_2 \backslash S$ is a stationary subset of ω_2.

The Solovay Splitting Theorem

Theorem (Solovay)

Suppose that λ is an uncountable regular cardinal and that $S \subset \lambda$ is stationary.

- *Then there is a partition*

$$\langle S_\alpha : \alpha < \lambda \rangle$$

  of S into λ-many pairwise disjoint stationary subsets of λ.

But suppose $S \in \mathrm{HOD}$.

- Can one require

$$S_\alpha \in \mathrm{HOD}$$

  for all $\alpha < \lambda$?

- Or just find a partition of S into 2 stationary sets, each in HOD?

Lemma

Suppose that λ is an uncountable regular cardinal and that:

- *$S \subset \lambda$ is stationary.*
- *$S \in \mathrm{HOD}$.*
- *$\kappa < \lambda$ and $(2^\kappa)^{\mathrm{HOD}} \geq \lambda$.*

Then there is a partition

$$\langle S_\alpha : \alpha < \kappa \rangle$$

of S into κ-many pairwise disjoint stationary subsets of λ such that

$$\langle S_\alpha : \alpha < \kappa \rangle \in \mathrm{HOD}.$$

But what if:

- $S = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \omega\}$ and $(2^\kappa)^{\mathrm{HOD}} < \lambda$?

Definition

Let λ be an uncountable regular cardinal and let

$$S = \{\alpha < \lambda \,|\, \mathrm{cof}(\alpha) = \omega\}.$$

Then λ is ω-**strongly measurable in** HOD if there exists $\kappa < \lambda$ such that:

1. $(2^{\kappa})^{\mathrm{HOD}} < \lambda$,
2. there is no partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of S into stationary sets such that
$$S_\alpha \in \mathrm{HOD}$$
for all $\alpha < \lambda$.

A simple lemma

Suppose $\mathbb{B}$ is a complete Boolean algebra and γ is a cardinal.

- $\mathbb{B}$ is γ-cc if

$$|\mathcal{A}| < \gamma$$

  for all $\mathcal{A} \subset \mathbb{B}$ such that $\mathcal{A}$ is an antichain:
  - $a \wedge b = 0$ for all $a, b \in \mathcal{A}$ such that $a \neq b$.

Lemma

Suppose that λ is an uncountable regular cardinal and that $\mathcal{F}$ is a λ-complete uniform filter on λ. Let

$$\mathbb{B} = \mathcal{P}(\lambda)/I$$

where I is the ideal dual to $\mathcal{F}$. Suppose that $\mathbb{B}$ is γ-cc for some γ such that $2^\gamma < \lambda$.

- *Then $|\mathbb{B}| \leq 2^\gamma$ and $\mathbb{B}$ is atomic.*

Lemma

Assume λ *is* ω*-strongly measurable in* HOD. *Then*

$$\mathrm{HOD} \models \lambda \textit{ is a measurable cardinal.}$$

Proof.

Let $S = \{\alpha < \lambda \mid (\mathrm{cof}(\alpha))^V = \omega\}$ and let

$\mathcal{F} = \{A \in \mathcal{P}(\lambda) \cap \mathrm{HOD} \mid S \backslash A$ is not a stationary subset of λ in $V\}$.

Thus $\mathcal{F} \in \mathrm{HOD}$ and in HOD, $\mathcal{F}$ is a λ-complete uniform filter on λ.

- Since λ is ω-strongly measurable in HOD, there exists $\gamma < \lambda$ such that in HOD:
  - $2^\gamma < \lambda$,
  - $\mathcal{P}(\lambda)/I$ is γ-cc where I is the ideal dual to $\mathcal{F}$.

Therefore by the simple lemma (applied within HOD), the Boolean algebra

$$(\mathcal{P}(\lambda) \cap \mathrm{HOD})/I$$

is atomic. □

Extendible cardinals

Lemma

Suppose that

$$\pi : V_{\alpha+1} \to V_{\pi(\alpha)+1}$$

is an elementary embedding and π is not the identity.

