Error estimates for a energy stable scheme to a Cahn-Hilliard phase-field model for two-phase incompressible flows

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2 Numerical methods
- Cahn-Hilliard (Allen-Cahn) Solver
- Navier-Stokes solver
- Phase-field equation solver

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Phase field for two immiscible fluids

- Phase field (diffusive interface)
- order parameter ϕ

$$\phi = \begin{cases} 1, & \text{fluid1}, \\ -1, & \text{fluid2}. \end{cases} \tag{2.1}$$

- interface $\{x : \phi(x,t) = 0\}$ described by a mixed energy

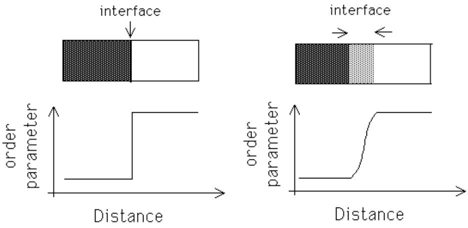

Fig. 1: (a) Sharp interface. (b) Diffuse interface.

Mixing energy

- Assume densities of the fluids are the same ρ_0
- Ginzburg- Landau double-well: $F(\phi) = \frac{(\phi^2-1)^2}{4\varepsilon^2}$, $f(\phi) = F'(\phi)$
- Energy:

$$W(\phi, \nabla\phi) = \lambda \int_\Omega \left(\frac{1}{2} |\nabla\phi|^2 + F(\phi) \right) dx \qquad (2.2)$$

- λ: mixing energy density
- ε: capillary width of interface thickness
- ratio λ/ε is related to surface tension coefficients in the sharp interface limit ($\varepsilon \to 0$)

Phase field equation

- Phase field equation governed by gradient flow

$$\phi_t + (u \cdot \nabla)\phi = -\gamma \frac{\delta W}{\delta \phi} \tag{2.3}$$

where $\frac{\delta W}{\delta \phi}$ in H^{-1} is Cahn-Hilliard:

$$\phi_t + (u \cdot \nabla)\phi = -\gamma \Delta(\Delta\phi - f(\phi)) \tag{2.4}$$

in L^2 is Allen-Cahn

$$\phi_t + (u \cdot \nabla)\phi = \gamma(\Delta\phi - f(\phi)) \tag{2.5}$$

For immiscible fluids, a Lagrangian multiplier introduced in Allen-Cahn for volume conservation $\int_\Omega \phi \equiv const$

Phase field equation

- Cahn-Hilliard Phase field equation for interfacial dynamics of two immiscible fluid

$$
\begin{cases}
\phi_t + u \cdot \nabla\phi - \gamma\Delta w = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \\
w = -\Delta\phi + f(\phi), \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \\
\rho_0(u_t + (u \cdot \nabla)u) - \mu_0\Delta u + \nabla p - \lambda w\nabla\phi = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \\
\nabla \cdot u = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \\
u|_{\partial\Omega} = 0, \dfrac{\partial\phi}{\partial\vec{n}}|_{\partial\Omega} = 0, \dfrac{\partial w}{\partial\vec{n}}|_{\partial\Omega} = 0,
\end{cases}
$$

  with given initial data $u(0) = u_0$, $\phi(0) = \phi_0$, $w(0) = w_0$.
- Ginzburg- Landau double-well: $F(\phi) = \frac{(\phi^2-1)^2}{4\varepsilon^2}$, $f(\phi) = F'(\phi)$
- Splitting of Cahn-Hilliard equation: Elliott, D.A French, and F.A. Milner, 89'
- Allen-Cahn: $\phi_t + u \cdot \nabla\phi + \gamma w = 0$

Numerical methods

- Cahn-Hilliard (Allen-Cahn) Solver + Navier-Stokes solver
- stiffness in the phase field equation, small ε
- Navier-Stokes equation
- mimic the energy law

$$\frac{d}{dt}\int_\Omega (\frac{\rho_0}{2}|u|^2+\frac{\lambda}{2}|\nabla\phi|^2+\lambda F(\phi))dx = -\int_\Omega (\mu_0|\nabla u|^2+\lambda\gamma|\nabla w|^2)dx$$

