LS Augmented Methods for Fluid and Porous Media Couplings

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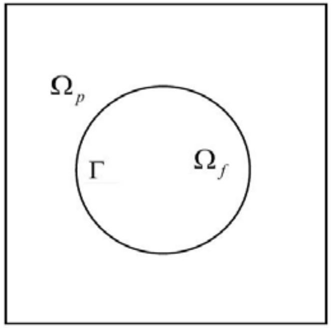
Key Words & Related to Others

- IB (Immersed Boundary) →IIM (Immersed Interface) 1st to 2nd velocity & pressure, *e.g.*, incompressible vesicle in incompressible fluid (2D, CAF). *Smoothing method* →***sharp interface method***
- Cartesian meshes & Fast Helmoltz/Poisson solvers
- Efficiency and accuracy; ***augmented IIM***
- Applications: Fluid and Porous media; moving contact lines (E, Ren, Li)
- AMS classification: ***65N***, …

Outline

- ***Introduction/Problem:***
  - **Fluid and porous media (Stokes/Darcy or *Navier-Stokes/Darcy*)**
  - Different governing equations on different regions
  - Interface conditions (including BJ or BJS)



- ***Augmented IIM for Stokes-Darcy coupling***
  - Stokes/Darcy → *three Poisson Solvers*
- ***Least squares augmented IIM for NS-D coupling***
  - Pressure incremental formulation
  - Least squares: more than augmented variables, why?
- ***Moving contact line*** sharp interface simulations (E/Ren's model)
- ***Numerical experiments & Conclusion***
  - *Analytic, flow problems, corners, …*

Fluid and Darcy Coupling

- Fluid flow: Stokes eqns or ***N-S eqns***

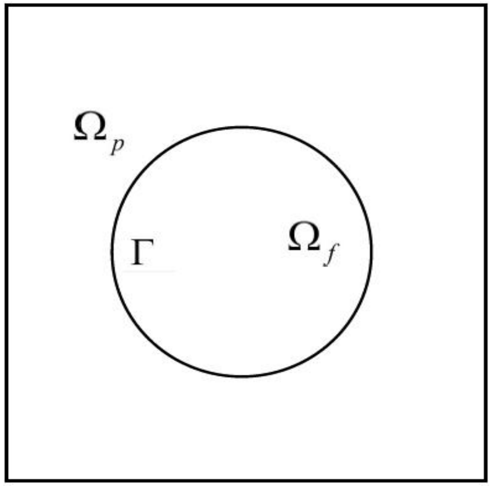
$$\nabla p = \nabla \bullet \mu(\nabla u + \nabla u^T) + g$$

$$\nabla \bullet u = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) + \nabla p = \mu \Delta u + g$$

$$\nabla \bullet u = 0$$

Navier-Stokes/Darcy



- Porous media flow: Darcy's law

$$u = -\frac{K}{\mu} \nabla p$$

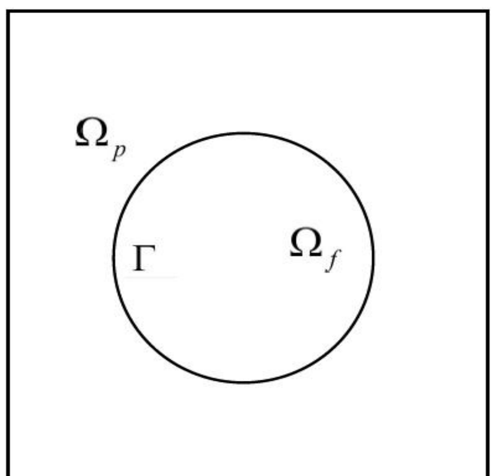
$$\nabla \bullet u = 0 \quad \text{or} \quad \nabla \bullet u = \phi$$

Interface Conditions

$$u_s \cdot n = u_D \cdot n \quad \text{(normal velocity is continuous)}$$

$$[p] = p_s - p_D = 2\mu (n \cdot T_s \cdot n), \quad T_s = (\nabla u_s + \nabla u_s^{\,T})/2$$

$$n \cdot T_s \cdot \tau = -\frac{\sqrt{K}}{\mu\alpha}(u_s - u_D)\cdot \tau \;\; \text{(BJ) or } -\frac{\sqrt{K}}{\mu\alpha} u_s \cdot \tau \;\; \text{(BJS)}$$



Applications

- Flows across interfaces between soil and surface
- Oil reservoir
- Bio-medicine & *cell deformation*
- Blood motion in lungs, solid tumors and vessels
- Heat transfer in walls with fibrous insulation (firefighter)

Literature Review

- Regularity: W. Layton (SINUM, 40, 2003) *et al.*
- Numerical methods
  - FEM with domain decomposition, X. He, M. Gunzburg,…

    $$\lambda_D K \nabla p_D \cdot n + g p_D = \eta_D$$

    Robin-Robin domain decomposition
  - Fictitious domain approach: P. Sun, … the interface conditions are built into the weak form & *affects convergence order, larger system of saddle point system*
  - Phase field method

Literature Review II

- ***Cartesian mesh*** method (BEM & Stokelet by R. Cortetz for a circular interface)
- Cartesian method with local modified mesh (Z. Wang/Li, 2015), free-fem
- Almost no ***non-trivial analytic solutions*** with curved interfaces
- Most simulations are for straight interfaces.

The weak form

Assume ϕ_p and $\mathbf{u}_f$ are 0 on the boundary $\partial\Omega$ and define the following functional spaces

$$
\begin{aligned}
H_f &= \{\mathbf{v}_f \in (H^1(\Omega_f))^d \,|\, \mathbf{v}_f = 0 \text{ on } \partial\Omega_f \backslash \Gamma\}, && (8)\\
Q &= L^2(\Omega_f), && (9)\\
H_p &= \{\psi_p \in H^1(\Omega_p) \,|\, \psi_p = 0 \text{ on } \partial\Omega_p \backslash \Gamma\}. && (10)
\end{aligned}
$$

The following bilinear forms are defined as

$$
\begin{aligned}
a_f(\mathbf{u}_f, \mathbf{v}_f) &= 2\nu\left(\frac{1}{2}(\nabla\mathbf{u}_f + \nabla^T\mathbf{u}_f), \frac{1}{2}(\nabla\mathbf{v}_f + \nabla^T\mathbf{v}_f)\right) \quad \text{on } \Omega_f, && (11)\\
a_p(\phi_p, \psi_p) &= (\mathbf{K}\nabla\phi_p, \nabla\psi_p) \qquad \text{on } \Omega_p, && (12)\\
b_f(\mathbf{v}_f, p_f) &= -(\nabla\cdot\mathbf{v}_f, p_f) \qquad \text{on } \Omega_f. && (13)
\end{aligned}
$$

Idea for Stokes-Darcy coupling

- **Step 1**: Get Poisson equation for the pressure.

$$\nabla p = \nabla \bullet \mu(\nabla u + \nabla u^T) + g;$$

$$\nabla \bullet u = 0$$

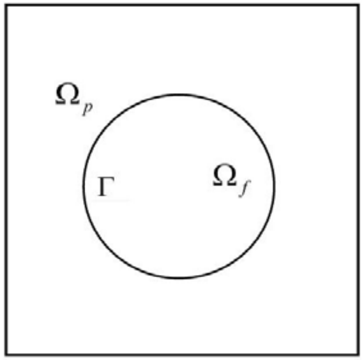
$$u = -\frac{K}{\mu}\nabla p$$

$$\nabla \bullet u = 0$$

$$\Delta p = \nabla g \qquad \text{in Stokes}$$

$$\Delta p = 0 \qquad \text{in Darcy}$$

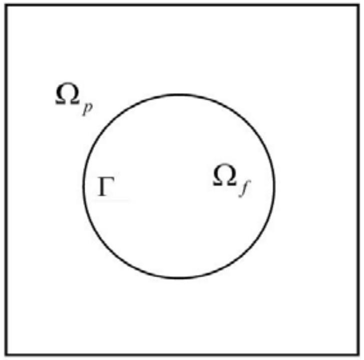
$$[p] = q_1, \qquad [p_n] = q_2, \quad \text{(new unknown)}$$



$$\Delta p = f(x) + \int_\Gamma q_2(s)\delta(x - X(s))\,ds + \int_\Gamma q_1(s)\delta'(x - X(s))\,ds$$

Idea for Stokes-Darcy coupling, II

- Step 2: Solve for the velocity



$$\nabla p = \nabla \bullet \mu(\nabla u + \nabla u^T) + g \qquad u = -\frac{K}{\mu}\nabla p$$

$$\nabla \bullet u = 0 \qquad \nabla \bullet u = 0$$

$$\Delta u = \begin{cases} (p_x - g_1)/\mu \\ -K\Delta p_x/\mu \end{cases}; \quad \Delta v = \begin{cases} (p_y - g_2)/\mu \\ -K\Delta p_y/\mu \end{cases},$$

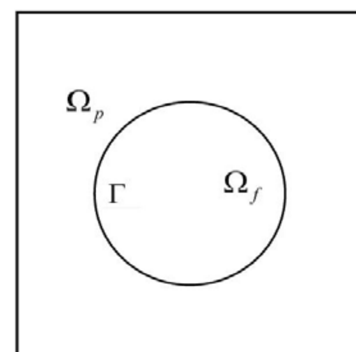
$$[u_n] = q_3, \quad [v_n] = q_4, \quad \text{(new unknown)}$$

$$[u \cdot n] = 0, \quad [u \cdot \tau] = q_5 \Rightarrow \text{ get } [u] \ \& \ [v]$$

Idea for Stokes-Darcy coupling III

- Introduce 5 (or 6) interface variables (from the primary variables), we get ***three Poisson equations*** for the pressure and velocity (***decoupled*** the PDEs)
- interface (augmented) variables

$$[p] = q_1, \quad [p_n] = q_2,$$
$$[u_n] = q_3, \quad [v_n] = q_4,$$
$$[u_\tau] = q_5$$



Other Interface Conditions

- We set up ***5*** augmented variables, q_1-q_5, we need 5 augmented equations to close the system

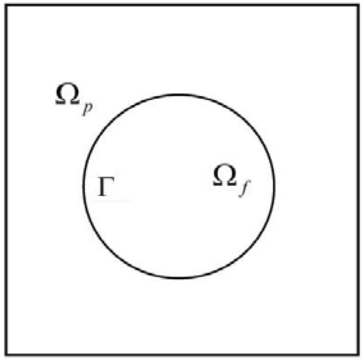
$$[p] = 2\mu(n \cdot T_s \cdot n)$$

$$u_s \cdot n = u_D \cdot n$$

$$n \cdot T_s \cdot \tau = -\frac{\sqrt{K}}{\mu\alpha}(u_s - u_D) \cdot \tau \quad \text{or} \quad -\frac{\sqrt{K}}{\mu\alpha} u_s \cdot \tau$$

$$\frac{\partial p_s}{\partial n} = \mu \Delta u_s \cdot n + g \cdot n$$

$$\frac{\partial p_D}{\partial n} = -\frac{\mu}{K} u_D \cdot n$$



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Outer Boundary Conditions

Darcy-Stokes (or Navier-Stokes)

