Two-gird methods for semilinear elliptic interface problems by immersed finite element methods

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Overview

Introduction

Interface problems often occur when the computational domain consists of two different media, which have applications in many physical and engineering problems, such as fluid dynamics, materials science, electromagnetics, seismo-acoustics.

In many applications, a simulation domain is often formed by several materials separated by curves or surfaces from each other, and this often leads to differential equations on irregular domain consisting of the usual boundary condition, plus jump conditions across the material interface required by pertinent physics. Solving interface problems efficiently is critical, as:

- discontinuity across the interfaces;
- interfaces with complicated geometries or moving with time;
- low global regularity.

Early works

Finite element methods using body fitting grids:

- Ivo Babuška [Computing, 1970] introduced an equivalent minimization problem to handle the jump interface condition using fitted finite element method.
- James H. Bramble and J. Thomas King [Adv. Comput. Math., 1996] derived a finite element method in which the smooth boundary and interface of the problem are approximated by polygonal domain and interface.
- Zhiming Chen and Jun Zou [Numer. Math., 1998] shown that for C^2 interfaces in 2D convex polygonal domains, the linear finite element approximation u_h has suboptimal standard error estimates of orders $\mathcal{O}(h|logh|^{1/2})$ and $\mathcal{O}(h^2|logh|^{1/2})$ in H^1 and L^2 norms, respectively.
- Sinha and Deka [Appl. Numer. Math., 2009] studied linear finite element approximation of semi-linear elliptic interface problems in two-dimensional convex polygonal domains.

- Hui Xie, Zhilin Li, and Zhonghua Qiao, [Int. J. Numer. Anal. Model., 2013] studied a finite element method for elasticity interface problems based on a Cartesian mesh with local modifications.

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- However, it is difficult and time consuming to generate a body fitting grid for an interface problem in which the interface is complicated geometries or moves with time.
- Moreover, due to the discontinuity of the coefficients along the interface and the low global regularity of the solution, it is difficult to achieve higher order accuracy with standard finite element methods.

Therefore, few publications can be found on using body fitting grids to solve moving interface problems with topological changes such as merging and splitting.

Unfitted finite element method

The first immersed finite element method was developed by Li [Appl. Numer. Math.1998] for solving the one-dimensional two-point boundary value problem. Following this idea,

- Zhilin Li et al. [Numer. Math., 2003; Numerical Methods for PDEs, 2004] proposed an immersed finite element method using uniform Cartesian triangular grids and their numerical examples demonstrated an optimal order of the errors.
- So-Hsiang Chou [Adv. Comput. Math., 2010] derived optimal error estimates of immerse finite element method for second order elliptic equations with discontinuous coefficients.
- Jinru Chen and Zhilin Li [J. Sci. Comput., 2014; Numerical Algorithms, 2016] proposed a symmetric and consistent, and a new augmented immersed finite element method for interface problems.

- Wenqiang Feng, Xiaoming He, Yanping Lin, and Xu Zhang, [Communications in Computational Physics, 2014]proposed the IFE-AMG algorithm to solve the linear systems of the bilinear and linear IFE methods for both stationary and moving interface problems.
- Tao Lin, Yanping Lin and Xu zhang, [SIAM J. Numer. Amal., 2015, etc] studied DG immersed finite element methods.
- Zhimin Zhang, [Adv. Comput. Math., 2017; IMA J. Numer. Anal., 2017] studied superconvergence properties.

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Other unfitted finite element methods

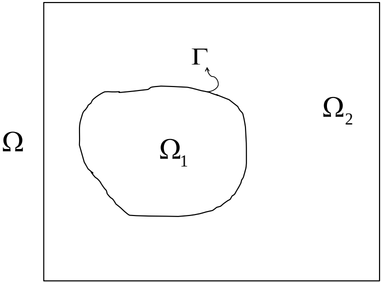
- Jianguo Huang and Jun Zou, [IMA J. Numer. Anal., 2002] a mortar element method.
- Peter Hansbo et al., [Comput. Method Appl. M., 2002] An unfitted finite element method, based on Nitsche's method.
- Haijun Wu and Yuanming Xiao, [arXiv:1007.2893] an unfitted *hp*-interface penalty finite element method.

Semi-linear interface model

We consider the following semi-linear elliptic interface problems with discontinuous diffusion coefficients:

$$\nabla \cdot (\beta \nabla u) = f(x, u), \quad x \in \Omega. \tag{1}$$

Let Ω be a convex polygonal domain in $\mathcal{R}^2$, $\Omega_1 \subset \Omega$ be an open domain with C^2 boundary $\Gamma = \partial\Omega_1 \subset \Omega$, and $\Omega_2 = \Omega \backslash \Omega_1$.



The system is subjected to the boundary condition:

$$u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

and the jump conditions on the interface

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \ \text{ across } \ \Gamma, \tag{3}$$

where $[v]$ is the jump of a quantity v across the interface Γ and $\mathbf{n}$ the unit outward normal to the boundary $\partial\Omega_1$. For ease of exposition, we assume that the coefficient function β is positive and piecewise constant, i.e.

$$\beta(x) = \beta_1 \ \text{ for } \ x \in \Omega_1; \quad \beta(x) = \beta_2 \ \text{ for } \ x \in \Omega_2. \tag{4}$$

We will also write $f(x,\xi) := f(\xi)$ and $\partial f(x,\xi)/\partial\xi := f'(\xi)$ for simplicity.

Weak formulation

The weak form for the interface problems (1)-(3) is stated as: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f(u),v), \quad \forall v \in H_0^1(\Omega), \tag{5}$$

where

$$a(u,v) = \int_\Omega \beta \nabla u \cdot \nabla v dx, \quad \forall u,v \in H_0^1(\Omega).$$

For the analysis, we introduce the following space

$$\widetilde{H}^2(\Omega) := \{u \in H^1(\Omega) : u \in H^2(\Omega_s), \quad s = 1,2\}$$

equipped with the norm

$$\|u\|^2_{\widetilde{H}^2(\Omega)} := \|u\|^2_{H^2(\Omega_1)} + \|u\|^2_{H^2(\Omega_2)}, \quad \forall u \in \widetilde{H}^2(\Omega).$$

Regularity theorem

Then we have the following regularity theorem for the weak solution u of the variational problem (5); see [Bramble and King, Adv. Comput. Math., 1996; Pierre Grisvard, SIAM, 2011].

Lemma 1.1

If $u \in H_0^1(\Omega)$ and $f \in L^q(\Omega)$, for $1 < q \leq 2$. Then the variational problem (5) has a unique solution $u \in \widetilde{H}^2(\Omega)$, for some constant $C > 0$, which satisfies

$$\|u\|_{\widetilde{H}^2(\Omega)} \leq C\|f\|_{L^q(\Omega)}. \tag{6}$$

where the constant C depends on q, and the domain Ω.

Some assumptions of $f(u)$

The weak assumption on the nonlinearity allows for a large class of nonlinear problems containing both monotone and nonmonotone nonlinearities.