Cahn-Hilliard Solver

- Cahn-Hilliard equation

$$\begin{cases} \phi_t - \Delta w = 0, \quad \partial_{\vec{n}} w|_{\partial\Omega} = 0, \\ w = -\Delta\phi + f(\phi), \quad \partial_{\vec{n}}\phi|_{\partial\Omega} = 0, \\ f(\phi) = F'(\phi) = \dfrac{(\phi^2 - 1)\phi}{2} \end{cases}$$

- Gradient flow of $E(\phi) = \int_\Omega (\frac{1}{2}|\nabla\phi|^2 + F(\phi))\, dx$ in H^{-1}
- Energy law

$$\partial_t E(\phi(t)) = -\int_\Omega |\nabla(-\Delta\phi + f(\phi))|^2 dx$$

- Numerical method: Elliott, D.A French, and F.A. Milner, 89'; Q. Du and R. Nicolaides, 91'; Chen and Shen, 98'; . . .

Energy stable scheme

- For gradient flow system
  - convex splitting $F(\phi) = F_c(\phi) - F_e(\phi)$ (Eyre 98' ...)
  - semi-implicit, linearly stabilized method (Shen, 98')
  - ETD (Qiao, Ju, Zhang, Li, 17')
  - Invariant energy quadratization (Yang, Shen, Wang, 16',17')
  - Scalar Auxiliary Variable (Yang, Shen, Xu, 17') ...

Convex splitting

- Split $F(\phi)$ into two convex functions, such that

$$F(\phi) = F_c(\phi) - F_e(\phi), \quad F_c''(\phi), F_e''(\phi) \geq 0$$

  where $f_c = F_c'$, $f_e = F_e'$
- First order scheme

$$\begin{cases} \frac{\phi^{n+1}-\phi^n}{\delta t} + \Delta w^{n+1} = 0, \\ w^{n+1} = -\Delta\phi^{n+1} + \frac{1}{\varepsilon^2} f_c(\phi^{n+1}) - \frac{1}{\varepsilon^2} f_e(\phi^n) \\ \partial_{\vec{n}} w^{n+1} = \partial_{\vec{n}}\phi^{n+1} = 0 \end{cases}$$

Convex Splitting continued

Theorem

For first order convex splitting scheme,

$$E(\phi^{n+1}) + \delta t \|\nabla w^{n+1}\|^2 + \frac{1}{2}\|\nabla(\phi^{n+1} - \phi^n)\|^2 \leq E(\phi^n)$$

Stabilized scheme

- Convex splitting unconditional energy stable, but implicit nonlinear term
- Stabilized scheme, explicit nonlinear term (Shen, Yang 07', 10')
- need assumptions on the nonlinear term $f(\phi)$ ($|f'| \leq L$) (truncate the double-well potential as)

$$F(\phi) = \begin{cases} \frac{1}{2\varepsilon^2}(\phi - 1)^2 & \phi > 1 \\ \frac{1}{4\varepsilon^2}(\phi^2 - 1)^2 & \phi \in [-1, 1] \\ \frac{1}{2\varepsilon^2}(\phi + 1)^2 & \phi < -1 \end{cases}$$

Stabilized scheme continued

- Explicit nonlinear term

$$\begin{cases} \frac{\phi^{n+1}-\phi^n}{\delta t} - \Delta w^{n+1} = 0, \\ w^{n+1} - \frac{S}{\varepsilon^2}(\phi^{n+1} - \phi^n) = -\Delta\phi^{n+1} + \frac{1}{\varepsilon^2} f(\phi^n) \end{cases}$$

- For $S \geq L/2$, energy stable:

$$E(\phi^{n+1}) + \delta t \|\nabla w^{n+1}\|^2 \leq E(\phi^n)$$

- stabilized term introduces an error of order $\frac{S\delta t}{\varepsilon^2}\partial_t \phi$

Navier-Stokes solver

- Incompressible N-S

$$\begin{cases} u_t + (u\cdot\nabla)u = \nu\Delta u - \nabla p + g, \\ \text{div } u = 0, \quad u|_{\partial\Omega} = 0 \end{cases}$$

- Projection method for incompressible constraint [1]

[1] Chorin, Temam

Navier-Stokes solver

- First order projection method
- intermediate velocity

$$\begin{cases} \frac{\tilde{u}^{n+1}-u^n}{\delta t} + (u^n \cdot \nabla)\tilde{u}^{n+1} = \nu\Delta\tilde{u}^{n+1} - \nabla p^n + g(t_{n+1}), \\ \tilde{u}|_{\partial\Omega} = 0 \end{cases}$$

