Thereshold dynamics method and applications

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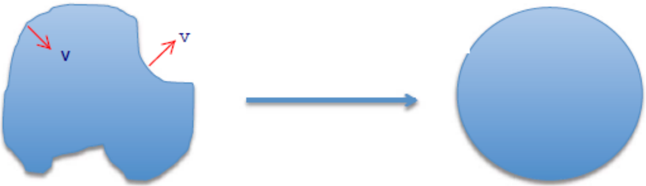
Outline

- A threshold dynamics method for (sharp) interface motion on solid surface (with XM Xu, D. Wang, JCP 2017)
- The contact line behavior and its dynamics (with D. Wang and XP Wang)
- An efficient implementation of the threshold dynamics method using boundary integral and NUFFT (with SD Jiang, D. Wang, JSC 2018)
- An efficient threshold dynamics method for image segementation (with D. Wang, H. Li and X. Wei, JCP 2017)

Motion by mean curvature



MBO Threshold Dynamics Method

Merriman, Bence, Osher (1992) introduced a threshold dynamics method for interface motion by mean curvature. Consider the Allen-Chan equation

$$\begin{cases} \phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f'(\phi) = 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}}\phi = 0, & \text{on } \partial\Omega, \end{cases} \qquad f(\phi) = \frac{(\phi^2 - \phi)^2}{4}$$

Operator splitting:

- Step 1: Solve a heat equation for δt:

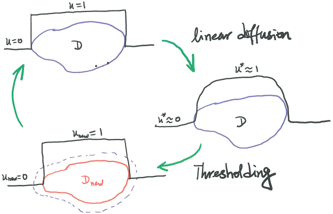
$$\begin{cases} \phi_t - \Delta\phi = 0, & \text{in } \Omega, \\ \phi(0) = \chi_D. & \\ \partial_{\mathbf{n}}\phi = 0, & \text{on } \partial\Omega, \end{cases}$$

- Step 2: Solve the equation:

$$\phi_t = -f'(\phi)/\varepsilon^2, \quad \text{in } \Omega.$$

- When $\epsilon \to 0$, Step 2 turns into thresholding

$$\phi(x) \approx \begin{cases} 1, & \text{if } \phi(x) > 1/2, \\ 0, & \text{if } \phi(x) < 1/2. \end{cases},$$



MBO converges to moton by mean curvature as $\delta t \to 0$ (Barles-Georgelin, 95).

Threshold Dynamics

Fix a time step size $\delta t > 0$ and generate a discrete in time approximation $\{D^k\}_{k=0}^{\infty}$ to the flow domain as follows:

MBO Algorithm:

- Convolution Step:

$$u(x,t) = \chi_{D^k} \star G_{\delta t} \quad \text{where} \quad G_{\delta t}(x) = \frac{1}{(4\pi\delta t)^{n/2}} e^{-\frac{|x|^2}{4\delta t}}$$

- Thresholding Step:

$$D^{k+1} = \{x \in R^n \ : \ u(x) > \frac{1}{2}\}$$

Threshold Dynamics

Benefits:

- Unconditionally stable: Accuracy only concern in choosing time step size.
- Fast implementation:
  Complexity only $O(N \log N)$, where N is the total number of grid points.
- Implicit representation of the front, allowing automatic topology changes.

Threshold Dynamics Method

Ginzburg-Landau Type Models and Dynamics

Ginzburg-Landau free energy functional

$$\mathcal{F}_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)dx, \qquad \Omega \subseteq R^n$$

Consider the gradient flow

$$u_t = -\frac{\delta \mathcal{F}_\varepsilon(u)}{\delta u} = \varepsilon \Delta u - \frac{1}{\varepsilon}W'(u)$$

where

$$u : \Omega \subseteq R^n \to R^m, \quad (n = 2, 3, m = 1, 2)$$

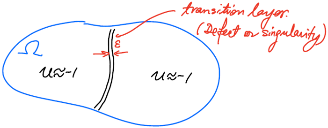
$$W : R^m \to R_+, \quad \text{multi-well potential}$$

Asymptotic limit ($\varepsilon \to 0$) of structures and dynamics of defects or singularities

(1) $m = 1$, i.e. u is a scalar valued

$$W(u) = (1 - u^2)^2$$

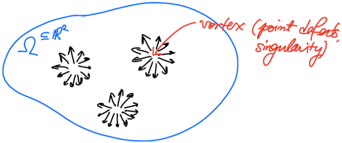
$$(W(u) \geq 0, = 0 \text{ iff } u = 1, -1)$$