- *Then there exists an ordinal η that $\pi(\eta) \neq \eta$.*

- $\mathrm{CRT}(\pi)$ denotes the least η such that $\pi(\eta) \neq \eta$.

Definition (Reinhardt)

Suppose that δ is a cardinal.

- Then δ is an **extendible cardinal** if for each $\lambda > \delta$ there exists an elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

  such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.

Extendible cardinals and a dichotomy theorem

Theorem (HOD Dichotomy Theorem (weak version))

Suppose that δ is an extendible cardinal. Then one of the following holds.

(1) **No** *regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

Further, suppose γ is a singular cardinal and $\gamma > \delta$.

- *Then γ is singular cardinal in HOD and $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$.*

(2) **Every** *regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.*

- If there is an extendible cardinal then HOD must be either **close** to V or HOD must be **far** from V.
- This is just like the Jensen Dichotomy Theorem but with HOD in place of L.

Supercompactness

Definition

Suppose that κ is an uncountable regular cardinal and that $\kappa < \lambda$.

1. $\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \mid |\sigma| < \kappa\}$.
2. Suppose that $U \subseteq \mathcal{P}(\mathcal{P}_\kappa(\lambda))$ is an ultrafilter.
   - U is **fine** if for each $\alpha < \lambda$,
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in \sigma\} \in U.$$
   - U is **normal** if for each function
     $$f : \mathcal{P}_\kappa(\lambda) \to \lambda$$
     such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) \in \sigma\} \in U,$$
     there exists $\alpha < \lambda$ such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) = \alpha\} \in U.$$

The original definition of supercompact cardinals

Definition (Solovay, Reinhardt)

Suppose that κ is an uncountable regular cardinal.

- Then κ is a **supercompact cardinal** if for each $\lambda > \kappa$ there exists an ultrafilter *U* on $\mathcal{P}_\kappa(\lambda)$ such that:
  - *U* is κ-complete, normal, fine ultrafilter.

Lemma (Magidor)

Suppose that δ is strongly inaccessible. Then the following are equivalent.

(1) *δ is supercompact.*

(2) *For all $\lambda > \delta$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that $\mathrm{CRT}(\pi) = \bar{\delta}$ and such that $\pi(\bar{\delta}) = \delta$.

Solovay's Lemma

Theorem (Solovay)

Suppose $\kappa < \lambda$ are uncountable regular cardinals and that U is a κ-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

- *Then there exists $Z \in U$ such that the function*

  $$f(\sigma) = \sup(\sigma)$$

  is 1-to-1 on Z.

- There is **one** set $Z \subset \mathcal{P}_\kappa(\lambda)$ which works for **all** U.

Supercompact cardinals and a dichotomy theorem

Theorem

Suppose that δ is an supercompact cardinal, $\kappa > \delta$ is a regular cardinal, and that κ is ω-strongly measurable in HOD.

- *Then every regular cardinal $\lambda > 2^\kappa$ is ω-strongly measurable in HOD.*

- Assuming δ is an extendible cardinal then one obtains a much stronger conclusion.

Supercompact cardinals and the Singular Cardinals Hypothesis

Theorem (Solovay)

Suppose that δ is a supercompact cardinal and that $\gamma > \delta$ is a singular strong limit cardinal.

- *Then $2^\gamma = \gamma^+$.*

Theorem (Silver)

Suppose that δ is a supercompact cardinal. Then there is a generic extension $V[G]$ of V such that in $V[G]$:

- *δ is a supercompact cardinal.*
- *$2^\delta > \delta^+$.*

- Solovay's Theorem is the strongest possible theorem on supercompact cardinals and the Generalized Continuum Hypothesis.

The δ-covering and δ-approximation properties

Definition (Hamkins)

Suppose N is a transitive class, $N \models \mathrm{ZFC}$, and that δ is an uncountable regular cardinal of V.

1. N has the δ-**covering property** if for all $\sigma \subset N$, if $|\sigma| < \delta$ then there exists $\tau \subset N$ such that:
   - $\sigma \subset \tau$,
   - $\tau \in N$,
   - $|\tau| < \delta$.
2. N has the δ-**approximation property** if for all sets $X \subset N$, the following are equivalent.
   - $X \in N$.
   - For all $\sigma \in N$ if $|\sigma| < \delta$ then $\sigma \cap X \in N$.