Assume $f : \Omega \times \mathcal{R} \rightarrow \mathcal{R}$ is a Carathéodory function, which satisfies the barrier-sign conditions in its second argument: there exist constants $\rho,\ \sigma \in \mathcal{R}$, with $\rho \leq \sigma$, such that

$$\begin{aligned} f(x,\xi) \leq 0, \quad \forall \xi \geq \sigma, \quad a.e. \ \ in \ \ \Omega, \\ f(x,\xi) \geq 0, \quad \forall \xi \leq \rho, \quad a.e. \ \ in \ \ \Omega. \end{aligned}$$

This assumption is somewhat different but contains larger class of nonlinear problems than [Sinha and Deka, Appl. Numer. Math., 2009], where their assumption is

$$|f'(\xi)| \leq C|\xi| \ \text{ and } \ |f''(\xi)| \leq C, \quad \forall \xi \in \mathcal{R}.$$

Then we have the following priori L^∞ bounds (see [Michael Holst et al., Numerical Methods for PDEs, 2013], Theorem 2.3).

Lemma 1.2

Let the assumption of f above hold. Let $u \in H_0^1(\Omega)$ be any weak solution to (1). Then, we have

$$u_1 \leq u \leq u_2, \quad a.e. \quad in \quad \Omega, \tag{7}$$

for the constants u_1 and u_2 defined by

$$u_1 = \min\{\rho, 0\}, \quad u_2 = \max\{\sigma, 0\}.$$

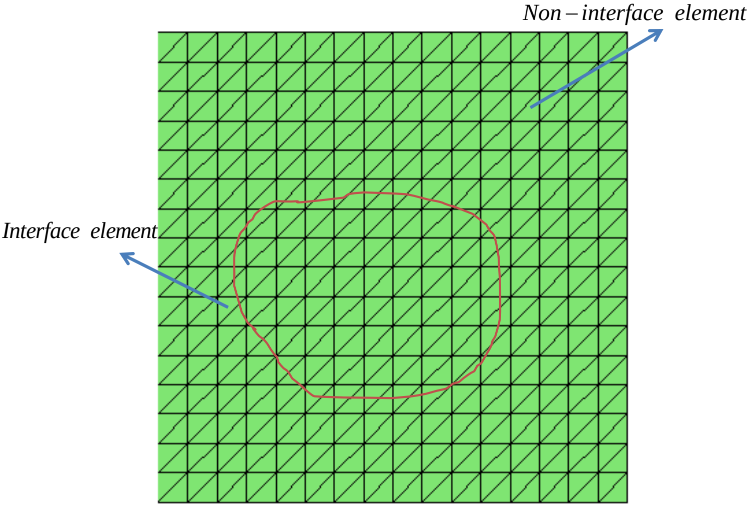
The priori L^∞ bounds play crucial roles in controlling the nonlinearity ensuring that the nonlinearity f has a certain "local Lipschitz" property. With this property, we can prove the error estimates later.

Immersed interface finite element space

Let $\{\mathcal{T}_h\}$ be the usual shape regular finite element triangulations with mesh size h that covers the domain Ω. Remark that in practice the computational domain Ω can often be chosen as a rectangular domain with sides parallel to the coordinate axes, and the finite element mesh can be uniform.

An element $T \in \mathcal{T}_h$ is an interface element if the interface Γ passes through the interior of T, otherwise the element T is a non-interface element. Let $\mathcal{T}_h^*$ be the collection of all interface elements. Note that if one of its edges is part of the interface, the element is a non-interface element. Assume that the interface meets the edges of an interface element at no more than two points.

Triangular Cartesian meshes independent of the interface



Immersed interface finite element space

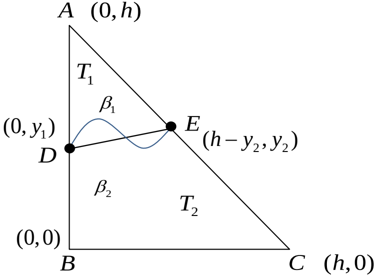
An immersed interface finite element space V_h can be defined with respect to the mesh $\mathcal{T}_h$ to be a finitely dimensional subspace of $L^2(\Omega)$ that consists of all the linear combinations of the corresponding basis functions $\phi_1, \phi_2, \cdots, \phi_N$ for some integer $N \geq 1$:

$$V_h = \text{span}\{\phi_1, \phi_2, \cdots, \phi_N\}. \tag{8}$$

As usual, we want to construct local basis functions on each element T of the partition $\mathcal{T}_h$. For a non-interface element $T \in \mathcal{T}_h$, we simply use the standard linear shape functions on T, and use $S_h(T)$ to denote the spaces spanned by the three nodal basis functions on T.

Interface element

Consider a typical interface element T whose geometric configuration is given in the following Figure in which three vertices are given by $A = (0, h)$, $B = (0, 0)$, $C = (h, 0)$, and the curve between points D and E is a part of the interface across which quantity β has a jump. Let $\overline{DE}$ be the line segment connecting the intersections of the interface and the edges of a triangle T. Assume that the coordinates at D and E are $(0, y_1)$ and $(h - y_2, y_2)$, respectively. The line segment divides T into two parts T_1 and T_2, one triangular and the other quadrilateral. The local basis function for a general interface element in the mesh $\mathcal{T}_h$ can be defined through the usual affine transformation.



Local basis functions on interface element

Each of the three local basis functions corresponding to the nodes A, B, or C takes value 1 at one node and 0 at the other two. Once the values at nodes A, B, and C are specified, a local nonconforming finite element basis function φ for this interface triangle is determined by

$$\varphi(x,y) = \begin{cases} \varphi_1(x,y) = a_0 + a_1 x + a_2(y-h) & in \quad T_1, \\ \varphi_2(x,y) = b_0 + b_1 x + b_2 y & in \quad T_2, \end{cases} \tag{9}$$

The coefficients a_i and b_i $(i = 0,1,2)$ are determined by the conditions

$$\varphi_1(D) = \varphi_2(D), \quad \varphi_1(E) = \varphi_2(E), \quad \beta_1 \frac{\partial \varphi_1}{\partial \mathbf{n}} = \beta_2 \frac{\partial \varphi_2}{\partial \mathbf{n}}, \tag{10}$$

where $\mathbf{n}$ is the unit normal direction of the segment $\overline{DE}$.

Theorem 2.1 (Li et al., Numer. Math., 2003; Chou et al., Adv. Comput. Math., 2010)

Given a right triangle ABC as indicated in the above figure. The piecewise linear function $\varphi(x, y)$ defined by (9) and (10) is uniquely determined by $\varphi(A)$, $\varphi(B)$, $\varphi(C)$.

Remark 2.1

Note that basis functions defined in this way can be discontinuous across edges of interface elements. So, this defines a nonconforming finite element. We note that the immersed finite element space V_h is a modification to the standard piecewise linear conforming finite element space when the coefficient β is discontinuous; the two spaces are the same when $\beta_1 = \beta_2$.