- projection step

$$\begin{cases} \frac{u^{n+1}-\tilde{u}^{n+1}}{\delta t} + \nabla(p^{n+1} - p^n) = 0, \\ \text{div}(u^{n+1}) = 0, \quad \vec{n} \cdot u^{n+1}|_{\partial\Omega} = 0 \end{cases}$$

- u^{n+1} and $\tilde{u}^{n+1}$ can be in different spaces

Navier-Stokes solver

- The projection scheme is unconditional stable in the sense that

$$\|u^{n+1}\|^2 + \nu\delta t \sum_{k=0}^{n} \|\nabla \tilde{u}^{k+1}\|^2 \leq \|u^0\|^2 + C\|f\|_{C(0,T;H^{-1})}$$

- First order accurate for velocity in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, pressure in $L^2(0, T; L^2)$

Cahn-Hilliard phase field solver

- First order stabilized scheme

$$\begin{cases} \frac{\phi^{n+1}-\phi^{n}}{\delta t} + (\tilde{u}^{n+1}\cdot\nabla)\phi^{n} - \gamma\Delta w^{n+1} = 0, \\ w^{n+1} - \frac{S}{\epsilon^2}(\phi^{n+1}-\phi^{n}) = -\Delta\phi^{n+1} + f(\phi^{n}), \\ \rho_0(\frac{\tilde{u}^{n+1}-u^{n}}{\delta t} + (u^{n}\cdot\nabla)\tilde{u}^{n+1}) - \mu_0\Delta\tilde{u}^{n+1} + \nabla p^{n} - \lambda w^{n+1}\nabla\phi^{n} = 0, \\ \frac{\partial\phi^{n+1}}{\partial\vec{n}}|_{\partial\Omega} = 0, \tilde{u}^{n+1}|_{\partial\Omega} = 0, \end{cases}$$

- Pressure correction

$$\begin{cases} \rho_0(\frac{u^{n+1}-\tilde{u}^{n+1}}{\delta t}) + \nabla(p^{n+1}-p^{n}) = 0 \\ \nabla\cdot u^{n+1} = 0, \vec{n}\cdot u^{n+1}|_{\partial\Omega} = 0 \end{cases}$$

Unconditional stability

- Energy stable in the sense that

$$\left[\frac{\rho_0}{2}\|u^{n+1}\| + \frac{\lambda}{2}\|\nabla\phi^{n+1}\|^2 + \lambda(F(\phi^{n+1}),1)\right] + \frac{\delta t^2}{2\rho_0}\|\nabla p^{n+1}\|^2 + \mu_0\delta t\|\nabla\tilde{u}^{n+1}\|^2 + \lambda\gamma\delta t\|\nabla w^{n+1}\|^2 \le \left[\frac{\rho_0}{2}\|u^n\|^2 + \frac{\lambda}{2}\|\nabla\phi^n\|^2 + \lambda(F(\phi^n),1)\right] + \frac{\delta t^2}{2\rho_0}\|\nabla p^n\|^2$$

Error estimates

- For Cahn-Hilliard/ Allen Cahn
  - Convex splitting/ semi-implicit/...
  - FEM/Spectral/Finite difference...
  - First order/ second order/... in time
- For Navier stokes equation
  - Projection method (Shen, Guermond, Minev 07')
  - FEM/Spectral method...
  - First order/ second order in time

Small ε

- For pure Cahn-Hilliard (Allen-Cahn) equation with small ε, the dependence can be reduced to polynomial[2]
- Key point: spectral gap
  - Let ϕ solve Cahn-Hilliard equation $\phi_t = -\Delta(\varepsilon\Delta\phi - \frac{1}{\varepsilon}f(\phi))$ linearized Cahn-Hilliard operator: $L_{CH} = \Delta(\varepsilon\Delta - \frac{1}{\varepsilon}f'(\phi)I)$

$$\lambda_{CH} = \inf_{0\neq\psi\in H^1(\Omega),\Delta\psi=w} \frac{\varepsilon\|\nabla\psi\|_{L^2}^2 + \frac{1}{\varepsilon}(f'(\phi)\psi,\psi)}{\|\nabla w\|_{L^2}^2} \geq -C_0$$

  - holds for ϕ with special profile (X. Chen,1994), solutions to Cahn-Hilliard equation

[2] X. Feng, et. al.

