Asymptotic analysis and numerical method for singularly perturbed eigenvalue problems

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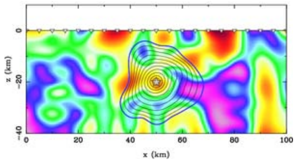
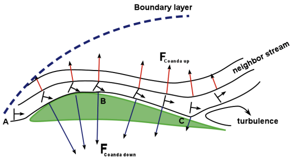
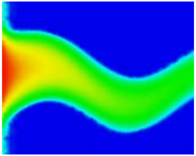
Motivation

Recently, we are interested in the high efficiency numerical methods for problems with multi-scale phenomena and singularities, such as the high frequency waves propagation, the singular perturbation problems with boundary/ interior layers. These problems arise in many scientific fields, such as

- the elastic/electromagnetic wave propagation in heterogeneous media,
- the seismic wave propagation in geophysics,
- aerodynamics,
- multi-phase flow,
- ...







Motivation

To solve these multi-scale problems efficiently, we usually need to solve the following eigenvalue problem first,

$$\begin{cases} -\varepsilon^2 \Delta\phi(\mathbf{x}) + V(\mathbf{x})\phi(\mathbf{x}) = \lambda\phi(\mathbf{x}), & \forall \mathbf{x} \in \Omega \subset \mathbb{R}^N, \\ \int_\Omega |\phi(\mathbf{x})|^2 d\mathbf{x} = 1, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ or the whole space, $V(\mathbf{x}) \geq 0$, $0 < \varepsilon \ll 1$. If Ω is a bounded domain, we usually add the following boundary condition for the eigenfunction ϕ,

$$\phi|_{\partial\Omega} = 0. \tag{2}$$

If $\Omega = \mathbb{R}^N$, we have the boundary condition at infinity,

$$\phi(\mathbf{x}) \to 0, \quad \text{as } |\mathbf{x}| \to \infty, \tag{3}$$

and usually we need

$$V(\mathbf{x}) \to \infty \text{ as } |\mathbf{x}| \to \infty. \tag{4}$$

Motivation

As $0 < \varepsilon \ll 1$ is a small parameter, this problem (1) is usually called *singularly perturbed eigenvalue problem* (SPEP).

The SPEP can be regarded as a semi-classical limit of a Schrödinger type eigenvalue problem which has been studied by

- Barry Simon, 1983-85;
- Helffer-Sjöstrand, 1984, 2006;
- Martinez-Rouleux, 1988;
- Dancer-López-Gómez, 2000;
- Reyes-Sweers, 2016, $\cdots$

Previous Asymptotic Analysis Results

Barry Simon (1983-85) studied a special case and obtained that if $V(\mathbf{x}) \geq 0$ is a C^∞ potential bounded away from zero at $\mathbf{x} = \infty$ and $V(\mathbf{x})$ has finite non-degenerate zeros, then

$$\lambda_0^\varepsilon = e_0\varepsilon + O(\varepsilon^2), \quad \varepsilon \to 0^+. \tag{5}$$

where e_0 is the ground state energy of the associated harmonic oscillator obtained by localization near the zeros of $V(\mathbf{x})$.

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Martinez and Rouleux (1988) showed that if $V(\mathbf{x}) \in C^\infty(\mathbb{R})$ is bounded away from zero at $\mathbf{x} = \infty$, $V(\mathbf{x}) \geq 0$, $V^{-1}(0) = 0$ and for $\mathbf{x} \simeq 0$

$$C^{-1}\|\mathbf{x}\|^{2\alpha} \leq V(\mathbf{x}) \leq C\|\mathbf{x}\|^{2\alpha},$$

then

$$\lambda_k^\varepsilon = O(\varepsilon^{\frac{2\alpha}{\alpha+1}}), \quad \varepsilon \to 0^+, \quad k = 0, 1, 2, \cdots.$$

Numerical Challenges

For the numerical solution of some special SPEPs, many mathematicians have established some numerical methods, for example,

- direct numerical integration techniques (Cooley, 1961),
- Rayleigh-Ritz methods (Mitra, 1978),
- perturbation methods (Bessis, 1980),
- the Fourier grid Hamiltonian methods (Marston, 1989),
- the gradient flow based method (Bao, 2003-)and so on.