We need introduce the following spaces. For any $T \subset \Omega$,

$$\widetilde{W}^{m,p}(T) = \{u : \quad u|_{T_s} \in W^{m,p}(T_s), \quad s = 1, 2\}, \quad p \geq 1, m \geq 0,$$

$$\widetilde{H}^2_{int}(T) = \left\{u \in H^1(T) : \quad u|_{T_s} \in H^2(T_s), \quad s = 1, 2, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \ \ on \ \ \Gamma \cap T\right\}$$

and for any $u \in \widetilde{W}^{m,p}(T)$,

$$\|u\|^2_{m,p,T} = \|u\|^2_{m,p,T_1} + \|u\|^2_{m,p,T_2}, \quad |u|^2_{m,p,T} = |u|^2_{m,p,T_1} + |u|^2_{m,p,T_2},$$

where $\|\cdot\|_{m,p,T_s}$ is the norm of $W^{m,p}(T_s)$, $s = 1, 2$. When $p = 2$, we define $\widetilde{H}^m(T) = \widetilde{W}^{m,2}(T)$ as usual and denote its norm by $\|\cdot\|_{m,T}$.

Furthermore, we define $H^{1/2}(e)$ as the trace space on an edge e of T of all functions in $H^1(T)$ with the norm

$$\|v\|_{1/2,e} = \inf_{\substack{u \in H^1(T) \\ u|_e = v}} \|u\|_{1,T} \tag{11}$$

and $H^{-1/2}(e)$ as the dual space of $H^{1/2}(e)$, where the norm is given by

$$\|u\|_{-1/2,e} = \sup_{v \in H^{1/2}(e)} \frac{\langle u, v \rangle_e}{\|v\|_{1/2,e}}, \tag{12}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing.

Next, we introduce a transfer operator $\gamma : V_h(T) \to S_h(T)$. Define two trace spaces on an edge e of T:

$$V_h(e) = \{\hat{\phi}|_e : \hat{\phi} \in V_h(T)\}, \quad S_h(e) = \{\phi|_e : \phi \in S_h(T)\}. \tag{13}$$

Now we define $\gamma_e : V_h(e) \to S_h(e)$ by $\gamma_e \hat{\phi}|_e := (\gamma \hat{\phi})|_e$. We introduce the following estimate [Chou et al., Adv. Comput. Math., 2010].

Lemma 2.1

Let T be an interface element and e an edge of T. Then the following inequality holds for all $\hat{\phi} \in V_h(T)$:

$$\left\| \hat{\phi}|_e - \gamma_e \hat{\phi}|_e \right\|_{-1/2,e} \le C_\gamma h |\hat{\phi}|_{1,T}, \tag{14}$$

where C_γ is a constant independent of h and interface points.

For any $u \in \widetilde{H}^2_{int}(T)$, let $\Pi_h u \in V_h$. The interpolation Π_h can be naturally extended such that Π_h: $\widetilde{H}^2_{int}(\Omega) \to V_h(\Omega)$.

Then, an estimate of the interpolation was given in the following theorem; see [Li et al., Numer. Math., 2003].

Theorem 2.2

Let T be an interface element. Then there exists a constant $C > 0$ such that

$$\|u - \Pi_h u\|_{m,T} \leq Ch^{2-m}\|u\|_{2,T}, \quad m = 0, 1, \tag{15}$$

for any $u \in \widetilde{H}^2_{int}(T)$.

Immersed finite element method

We now consider the immersed interface finite element method for semi-linear interface problems: find $u_h \in V_h(\Omega)$ such that

$$a_h(u_h, \phi) = (f(u_h), \phi), \quad \forall \phi \in V_h, \tag{16}$$

where

$$a_h(u, v) = \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla v dx, \quad \forall u, v \in H_h(\Omega), \tag{17}$$

where $H_h(\Omega) := \{v| \ v|_T \in H^1(T), \quad \forall T \in \mathcal{T}_h\}$ and $H_h(\Omega)$ is endowed with the broken H^1 semi-norm as $\|v\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} |v|_{1,T}^2$.

Remark 3.1

Note that the bilinear operator $a_h(\cdot, \cdot)$ is bounded and coercivity (see [Li et al., Numer. Math., 2003; Chou et al., Adv. Comput. Math., 2010]).

For the energy norm error estimate of the immersed interface finite element method for semi-linear elliptic interface problems, we can prove the following lemma similar to the well-known second Strang lemma.

Lemma 3.1

Let $u \in \widetilde{H}^2(\Omega)$ and $u_h \in V_h$ be the solutions of the problem (5) *and the finite element equations* (16)*, respectively. Then there exists a constant $C > 0$ such that*

$$\|u - u_h\|_{1,h} \le C \left\{ \inf_{v_h \in V_h(\Omega)} \|u - v_h\|_{1,h} + \sup_{w_h \in V_h(\Omega)} \frac{|a_h(u, w_h) - (f(u_h), w_h)|}{\|w_h\|_{1,h}} \right\}, \tag{18}$$

where C depends on α and M.

Proof. By the coercivity of $a_h(\cdot,\cdot)$,

$$
\begin{aligned}
&\alpha\|u_h - v_h\|_{1,h}^2 \\
\leq &a_h(u_h - v_h, u_h - v_h) \\
= &a_h(u - u_h, v_h - u_h) + a_h(v_h - u, v_h - u_h) \\
\leq &a_h(u, v_h - u_h) - (f(u_h), v_h - u_h) + M\|u - v_h\|_{1,h}\|v_h - u_h\|_{1,h},
\end{aligned}
$$

where α is the coercivity constant.

Next, we deduce the left-hand side bound

$$
a_h(u - u_h, w_h) \leq M\|u - u_h\|_{1,h}\|w_h\|_{1,h}, \quad \forall w_h \in V_h.
$$

Thus,

$$
\|u - u_h\|_{1,h} \geq M^{-1} \sup_{w_h \in V_h} \frac{a_h(u, w_h) - (f(u_h), w_h)}{\|w_h\|_{1,h}}.
$$

where M is boundness constant of $a_h(\cdot,\cdot)$.

□

We now use the above lemma to prove the following broken H^1-norm error estimate.

Theorem 3.1

Let $u \in \widetilde{H}^2(\Omega)$ and $u_h \in V_h$ be the solutions of the problem (5) and the finite element equations (16), respectively. Then we have

$$\|u - u_h\|_{1,h} \leq Ch\|u\|_{\widetilde{H}^2(\Omega)}. \tag{19}$$

where $C = \max\{\|f\|_{1,\infty}, C_\gamma\}$ is independent of h and the location of the interface.

Sketch of proof

Proof. The first term in (18) is trivial through Theorem 2.2:

$$\inf_{v_h \in V_h(\Omega)} \|u - v_h\|_{1,h} \leq Ch\|u\|_{\widetilde{H}^2(\Omega)}. \tag{20}$$

For the second term in (18), we have

$$\begin{aligned}
& a_h(u, w_h) - (f(u_h), w_h) \\
=& \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla w_h dx - \int_\Omega f(u_h) w_h dx \\
=& (f(u) - f(u_h), w_h) + \sum_{T \in \mathcal{T}_h} \langle \beta \frac{\partial u}{\partial \mathbf{n}}, w_h \rangle_{\partial T} \\
=& (I)_1 + (I)_2,
\end{aligned} \tag{21}$$

where $\mathbf{n}$ is the unit outward normal vector on each ∂T, and $w_h \in V_h(\Omega)$.