Phase-field equations

- Allen-Cahn-Navier-Sotkes (Feng 06') (First order in time)
- Cahn-Hilliard-Navier-Sotkes (Feng, He, Liu 07') first order in time
- Cahn-Hilliard-Navier-Sotkes (Diegel, Wang, Wang, Wise 17') second order in time
- Variable density, (Grun 13')
- Pressure is coupled with other variables in the scheme
- Goal: error analysis with projection type decoupled scheme

FEM discretization of phase-field equation

- $$X = H_0^1(\Omega)^d, \qquad \tilde{X} = \left\{\psi \in H^1(\Omega), \partial_{\vec{n}}\psi|_{\partial\Omega} = 0\right\},$$
  $$M = L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q = 0\right\}.$$
- Incompressible vector field
  $$H = \left\{v \in L^2(\Omega)^d; \nabla \cdot v = 0;\ v \cdot \vec{n}|_{\partial\Omega} = 0\right\},$$
  $$V = \left\{v \in H^1(\Omega)^d; \nabla \cdot v = 0;\ v|_{\partial\Omega} = 0\right\},$$
  and it holds
  $$L^2(\Omega)^d = H \oplus \nabla(H^1(\Omega)).$$
- $\mathcal{T}_h$ be a regular, quasi-uniform triangulation of Ω of mesh size $0 < h < 1$ and $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} \overline{K}$
- denote $P_r(K)$ as the space of polynomials of degree less than or equal to r on K.

FEM spaces

- Introduce mixed finite element space X_h, M_h as conformal approximations of X and M respectively, based on the triangulation $\mathcal{T}_h$. Define Y_h a finite dimensional subspace of $L^2(\Omega)^d$, and we assume that either Y_h is conformal in

$$H_0^{\text{div}}(\Omega) = \left\{ v \in L^2(\Omega)^d, \quad \nabla \cdot v \in L^2(\Omega), \quad v \cdot \vec{n}|_{\partial\Omega} = 0 \right\}$$

  or M_h is conformal in $H^1(\Omega)$.
- In particular, we assume $X_h \subseteq Y_h$ for simplicity. Define Φ_h a fine dimensional subspace of $H^1(\Omega)$ and we assume that $1 \in \Phi_h$.
- A possible choice (X. Feng et al., 2007)

$$X_h = Y_h = \{v_h \in [C^0(\overline{\Omega})]^d \cap [H_0^1(\Omega)]^d; v_h|_K \in [P_2(K)]^d\},$$
$$V_h = \{v_h \in X_h; (\text{div} v_h, q_h) = 0. \quad \forall q_h \in M_h\},$$
$$M_h = \{q_h \in L_0^2(\Omega); q_h|_K \in P_0(K)\}, \quad \Phi_h = \{\psi_h \in C^0(\overline{\Omega}); \psi_h|_K \in P_2(K)\}.$$

Find $\{(u_h^n, \tilde{u}_h^n, \phi_h^n, w_h^n, p_h^n)\}_{n=1}^N \in Y_h \times X_h \times \Phi_h \times \Phi_h \times M_h$ such that

$$
\begin{cases}
\dfrac{\phi_h^{n+1} - \phi_h^n}{\delta t} + P_{\Phi_h}\left((\tilde{u}_h^{n+1} \cdot \nabla)\phi_h^n\right) + \gamma \tilde{A}_h w^{n+1} = 0, \\
w_h^{n+1} - \dfrac{S}{\epsilon^2}(\phi_h^{n+1} - \phi_h^n) = \tilde{A}_h \phi_h^{n+1} + P_{\Phi_h} f(\phi_h^n), \\
\dfrac{\rho_0}{\delta t}\left(\tilde{u}_h^{n+1} - i_h^T u_h^n\right) + \rho_0 P_{X_h}\left((u_h^n \cdot \nabla)\tilde{u}_h^{n+1}\right) + \mu_0 A_h \tilde{u}_h^{n+1} - B_h^T p_h^n \\
\qquad\qquad -\lambda P_{X_h}\left(w_h^{n+1} \nabla \phi_h^n\right) = 0, \\
\dfrac{\rho_0}{\delta t}(u_h^{n+1} - i_h \tilde{u}_h^{n+1}) + C_h^T(p_h^{n+1} - p_h^n) = 0, \\
C_h u_h^{n+1} = 0,
\end{cases}
$$