(2) $n = 2, m = 2$, i.e $u : \Omega \subseteq R^2 \to R^2 \cong C$

$$W(u) = (1 - |u|^2)^2$$

$$(W(u) \geq 0, = 0 \text{ iff } |u| = 1, \text{ i.e. } u \in S^1)$$



F. Lin (1998)

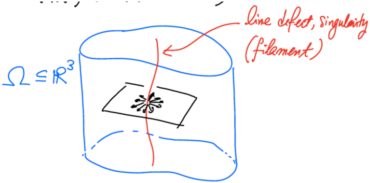
Typical examples

(3) $n = 3, m = 2$, i.e. $u : \Omega \subseteq R^3 \to R^2 \cong C$

$$W(u) = (1 - |u|^2)^2$$

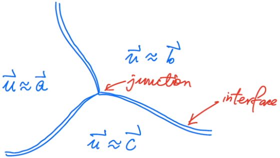
$$(W(u) = 0 \text{ iff } u \in S^1)$$

Ruuth, Merriman, Xin, Osher (2001)

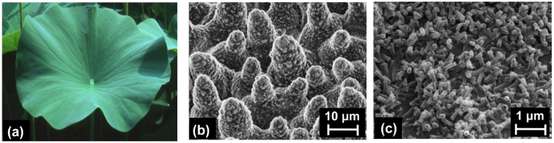


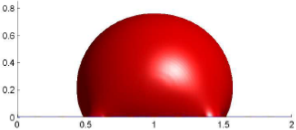
(4) Multiphase (e.g. $m = 3$)

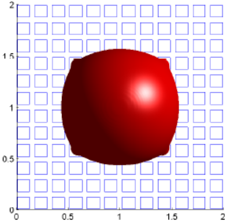
$(W(u) = 0 \text{ iff } u = \vec{a}, \vec{b}, \vec{c})$



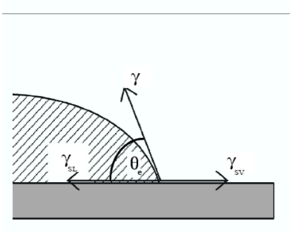
Wetting on rough surface

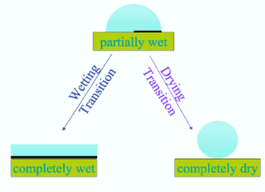






Young's equation





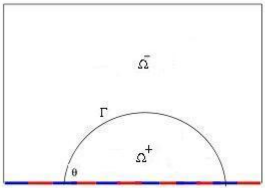
Young's Equation:

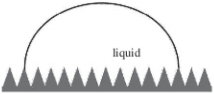
$$\gamma_{SV} = \gamma_{SL} + \gamma \cos\theta_e,$$

Contact angle $\theta_e < 90^o$, the surface is hydrophilic (亲水)
Contact angle $\theta_e > 90^o$, the surface is hydrophobic（疏水）

Minimizing the total interface energy

The equilibrium state of a liquid drop is determined by minimizing the total surface energy with a given droplet volume.





$$I(\phi) = \gamma_{LV}|\Gamma| + \int_{\partial\Omega^+ \cap \partial\Omega} \gamma_{SL}(\mathbf{x})ds + \int_{\partial\Omega \setminus \partial\Omega^+} \gamma_{SV}(\mathbf{x})ds \quad (1)$$

$$= \gamma_{LV}\left(|\Gamma| - \int_{\partial\Omega^+ \cap \partial\Omega} \cos\theta_Y(\mathbf{x})ds\right) + C \quad (2)$$

Here $\phi = \chi_{\Omega^+}$, $\int_\Omega \phi d\mathbf{x} = V_0$.

- Energy minimizing problem: $\min_{\phi \in BV(\Omega)} I(\phi)$

A diffuse interface model

- Total surface energy minimization

$$I_\varepsilon(\phi^\varepsilon) = \int_\Omega \frac{\varepsilon}{2}|\nabla\phi^\varepsilon|^2 + \frac{1}{\varepsilon}f(\phi^\varepsilon)dxdy + \int_{\partial\Omega}\gamma(\phi^\varepsilon)ds, \tag{3}$$

where $f(\phi) = \frac{(\phi^2-\phi)^2}{4}$.

$$\min_{\int_\Omega \phi^\varepsilon = V_0} I_\varepsilon(\phi^\varepsilon)$$

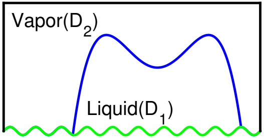
- In the sharp interface limit $I_\varepsilon(\phi^\varepsilon) \to I(\phi)$ (L. Modica, ARMA. 1987,1989)

Threshold dynamics method for wetting problem

Can we use the method to solve Allen-Cahn equation with volume conservation and wetting boundary condition?

$$\begin{cases} \phi_t - \varepsilon\Delta\phi + f'(\phi)/\varepsilon = \lambda, & \text{in } \Omega, \\ \varepsilon\partial_{\mathbf{n}}\phi + \partial_\phi\gamma(\mathbf{x},\phi) = 0, & \text{on } \partial\Omega, \\ \int_\Omega \phi = V_0. \end{cases}$$

$\Omega = D_1 \cup D_2$



Given two initial domain D_1, D_2 such that $|D_1| = V_0$.