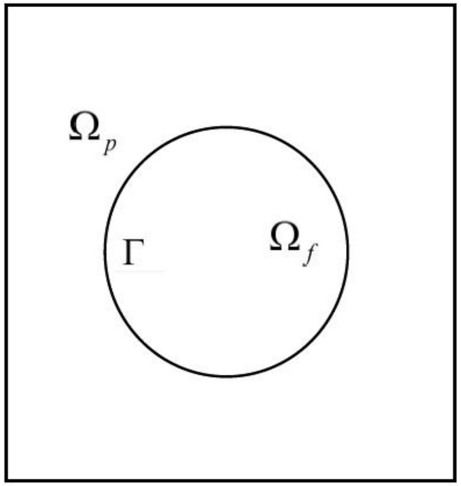
$$u_D \cdot n \ \text{ is given,} \quad \frac{\partial p_D}{\partial n} = -\frac{u_D \cdot n}{K} \ \text{ on } \partial\Omega$$

$$u_D = -K \nabla p_D, \quad \text{Dirichlet BC for } (u, v) \ \text{ on } \partial\Omega$$

Stokes (or Navier-Stokes)-Darcy

$$u_S \ \text{ is given,} \quad \frac{\partial p_S}{\partial n} = \mu \Delta u_S \cdot n + F \cdot n \ \text{ on } \partial\Omega$$

$$\text{set } \ \frac{\partial p_S}{\partial n} = q_6 \quad \text{another augmented variable}$$



Discretization & Schur Complement

- Discretization of ***three Poisson Eqns*** with jump conditions (linear problem)

$$AU + BQ = F$$

- Discretization of the *physical interface conditions*

$$CU + DQ = F_2$$

- Q is along the boundary ***O(5N)***
- Schur Complement (direct or *GMRES*)

$$(D - C A^{-1} B)Q = F_2 - C A^{-1}F, \quad SQ = F_3$$

Matrix-vector multiplication

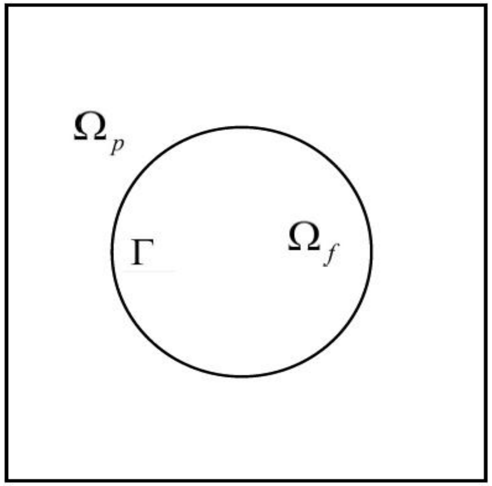
- It is easy to get the matrix multiplication for a given augmented vector ***Q*** to get ***SQ***
  - Solve three Poisson equations $AU + BQ = F$
  - Find the residual of interface conditions

    $R(Q)=CU + DQ - F_2$ ***via some interpolation***
- LU (SVD) or GMRES
  - Fixed interface, time independent, form the matrix S and use LU (SVD)
  - Moving interface *GMRES + precondioning*

Validation for Stokes/Darcy

- It is challenging to construct non-trivial analytic soln for curved interface (***r=1***). In Darcy region, ***u=0, v=0, p=1, with slip jump***. In fluid:

$$
\begin{aligned}
u(x,y) &= y(x^2+y^2-1)-2y,\\
v(x,y) &= -x(x^2+y^2-1)+2x,\\
p(x,y) &= x^2+y^2,\\
F_1(x,y) &= -8y+2x\\
F_2(x,y) &= 8x+2y,
\end{aligned}
$$



Validation for Stokes/Darcy II

- Average convergence rate: 1.8334 & 2.1388

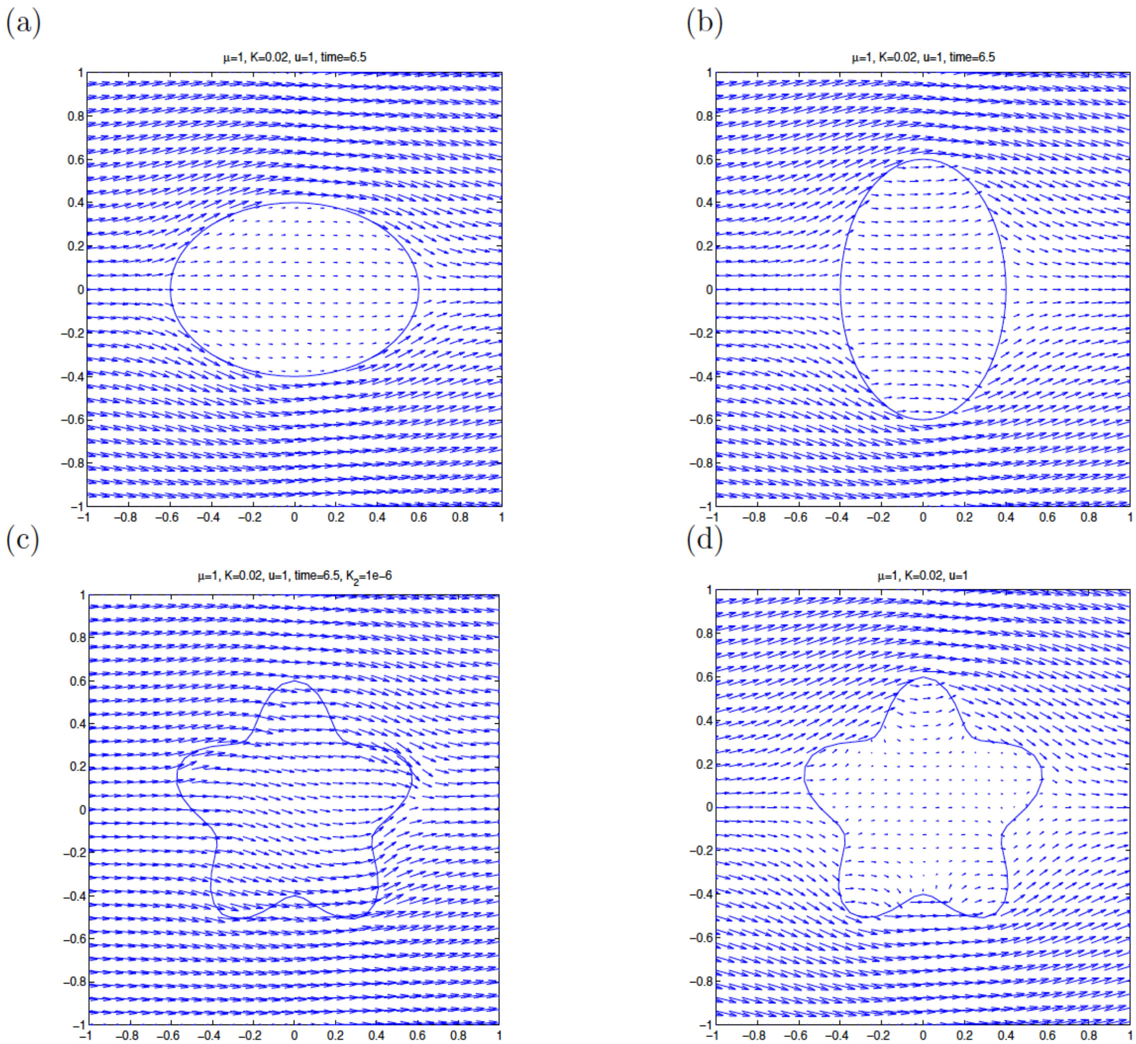
$$\mathbf{n} = [x,\ y]^T, \quad \boldsymbol{\tau} = [-y,\ x]^T, \quad p_s = 1, \quad p_D = 1, \quad u_s = -2y, \quad v_s = 2x,$$

$$u_D = 0, \quad v_D = 0, \quad \frac{\partial u_s}{\partial n} = 0, \quad \frac{\partial v_s}{\partial n} = 0, \quad \frac{\partial u_s}{\partial \boldsymbol{\tau}} = -2x, \quad \frac{\partial v_s}{\partial \boldsymbol{\tau}} = -2y,$$

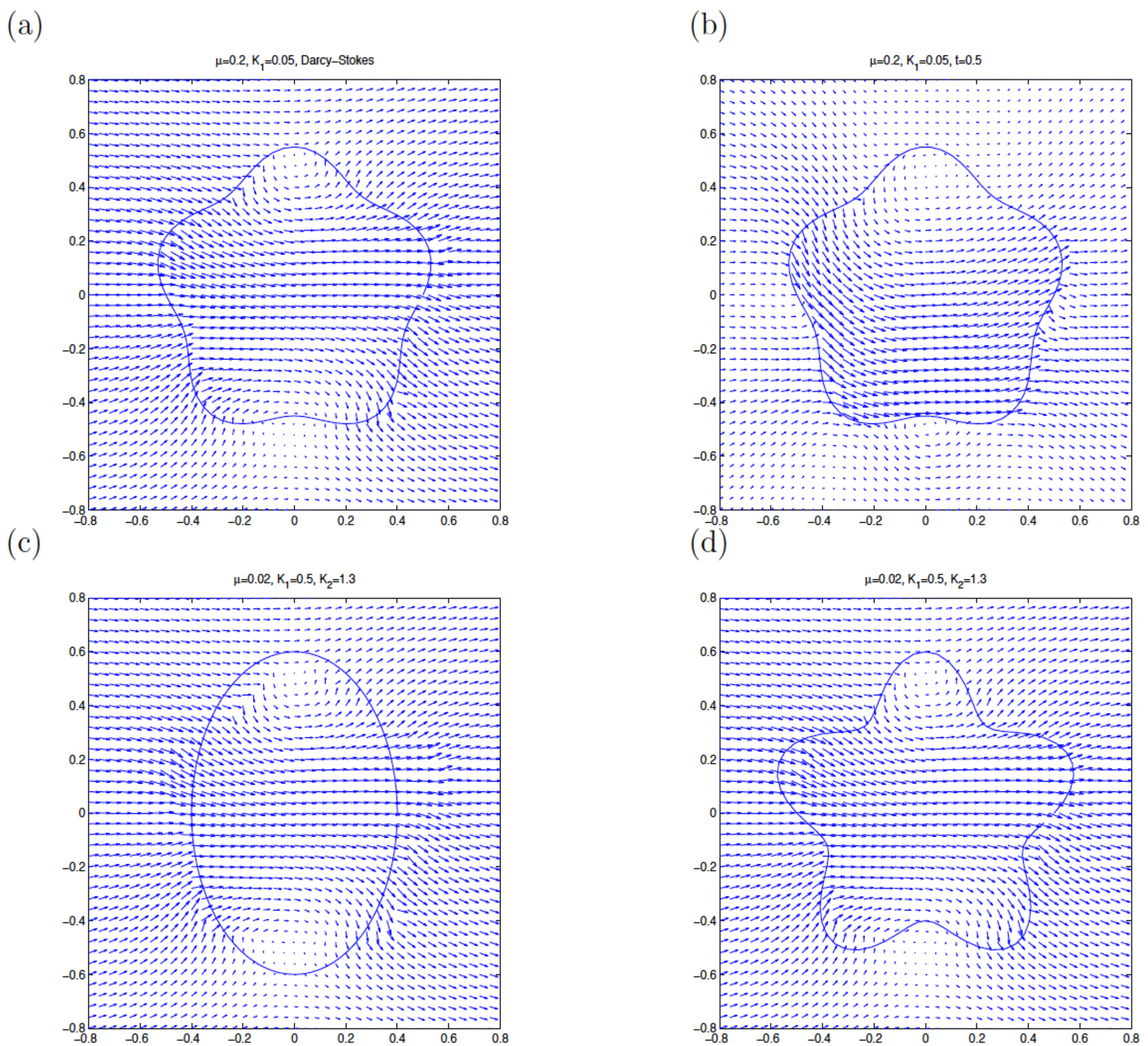
$$\mathbf{D}_s\mathbf{n}\cdot\mathbf{n} = 0 \qquad \mathbf{D}_s\boldsymbol{\tau}\cdot\mathbf{n} = -2.$$