Recently, Han, Shih and Tsai (2014), Han, Shih and Yin (2017) studied the tailored finite point methods for solving the Problem (1) in some special cases.

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However, how to design an efficient method with high accuracy for this kind of singularly perturbed eigenvalue problem (1) is still a challenge.

Outline

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Singularly perturbed eigenvalue problem

Let us consider the following singularly perturbed eigenvalue problem on a domain $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} -\varepsilon^2 \Delta \phi(\mathbf{x}) + V(\mathbf{x})\phi(\mathbf{x}) = \lambda \phi(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega \subset \mathbb{R}^N, \\ \int_\Omega |\phi(\mathbf{x})|^2 d\mathbf{x} = 1, \end{cases} \tag{6}$$

$V(\mathbf{x}) \geq 0$, $0 < \varepsilon \ll 1$. If Ω is a bounded domain, we usually add the following boundary condition for the eigenfunction ϕ,

$$\phi\big|_{\partial\Omega} = 0. \tag{7}$$

If $\Omega = \mathbb{R}^N$, we have the boundary condition at infinity,

$$\phi(\mathbf{x}) \to 0, \quad \text{as } |\mathbf{x}| \to \infty, \tag{8}$$

and usually we need

$$V(\mathbf{x}) \to \infty \text{ as } |\mathbf{x}| \to \infty. \tag{9}$$

Weak form of SPEP

Define a differential operator $\mathcal{L}^\varepsilon$ in $C_0^\infty(\Omega)$ by

$$\mathcal{L}^\varepsilon \phi(\mathbf{x}) = -\varepsilon^2 \Delta \phi(\mathbf{x}) + V(\mathbf{x})\phi(\mathbf{x}), \quad \forall \phi(\mathbf{x}) \in C_0^\infty(\Omega). \tag{10}$$

The differential operator $\mathcal{L}^\varepsilon$ can be extended to $H_0^1(\Omega) \bigcap H^2(\Omega)$ by the following way, $\forall \phi(\mathbf{x}) \in H_0^1(\Omega)$,

$$\big(\mathcal{L}^\varepsilon \phi(\mathbf{x}), \psi(\mathbf{x})\big) := \int_\Omega [\varepsilon^2 \nabla \phi(\mathbf{x}) \cdot \nabla \psi(\mathbf{x}) + V(\mathbf{x})\phi(\mathbf{x})\psi(\mathbf{x})] d\mathbf{x}, \quad \forall \psi(\mathbf{x}) \in C_0^\infty(\Omega), \tag{11}$$

While $V(\mathbf{x}) \in L^\infty(\Omega)$, we can obtain a weak form of the SPEP (6): Find $\lambda \in \mathbb{R}$ and $\phi(\mathbf{x}) \in H_0^1(\Omega)$, such that

$$\big(\mathcal{L}^\varepsilon \phi(\mathbf{x}), \psi(\mathbf{x})\big) = \lambda\big(\phi(\mathbf{x}), \psi(\mathbf{x})\big), \quad \forall \psi(\mathbf{x}) \in H_0^1(\Omega). \tag{12}$$

Preliminary results of eigenvalue problem

Furthermore, we can define an energy functional $\mathcal{F}$ in $H_0^1(\Omega)$,

$$\mathcal{F}(\phi) := \int_{\Omega} \left[\varepsilon^2 |\nabla \phi(\mathbf{x})|^2 + V(\mathbf{x})\phi^2(\mathbf{x})\right] d\mathbf{x}, \quad \forall \phi(\mathbf{x}) \in H_0^1(\Omega). \tag{13}$$

Now we can analyze the asymptotic behavior of the eigenvalues/ eigenfunctions of $\mathcal{L}^{\varepsilon}$ as $\varepsilon \to 0$ with $V(x) \in C_p(\Omega) \subset L^{\infty}(\Omega)$.