For $(I)_1$, we have used $\|u-\Pi_h u\|_{L^2(T)}$, and note that u_h is a piecewise linear function in element T.

As for $(I)_2$, we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\langle\beta\frac{\partial u}{\partial\mathbf{n}},w_h\rangle_{\partial T} &= \sum_{T\in\mathcal{T}_h^*}\langle\beta\frac{\partial u}{\partial\mathbf{n}},w_h-\gamma_e w_h\rangle_{\partial T}\\ &= \sum_{T\in\mathcal{T}_h^*}\sum_{e\subset\partial T}\langle\beta\frac{\partial u}{\partial\mathbf{n}},w_h-\gamma_e w_h\rangle_{e}\\ &= \sum_{T\in\mathcal{T}_h^*}\sum_{e\subset\partial T}\sum_{s=1}^{2}\langle\beta\frac{\partial u}{\partial\mathbf{n}},w_h-\gamma_e w_h\rangle_{e_s}\end{aligned}$$

where $e_s=e\cap\Omega_s$, $s=1,2$ and note that

$$\sum_{T\in\mathcal{T}_h}\langle\beta\frac{\partial u}{\partial\mathbf{n}},\gamma_e w_h\rangle_{\partial T}=0.$$

From Lemma 2.1, it follows that

$$
\begin{aligned}
\langle \beta \frac{\partial u}{\partial \mathbf{n}}, w_h - \gamma_e w_h \rangle_{e_s} \leq & \left\| \beta \frac{\partial u}{\partial \mathbf{n}} \right\|_{1/2, e_s} \| w_h - \gamma_e w_h \|_{-1/2, e_s} \\
\leq & C_\gamma h \| u \|_{\widetilde{H}^2(\Omega)} | w_h |_{1,T}.
\end{aligned}
\tag{22}
$$

Thus summing over all the elements $T \in \mathcal{T}_h$, we get

$$
\sum_{T \in \mathcal{T}_h} \langle \beta \frac{\partial u}{\partial \mathbf{n}}, w_h \rangle_{\partial T} \leq Ch \| u \|_{\widetilde{H}^2(\Omega)} | w_h |_{1,h}. \tag{23}
$$

Now the desired estimate (19) follows from (18), (20), (21) and (23). This finishes the proof of Lemma 3.1.

□

Auxiliary problem

We now apply the duality argument to obtain L^p $(2 \le p < \infty)$ norm error estimates of the immersed finite element method for semi-linear interface problems.

Let $\omega \in \widetilde{H}^2(\Omega)$ be the solution of the following auxiliary problem: find $\omega \in \widetilde{H}^2(\Omega)$ satisfying

$$\begin{aligned} -\nabla \cdot (\beta \nabla \omega) - f'(u)\omega = u - u_h, \quad in \;\; \Omega, \\ w = 0, \quad on \;\; \partial\Omega, \end{aligned} \tag{24}$$

with jump conditions $[\omega] = 0$ and $\left[\beta \frac{\partial \omega}{\partial \mathbf{n}}\right] = 0$ across the interface Γ. Given $2 \le p < \infty$, set $q = p/(p-1) \in (1, 2]$. By Lemma 1.1, we know that

$$\|\omega\|_{\widetilde{H}^2(\Omega)} \le C \|u - u_h\|_{L^q(\Omega)}. \tag{25}$$

Along with (24), let us also introduce the immersed interface finite element approximation: find $\omega_h \in V_h$ satisfying

$$a_h(\omega_h, v_h) - (f'(u)\omega_h, v_h) = (u - u_h, v_h), \quad \forall v_h \in H_h. \tag{26}$$

Now noting the fact the jump $[\beta \frac{\partial \omega}{\partial \mathbf{n}}] = 0$ across Γ, and by immersed finite element approximation theory for the linear interface problem (24)-(26) (cf. [Chou, Adv. Comput. Math., 2010]), we have

$$\|\omega - \omega_h\|_{1,h} \leq C_1 h \|\omega\|_{\widetilde{H}^2(\Omega)} \leq Ch \|u - u_h\|_{L^q(\Omega)}. \tag{27}$$

L^p error estimate

The main results of this section are stated in the following theorem:

Theorem 3.2

Let $u \in \widetilde{H}^2(\Omega)$ and $u_h \in V_h$ be the solutions of the problem (5) and the finite element equations (16), respectively. For $2 \le p < \infty$, we have

$$\|u - u_h\|_{L^p(\Omega)} \le Ch^2 \|u\|_{\widetilde{H}^2(\Omega)}. \tag{28}$$

where the constant C is positive and depends on f, u_1, u_2 and C_γ.

Sketch of proof

Proof. Multiply $v_h = u - u_h \in H_h(\Omega)$ to both sides of (24) and by using (26), (5) and (16), we get

$$
\begin{aligned}
&(u-u_h, u-u_h) \\
&= \sum_{T\in\mathcal{T}_h}\int_T \beta\nabla(u-u_h)\cdot\nabla\omega dx - (f'(u)\omega, u-u_h) \\
&\quad - \sum_{T\in\mathcal{T}_h}\int_{\partial T}(u-u_h)\beta\frac{\partial\omega}{\partial\mathbf{n}}ds \\
&= a_h(u-u_h,\omega-\omega_h) - (f'(u)(\omega-\omega_h), u-u_h) - (f'(u)(u-u_h),\omega_h) \\
&\quad + (f(u)-f(u_h),\omega_h) + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\beta\frac{\partial u}{\partial\mathbf{n}}\omega_h ds - \sum_{T\in\mathcal{T}_h}\int_{\partial T}(u-u_h)\beta\frac{\partial\omega}{\partial\mathbf{n}}ds \\
&=: \ (II)_1 + (II)_2 + (II)_3 + (II)_4 + (II)_5 + (II)_6. \qquad (29)
\end{aligned}
$$

We need use $\|u-u_h\|_{1,h}$, $\|\omega-\omega_h\|_{1,h}$ error estimate for $(II)_1$, (II_2). By Taylor expansion, we have

$$\begin{aligned}(II)_3+(II)_4 =&(f(u)-f(u_h)-f'(u)(u-u_h),\omega_h)\\ =&-(\frac{1}{2}f''(\xi_1)(u-u_h)^2,\omega_h),\end{aligned} \tag{30}$$

where ξ_1 is some function value.

Therefore, by Hölder inequality, the $(II)_3+(II)_4$ term in (29) can be bounded as:

$$\begin{aligned}|(II)_3+(II)_4| \leq&\|f\|_{2,\infty}\|u-u_h\|_{0,2m}^2\|\omega_h\|_{0,m/(m-1)}\\ \leq&Ch^2\|u\|_{\widetilde{H}^2(\Omega)}\|u-u_h\|_{0,q},\end{aligned} \tag{31}$$

where we choose $m>2p/(p+2)$.