- Solve the heat equation for δt:

$$\begin{cases} \phi_t - \varepsilon\Delta\phi = 0, & \text{in } \Omega, \\ \varepsilon\partial_{\mathbf{n}}\phi + \partial_\phi\gamma(\mathbf{x},\phi) = 0, & \text{on } \Gamma_\varepsilon, \\ \phi(0) = \chi_{D_1}. \end{cases}$$

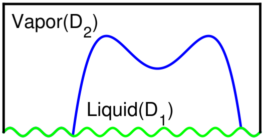
- The thresholding step (redefine D_1): find a δ, such that

$$D_1 = \{x : \phi(x) < 1/2 + \delta\} \qquad \text{and } |D_1| = V_0.$$

☐ Solving the heat equation is no longer efficient, especially when the boundary is rough.

The approximation of the interface energy

$$\Omega = D_1 \cup D_2$$



- The interface area $|\Sigma_{LV}|$ can be approximated by

$$|\Sigma_{LV}| \approx \frac{1}{\sqrt{\delta t}} \int \chi_{D_1} G_{\delta t} * \chi_{D_2} \mathrm{d}\mathbf{x}.$$

  where $G_{\delta t}(\mathbf{x}) = \frac{1}{(4\pi\delta t)^{n/2}} \exp(-\frac{|\mathbf{x}|^2}{4\delta t})$.

- Alberti & Bellettini(1998) give a proof by Γ-convergence theory.
- Esedoglu & Otto (2014) develop a threshold method for N-phase motion by mean curvature flow

Approximation of the wetting energy

- The total energy

$$\mathcal{E}^{\delta t} = \frac{\gamma_{LV}}{\sqrt{\delta t}} \int \chi_{D_1} G_{\delta t} * \chi_{D_2} + \frac{\gamma_{SL}}{\sqrt{\delta t}} \int \chi_{D_1} G_{\delta t} * \chi_{D_3} + \frac{\gamma_{SV}}{\sqrt{\delta t}} \int \chi_{D_2} G_{\delta t} * \chi_{D_3}$$

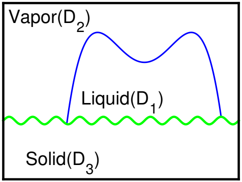
- The energy minimizing problem: Denote $u_1(x) = \chi_{D_1}, u_2(x) = \chi_{D_2}$,

$$\mathcal{B} = \{(u_1, u_2) \in BV(\Omega) \mid \; u_i(x) = 0, 1, u_1(x) + u_2(x) = 1, \int_\Omega u_1 \mathrm{d}\mathbf{x} = V_0\}$$

$$\min_{(u_1,u_2)\in\mathcal{B}} \mathcal{E}^{\delta t}(u_1, u_2). \qquad (4)$$

This is a non-convex problem.

$\tilde{\Omega} = D_1 \cup D_2 \cup D_3$



- An equivalent optimization problem on convex set:
  Denote

$$\mathcal{K} = \{(u_1, u_2) \in BV(\Omega) \mid 0 \le u_i \le 1, u_1(x) + u_2(x) = 1, \int_\Omega u_1 \mathrm{d}\mathbf{x} = V_0\}$$

$$\min_{(u_1,u_2)\in\mathcal{K}} \mathcal{E}^{\delta t}(u_1, u_2). \tag{5}$$

Lemma

For any given $\alpha, \beta \ge 0$ and any linear functional $\mathcal{L}(u_1, u_2)$, we have

$$\min_{(u_1,u_2)\in\mathcal{K}} (\alpha\mathcal{E}^{\delta t}(u_1, u_2) + \beta\mathcal{L}(u_1, u_2)) = \min_{(u_1,u_2)\in\mathcal{B}} (\alpha\mathcal{E}^{\delta t}(u_1, u_2) + \beta\mathcal{L}(u_1, u_2)).$$

Derivation of the threshold dynamics method

We now solve (5) iteratively

- For any given u_1^k, u_2^k, the energy functional $\mathcal{E}^{\delta t}(u_1, u_2)$ can be linearized near the point (u_1^k, u_2^k):

$$\mathcal{E}^{\delta t}(u_1, u_2) \approx \mathcal{E}^{\delta t}(u_1^k, u_2^k) + \hat{\mathcal{L}}(u_1 - u_1^k, u_2 - u_2^k, u_1^k, u_2^k) + h.o.t.$$

  with

$$\hat{\mathcal{L}}(u_1, u_2, u_1^k, u_2^k) = \frac{1}{\sqrt{\delta t}} \left(\int_{\tilde{\Omega}} u_1 G_{\delta t} * (\gamma_{LV} u_2^k + \gamma_{SL} \chi_{D_3}) + \int_{\tilde{\Omega}} u_2 G_{\delta t} * (\gamma_{LV} u_1^k + \gamma_{SV} \chi_{D_3}) \right).$$

- Solve

$$\min_{(u_1, u_2) \in \mathcal{K}} \hat{\mathcal{L}}(u_1, u_2, u_1^k, u_2^k) \tag{6}$$

  to get (u_1^{k+1}, u_2^{k+1})

The problem (6) can be solved easily by a thresholding step.