For each (infinite) cardinal γ:

- $H(\gamma)$ denotes the union of all transitive sets M such that $|M| < \gamma$.

The Hamkins Uniqueness Theorem

Theorem (Hamkins)

Suppose N_0 and N_1 both have the δ-approximation property and the δ-covering property. Suppose

- $N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+)$.

Then:

- $N_0 = N_1$.

Corollary

Suppose N has the δ-approximation property and the δ-covering property. Let $A = N \cap H(\delta^+)$.

- *Then $N \cap H(\gamma)$ is (uniformly) definable in $H(\gamma)$ from A,*
  - *for **all** strong limit cardinals $\gamma > \delta^+$.*

- N is a Σ_2-definable class from parameters.

Set Theoretic Geology

Definition (Hamkins)

A transitive class N is a **ground** of V if

- $N \models \mathrm{ZFC}$.
- There is a partial order $\mathbb{P} \in N$ and an N-generic filter $G \subseteq \mathbb{P}$ such that $V = N[G]$.
  - G is allowed to be trivial in which case $N = V$.

Lemma (Hamkins)

Suppose N is a ground of V. Then for all sufficiently large regular cardinals δ:

- *N has the δ-approximation property.*
- *N has the δ-covering property.*

- Simply take δ be any regular cardinal of N such that $|\mathbb{P}|^N < \delta$.

Corollary

The grounds of V are Σ_2-definable classes from parameters.

- By the Hamkins Uniqueness Theorem.

Set Theoretic Geology (Hamkins)

What is the possible structure of the grounds of V?

- This is part of the first order theory of V.
- Suppose $N \subseteq M \subseteq V$, N is a ground of V, and $M \models \mathrm{ZFC}$.
  - Then M is a ground of V and N is a ground of M.

Definition (Hamkins)

The **mantle** of V is the intersection of all the grounds of V.

Let $\mathbb{M}$ be the mantle of V.

- (Hamkins) If $\mathbb{M}$ is a ground of V then $\mathbb{M}$ has no nontrivial grounds.
- (Hamkins) $\mathbb{M} \models \mathrm{ZF}$ but must $\mathbb{M} \models \mathrm{ZFC}$?

The Directed Grounds Problem

- For each uncountable regular cardinal λ, there is a canonical forcing notion for adding a *fast club* at λ.

Theorem (after Hamkins et al)

Fix an ordinal α. Suppose $V[G]$ is an Easton extension of V where for each strong limit cardinal γ, if $\gamma > \alpha$ then G adds a fast club at γ^+. Then:

- *The grounds of $V[G]$ are downward set-directed.*
- *V is not a ground of $V[G]$ and $V_\alpha = (V[G])_\alpha$.*
- *V is the mantle of $V[G]$ and $\mathrm{HOD}^V = \mathrm{HOD}^{V[G]}$.*

- The same example but with Backward Easton forcing yields $V[G]$ for which there are no non-trivial grounds:
  - $V[G]$ is the mantle of $V[G]$.

Question (Hamkins)

Are the grounds of V downward set-directed under inclusion?

When the grounds of V are downwards set-directed

Claim

Suppose that grounds of V are downwards set-directed. Then the following are equivalent.

1. *The mantle of V is a ground of V.*
2. *There are only set-many grounds of V.*
3. *This is a minimum ground of V.*

Claim

Suppose that grounds of V are downwards set-directed and let $\mathbb{M}$ be the mantle of V. Then

$$\mathbb{M} \models \mathrm{ZFC}.$$

Bukovsky's Theorem and Usuba's Solution

Theorem (Bukovsky)

Suppose that κ is a regular cardinal and $N \subset V$ is a transitive inner model of ZFC. Then the following are equivalent.

1. *For each $\theta \in \mathrm{Ord}$ and for each function $F : \theta \to N$ there exists a function*
$$H : \theta \to \mathcal{P}_\kappa(N)$$
such that $H \in N$ and such that $F(\alpha) \in H(\alpha)$ for all $\alpha < \theta$.
2. *V is a κ-cc generic extension of N.*

Theorem (Usuba)

The grounds of V are downward set-directed under inclusion.