As for $(II)_5$, we introduce the extended transfer operator $\bar{\gamma}$: $V_h(T) \oplus span\{\omega\} \to S_h(T) \oplus span\{\omega\}$ defined by

$$\bar{\gamma}(\hat{\phi} + c\omega) := \gamma\hat{\phi} + c\omega, \quad \text{for } \hat{\phi} \in V_h, \quad c \in \mathcal{R}.$$

Define $\bar{\gamma}_e$ on each edge of T by

$$\bar{\gamma}_e(\hat{\phi} + c\omega)|_e := (\gamma\hat{\phi} + c\omega)|_e.$$

It is clear that $\|\bar{\gamma}_e\|$ is bounded above by a constant independent of h and the location of the interface for any norm $\|\cdot\|$. Now applying an analysis similar to obtain (23), we rewrite $(II)_5$ as

$$\begin{aligned}(II)_5 &= \sum_{T\in\mathcal{T}_h} \int_{\partial T} \beta \frac{\partial u}{\partial \mathbf{n}}(\omega_h - \omega)ds \\ &= \sum_{T\in\mathcal{T}_h} \sum_{e\subset\partial T} \int_e \beta \frac{\partial u}{\partial \mathbf{n}}[(\omega_h - \omega) - \bar{\gamma}_e(\omega_h - \omega)]ds \qquad (32)\end{aligned}$$

Hence by Lemma 2.1, we get

$$\begin{aligned}|(II)_5| \leq& Ch\|u\|_{\widetilde{H}^2(\Omega)}\|\omega - \omega_h\|_{1,h} \leq Ch^2\|u\|_{\widetilde{H}^2(\Omega)}\|\omega\|_{\widetilde{H}^2(\Omega)}\\ \leq& Ch^2\|u\|_{\widetilde{H}^2(\Omega)}\|u - u_h\|_{L^q(\Omega)}.\end{aligned} \tag{33}$$

Arguing as deriving (32), we can deduce $(II)_6$ by interchanging the role of ω and u,

$$\begin{aligned}|(II)_6| \leq& Ch\|\omega\|_{\widetilde{H}^2(\Omega)}\|u - u_h\|_{1,h}\\ \leq& Ch^2\|\omega\|_{\widetilde{H}^2(\Omega)}\|u - u_h\|_{L^q(\Omega)},\end{aligned} \tag{34}$$

where we have used (27) in the last step.

Now the L^p-error estimate (28) follows from the error estimates of $(II)_1$, $(II)_2$, $(II)_3$, $(II)_4$, $(II)_5$ and $(II)_6$.

□

Two-grid Methods

Immersed finite element approximation of semi-linear problems results in the need to solve systems of nonlinear algebraic equations, and the numbers of unknowns in these systems can be extraodinary large. So, it is necessary to study highly efficient and highly accurate algorithms for non-linear systems.

The two-grid finite element method based on two finite element spaces on one coarse and one fine grid was first introduced by Xu in 1994, 1996 for the nonsymmetric and nonlinear elliptic problems.

- J. Xu, A novel two-grid method for semilinear equations, SIAM Journal on Scientific Computing, 15:231-237, 1994.
- J. Xu, Two-grid discretization techniques for linear and non-linear PDEs, SIAM Journal on Numerical Analysis, 33:1759-1777, 1996.

As almost same time, we have proposed a multilevel iterative method that not only reduces the computing work but also preserves all of the high accuracy properties such as superconvergence, extrapolation, etc, for finite element solutions to singular problems.

- Y. Chen and Y. Huang, A multilevel iterative method for finite element solutions to singular two-point boundary value problems, Natural Science J. of Xiangtan University, 1994.

With the development of this highly efficient method, the two-grid method was further investigated by many author. For instance,

- C. N. Dawson, M. F. Wheeler and C. S. Woodward, nonlinear parabolic equations, SIAM J. Numer. Anal., 1998;
- J. Xu and A. Zhou, eigenvalue problems, Math. Comp., 2001;
- Yinnian He, Navier-Stokes equations , SIAM J. Numer. Anal., 2003;
- Y. Chen, nonlinear reaction-diffusion problems, Int. J. Numer. Met. Eng., 2003, 2007; miscible displacement problem in a porous media, Commun. Comput. Phys., Sci. Comput., 2016; compressible miscible displacement problem, J. Comput. Appl. Math., 2018;
- Jicheng Jin, nonlinear schrödinger equation, Math. Comput., 2006; Journal of Computational Mathematics, 2015;

- M. Mu and J. Xu, coupling fluid flow with porous media flow, SIAM Journal on Numerical Analysis, 2007.

  ... ...

Now, however, there are few results about two-grid methods for semi-linear interface problems by finite element methods (or immersed finite element methods). Only one work by Michael Holst using fitted finite element method:

- M. Holst, R. Szypowski, and Y. Zhu, Two grid methods for semilinear interface problems, Numerical Methods for Partial Differential Equations, 2013.

The main idea of two-grid method is to reduce the solution of a semi-linear elliptic problem on a given fine grid with mesh size h to the solution of the same elliptic problem on a much coarser grid with mesh size $h \ll H$, which can be easily solved as the size of the discrete elliptic problem is significantly smaller than the original elliptic problem on the fine grid, and the solution of a linear problem on the same fine grid, which can be solved by mature and efficient numerical algorithms.

Next, we will present the two-grid algorithms and analyze the convergence accuracy. The fundamental ingredient in these schemes is another immersed finite element space $V_H \subset V_h$ $(h \ll H < 1)$, defined on a coarser uniform triangulation of Ω.

To estimate our two-grid methods, we need the help of a local monotonicity assumption on the nonlinearity. This approach was used in [Long Chen, M. Holst, and J. Xu, SIAM J. Numer. Anal., 2007; M. Holst et al., Commun. Comput. Phys., 2012] for analyzing the convergence of adaptive methods for the Poisson-Boltzmann equation.

Assume f is locally monotone, namely,

$$f'(\xi) \leq 0, \quad \forall \xi \in [u_1, u_2].$$

where u_1 and u_2 are the barriers defined in (7).

For simplicity, let us denote

$$N(u; v, \varphi) = \left(f(u) + f'(u)(v - u), \varphi \right). \tag{35}$$

Two-grid method: Algorithm 1

Now, we present the two-grid method which has two steps as follows.

Algorithm 1

Step 1: On the coarse grid $\mathcal{T}_H$, we solve the nonlinear system (16) *to compute $u_H \in V_H$,*

$$a_H(u_H, v_H) = (f(u_H), v_H), \quad \forall v_H \in V_H. \tag{36}$$

Step 2: On the fine grid $\mathcal{T}_h$, we compute $U_h \in V_h$ to satisfy the following linear system:

$$a_h(U_h, v_h) = N(u_H; U_h, v_h), \quad \forall v_h \in V_h. \tag{37}$$

Algorithm 1 solves the original semi-linear interface problem on the coarse grid $\mathcal{T}_h$, and then performs one Newton iteration on the fine grid.