Derivation of the threshold dynamics method

Lemma

Denote

$$\phi_1 = \frac{1}{\sqrt{\delta t}} G_{\delta t} * (\gamma_{LV} u_2^k + \gamma_{SL} \chi_{D_3}),$$

$$\phi_2 = \frac{1}{\sqrt{\delta t}} G_{\delta t} * (\gamma_{LV} u_1^k + \gamma_{SV} \chi_{D_3}).$$

Let

$$D_1^{k+1} = \{x \in \Omega | \ \phi_1 < \phi_2 + \delta\} \tag{7}$$

for some δ *such that* $|D_1^{k+1}| = V_0$*. Let* $D_2^{k+1} = \Omega \setminus D_1^{k+1}$*. Then* $(u_1^{k+1}, u_2^{k+1}) = (\chi_{D_1^{k+1}}, \chi_{D_2^{k+1}})$ *is a solution of* (6)*. That is*

$$\hat{\mathcal{L}}(u_1^{k+1}, u_2^{k+1}, u_1^k, u_2^k) \leq \hat{\mathcal{L}}(u_1, u_2, u_1^k, u_2^k), \tag{8}$$

for all $(u_1, u_2) \in \mathcal{B}$*.*

Step 0. Given initial $D_1^0, D_2^0 \subset \Omega$, such that $D_1^0 \cap D_2^0 = \emptyset$, $D_1^0 \cup D_2^0 = \Omega$ and $|D_1^0| = V_0$. Set a tolerance parameter $\varepsilon > 0$.

Step 1. For given set (D_1^k, D_2^k), we define two functions

$$\phi_1 = \frac{1}{\sqrt{\delta t}} G_{\delta t} * (\gamma_{LV}\chi_{D_2^k} + \gamma_{SL}\chi_{D_3}),$$

$$\phi_2 = \frac{1}{\sqrt{\delta t}} G_{\delta t} * (\gamma_{LV}\chi_{D_1^k} + \gamma_{SV}\chi_{D_3}).$$

Step 2. Find a δ so that the set

$$\tilde{D}_1^\delta = \{x \in \Omega | \phi_1 < \phi_2 + \delta.\} \tag{9}$$

satisfies $|\tilde{D}_1^\delta| = V_0$. Denote $D_1^{k+1} = \tilde{D}_1^\delta$ and $D_2^{k+1} = \Omega \setminus D_1^{k+1}$.

Step 3. If $|D_1^k - D_1^{k+1}| \leq \varepsilon$, stop. Otherwise, go back to Step 1.

Stability

Theorem

Let (D_1^k, D_2^k), $k = 0, 1, 2, \ldots$ be the sets obtained by the above process. Denote $(u_1^k, u_2^k) = (\chi_{D_1^k}, \chi_{D_2^k})$, we have

$$\mathcal{E}^{\delta t}(u_1^{k+1}, u_2^{k+1}) \leq \mathcal{E}^{\delta t}(u_1^k, u_2^k), \tag{10}$$

for all $\delta t > 0$.

Behaviour near the contact point

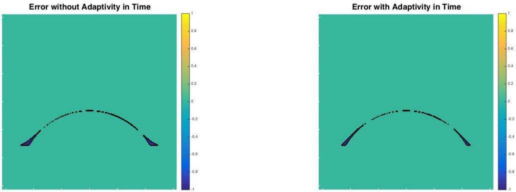


Figure: 256×256 grid points

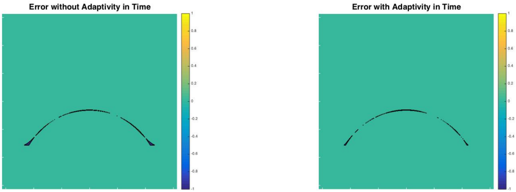


Figure: 512×512 grid points

Behaviour near the contact point

The total surface energy

$$\mathcal{E} = \gamma_{LV}|\Sigma_{LV}| + \gamma_{SL}|\Sigma_{SL}| + \gamma_{SV}|\Sigma_{SV}|, \tag{11}$$

where $|\Sigma_{SV}|, |\Sigma_{SL}|$ and $|\Sigma_{LV}|$ are the solid-vapor, solid-liquid and liquid-vapor interface areas. (11) is now approximated by

$$\mathcal{E}^h = \mathcal{E}^h_{LV} + \mathcal{E}^h_{SV} + \mathcal{E}^h_{SL} \tag{12}$$

where, as $h \to 0$,

$$\mathcal{E}^h_{LV} = \gamma_{LV}\frac{\sqrt{\pi}}{\sqrt{h}}\int_{\tilde{\Omega}} \chi_{D_1} G_h * \chi_{D_2}\mathrm{d}\mathbf{x} \quad \to \quad \gamma_{LV}|\Sigma_{LV}| \tag{13}$$