Corollary (Usuba)

Let $\mathbb{M}$ be the mantle of V.

- *Then $\mathbb{M} \models$ The Axiom of Choice.*

Usuba's Mantle Theorem

Theorem (Usuba)

Suppose that there is an extendible cardinal. Let $\mathbb{M}$ be the mantle of V.

- *Then $\mathbb{M}$ is a ground of V.*

Corollary

Suppose that there is an extendible cardinal. Let $\mathbb{M}$ be the mantle of V and suppose that $\mathbb{M} \subseteq \mathrm{HOD}$.

- *Then HOD is a ground of V.*

- In this case, the **far** option in the HOD Dichotomy Theorem **cannot hold**.

A natural conjecture

Assuming sufficient large cardinals exist, then **provably** the far option in the HOD Dichotomy Theorem cannot hold.

The HOD Hypothesis

Definition (The HOD Hypothesis)

There exists a proper class of regular cardinals λ which are **not** ω-strongly measurable in HOD.

- It is not known if there can exist 4 regular cardinals which **are** ω-strongly measurable in HOD.
- It is not known if there can exist 2 regular cardinals above $2^{\aleph_0}$ where are ω-strongly measurable in HOD.
- Suppose γ is a singular cardinal of uncountable cofinality.
  - It is not known if γ^+ **can ever be** ω-strongly measurable in HOD.

Conjecture

Suppose $\gamma > 2^{\aleph_0}$ and that γ^+ is ω-strongly measurable in HOD.

- Then γ^{++} is not ω-strongly measurable in HOD.

The HOD Conjecture

Definition (HOD Conjecture)

The theory

$$\mathrm{ZFC} + \text{“There is a supercompact cardinal”}$$

proves the HOD Hypothesis.

- Assume the HOD Conjecture and that there is an extendible cardinal. Then:
  - The far option in the HOD Dichotomy Theorem is **vacuous**:
    - HOD **must** be close to V.
- The HOD Conjecture is a **number theoretic statement**.

The Weak HOD Conjecture and the $\mathrm{Ultimate}$-L Conjecture

Definition (Weak HOD Conjecture)

The theory

$$\mathrm{ZFC} + \text{“There is a extendible cardinal”}$$

proves the HOD Hypothesis.

$\mathrm{Ultimate}$-L Conjecture

(ZFC) *Suppose that δ is an extendible cardinal. Then* (*provably*) *there is a transitive class N such that:*

1. *N is a weak extender model of δ is supercompact.*
2. *$N \models$ “$V = \mathrm{Ultimate}$-L”.*

Theorem

The $\mathrm{Ultimate}$-L Conjecture implies the Weak HOD Conjecture.

An equivalence

Theorem

Suppose there is a proper class of extendible cardinals. Then following are equivalent.

(1) *The* HOD *Hypothesis holds.*

(2) *For some* δ*, there is a weak extender model* N *of* δ *is supercompact such that*

$$N \models \text{“The HOD Hypothesis”}.$$

Defining large cardinals in ZF

Definition

Suppose λ is a cardinal. Then

$$V_\lambda \prec_{\Sigma_1^*} V$$

if for all $a \in V_\lambda$, for all $\alpha < \lambda$, and all Σ_1-formulas, $\varphi(x_0)$;

- if there exists transitive set M such that
  - $M \models \varphi[a]$,
  - $M^{V_\alpha} \subset M$;

Then there exists such a transitive set $M \in V_\lambda$.

Lemma

Assume the Axiom of Choice. Then the following are equivalent.

1. $|V_\lambda| = \lambda$.
2. $V_\lambda \prec_{\Sigma_1} V$.
3. $V_\lambda \prec_{\Sigma_1^*} V$.

Defining extendible cardinals in ZF

Definition

Suppose $\delta > \omega$ is a cardinal. Then δ is **weakly extendible** if for all $\lambda > \delta$, there exists an elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$ and such that $\pi(\delta) > \lambda$.