Stability

Theorem

The fully discrete scheme is unconditionally stable and satisfies the following discrete energy law:

$$\left[\frac{\rho_0}{2}\|u_h^{n+1}\|^2 + \frac{\lambda}{2}\|\nabla\phi_h^{n+1}\|^2 + \lambda(F(\phi_h^{n+1}),1)\right] + \frac{\delta t^2}{2\rho_0}\|C_h^T p_h^{n+1}\|^2 + \mu_0\delta t\|\nabla\tilde{u}_h^{n+1}\|^2$$

$$+\lambda\gamma\delta t\|\nabla w_h^{n+1}\|^2 \le \left[\frac{\rho_0}{2}\|u_h^{n}\|^2 + \frac{\lambda}{2}\|\nabla\phi_h^{n}\|^2 + \lambda(F(\phi_h^{n}),1)\right] + \frac{\delta t^2}{2\rho_0}\|C_h^T p_h^{n}\|^2$$

Convergence

Theorem

Assume initial data $(\phi_0, u_0) \in H^1(\Omega) \times V$ *and* $(\phi_h^0, w_h^0, u_h^0, p_h^0)^T$ *is initialized such that*
$\frac{\rho_0}{2}\|u_h^0\|^2 + \frac{\lambda}{2}\|\phi_h^0\|^2 + \lambda(F(\phi_h^0), 1) + \frac{\delta t^2}{\rho_0}\|C_h^T p_h^0\|^2 \leq c_0.$
Furthermore, we require $h^2 \lesssim \delta t$ *if* $Y_h \neq X_h$. *Let* (u, ϕ, w, p) *denote the unique solution of NSCH system. Then there hold*

$$\lim_{h,\delta t \to 0} \left(\|\overline{U}_{h,\delta t} - u\|_{L^2(L^2)} + \|\overline{\Phi}_{h,\delta t} - \phi\|_{L^2(L^2)}\right) = 0,$$

$$\overline{W}_{h,\delta t} \to w, \quad \text{weakly} \quad \text{in} \quad L^2([0.T]; H^1(\Omega)),$$

$$\overline{P}_{h,\delta t} \to p, \quad \text{weakly} \quad \text{in} \quad L^{\frac{12}{6+d}}([0, T]; L^2(\Omega)).$$

- Estimates

$$\max_{0\le n\le N}\left\{\|u_h^n\|^2+\|\nabla\phi_h^n\|^2+(F(\phi_h^n),1)+\delta t^2\|C_h^T p_h^n\|^2\right\}\le c,$$

$$\sum_{n=0}^{N-1}\left(\|\delta_t\phi_h^n\|_1^2+\|\tilde u_h^{n+1}-i_h^T u_h^n\|^2+\|u_h^{n+1}-i_h i_h^T u_h^{n+1}\|^2+\|\delta_t u_h^n\|^2+\|\delta_t\tilde u_h^n\|^2\right)\le c,$$

$$\delta t\sum_{n=0}^{N}\left(\|\nabla w_h^n\|^2+\|\nabla\tilde u_h^n\|^2\right)\le c,$$

$$\max_{0\le n\le N}\left\{\|\phi_h^n\|^2+\|\tilde u_h^n\|^2\right\}+\delta t\sum_{n=0}^{N}\|w_h^n\|^2\le c,$$

$$\delta t\sum_{n=0}^{N-1}\left\|\frac{1}{\delta t}\delta_t i_h^T u_h^n\right\|_{-1}^{\frac{12}{6+d}}\le c\left(1+\left(\frac{h^2}{\delta t}\right)^{\frac{6}{d}}\right),$$

$$\delta t\sum_{n=0}^{N}\|p_h^n\|^{\frac{12}{6+d}}\le c\left(1+\left(\frac{h^2}{\delta t}\right)^{\frac{6}{d}}\right),$$

$$\delta t\sum_{n=0}^{N-1}\left\|\frac{1}{\delta t}\delta_t\tilde u_h^n\right\|_{-1}^{\frac{12}{6+d}}\le c\left(1+\left(\frac{h^2}{\delta t}\right)^{\frac{6}{d}}\right),$$

$$\delta t\sum_{n=0}^{N-1}\left\|\frac{1}{\delta t}\delta_t\phi_h^n\right\|_{-1}^2\le c,$$