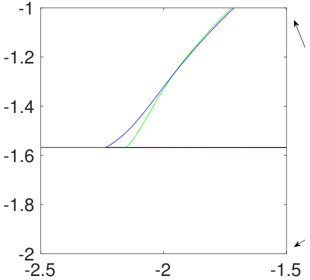
$$\mathcal{E}^h_{SL} = \gamma_{SL}\frac{\sqrt{\pi}}{\sqrt{h}}\int_{\tilde{\Omega}} \chi_{D_1} G_h * \chi_{D_3}\mathrm{d}\mathbf{x}. \quad \to \quad \gamma_{SL}|\Sigma_{SL}| \tag{14}$$

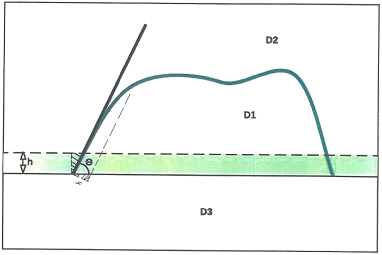
$$\mathcal{E}^h_{SV} = \gamma_{SV}\frac{\sqrt{\pi}}{\sqrt{h}}\int_{\tilde{\Omega}} \chi_{D_2} G_h * \chi_{D_3}\mathrm{d}\mathbf{x}, \quad \to \quad \gamma_{SV}|\Sigma_{SV}| \tag{15}$$

Behaviour near the contact point

As indicated by the two shaded regions, if the contact angle is less than 90^o,

- $|\Sigma_{LV}|$ and $|\Sigma_{SL}|$ are underestimated by $\mathcal{E}^h_{LV}$ and $\mathcal{E}^h_{SL}$ respectively,
- $|\Sigma_{SV}|$ is overestimated by $\mathcal{E}^h_{SV}$.
- The effect is the opposite if the contact angle is greater than 90^o.





An improved threshold dynamics method

The total energy

$$\mathcal{E} = \frac{\gamma_{LV}}{\sqrt{h_1}} \int \chi_{D_1} G_{h_1} * \chi_{D_2} + \frac{\gamma_{SL}}{\sqrt{h_2}} \int \chi_{D_1} G_{h_2} * \chi_{D_3} + \frac{\gamma_{SV}}{\sqrt{h_2}} \int \chi_{D_2} G_{h_2} * \chi_{D_3}$$

where $h_2 = \lambda h_1$.

An improved threshold dynamics method

$$\phi(x_1, x_2) = \frac{1}{\sqrt{h_1}} G_{h_1} * (\chi_{D_2} - \chi_{D_1}) - \frac{\cos\theta_Y}{\sqrt{h_2}} G_{h_2} * \chi_{D_3}$$

Asymptotic behavior of $\phi(0, x_2)$ for $\epsilon = \sqrt{h_1}$ gives

$$\phi(0, x_2) = \frac{1}{\epsilon}\left(\frac{\frac{\pi}{2} - \Theta}{\pi} - \frac{\cos\theta_Y \sqrt{h_1}}{2\sqrt{h_2}} - \frac{\epsilon}{2\sqrt{\pi}}\left(\frac{x_2/h_1}{\sqrt{1 + (g'(0))^2}} - \frac{g''(0)}{(1 + (g'(0))^2)^{\frac{3}{2}}}\right) + o(\epsilon)\right).$$

Thresholding at $\phi(0, x_2) = \delta$ and let $x_2 = v \cdot h_1$, v is the contact point velocity, we have

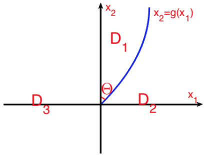
$$\lambda = \left(\frac{\pi\cos\theta_Y}{\pi - 2\theta_Y}\right)^2, \quad \theta_Y \in (0, \pi)$$