N	$\|E_p\|_\infty$	$order$	$\|E_u\|_\infty$	order
16	$2.2299e-01$		$2.8516e-01$	
32	$2.7892e-02$	2.9991	$3.2203e-02$	3.1465
64	$7.8690e-03$	1.8256	$1.4188e-02$	1.1825
128	$6.0515e-03$	0.37889	$2.6563e-03$	2.4172
256	$3.2848e-03$	0.88149	$1.2963e-03$	1.0350
512	$9.6182e-04$	1.7720	$2.5613e-04$	2.3395

Flow Test I


(a)
μ=1, K=0.02, u=1, time=6.5
(b)
μ=1, K=0.02, u=1, time=6.5
(c)
μ=1, K=0.02, u=1, time=6.5, K2=1e−6
(d)
μ=1, K=0.02, u=1

Flow Test (fluid inside)



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Flow tests: Stokes-Darcy

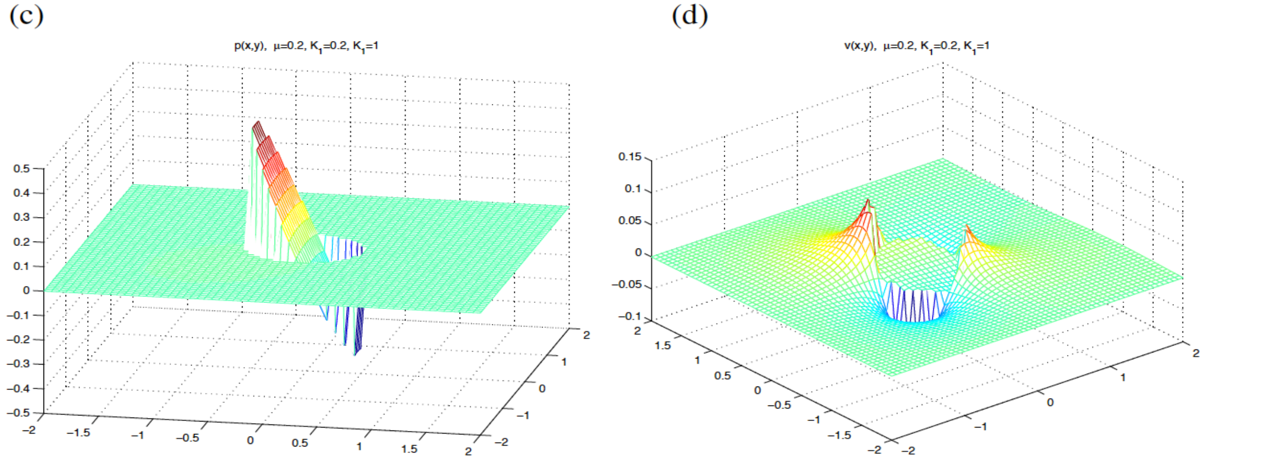
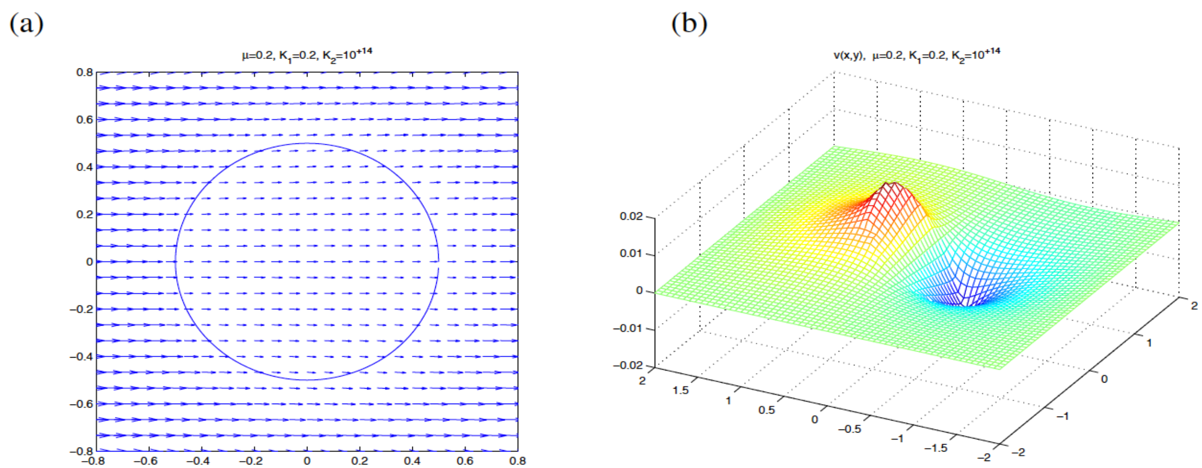


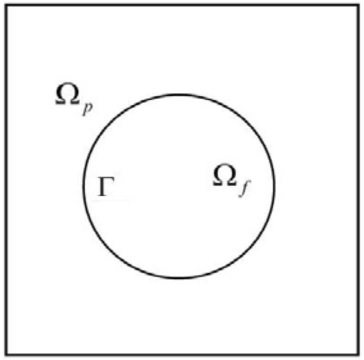
Figure 2. (a)-(b): Velocity plot of a porous media and a fluid (outside the interface $x^2+y^2=0.5^2$) interaction with different parameters. (a): $K_1=1$, $\mu=1$, $K_2=1$; (b): $K_1=0.2$, $\mu=0.2$, $K_2=1$. (c): The mesh plot of the pressure corresponding to (b). (d): The mesh plot of the v component of the velocity corresponding to (b).



Equivalence of two systems

- **Original system** (Stokes/Darcy) —> via augmented (interface) variables —> **New system** (*three Poisson eqns*):

$$[p] = q_1, \quad [p_n] = q_2,$$
$$[u_n] = q_3, \quad [v_n] = q_4,$$
$$[u \;\mathrm{i}\; \tau] = q_5$$

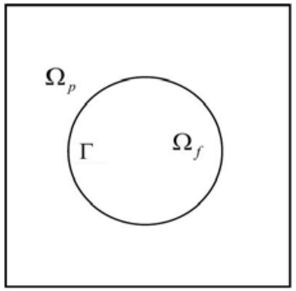


- The soln to the ***original*** is also a soln to the ***new.***

Equivalence of two systems

Is the solution to the new system also the soln to the original system?

- The momentum equations is satisfied
- The Darcy's law in porous media region is also enforced.
- For Stokes and Darcy's coupling, it works fine



Why not work well for NSE/Darcy?

(c) Comparison of the condition number.

$N = N_b$	$cond^{6eq}$ (present)	$cond^{5eq}$ ([27])
32	$5.1733e+02$	$1.6935e+06$
64	$3.1495e+03$	$1.8021e+06$
128	$3.3617e+04$	$1.2956e+08$
256	$2.5181e+05$	$2.2210e+09$
512	$2.3641e+06$	$1.5831e+11$

Why & where Least Squares

- The method (5 aug. variables) works fine for Stokes-Darcy's coupling, but barely works for Navier-Stokes & Darcy's coupling.
- Why? Is the velocity divergence free in flow region? We have ***Δ(div(u))=0 → div(u)=0****? Yes, if it true along* Γ & $\partial\Omega$.
- Our solution: enforce the divergence condition on Γ & $\partial\Omega$: $5\ aug\ viariables, six\ eqn.$ Least squares!
- Soln exists & it is unique if the original problem is well-posed!

Algorithm for NSE/Darcy

- Time marching + pressure incremental

$$\Delta p^{k+1} = \begin{cases} \nabla \cdot \mathbf{F}^{k+1/2} - \rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})^{k+1/2}, & \mathbf{x} \in \Omega_f, \\ 0 & \mathbf{x} \in \Omega_D, \end{cases}$$

$$[p^{k+1}] = q_1^{k+1}, \qquad \left[\frac{\partial p^{k+1}}{\partial n} \right] = q_2^{k+1}, \qquad \text{on } \Gamma.$$

Algorithm for NSE/Darcy II

$$\Delta u^* - \frac{2\rho}{\mu\Delta t} u^* = \begin{cases} \frac{2}{\mu}\left(p_x^{k+1} - \frac{\rho}{\Delta t} u^k\right) - \Delta u^k + \frac{2}{\mu}\left(\rho(\mathbf{u}\cdot\nabla u)^{k+1/2} - F_1^{k+1/2}\right), & \mathbf{x}\in\Omega_f, \\ -K_1 \Delta p_x^{k+1} + \frac{2K_1\rho}{\mu\Delta t} p_x^{k+1}, & \mathbf{x}\in\Omega_D, \end{cases} \quad (26)$$

$$[u^*] = -q_5^{k+1}\sin\theta, \qquad \left[\frac{\partial u^*}{\partial n}\right] = q_3^{k+1}, \qquad \text{on } \Gamma;$$

$$\Delta v^* - \frac{2\rho}{\mu\Delta t} v^* = \begin{cases} \frac{2}{\mu}\left(p_y^{k+1} - \frac{\rho}{\Delta t} v^k\right) - \Delta v^k + \frac{2}{\mu}\left(\rho(\mathbf{u}\cdot\nabla v)^{k+1/2} - F_2^{k+1/2}\right), & \mathbf{x}\in\Omega_f, \\ -K_1 \Delta p_y^{k+1} + \frac{2K_1\rho}{\mu\Delta t} p_y^{k+1}, & \mathbf{x}\in\Omega_D, \end{cases} \quad (27)$$

$$[v^*] = -q_5^{k+1}\cos\theta, \qquad \left[\frac{\partial v^*}{\partial n}\right] = q_4^{k+1}, \qquad \text{on } \Gamma;$$

Important Details

- How to make the Helmholtz *eqn* have the same magnitude in both fluid & porous media region? Add an artificial term in the Darcy's domain $\boldsymbol{u^*/\Delta t}$
- Reinforce the Darcy's law after solving the Helmholtz eqns.
- With and without projection step *(O(h²))*.
- Given $\boldsymbol{Q^{k+1}}$, solve to get:

$$\boldsymbol{A U^{k+1} + B\, Q^{k+1} = F^k}$$

Algorithm Revisit II

- Discretization of ***three Poisson Eqns (ignore k)***

$$AU + BQ = F$$

- Discretization of the *physical interface conditions*

$$CU + DQ = F_2$$

- Q is along the boundary ***O(5N)***
- Schur Complement (direct or *GMRES*)

$$(D - C A^{-1} B)Q = F_2 - C A^{-1} F, \quad SQ = F_3$$

- For fixed Γ, h, S is a constant matrix.

Matrix-vector form at one step

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{u}^{\,k+1} \\ \mathbf{Q}^{k+1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{F}}_1^{k+1} \\ \tilde{\mathbf{F}}_2^{k+1} \end{bmatrix}.$$

Therefore, the Schur complement for $\mathbf{Q}^{k+1}$ is

$$(D - CA^{-1}B)\mathbf{Q}^{k+1} = \tilde{\mathbf{F}}_2^{k+1} - CA^{-1}\tilde{\mathbf{F}}_1^{k+1} = \bar{\mathbf{F}}^{k+1}, \qquad \text{or} \qquad S\mathbf{Q}^{k+1} = \bar{\mathbf{F}}^{k+1}.$$

Validation of NSE/Darcy

$$
\begin{aligned}
u_f &= g(t)\left(y(x^2+y^2-1)+2y\right), \\
v_f &= g(t)\left(-x(x^2+y^2-1)-2x\right), \\
p_f &= g(t)\left(x^2+y^2\right),
\end{aligned}
$$

$$
\mathbf{n}=[x,\ y]^T, \quad \boldsymbol{\tau}=[-y,\ x]^T, \quad p_f=g(t), \quad p_D=g(t), \quad u_f=2y\,g(t), \quad v_f=-2x\,g(t),
$$

$$
u_D=0, \quad v_D=0, \quad \frac{\partial u_f}{\partial n}=4y\,g(t), \quad \frac{\partial v_f}{\partial n}=-4x\,g(t), \quad \frac{\partial u_f}{\partial \tau}=2x\,g(t), \quad \frac{\partial v_f}{\partial \tau}=2y\,g(t),
$$

$$
\mathbf{n}\cdot\mathbf{D}_f\cdot\mathbf{n}=0 \qquad \boldsymbol{\tau}\cdot\mathbf{D}_f\cdot\mathbf{n}=0.
$$

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Grid refinement analysis

- Tangent slip, pressure is constant in Darcy. Average convergence: 2.0221 & 3.2110

(a) Comparison of the pressure error and accuracy order.