The remainder term of Algorithm 1

Fix any $\xi_h \in V_h$, let

$$\eta(t) =: a_h(\psi(t), \xi_h) - (f(\psi(t)), \xi_h), \tag{38}$$

with $\psi(t) = u_H + t(u_h - u_H)$.
Then, by Taylor expansion and (16), we have

$$\begin{aligned} 0 =& a_h(u_h, \xi_h) - (f(u_h), \xi_h) = \eta(1) \\ =& \eta(0) + \eta'(0) + \int_0^1 \eta''(t)(1-t)dt \\ =& a_h(u_H, \xi_h) - (f(u_H), \xi_h) + a_h(u_h - u_H, \xi_h) \\ & - (f'(u_H)(u_h - u_H), v_h) + \int_0^1 \eta''(t)(1-t)dt. \end{aligned} \tag{39}$$

By direct calculation, the remainder term, denoted by $R(u_H, u_h, \xi_h)$, has the following form:

$$\begin{aligned} R(u_H, u_h, \xi_h) =: & \int_0^1 \eta''(t)(1-t)dt \\ = & -\int_0^1 \left(f''(\psi(t))(u_h - u_H)^2, \xi_h\right)(1-t)dt. \end{aligned}$$

Therefore, we have the following estimate

$$|R(u_H, u_h, \xi_h)| \leq C_R \|u_h - u_H\|_{0,2p_1}^2 \|\xi_h\|_{0,q_1}, \tag{40}$$

for any p_1, $q_1 \geq 1$ with $1/p_1 + 1/q_1 = 1$, where C_R depends on f, u_1 and u_2.

Lemma 4.1

Let $u_h \in V_h$ be the solutions to (16) on $\mathcal{T}_h$ and $U_h \in V_h$ be the approximated solution obtained by Algorithm 1. Then we have the following estimate

$$\|u_h - U_h\|_{1,h} \leq C(h^4 + H^4)\|u\|_{\widetilde{H}^2(\Omega)}. \tag{41}$$

for some positive constant C depending on C_R, f, u_1 and u_2.

Proof. From Algorithm 1, we have

$$\begin{aligned} & a_h(u_h - U_h, \xi_h) - (f'(u_H)(u_h - U_h), \xi_h) \\ =& a_h(u_h, \xi_h) - (f(u_H), \xi_h) - (f'(u_H)(u_h - u_H), \xi_h) \\ =& -R(u_H, u_h, \xi_h), \quad \forall \xi_h \in V_h. \end{aligned} \tag{42}$$

Then, taking $\xi_h = u_h - U_h$ in (42), we obtain that

$$\begin{aligned} \|u_h - U_h\|_{1,h}^2 \lesssim & a_h(u_h - U_h, u_h - U_h) \\ =& (f'(u_H)(u_h - U_h), u_h - U_h) - R(u_H, u_h, u_h - U_h) \\ \leq & -R(u_H, u_h, u_h - U_h) \\ \leq & C_R \|u_h - u_H\|_{0,2p_1}^2 \|u_h - U_h\|_{1,h}, \end{aligned} \tag{43}$$

where in the third step we used the assumption of f, and in the last step we used (40).

Then, Lemma 4.1 follows immediately from the error estimate $\|u - u_H\|_{L^p}$, $\|u - u_h\|_{L^p}$ and the triangle inequality.

Then, we have the following theorem.

Theorem 4.1

Let $u \in H_0^1$ be the solution of (5), and $U_h \in V_h$ be the solution of Algorithm 1. Then, We have

$$\|u - U_h\|_{1,h} \leq C(h + H^4)\|u\|_{\widetilde{H}^2(\Omega)}. \tag{44}$$

for some positive constant C.

Sketch of proof

Proof. Since the immersed finite element space V_h is nonconforming, by using Lemma 3.1, we have

$$\|u - U_h\|_{1,h} \leq C\left\{ \inf_{v_h \in V_h(\Omega)} \|u - v_h\|_{1,h} + \sup_{\varphi \in V_h(\Omega)} \frac{|a_h(u,\varphi) - N(u_H; U_h, \varphi)|}{\|\varphi\|_{1,h}} \right\}. \tag{45}$$

From (16), (35) and Green's formula, we have

$$\begin{aligned} & a_h(u,\varphi) - N(u_H; U_h, \varphi) \\ & = \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla \varphi dx - \int_\Omega (f(u_H) + f'(u_H)(U_h - u_H))\varphi dx \\ & = (f(u) - f(u_H) - f'(u_H)(U_h - u_H), \varphi) + \sum_{T \in \mathcal{T}_h} \langle \beta \frac{\partial u}{\partial \mathbf{n}}, \varphi \rangle_{\partial T}. \end{aligned} \tag{46}$$

By Taylor expansion,

$$
\begin{aligned}
&(f(u) - f(u_H) - f'(u_H)(U_h - u_H), \varphi) \\
=&(f(u) - f(u_h), \varphi) + (\frac{1}{2} f''(\xi_2)(u_h - u_H)^2, \varphi) \\
&+ (f'(u_H)(u_h - U_h), \varphi) \\
=:& (III)_1 + (III)_2 + (III)_3
\end{aligned}
\tag{47}
$$

where ξ_2 is some function value between u_h and u_H. We need use the error estimates $\|u - u_H\|_{L^4}$, $\|u - u_h\|_{L^p}$ and $\|u_h - U_h\|_{1,h}$ for $(III)_1$, $(III)_2$ and $(III)_3$.

For the last term in (46), arguing as deriving (23), we can deduce

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\langle\beta\frac{\partial u}{\partial\mathbf{n}},\varphi\rangle_{\partial T} &= \sum_{T\in\mathcal{T}_h^*}\sum_{e\subset\partial T}\sum_{s=1}^{2}\langle\beta\frac{\partial u}{\partial\mathbf{n}},\varphi-\gamma_e\varphi\rangle_{e_s}\\ &\le \sum_{T\in\mathcal{T}_h^*}\sum_{e\subset\partial T}\sum_{s=1}^{2}\left\|\beta\frac{\partial u}{\partial\mathbf{n}}\right\|_{1/2,e_s}\|\varphi-\gamma_e\varphi\|_{-1/2,e_s}\\ &\le Ch\|u\|_{\widetilde{H}^2(\Omega)}\|\varphi\|_{1,h}. \qquad (48)\end{aligned}$$

where $e_s=e\cap\Omega_s$, $s=1,2$.

Then, Theorem 4.1 follows immediately from (45), (20), (48) and the previous estimates for $(III)_1$, $(III)_2$ and $(III)_3$. This finishes the proof of Theorem 4.1.

□

Remark 4.1

According to Theorem 4.1, in order to obtain the optimal (or nearly optimal) approximation for the discretization U_h in H^1 norm, it suffices to take $H = h^{1/4}$. To get an idea numerically if the fine mesh size $h = 2^{-24}$ gives $\dim V_h \approx 2.8 \times 10^{14}$, the coarse mesh size H could be $H = 1/64$ which gives $\dim V_H \approx 3.9 \times 10^3$.

Algorithm 1 presented above can be greatly improved if one further correction step is carried out on V_h. The third correction step (which needs very little extra work) improves the accuracy of Algorithm 1. We have the following three step two-grid algorithm.