so that the correct contact angle

$$\Theta = \theta_Y$$

and the contact point velocity is given by

$$v = \frac{1}{\sin\Theta}(\kappa - 2\sqrt{\pi}\delta).$$



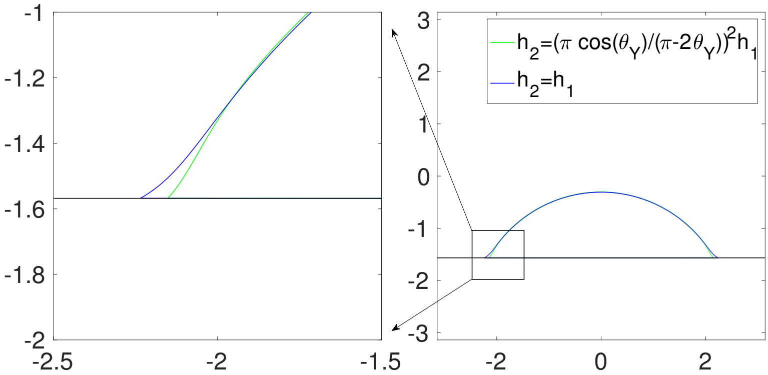


Figure: Comparison of the equilibrium states computed with $h_2 = h_1$ and $h_2 = \left(\dfrac{\pi \cos\theta_Y}{\pi - 2\theta_Y}\right)^2 h_1$ with $\theta_Y = \dfrac{\pi}{3}$.

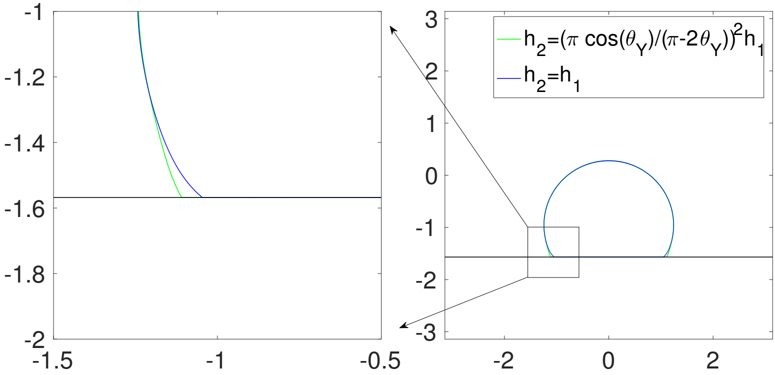


Figure: Comparison of the equilibrium states of a droplet computed with $h_2 = h_1$ and $h_2 = \left(\frac{\pi \cos \theta_Y}{\pi - 2\theta_Y}\right)^2 h_1$ with $\theta_Y = \frac{2\pi}{3}$.

Summary

- Contact line behavior is studied. Different scaling factors for the interface energies are needed to obtain the correct microscopic contact angle
- Contact line dynamics is derived
- An improved threshold dynamics method is proposed for wetting on rough surfaces

An efficient implementation using boundary integral and NUFFT
(S.D. Jiang, D. Wang and XP. Wang)

Approximation of Heat Kernel

It is well know that $G_{\delta t}$ admits the following Fourier representation

$$G_{\delta t}(\mathbf{x}, \delta t) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{-\|\mathbf{k}\|^2 \delta t + i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}.$$

Using the Poisson summation formula, we have

Theorem (Spectral Fourier Approximation of the Heat Kernel)

Suppose that $\epsilon < 1$ is the prescribed accuracy. Let $h = \min\left(\frac{\pi}{R}, \frac{\pi}{2\sqrt{\Delta t |\ln \epsilon|}}\right)$, and $M = \frac{1}{h}\sqrt{\frac{|\ln \epsilon|}{\Delta t}}$. Then for all $\|\mathbf{x} - \mathbf{y}\| \leq R$,

$$\left| G_{\delta t}(\mathbf{x} - \mathbf{y}, \Delta t) - \frac{h^2}{(2\pi)^2} \sum_{m_2=-M}^{M-1} \sum_{m_1=-M}^{M-1} e^{-\|\mathbf{m}\|^2 h^2 \Delta t + ih\mathbf{m}\cdot(\mathbf{x}-\mathbf{y})} \right| \leq \frac{2\epsilon}{\pi \Delta t}. \quad (16)$$

Motion by Mean Curvature Under Volume Preserving

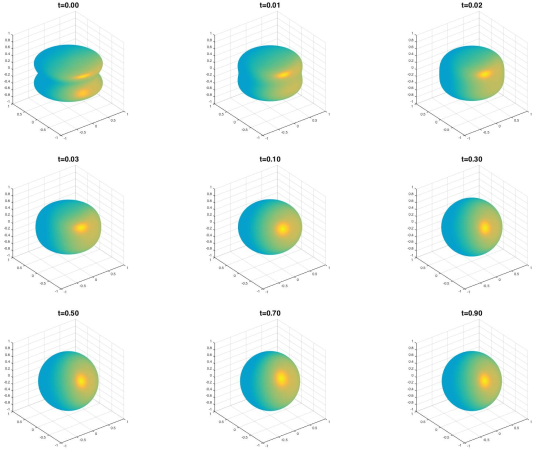


Figure: Double Biscuit Shaped Region Moving by Mean Curvature Motion under Volume Preserving. We set $dt = 0.002$ and use 96×96 points to descretize the interface. The time for the whole process is 3163s on a Laptop.