$N = N_b$	$\|E_p^{6eq}\|_\infty^N$ (present)	order	$\|E_p^{5eq}\|_\infty^N$ ([27])	order
32	$9.1289e-03$		$2.7892e-02$	
64	$1.9969e-03$	2.1927	$6.0515e-03$	0.37889
128	$9.9901e-04$	0.9918	$3.2848e-03$	0.88149
256	$3.6806e-04$	1.4406	$3.2848e-03$	0.88149
512	$6.4588e-05$	2.5106	$9.6182e-04$	1.7720

(b) Comparison of velocity error and accuracy order.

N	$\|E_{\mathbf{u}}^{6eq}\|_\infty^N$ (present)	order	$\|E_{\mathbf{u}}^{5eq}\|_\infty^N$ ([27])	order
32	$2.1160e-02$		$3.2203e-02$	
64	$5.8094e-03$	1.8649	$1.4188e-02$	1.1825
128	$1.4762e-03$	1.9765	$2.6563e-03$	2.4172
256	$3.3356e-04$	2.1459	$1.2963e-03$	1.0350
512	$9.0142e-05$	1.8877	$2.5613e-04$	2.3395

Another example

A *continuous tangential* velocity but *discontinuous pressure* along the interface. More importantly, the velocity and pressure are *non-trivial* in both regions and the normal derivatives of the velocity components are also discontinuous across the interface.

$$u_f = g(t)\left(y(x^2+y^2-1)+2x \right),$$

$$v_f = g(t)\left(-x(x^2+y^2-1)-2y \right),$$

$$p_f = 3g(t)\left(x^2-y^2 \right),$$

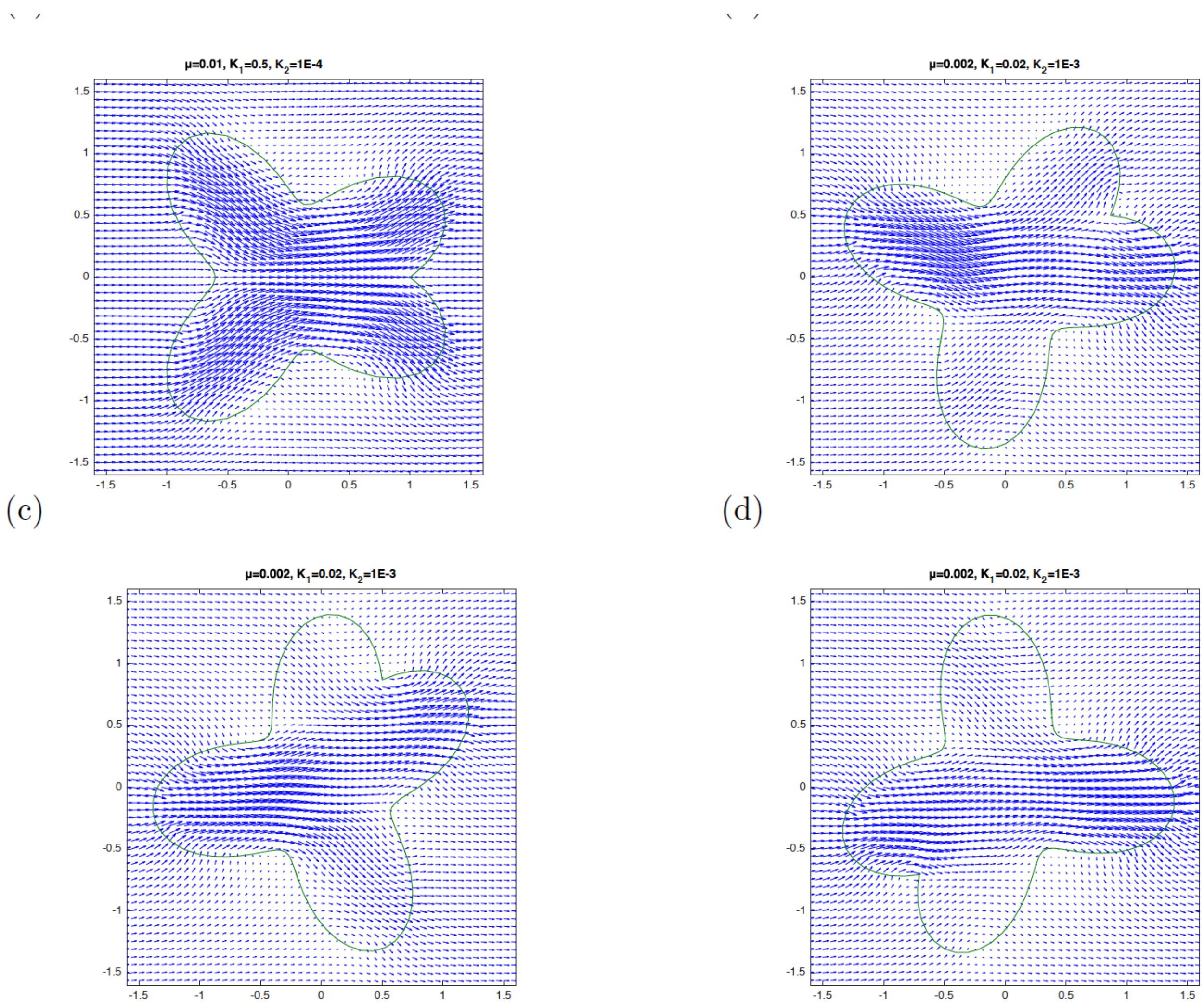
Grid Refinement Analysis

N	N_b	$\|\|E_p\|\|_\infty^N$	order	$\|\|E_{\mathbf{u}}\|\|_\infty^N$	order	cond-6eq	cond-5eq
16	16	$1.8890e-00$		$1.9839e-01$		$8.0806e+01$	$1.2104e+02$
32	22	$5.7256e-01$	1.7221	$2.2640e-02$	3.1314	$2.0234e+02$	$7.3228e+02$
64	30	$1.4790e-01$	1.9528	$1.7365e-03$	3.7046	$7.7313e+02$	$6.7254e+03$
128	42	$3.5359e-02$	2.0645	$1.5333e-04$	3.5015	$3.0480e+03$	$8.4892e+04$
256	68	$6.9394e-03$	2.3492	$2.6986e-05$	2.5064	$1.0844e+04$	$1.0888e+06$

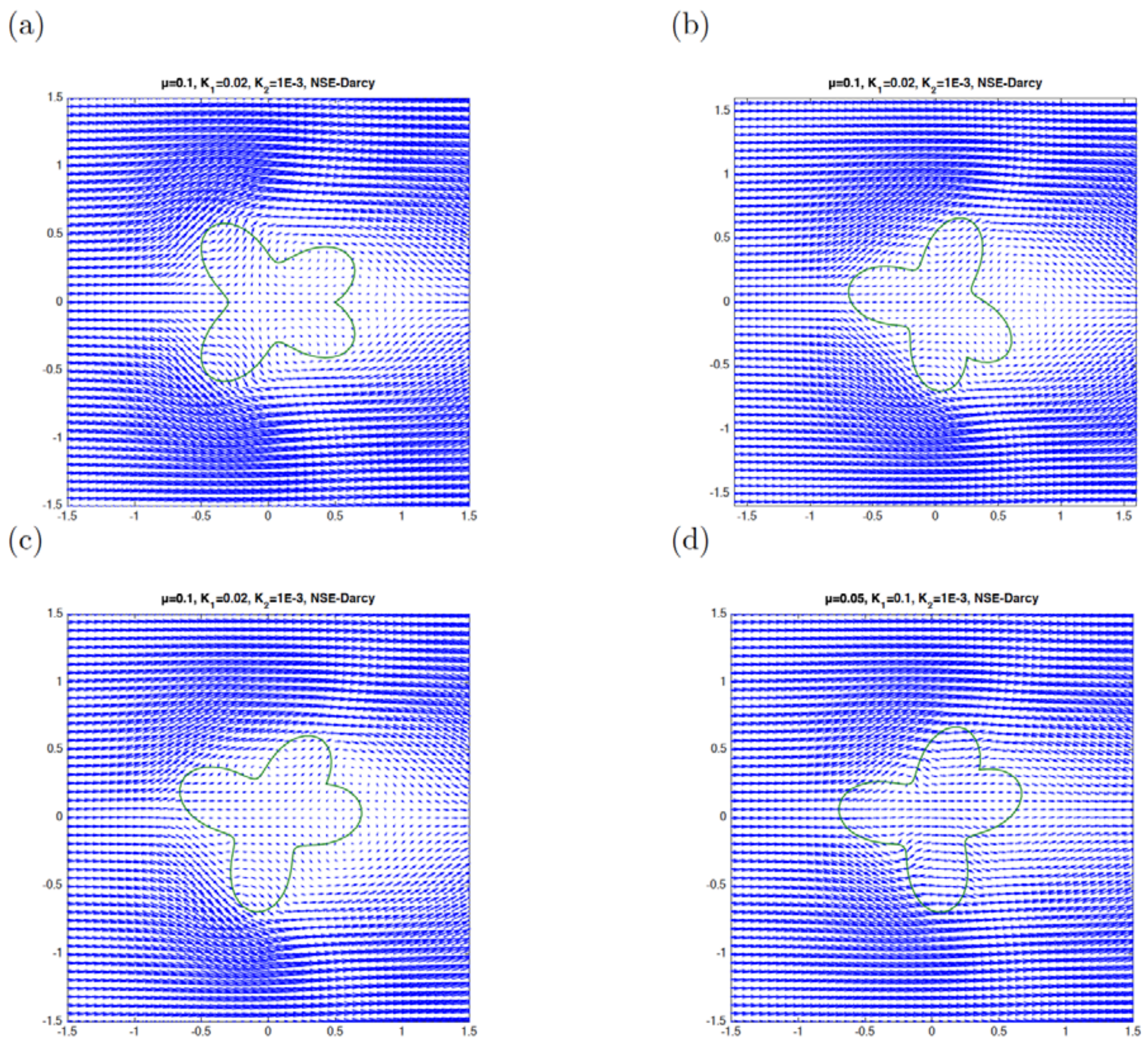
N	$\|\|E_1\|\|_\infty^N$	$\|\|E_2\|\|_\infty^N$	$\|\|E_3\|\|_\infty^N$	$\|\|E_4\|\|_\infty^N$	$\|\|E_5\|\|_\infty^N$	$\|\|E_6\|\|_\infty^N$
32	$3.4972e-03$	$1.8503e-03$	$2.8119e-03$	$4.7202e-03$	$9.7248e-04$	$3.3883e-03$
64	$1.2280e-03$	$5.2899e-04$	$5.1238e-04$	$2.1042e-03$	$2.7033e-04$	$1.0480e-03$
128	$2.0427e-04$	$8.3986e-05$	$8.6997e-05$	$5.9738e-04$	$7.5299e-05$	$1.5937e-04$
256	$2.6319e-05$	$1.2165e-05$	$2.1739e-05$	$9.6036e-05$	$1.5473e-05$	$1.8553e-05$
order	2.5026	2.5091	2.2540	2.0640	2.0642	2.6538

Table 2: The residual of the six interface conditions (15)-(20) of the computed solution. The last row is the average convergence order of the six interface equations.