- If Y_h is chosen as $Y_h=X_h$, then

$$\frac{1}{\delta t}\sum_{n=0}^{N-1}\|\delta_t i_h^T u_h^n\|_{-1}^{\frac{12}{6+d}}+\frac{1}{\delta t}\sum_{n=0}^{N-1}\|\delta_t\tilde u_h^n\|_{-1}^{\frac{12}{6+d}}+\delta t\sum_{n=0}^{N}\|p_h^n\|^{\frac{12}{6+d}}\le c.$$

Convergence order

Theorem

For sufficiently small δt, the finite element approximate solution satisfies

$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(L^2)} + \|u - u_{h,\delta t}\|_{l^\infty(L^2)} + \|u - \tilde{u}_{h,\delta t}\|_{l^\infty(L^2)} \lesssim \delta t + h^l,$$
$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(H^1)} + \|u - \tilde{u}_{h,\delta t}\|_{l^2(H^1)} + \|w - w_{h,\delta t}\|_{l^2(H^1)} \lesssim \delta t + h^l,$$
$$\|p - p_{h,\delta t}\|_{l^2(L^2(\Omega))} \lesssim (1 + h/\sqrt{\delta t})(\delta t + h^l).$$

If $X_h = Y_h$, the error on pressure becomes

$$\|p - p_{h,\delta t}\|_{l^2(L^2(\Omega))} \lesssim \delta t + h^l.$$

Lemma

For sufficiently small δt, the finite element approximate solution satisfies

$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(L^2)} + \|u - u_{h,\delta t}\|_{l^\infty(L^2)} + \|u - \tilde{u}_{h,\delta t}\|_{l^\infty(L^2)} \lesssim \delta t + h^l,$$
$$\|\phi - \phi_{h,\delta t}\|_{l^\infty(H^1)} + \|w - w_{h,\delta t}\|_{l^2(H^1(\Omega))} + \|u - \tilde{u}_{h,\delta t}\|_{l^2(H^1)} \lesssim \delta t + h^l.$$

By energy type computation for the error functions.

$$\|\delta_t e_u^n\| \lesssim \delta t + h^l, \quad \left(\delta t \sum_{k=0}^{[T/\delta t]-1} \|\delta_t e_u^k\|^2\right)^{1/2} \lesssim \delta t^{1/2}(\delta t + h^l).$$

Estimates for pressure

- Inf-sup condition and Inverse Stokes operator (time increment)
- $|v|_{*,h} = \sup\limits_{v_h \in X_h} \frac{\langle v, v_h \rangle}{\|v_h\|_1}$
- $\mathcal{S} : H^{-1}(\Omega)^d \to V$ as follows. For $v \in H^{-1}(\Omega)^d$, $(\mathcal{S}(v), r) \in V \times L_0^2(\Omega)$

$$\begin{cases} (\nabla \mathcal{S}(v), \nabla w) - (r, \nabla \cdot w) = \langle v, w \rangle, & \forall w \in H_0^1(\Omega)^d, \\ (q, \nabla \cdot \mathcal{S}(v)) = 0, & \forall q \in L_0^2(\Omega), \end{cases} \tag{4.1}$$

- $|v|_{*,h} \leq c(\|\nabla \mathcal{S}_h(v)\| + h\|v\|)$
  if $X_h = Y_h$, $|v|_{*,h} \leq c\|\nabla \mathcal{S}_h(v)\|$

Estimates for pressure...

- Instead of improving L^2 norm of time increment $\delta_t e_u^n$, use

$$\|\nabla \mathcal{S}_h(\delta_t e_u^n)\| \lesssim \delta t^{1/2}(\delta t + h^l), \qquad \left(\delta t \sum_{n=0}^{N} \|\nabla \mathcal{S}_h(\delta_t e_u^n)\|\right)^{1/2} \lesssim \delta t(\delta t + h^l)$$

- By inf-sup condtion

$$\|e_p^{n+1}\| \leq c \sup_{v_h \in X_h} \frac{\langle B_h^T e_p^{n+1}, v_h \rangle}{\|v_h\|_1} \leq c(\frac{1}{\delta t} |i_h^T \delta_t e_u^n|_{*,h} + \ldots)$$
$$\leq c \left(\frac{1}{\delta t} \|\nabla \mathcal{S}_h(\delta_t e_u^n)\| + \frac{h}{\delta t} \|\delta_t e_u^n\| \right)$$

conclusion

- Cahn-Hilliard (Allen-Cahn) phase field equation
- energy stable time discrete method
- FEM discretization with pressure decoupled
- Convergence and order of convergence

THANK YOU!