Image segmentation

Piecewise constant Mumford-Shah model

Chan-Vese, 2001

$$E_{CV}(\Sigma, C_1, C_2) = \lambda Per(\Sigma; D) + \int_{\Sigma} (C_1 - f)^2 dx + \lambda \int_{D/\Sigma} (C_2 - f)^2 dx \qquad (17)$$

where Σ is the interior of a closed curve and $Per(\Delta)$ denotes the perimeter. C_1 and C_2 are averages of f within Σ and D/Σ respectively.

Methods for solving the optimization problem: Split-Bregman algorithm, Augmented Lagrangian method, Primal-dual method, threshold dynamics method, framelet
Chan, Chan, Esedoglu, Osher, Shen, Tai, Tsai,......

A diffuse interface approximation for (17) is given by

$$E^{\varepsilon}_{MS}(u, c_1, c_2) = \int_D \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u) + \lambda\{u^2(c_1 - f)^2 + (1 - u)^2(c_2 - f)^2\}dx \qquad (18)$$

Variation of (18) with respect to u yields the following gradient descent equation:

$$u_t = 2\epsilon\Delta u - \frac{1}{\epsilon}W'(u) - 2\lambda\{u(c_1 - f)^2 + (u - 1)(c_2 - f)^2\}$$

which can be solved by MBO method:

- Step 1: Let $v(x) = S(\delta t)u^k(x)$, where $S(\delta t)$ is the propagator (by δt) of the linear heat equation

$$w_t = \Delta w - \frac{\lambda}{\sqrt{\pi\delta t}}(w(c_1 - f)^2 + (w - 1)(c_2 - f)^2$$

- Step 2: Set

$$u^{k+1}(x) = \begin{cases} 1 & \text{if } v(x) \in (-\infty, \frac{1}{2}] \\ 0 & \text{if } v(x) \in (\frac{1}{2}, \infty) \end{cases} \qquad (19)$$

Modified energy functional

- The perimeter of $\partial\Omega_i \cap \partial\Omega_j$ can be approximated by

$$|\partial\Omega_i \cap \partial\Omega_j| \approx \frac{\sqrt{\pi}}{\sqrt{\delta t}} \int_{\Omega} u_i G_{\delta t} * u_j d\Omega \tag{20}$$

Then, the multiphase energy can be approximated by:

$$\mathcal{E}^{\delta t}(u_1, \cdots, u_n) = \sum_{i=1}^{n} \int_{\Omega} \left(\lambda \sum_{j=1, j\neq i}^{n} \frac{\sqrt{\pi}}{\sqrt{\delta t}} u_i G_{\delta t} * u_j + u_i f_i \right) d\Omega \tag{21}$$

where $f_i(x) = (f(x) - C_i)^2$ and C_i is average intensity of the region i. That is,

$$C_i = \frac{\int_{\Omega_i} f(x) d\Omega_i}{\int_{\Omega_i} 1 d\Omega_i}. \tag{22}$$

$(u_1, \cdots, u_n) \in \mathcal{S} =$
$\left\{ u = (u_1, \cdots, u_n) \in BV(\Omega) : u_i(x) = 0, 1, \text{and} \sum_{i=1}^{n} u_i = 1 \right\}.$

Again, this is a non-convex problem

Relaxation to a convex problem

- Define the convex hull of $\mathcal{S}$ as

$$\mathcal{K} = \left\{ u = (u_1, \cdots, u_n) \in BV(\Omega) : 0 \leq u_i(x) \leq 1, \text{and} \sum_{i=1}^{n} u_i = 1 \right\}. \tag{23}$$

We can prove:

Lemma

Let $\mathcal{L}$ be any linear functional defined on $\mathcal{K}$ and $u = (u_1, \cdots, u_n)$. Then:

$$\min_{u \in \mathcal{S}} (\mathcal{E}^{\delta t}(u) + \mathcal{L}(u)) = \min_{u \in \mathcal{K}} (\mathcal{E}^{\delta t}(u) + \mathcal{L}(u)) \tag{24}$$

Linearization of the energy

- Suppose we solve the minimisation problem by an iterative method. In the k step, we have an approximated solution $(u_1^k, \cdots, u_n^k)$. The energy functional $\mathcal{E}^{\delta t}(u_1, \cdots, u_n)$ can be linearized near the point $(u_1^k, \cdots, u_n^k)$ by

$$\mathcal{L}(u_1, \cdots, u_n, u_1^k, \cdots, u_n^k) = \sum_{i=1}^{n} \int_\Omega \left(u_i f_i^k + 2\lambda \sum_{j=1, j\neq i}^{n} \frac{\sqrt{\pi}}{\sqrt{\delta t}} u_i G_{\delta t} * u_j^k \right) d\Omega \tag{25}$$

where $f_i^k = (C_i^k - f)^2$ and

$$C_i^k = \frac{\int_\Omega u_i^k f d\Omega}{\int_\Omega u_i^k d\Omega}.$$

Then, we minimize the linearised functional:

$$\min_{(u_1, \cdots, u_n) \in \mathcal{K}} \mathcal{L}(u_1, \cdots, u_n, u_1^k, \cdots, u_n^k) \tag{26}$$

and set the solution to be $(u_1^{k+1}, \cdots, u_n^{k+1})$.