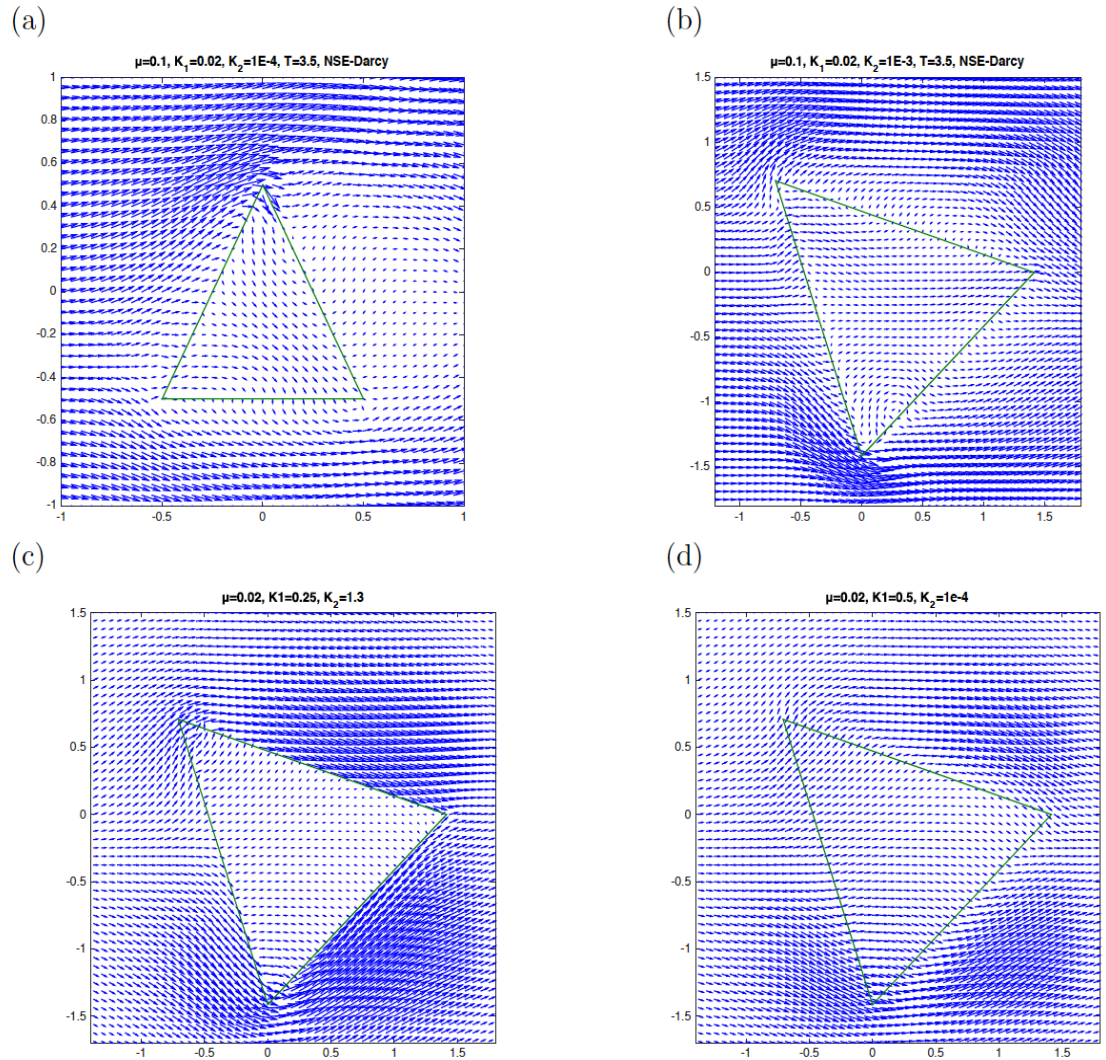
Orientation Effect (flow inside)



Orientation Effect (flow outside)

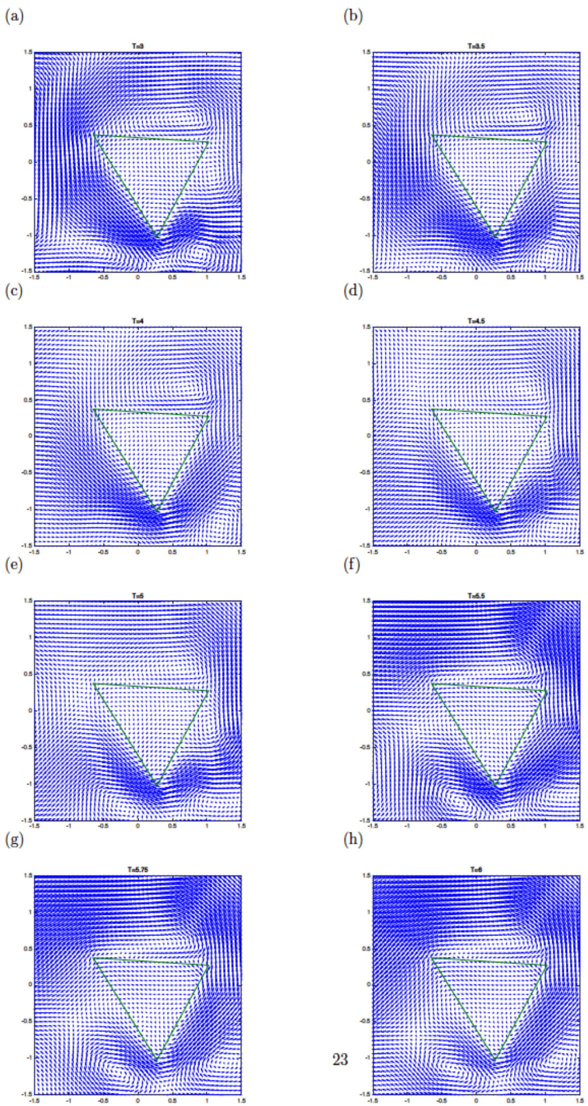


Corner Effect (flow outside)



Zhilin LI

Transient Behaviors



More on flow test with corners

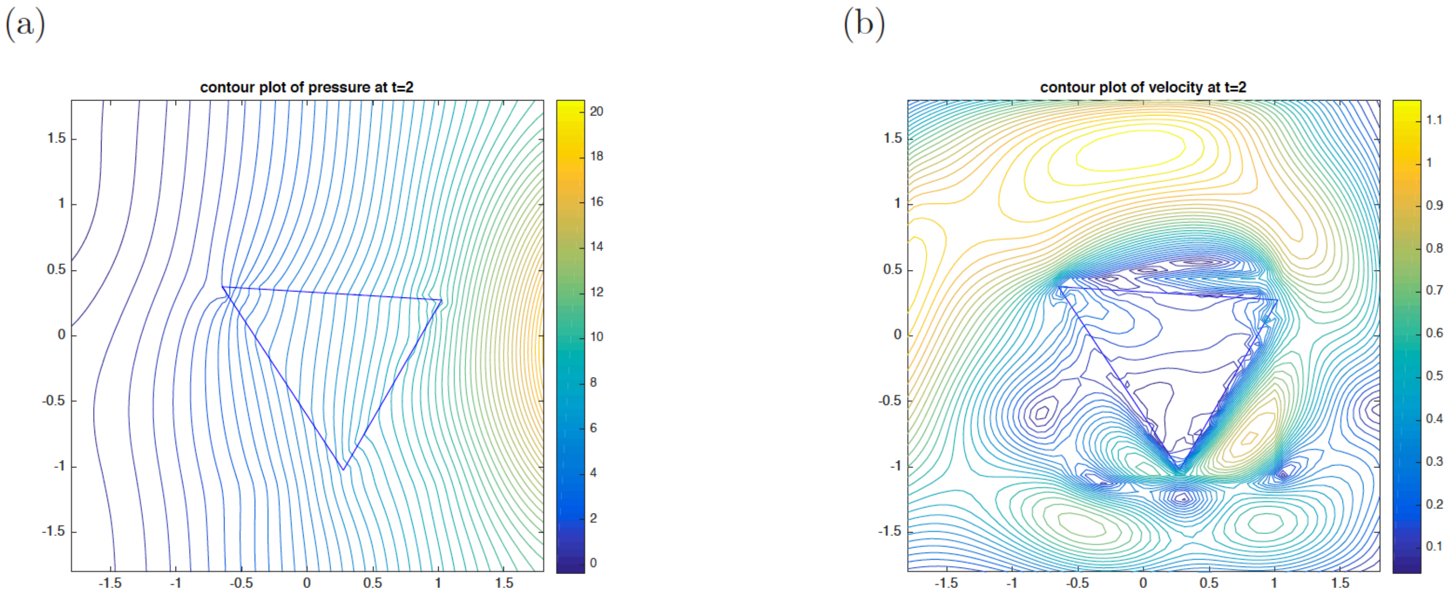
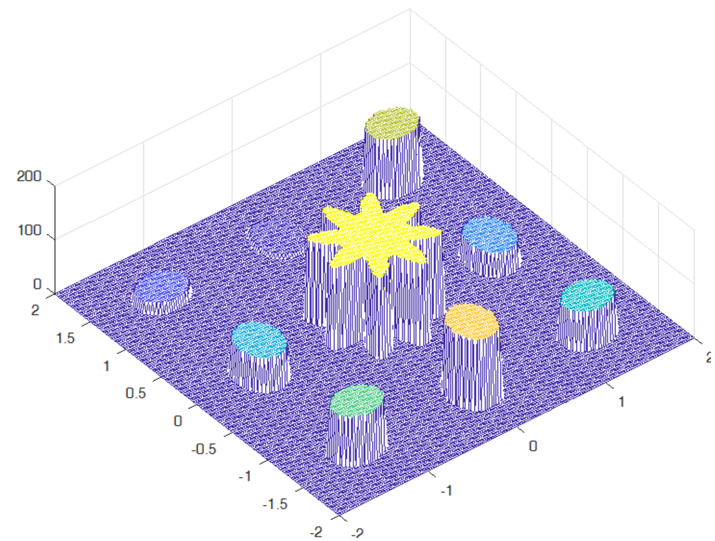
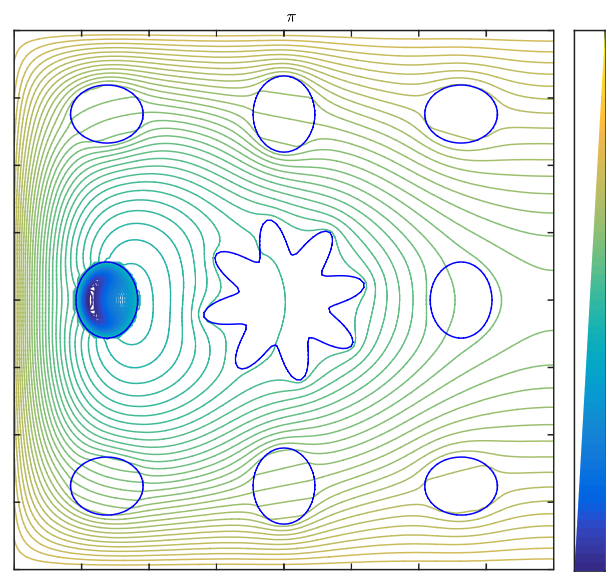


Figure 9: Contour plots of the pressure and the magnitude of the velocity at $t = 2$ with the set-up in Figure 8.