Two-grid method: Algorithm 2

Algorithm 2

Step 1: On the coarse grid $\mathcal{T}_H$*, we solve the nonlinear system* (16) *to compute* $u_H \in V_H$*,*

$$a_H(u_H, v_H) = (f(u_H), v_H), \quad \forall v_H \in V_H. \tag{49}$$

Step 2: On the fine grid $\mathcal{T}_h$*, we compute* $U_h \in V_h$ *to satisfy the following linear system:*

$$a_h(U_h, v_h) = N(u_H; U_h, v_h), \quad \forall v_h \in V_h. \tag{50}$$

Step 3: On the fine grid $\mathcal{T}_h$*, solving the following linear system for* $e_h \in V_h$*:*

$$a_h(e_h, v_h) = (f'(u_H)e_h + \frac{1}{2}f''(u_H)(U_h - u_H)^2, v_h), \quad \forall v_h \in V_h. \tag{51}$$

Set $u^h = U_h + e_h$*.*

We define the same remainder term as Algorithm 1. We have the following error estimate:

Lemma 4.2

Let $u_h \in V_h$ be the solutions of (16) on $\mathcal{T}_h$ and $u^h \in V_h$ be the approximated solution obtained by Algorithm 2. Then we have the following estimate

$$\|u_h - u^h\|_{1,h} \leq C(h^4 + H^4)\|u\|_{\widetilde{H}^2(\Omega)}. \tag{52}$$

Proof. From Algorithm 2, we have

$$
\begin{aligned}
&a_h(u_h - u^h, \zeta_h) - (f'(u_H)(u_h - u^h), \zeta_h) - (\frac{1}{2} f''(u_H)(U_h - u_H)^2, \zeta_h) \\
=&a_h(u_h, \zeta_h) - (f(u_H), \zeta_h) - (f'(u_H)(u_h - u_H), \zeta_h) \\
=&- R(u_H, u_h, \zeta_h), \quad \forall \zeta_h \in V_h.
\end{aligned} \tag{53}
$$

Then, taking $\zeta_h = u_h - u^h$, we obtain that

$$
\begin{aligned}
\|u_h - u^h\|_{1,h}^2 \lesssim & a_h(u_h - u^h, u_h - u^h) \\
=&(f'(u_H)(u_h - u^h), u_h - u^h) + (\frac{1}{2} f''(u_H)(U_h - u_H)^2, \zeta_h) \\
&- R(u_H, u_h, u_h - u^h) \\
\leq & C\|f\|_{2,\infty}\|U_h - u_H\|_{L^4}^2 \|u_h - u^h\|_{1,h} + C\|u_h - u_H\|_{L^4(\Omega)}^2 \|u_h - u^h\|_{1,h}.
\end{aligned}
$$

We need use the error estimates of $\|u - U_h\|_{1,h}$, $\|u - u_H\|_{L^4(\Omega)}$ and $\|u - u_h\|_{L^4(\Omega)}$. □

Thus, we have the following Theorem.

Theorem 4.2

Let $u \in \widetilde{H}^2(\Omega)$ be the solution of (5), and $u^h \in V_h$ be the solution of Algorithm 2. We have the following estimate

$$\|u - u^h\|_{1,h} \leq C(h + H^6). \tag{54}$$

Sketch of proof

Proof. Following Lemma 3.1, we have

$$\|u - u^h\|_{1,h} \leq C\Bigg\{ \inf_{v_h \in V_h(\Omega)} \|u - v_h\|_{1,h} + \sup_{\zeta_h \in V_h(\Omega)} \frac{|a_h(u, \zeta_h) - N(u_H; u^h, \zeta_h) - (\frac{1}{2} f''(u_H)(U_h - u_H)^2, \zeta_h)|}{\|\zeta_h\|_{1,h}} \Bigg\}. \quad (55)$$

By (18), (17), (50) and (51), we have

$$\begin{aligned} & a_h(u, \zeta_h) - N(u_H; u^h, \zeta_h) - (\frac{1}{2} f''(u_H)(U_h - u_H)^2, \zeta_h) \\ &= \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla \zeta_h dx - \int_\Omega (f(u_H) + f'(u_H)(u^h - u_H) \\ &\quad + \frac{1}{2} f''(u_H)(U_h - u_H)^2) \zeta_h dx \end{aligned}$$

$$
\begin{aligned}
=&(f(u) - f(u_H) - f'(u_H)(u^h - u_H) \\
&- \frac{1}{2} f''(u_H)(U_h - u_H)^2, \zeta_h) + \sum_{T \in \mathcal{T}_h} \langle \beta \frac{\partial u}{\partial \mathbf{n}}, \zeta_h \rangle_{\partial T}.
\end{aligned} \tag{56}
$$

By Taylor expansion,

$$
\begin{aligned}
&(f(u) - f(u_H) - f'(u_H)(u^h - u_H) - \frac{1}{2} f''(u_H)(U_h - u_H)^2, \zeta_h) \\
=&(f(u) - f(u_h), \zeta_h) + (\frac{1}{2} f''(u_H)((u_h - u_H)^2 - (U_h - u_H)^2), \zeta_h) \\
&+ (\frac{1}{6} f'''(\xi_4)(u_h - u_H)^3, \zeta_h) \\
=:& (IV)_1 + (IV)_2 + (IV)_3
\end{aligned} \tag{57}
$$

where ξ_4 is some function value between u_h and u_H.

We use the error estimate $\|u-u_H\|_{L^6}$, $\|u-u_h\|_{L^6}$, $\|u-u_H\|_{L^4}$, $\|u-U_h\|_{1,h}$ and $\|\zeta_h-\gamma_e\zeta_h\|_{-1/2,e}$ for $(IV)_1$, $(IV)_2$, $(IV)_3$ and the last term in (56).

□

Remark 4.2

According to Theorem 4.2, in order to obtain the optimal (or nearly optimal) approximation for the discretization u^h in H^1 norm, it suffices to take $H=h^{1/6}$. To get an idea numerically if the fine mesh size $h=2^{-24}$ gives dim $V_h\approx 2.8\times 10^{14}$, the coarse mesh size H could be $H=1/16$ which gives dim $V_H\approx 225$.

Algorithm 1 presented above can be greatly improved if one further Newton iteration is carried out on V_h. We have the following three step two-grid algorithm.

Two-grid method: Algorithm 3

Algorithm 3

Step 1: On the coarse grid $\mathcal{T}_H$*, we solve the nonlinear system* (16) *to compute* $u_H \in V_H$*,*

$$a_H(u_H, v_H) = (f(u_H), v_H), \quad \forall v_H \in V_H. \tag{58}$$

Step 2: On the fine grid $\mathcal{T}_h$*, we compute* $U_h \in V_h$ *to satisfy the following linear system:*

$$a_h(U_h, v_h) = N(u_H; U_h, v_h), \quad \forall v_h \in V_h. \tag{59}$$

Step 3: On the fine grid $\mathcal{T}_h$*, solving the following linear system for* $\mathcal{U}_h \in V_h$*:*

$$a_h(\mathcal{U}_h, v_h) = N(U_h; \mathcal{U}_h, v_h), \quad \forall v_h \in V_h. \tag{60}$$

Algorithm 3, roughly speaking, is to use the solution of Algorithm 1 as an initial guess for one Newton iteration on the fine grid.