Algorithm:

Step 0. *Given initial partition $\Omega_1^0, \cdots, \Omega_n^0 \subset \Omega$, to obtain partition $\Omega_1^{k+1}, \cdots, \Omega_n^{k+1}$ at time step $t = (k+1)\delta t$ from the partition $\Omega_1^k, \cdots, \Omega_n^k$ at time $t = k\delta t$. Set a tolerance parameter $\tau > 0$.*

Step 1. *Compute the following convolutions for $i = 1, \cdots, n$:*

$$\phi_i^k := f_i^k + \frac{2\lambda\sqrt{\pi}}{\sqrt{\delta t}}(1 - G_{\delta t} * u_i^k) \tag{27}$$

Step 2. *Thresholding:*

$$\Omega_i^{k+1} = \left\{ x : \phi_i^k(x) < \min_{j \neq i} \phi_j^k(x) \right\} \tag{28}$$

and let $u_i^{k+1} = \chi_{\Omega_i^{k+1}}$ where $\chi_{\Omega_i^{k+1}}$ represents the charecteristic function of region Ω_i^{k+1}

Step 3. *If $\int_\Omega \sum_{i=1}^{n} (u_i^{k+1} - u_i^k)^2 d\Omega \leq \tau$, stop. Otherwise, go back to step 1.*

Stability Analysis

- The following theorem shows that the algorithm is stable. In other words, the total energy $\mathcal{E}^{\delta t}$ always decrease in the algorithm for any $\delta t > 0$. We have the following theorem.

Theorem

Let $(u_1^k, \cdots, u_n^k)$, $k = 0, 1, 2, \cdots$, *obtained in the algorithm, we have*

$$\mathcal{E}^{\delta t}(u_1^{k+1}, \cdots, u_n^{k+1}) \leq \mathcal{E}^{\delta t}(u_1^k, \cdots, u_n^k) \tag{29}$$

Numerical Examples

Figure: Test with a real image of resolution 256×256. Left hand side figure has the initial contour. Right hand side image shows the final contour. Only 17 iterations are needed to achieve the finial steady with CPU time 0.1188 seconds on a MacBook Pro laptop with Intel Core i7 CPUs @ 3.0GHz

Numerical Examples

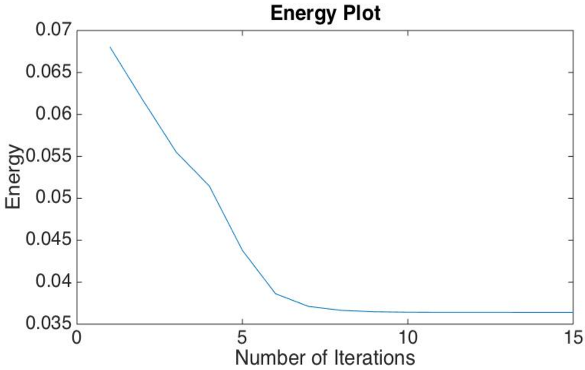


Figure: Energy curve for the iteration algorithm with $\delta t = 0.03$ and $\lambda = 0.01$.

Numerical Examples

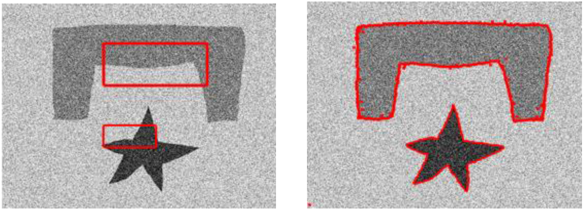


Figure: Test with a synthetic image with Gaussian noise of resolution 200×267. Left hand side figure has the initial contour to be taken. Right hand side image shows the final contour found. The experiment is carried out using the three-phase algorithm. Only 17 iterations are needed with CPU time 0.23 seconds.

Numerical Examples

Figure: Test with a real image of resolution $375 \times 500 \times 3$. Left hand side figure has the initial contour to be taken. Right hand side image shows the final contour found. The experiment is carried out using the two-phase vectored algorithm. Only 20 iterations are needed with CPU time 0.63 seconds.

Numerical Examples

Figure: Test with a real image of resolution $375 \times 500 \times 3$. Left hand side figure has the initial contour to be taken. Right hand side image shows the final contour found. The experiment is carried out using the four-phase vectored algorithm. Only 19 iterations are needed with CPU time 1.097 seconds.

Thank you!