FSI (Fluid and Porous Media)

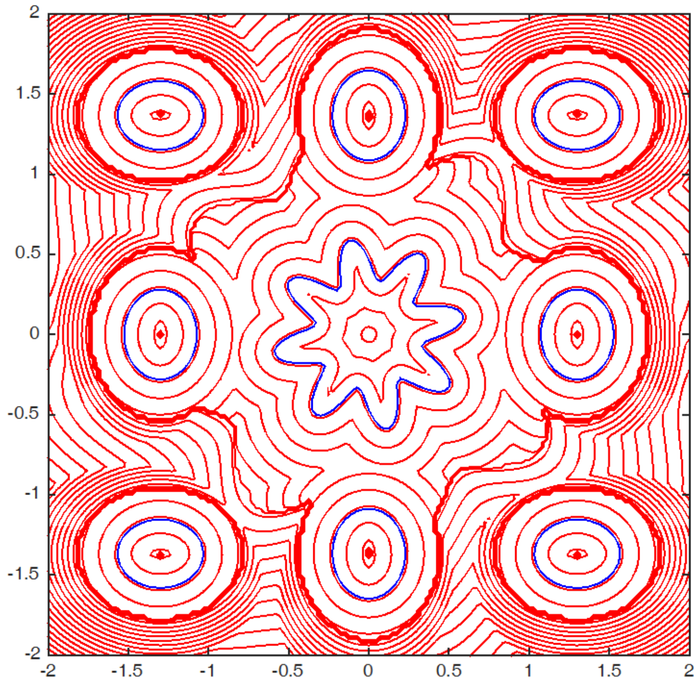
- Model: incompressible Navier-Stokes or Stokes equations coupled with Darcy's law. Multi-connected domain





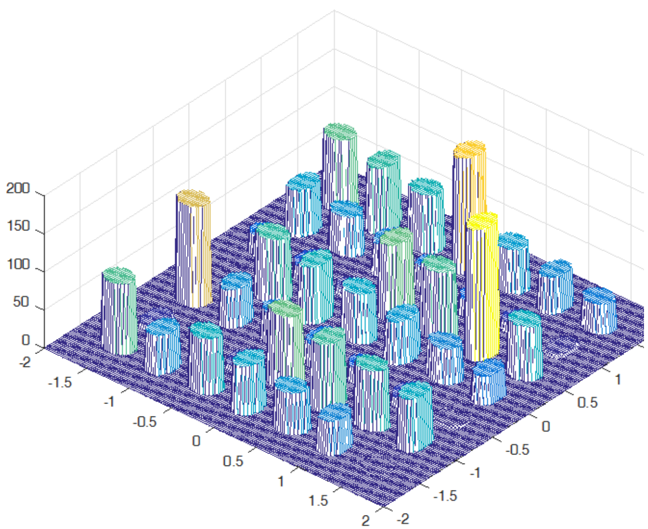
Signed distance function

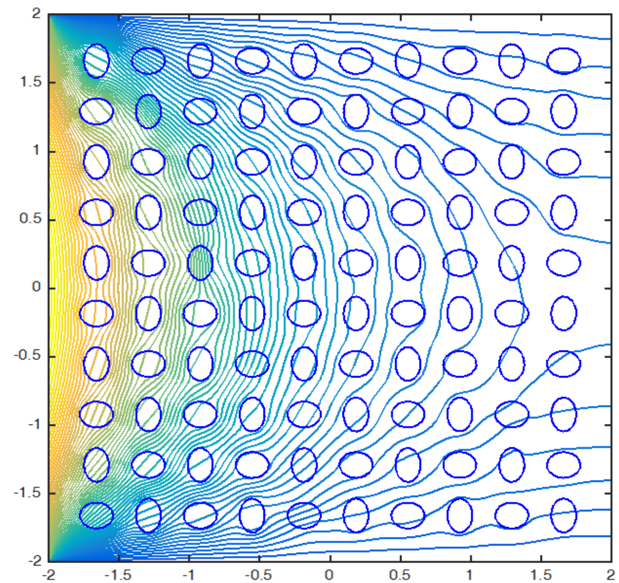
- Re-initialization of a 2D level set function in a tube: *blue line* is the interface. Treat boundary is challenging!



Multi-particles & multi-scales

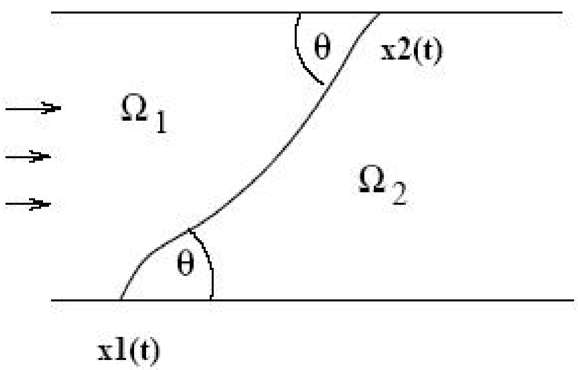
- Many inclusions with different permeability. Flow from right

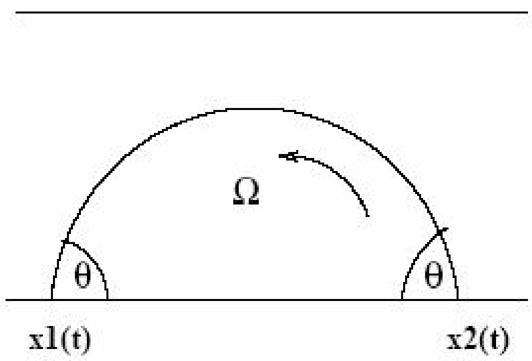




Moving Contact Line (E, Ren, Li)

- Free boundary problems (drop spreading)





- Navier Stokes equation for incompressible fluid(s)
- Two phase *or* one phase
- Navier BC along the contact line

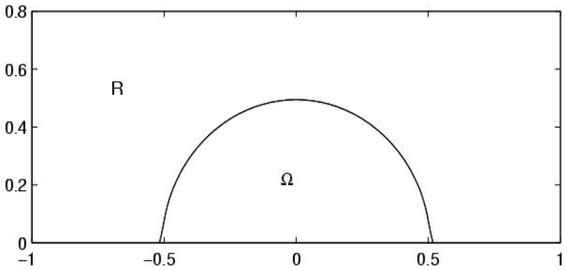
$$\beta_i u_i = -\mu_i \frac{\partial u_i}{\partial y}$$

Governing Eqns and BC

- Navier-Stokes equation for the fluid

$$\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)+\nabla p=\nabla\bullet\mu(\nabla u+\nabla u^T)+G$$

$$\nabla\bullet u=0$$



- Free boundary condition

$$-p+\mathbf{n}^T\cdot\mu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\cdot\mathbf{n}=-p_{air}+\sigma\kappa+f_n,\quad \mathbf{x}\in\partial\Omega$$

$$\boldsymbol{\tau}^T\cdot\mu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\cdot\mathbf{n}=f_\tau,\quad \mathbf{x}\in\partial\Omega$$

- Navier (slip) BC: $\beta_i u_i=-\mu_i\frac{\partial u_i}{\partial y}$

- Tip velocity (W. Ren & W. E): $\alpha u=\gamma(\cos\theta-\cos\theta^*)$
- What is the consistent boundary condition at the triple junctions?

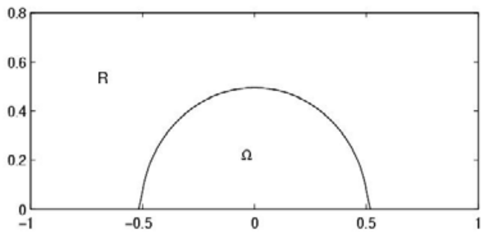
The Augmented Algorithm

$$\rho\frac{\mathbf{u}^*-\mathbf{u}^k}{\Delta t}=\begin{cases}-\nabla p^{k-\frac12}-\rho\,(\mathbf{u}\cdot\nabla\,\mathbf{u})^{k+\frac12}+\frac{\mu}{2}\left(\Delta\,\mathbf{u}^*+\Delta\,\mathbf{u}^k\right)+\mathbf{G}^{k+\frac12}, & \mathbf{x}\in(\Omega\cap Z^R)\\ -\nabla p^k-\rho\,(\mathbf{u}\cdot\nabla\,\mathbf{u})^k+\mu\Delta\,\mathbf{u}^*+\mathbf{G}^k, & \mathbf{x}\in(\Omega\cap Z^I)\\ -\rho\,(\mathbf{u}\cdot\nabla\,\mathbf{u})^{k+\frac12}+\frac{\mu}{2}\left(\Delta\,\mathbf{u}^*+\Delta\,\mathbf{u}^k\right), & \mathbf{x}\in(\Omega^c\cap Z^R)\\ -\rho\,(\mathbf{u}\cdot\nabla\,\mathbf{u})^k+\mu\Delta\,\mathbf{u}^*, & \mathbf{x}\in(\Omega^c\cap Z^I)\end{cases}\tag{2.9}$$

$$\frac{\partial\mathbf{u}^*}{\partial n}=0\quad\text{at } x=a,\ x=b,\ \text{and } y=d,\quad v=0\ \text{and}\ \beta u=-\mu\frac{\partial u}{\partial y}\ \text{at } y=c.\tag{2.10}$$

$$[\mathbf{u}^*]_{\partial\Omega}=0,\qquad\left[\frac{\partial\mathbf{u}^*}{\partial n}\right]_{\partial\Omega}=\mathbf{q}^{k+1},$$

$$\begin{bmatrix}A & B\\ S & E\end{bmatrix}\begin{bmatrix}U^{k+1}\\ G^{k+1}\end{bmatrix}=\begin{bmatrix}F_1\\ F_2\end{bmatrix}$$



$$\begin{cases}\Delta\,\phi^{k+1}=\dfrac{\nabla\cdot\mathbf{u}^*}{\Delta t}, & \mathbf{x}\in R,\\ \left.\dfrac{\partial\phi^{k+1}}{\partial\mathbf{n}}\right|_{\partial R}=0, \quad \left[\phi^{k+1}\right]_{\partial\Omega}=0, \quad \left[\dfrac{\partial\phi^{k+1}}{\partial n}\right]_{\partial\Omega}=0,\end{cases}$$

$$\mathbf{u}^{k+1}=\mathbf{u}^*-\Delta t\,\nabla\,\phi^{k+1},\qquad\mathbf{x}\in R,$$

$$\nabla p^{k+\frac12}=\nabla p^{k-\frac12}+\nabla\phi^{k+1},\qquad\mathbf{x}\in\Omega,$$

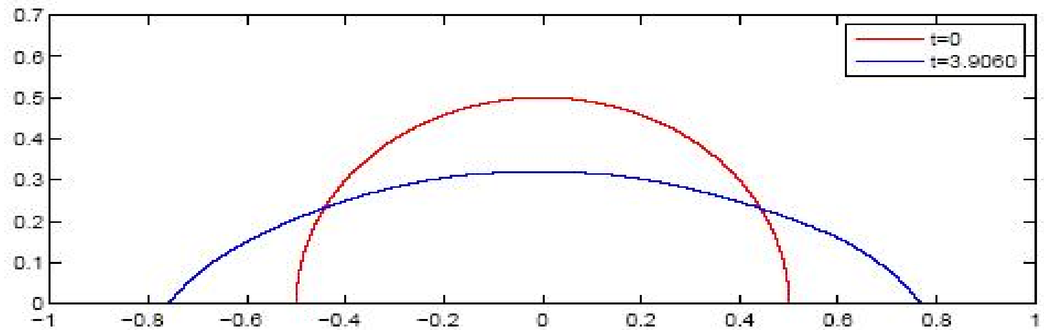
$$p+\mathbf{n}^T\cdot\mu\frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2}\cdot\mathbf{n}=p_{air}-\sigma\kappa,\quad\mathbf{x}\in\partial\Omega,$$

$$\boldsymbol{\tau}^T\cdot\mu\frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2}\cdot\mathbf{n}=0,\quad\mathbf{x}\in\partial\Omega,$$