Definition

Suppose $\delta > \omega$ is a cardinal. Then δ is **extendible** if for all $\lambda > \delta$ such that

$$V_\lambda \prec_{\Sigma_1^*} V,$$

there exists an elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$, $\pi(\delta) > \lambda$, and such that

$$V_{\pi(\lambda)} \prec_{\Sigma_1^*} V.$$

The Strong HOD Conjecture

Definition (Strong HOD Conjecture)

ZFC **proves** the HOD Hypothesis.

Theorem

Assume the Strong HOD Conjecture and that δ is a weakly extendible cardinal.

- *Then for all $\lambda > \delta$ the following are equivalent.*
  - *For all $\alpha < \lambda$, there is no surjection $\rho : V_\alpha \to \lambda$.*
  - $V_\lambda \prec_{\Sigma_1} V$.
  - $V_\lambda \prec_{\Sigma_1^*} V$.

Corollary (ZF)

Assume the Strong HOD Conjecture and that δ is a weakly extendible cardinal.

- *Then δ is an extendible cardinal.*

Applications of the HOD Conjecture in ZF

Theorem (ZF)

Assume the HOD Conjecture and that δ is an extendible cardinal.

- *Then for every cardinal $\lambda \geq \delta$, λ^+ is a regular cardinal.*

Theorem (ZF)

Assume the HOD Conjecture and that δ is an extendible cardinal.

- *Then for every regular cardinal $\lambda \geq \delta$, the Solovay Splitting Theorem holds at λ.*

- Assuming the HOD Conjecture:
  - Large cardinal axioms are trying to prove the Axiom of Choice.

Kunen's Theorem

Theorem (Kunen)

Suppose that λ is a cardinal.

- *Then there is no non-trivial elementary embedding*

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

- Kunen's Theorem is a ZFC theorem.

Theorem (ZF)

Assume the HOD Conjecture and that δ is an extendible cardinal.

- *Then for every cardinal $\lambda > \delta$, there is no nontrivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.*

Berkeley cardinals

Definition

A cardinal δ is a **Berkeley cardinal** if:

- For all $\alpha < \delta$ and for all transitive sets M with $\delta \subset M$, there exists a nontrivial elementary embedding
$$j : M \to M$$
such that $\alpha < \mathrm{CRT}(j) < \delta$.

- Assuming the Axiom of Choice, there are no Berkeley cardinals by Kunen's Theorem:
  - Just let $M = V_{\delta+2}$.

Theorem (ZF)

Assume the HOD *Conjecture. Then:*

- *There are no Berkeley cardinals.*

The inner model $L(\mathcal{P}(\mathrm{Ord}))$

Definition

$L(\mathcal{P}(\mathrm{Ord}))$ is the class of all sets X such that

$$X \in L(\mathcal{P}(\lambda))$$

for some ordinal λ.

Lemma (ZF)

The following are equivalent.

(1) *The Axiom of Choice.*

(2) $L(\mathcal{P}(\mathrm{Ord})) \models$ *The Axiom of Choice.*

HOD Conjecture and the Axiom of Choice

Theorem (ZF)

Assume the HOD *Conjecture. Suppose* δ *is an extendible cardinal. Then:*

- δ *is an extendible cardinal in* $L(\mathcal{P}(\mathrm{Ord}))$.
- *There exists* $\lambda < \delta$ *such that for all* $X \in L(\mathcal{P}(\mathrm{Ord}))$, *there exists an ordinal* η *and a surjection*

$$\pi : \mathcal{P}(\lambda) \times \eta \to X$$

  such that $\pi \in L(\mathcal{P}(\mathrm{Ord}))$.

- By using symmetric forcing extensions, the conclusion is best possible.

Summary

There is a progression of theorems from large cardinal hypotheses that suggest:

- Some version of $V = L$ is true.

Further:

- The theorems become much stronger as the large cardinal hypothesis is increased.

Large cardinals amplify structure.

- **They measure V and force the structure of V into discrete options.**

Perhaps this is all evidence that $V = \mathrm{Ultimate}\text{-}L$.