The remainder term of Algorithm 3

Fix any $\chi_h \in V_h$, let

$$\eta_1(t) =: a_h(\psi_1(t), \chi_h) - (f(\psi_1(t)), \chi_h), \tag{61}$$

with $\psi_1(t) = U_h + t(u_h - U_h)$. Then, by Taylor expansion and (16), we have

$$\begin{aligned} 0 =& a_h(u_h, \chi_h) - (f(u_h), \chi_h) = \eta_1(1) \\ =& \eta_1(0) + \eta_1'(0) + \int_0^1 \eta_1''(t)(1-t)dt \\ =& a_h(U_h, \chi_h) - (f(U_h), \chi_h) + a_h(u_h - U_h, \chi_h) \\ & - (f'(U_h)(u_h - U_h), \chi_h) + \int_0^1 \eta_1''(t)(1-t)dt. \end{aligned} \tag{62}$$

By direct calculation, the remainder term, denoted by $R_1(U_h, u_h, \chi_h)$, has the following form:

$$\begin{aligned} R_1(U_h, u_h, \chi_h) &:= \int_0^1 \eta_1''(t)(1-t)dt \\ &= -\int_0^1 (f''(\psi_1(t))(u_h - U_h)^2, \chi_h)(1-t)dt. \end{aligned}$$

Therefore, we have the following estimate

$$|R_1(U_h, u_h, \chi_h)| \leq C_{R_1} \|u_h - U_h\|_{0,2p_2}^2 \|\chi_h\|_{0,q_2}, \tag{63}$$

for any p_2, $q_2 \geq 1$ with $1/p_2 + 1/q_2 = 1$.

Lemma 4.3

Let $u_h \in V_h$ *be the solutions to* (16) *on* $\mathcal{T}_h$ *and* $\mathcal{U}_h \in V_h$ *be the approximated solution obtained by Algorithm 3. Then we have the following estimate*

$$\|u_h - \mathcal{U}_h\|_{1,h} \leq C(h^8 + H^8)\|u\|_{\widetilde{H}^2(\Omega)}. \tag{64}$$

Proof. From Algorithm 3, we have

$$
\begin{aligned}
& a_h(u_h - \mathcal{U}_h, \chi_h) - (f'(U_h)(u_h - \mathcal{U}_h), \chi_h) \\
= & a_h(u_h, \chi_h) - (f(U_h), \chi_h) - (f'(U_h)(u_h - U_h), \chi_h) \\
= & - R_1(u_H, u_h, \chi_h), \quad \forall \chi_h \in V_h.
\end{aligned} \tag{65}
$$

Then, taking $\chi_h = u_h - \mathcal{U}_h$ in (65), we obtain that

$$
\begin{aligned}
\|u_h - \mathcal{U}_h\|_{1,h}^2 \lesssim & a_h(u_h - \mathcal{U}_h, u_h - \mathcal{U}_h) \\
= & (f'(u_H)(u_h - \mathcal{U}_h), u_h - \mathcal{U}_h) - R_1(u_H, u_h, u_h - \mathcal{U}_h) \\
\le & C\|u_h - U_h\|_{1,h}^2 \|u_h - \mathcal{U}_h\|_{1,h},
\end{aligned} \tag{66}
$$

where the positive constant C depends on C_{R_1} and the Poincaré constant.

Then, Lemma 4.3 follows immediately from the error estimate of $\|u_h - U_h\|_{1,h}$. This completes the proof. □

A priori error estimate of Algorithm 3

Thus, we have the following Theorem.

Theorem 4.3

Let $u \in \widetilde{H}^2(\Omega)$ be the solution of (5), and $\mathcal{U}_h \in V_h$ be the solution of Algorithm 3. We have the following estimate

$$\|u - \mathcal{U}_h\|_{1,h} \leq C(h + H^8)\|u\|_{\widetilde{H}^2(\Omega)}. \tag{67}$$

for some positive constant C depending on f, u_1, u_2, C_{R_1} and C_γ.

Sketch of proof

Proof. Following Lemma 3.1, we have

$$\|u - \mathcal{U}_h\|_{1,h} \leq C\Bigg\{ \inf_{v_h \in V_h(\Omega)} \|u - v_h\|_{1,h} + \sup_{\chi_h \in V_h(\Omega)} \frac{|a_h(u, \varphi) - N(U_h; \mathcal{U}_h, \chi_h)|}{\|\chi_h\|_{1,h}} \Bigg\}. \tag{68}$$

By (18), (17) and (60), we have

$$\begin{aligned} & a_h(u, \chi_h) - N(U_h; \mathcal{U}_h, \chi_h) \\ =& \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u \cdot \nabla \chi_h dx - \int_\Omega (f(U_h) + f'(U_h)(\mathcal{U}_h - U_h))\chi_h dx \\ =& (f(u) - f(U_h) - f'(U_h)(\mathcal{U}_h - U_h), \chi_h) + \sum_{T \in \mathcal{T}_h} \langle \beta \frac{\partial u}{\partial \mathbf{n}}, \chi_h \rangle_{\partial T}. \end{aligned} \tag{69}$$

By Taylor expansion,

$$
\begin{aligned}
&(f(u)-f(U_h)-f'(U_h)(U_h-u_H),\chi_h)\\
=&(f(u)-f(u_h),\chi_h)+(\frac{1}{2}f''(\xi_3)(u_h-U_h)^2,\chi_h)\\
&+(f'(u_H)(u_h-\mathcal{U}_h),\chi_h)\\
=&:(V)_1+(V)_2+(V)_3,
\end{aligned}
\tag{70}
$$

where ξ_3 is some function value between u_h and U_h.

We need use error estimates $\|u-u_h\|_{L^2(\Omega)}$, $\|u-U_h\|_{1,h}$, $\|u_h-\mathcal{U}_h\|_{1,h}$ and $\|\chi_h-\gamma_e\chi_h\|_{-1/2,e}$ for $(V)_1$, $(V)_2$, $(V)_3$ and the last term in (69).

□

Remark 4.3

According to Theorem 4.3, in order to obtain the optimal (or nearly optimal) approximation for the discretization U_h in H^1 norm, it suffices to take $H=h^{1/8}$. To get an idea numerically if the fine mesh size $h=2^{-24}$ gives dim $V_h\approx 2.8\times 10^{14}$, the coarse mesh size H could be $H=1/8$ which gives dim $V_H\approx 49$.

Conclusions and future work

Conclusions

- The immersed interface finite element method is used to semi-linear interface problems, and the optimal error estimates in broken H^1 and L^p norm are derived.
- We propose three two-grid methods of the immersed interface finite element solutions for semi-linear interface problems and prove the priori error estimates.

Future work

- We will prove the L^p error estimates of the two-grid methods.
- Numerical experiments
- Two-grid methods for transient advection-diffusion equations with interfaces.
- Two-grid methods for semi-linear parabolic interface problems.

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Thank you!