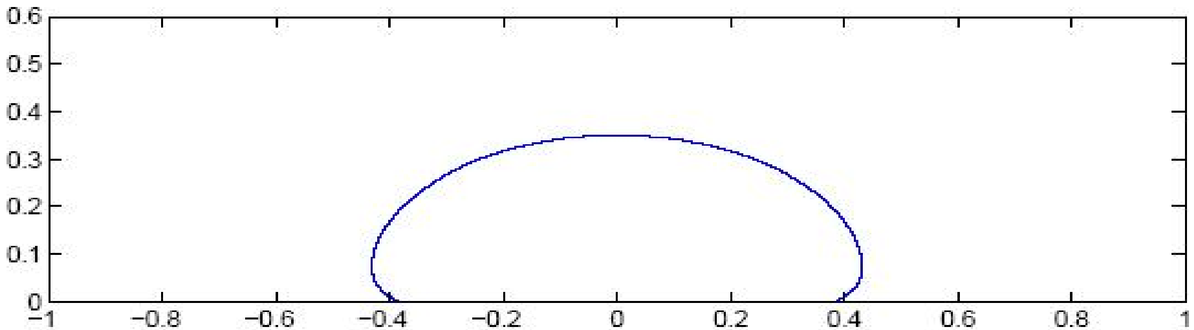
Drop spreading and contracting

(a), $\theta^* = \pi/4$, $\theta^0 = \pi/2$; (b), $\theta^* = 3\pi/4$, $\theta^0 = \pi/2$

(a).

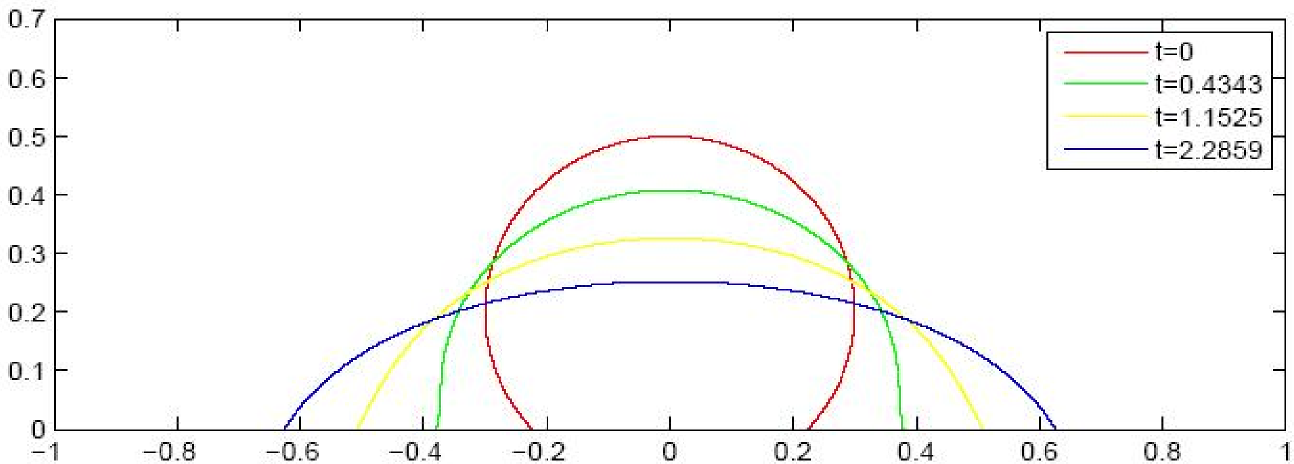


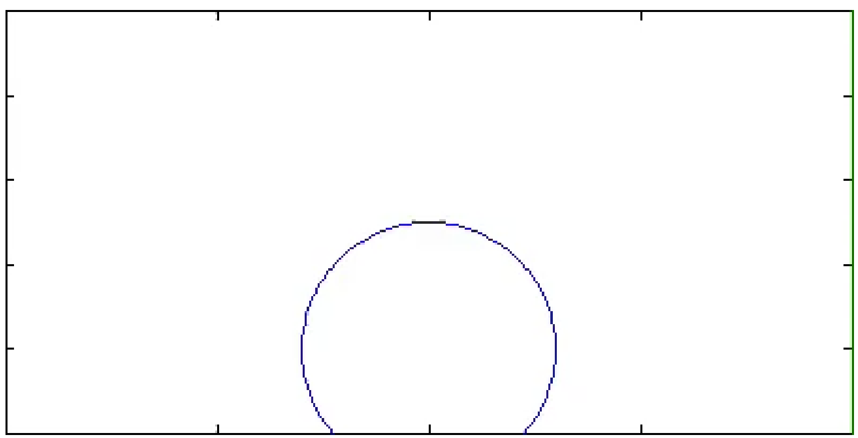
(b).



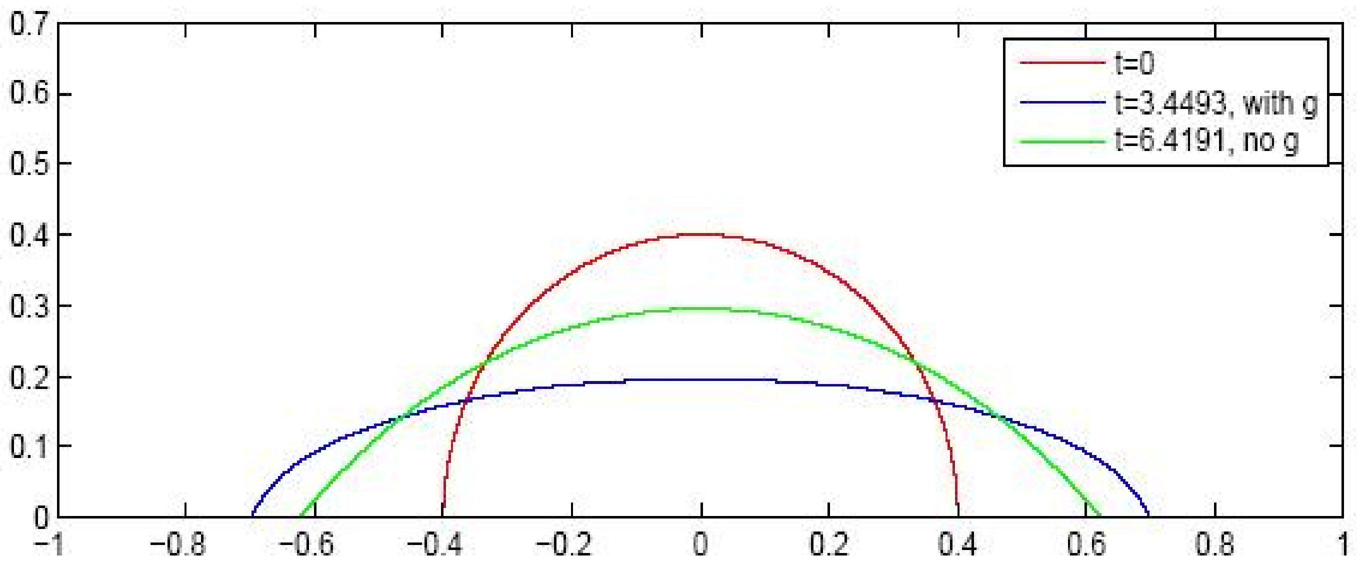
Another Example

For this one, the Crank-Nicholson approach fails



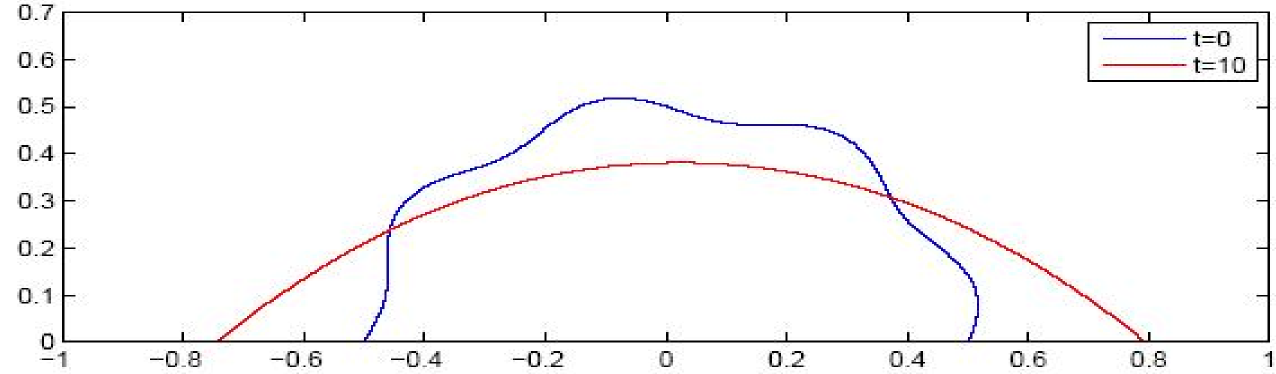


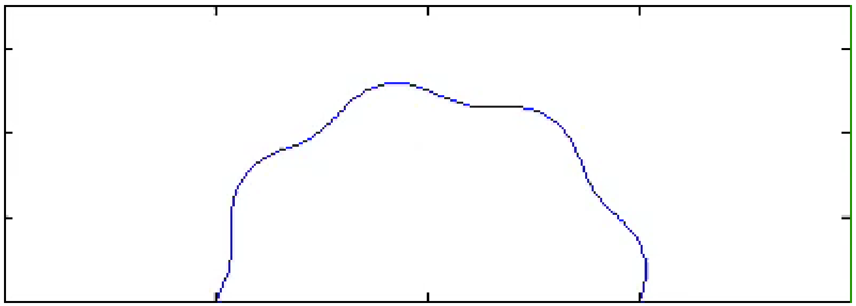
Effect of Gravity



Drop spreading with perturbation

Cu



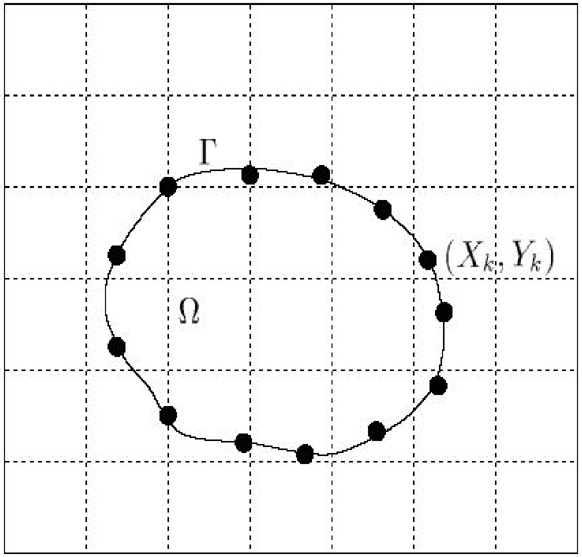


How to solve Poisson Eqn. (regular)

- Regular domain (rectangular, circles, spheres, ..), no interface/singularity

$$\Delta u = f(x)$$

BC (e.g. Dirichlet, Neuman, Mixed)



- The FD scheme at (x_i, y_j)

$$\frac{u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij}$$

- ***AU=F;*** ***A***: Discrete Laplacian. Can be solved by a fast Poisson solver (e.g. ***FFT***, *$O(N^2)log(N)$), e.g.,* Fish-pack, or structured multigrid

How to Solve Poisson Eqn. with Jumps

- **Interface problems**, simplified ***Peskin's IB*** model

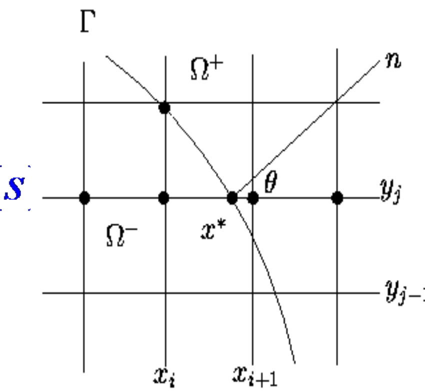
$$\Delta u = f(x) + \int_\Gamma c(s)\delta(x - X(s))\,ds + \int_\Gamma w(s)\delta'(x - X(s))\,ds$$

BC (e.g., Dirichlet, Neuman, Mixed)

- ***Equivalent Problem***

$$\Delta u = f(x), \quad x \in \Omega \setminus \Gamma, \quad [u]_\Gamma = w(s), \quad [\nabla u \cdot n]_\Gamma = \left[\frac{\partial u}{\partial n}\right]_\Gamma = C(s$$

BC (e.g., Dirichlet, Neuman, Mixed)



- FD scheme (x_i, y_j), regular/irregular

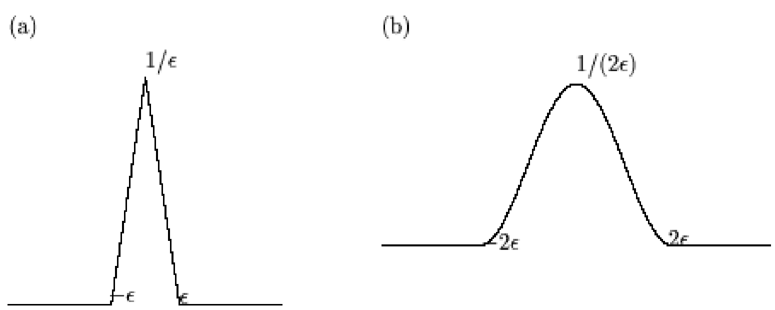
$$\frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij}$$

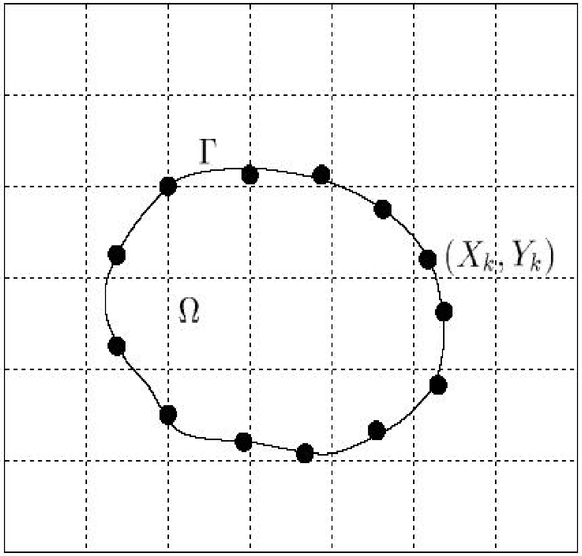
$$\frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} + C_{ij}$$

Zhilin LI

Solve Poisson Eqn. with Jumps using IB method

- Discrete delta function approach





$$\frac{u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}}{h^2}=L_h u_{i,j}=f_{ij}+C_{ij}$$

$$C_{ij}=f_{ij}+\sum_k c_k\delta_h(x_i-X_k)\delta_h(y_j-Y_k)\Delta s+\sum_k w_k\delta_h'(x_i-X_k)\delta_h'(y_j-Y_k)\Delta s$$

Analysis of IB Method

- It is inconsistent! The local truncation error is O(1/h)!

$$\Delta u = f(x) + \int_{\Gamma} c(s)\delta(x - X(s))\, ds + g$$

BC (e.g., Dirichlet, Neuman, Mixed)

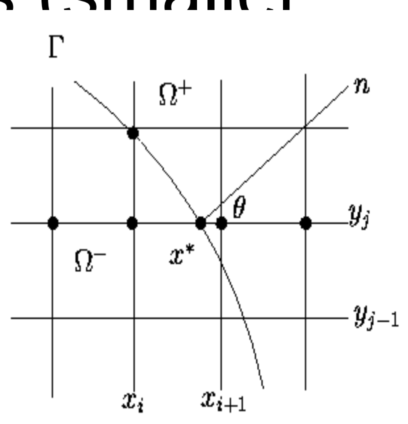
$$\frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} + C_{ij}$$

$$C_{ij} = \sum_k C_k \delta_h(x_i - X_k)\delta_h(y_j - Y_k)\Delta s_k$$

- But it is first order convergent in the infinity norm!
- It has better convergence order in ***L*²** (average) norm
- Rigorous proof by Z. Li, MathCom, 2015.
- Often local adaptive mesh is used

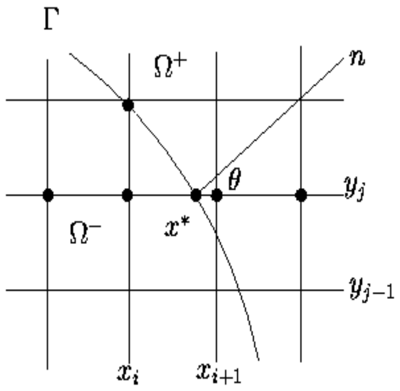
More Accurate Method: IIM

- IIM: Immersed Interface Method (LeVeque/Li)
  - Replace the Dirac delta function with jump conditions
  - For Poisson equations (coef=1), the FD scheme (LHS) is the same
  - Better correction terms only the 5-point FD stencil has points from both sides (smaller support than IB method



More Accurate Method: IIM

- Use Taylor expansion fro each side of the interface to minimize the local truncation error
- Second order accurate solution and ***gradient*** at all grid points (strictly proved, T. Beale, Li)
- One fast Poisson solver for one variable



An example

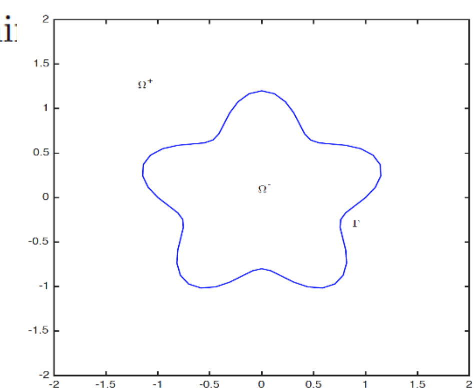
- A Poisson equation with general BC on a star region

Table 2: (a), A grid refinement analysis with $\beta^- = 1000$, $\beta^+ = 1$. The average convergence orders are 3.3288, 4.2931, and 4.2812, respectively for the three quantities.

N	$E(u)$	r	$E(u_n^-)$	r	$E(u_n^+)$	r
40	9.3120e-01		8.5460e-03		1.1640e+01	
80	5.2030e-02	4.1617	3.1310e-05	8.0925	4.2750e-02	8.0890
160	2.6230e-02	0.98813	1.7070e-05	0.87516	2.3290e-02	0.87621
320	2.3590e-04	6.7969	1.5360e-07	6.7961	2.3210e-04	6.6488
640	9.1360e-05	1.3685	5.7850e-08	1.4088	8.1450e-05	1.5108

(b) The number of iterations without and with the two preconditioni

N	40	80	160	320	640
No-Pre	64	102	182	330	512
Pre-(18)	60	88	127	191	277
New-Pre	17	26	37	54	87



Conclusions

New augmented methods for fluid flow and porous media (***Stokes-Darcy or NES-Darcy***)

- Different governing equations are transformed to the same type equations ***via augmented interface variables***
- ***Three Poisson equations*** with jump in the soln and normal derivative
- Use ***least squares*** to get equivalent systems
- Can utilize the FFT based ***fast Poisson solver***

Conclusions

- Second order accurate in both ***pressure*** and ***velocity***
- Equivalence has been proved under stronger regularity assumptions
- What are the best augmented variables that have the same No. of unknowns and equations?

Why augmented approach?

- Can make the solver ***faster***, e.g, fast IIM for elliptic interface problems with piecewise constant coefficient, IIM for irregular domains.
- Can ***decouple problems***, e.g., the Stokes or NSE equations with discontinuous viscosity, the augmented approach enable us to decouple the jump conditions in the pressure and the velocity
- For some problems, it is the only way to get accurate discritization
- No need to have the Green functions, independent of BC, source terms, domains *etc*.
- Can ***couple problems***: deal with Stokes-Darcy coupling

Key Words & Related to others

- IB (Immersed Boundary) → IIM (Immersed Interface) 1^{st} to 2^{nd} velocity & pressure
- Cartesian meshes & Fast Helmoltz/Poisson solvers
- Efficiency and accuracy; augmented IIM
- Applications: Fluid and Porous media; moving contact lines (E, Ren, Li)
- AMS classification: 65N, …

Thank you!

How to Solve Poisson Eqn. with jumps

- **Interface problems**, simplified ***Peskin's IB*** model

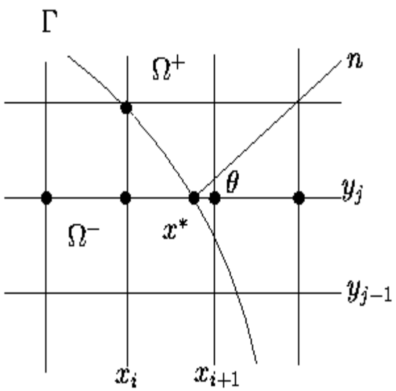
$$\Delta u = f(x) + \int_\Gamma c(s)\delta(x - X(s))ds + g$$

BC (e.g., Dirichlet, Neuman, Mixed)

- *Equivalent Problem*

$$\Delta u = f(x), \quad x \in \Omega \setminus \Gamma, \quad [u]_\Gamma = 0, \quad [\nabla u \cdot n]_\Gamma = \left[\frac{\partial u}{\partial n}\right]_\Gamma = C(s)$$

BC (e.g., Dirichlet, Neuman, Mixed)



- FD scheme (x_i, y_j), regular/irregular

$$\frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij}$$

$$\frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} + C_{ij}$$

- IB, first order, IIM second order (C's depend on curvature!)
- ***AU=F+BC;*** ***A***: Discrete Laplacian. Can be solved by fast Poisson solver
- IIM is *second order* both in solution & gradient (T. Beale …)

Algorithm Revisit I

- Given ***Q***, solve *p, u, v*: ***AU+BQ=F***

$$\nabla p = \nabla \bullet \mu(\nabla u + \nabla u^T) + g \qquad u = -\frac{K}{\mu}\nabla p$$

$$\nabla \bullet u = 0 \qquad \nabla \bullet u = 0$$

$$\Delta u = \begin{cases} (p_x - g_1)/\mu \\ -K\Delta p_x/\mu \end{cases}; \quad \Delta v = \begin{cases} (p_y - g_2)/\mu \\ -K\Delta p_y/\mu \end{cases},$$

$$[u_n] = q_3, \quad [v_n] = q_4, \quad \text{(new unknown)}$$

$$[u \cdot n] = 0, \quad [u \cdot \tau] = q_5 \Rightarrow \text{ get } [u] \;\&\; [v]$$