Asymptotic Analysis for SPEP

Definition 2.1

We denote by $C_p(\Omega)$ *a subspace of the piecewise continuous functions, that is:*

1. *There are a series of open sets* $\{\Omega_i \subset \Omega, \quad i = 1, 2, 3, \cdots\}$ *such that:*

$$\Omega \subset \cup_i \bar{\Omega}_i, \qquad \Omega_k \cap \Omega_j = \emptyset, \quad k \neq j.$$

   And for any $\mathbf{x} \in \Omega$*, there are only finite* Ω_i *such that* $\mathbf{x} \in \bar{\Omega}_i$*.*

2. $V(\mathbf{x}) \in C(\Omega_i), \quad i = 1, 2, 3, \cdots$*. And for any* $i \neq j$*,* $V(\mathbf{x})$ *is discontinuous on* $\bar{\Omega}_i \cap \bar{\Omega}_j$*.*
3. *And the restriction of* $V(\mathbf{x})$ *on* Ω_i *can be continuously extended to* $\partial\Omega_i$*.*

Denote the set of minima of $V(\mathbf{x}) \in C_p(\Omega)$ in Ω (*cf.* the assumption (7)) by

$$V^{(0)} = \left\{ \mathbf{y} \in \Omega \,\middle|\, \lim_{\mathbf{x} \to \mathbf{y}} V(\mathbf{x}) = \inf_{\mathbf{x} \in \Omega} V(\mathbf{x}) = 0 \right\}. \qquad (14)$$

Asymptotic Analysis for SPEP

Assume that $V(\mathbf{x})$ is smooth near its minima, *i.e.* for $\mathbf{x}_0 \in V^{(0)}$, there exists some $\gamma > 0$ and $r > 0$, *s.t.*

$$V(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|^\gamma \tilde{V}(\mathbf{x}), \quad \text{for } \mathbf{x} \in B_r(\mathbf{x}_0), \tag{15}$$

where $\tilde{V}(\mathbf{x})$ is bounded and positive in $\overline{B_r(\mathbf{x}_0)}$.

1. $V(\mathbf{x})$ has only isolated minima

Theorem 2.1

Assume that Ω is a bounded open domain with piecewise smooth boundary, and $V(\mathbf{x})$ has only isolated minima in Ω with the following assumption:

$$\lim_{x \in \Omega_i, x \to y} V(x) > 0, \quad \forall y \in \partial\Omega_i, \quad i = 1, 2, 3, \cdots \tag{16}$$

If (15) *holds true, we have that for some $\gamma > 0$,*

$$\lambda_n^\varepsilon = O\left(\varepsilon^{\frac{2\gamma}{\gamma+2}}\right), \quad \textit{as } \varepsilon \to 0^+, \quad n = 1, 2, 3, \cdots \tag{17}$$

Furthermore, if $V(\mathbf{x})$ has only one minimum $\tilde{\mathbf{x}}$, then we have

$$|\phi_n^\varepsilon(\mathbf{x})|^2 \xrightarrow{w} \delta(\mathbf{x} - \tilde{\mathbf{x}}), \quad \textit{as } \varepsilon \to 0^+, \quad n = 1, 2, 3, \cdots \tag{18}$$

in the weak sense.

Asymptotic Analysis for SPEP

2. The inner points set of $V^{(0)}$ is not empty

Theorem 2.2

Assume that the set of the inner points of $V^{(0)}$ is not empty and $\partial V^{(0)}$ is piecewise smooth. Then we have:

$$\lambda_n^\varepsilon = O(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0^+, \quad n = 1, 2, 3, \cdots, \tag{19}$$

and as $\varepsilon \to 0^+$, ϕ_n^ε will be almost concentrated in $V^{(0)}$, that is, for any $\delta > 0$, we have

$$\lim_{\varepsilon \to 0^+} \int_{\cup_i \{x \in \Omega_i | V(x) \geq \delta\}} |\phi_n^\varepsilon(x)|^2 dx = 0. \tag{20}$$

Asymptotic Analysis for SPEP

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$$\lim_{\varepsilon \to 0^+} \int_{\cup_i \{\mathrm{x} \in \Omega_i | V(\mathrm{x}) \geq \delta\}} |\phi_n^\varepsilon(\mathrm{x})|^2 d\mathrm{x} = 0. \tag{20}$$

Remark 2.1

Up to now we have studied the asymptotic behavior of the eigenvalues and eigenfunctions for the singularly perturbed eigenvalue problem (1) in bounded domain or $\mathbb{R}^N$ for a subspace of the piecewise continuous functions. However, it's still an open problem for more general potential $V(\mathbf{x})$ (*cf.* Helffer (2006)).

Outline

Approximation of SPEP

At first, we consider the numerical solution of the following 1D SPEP:

$$\begin{cases} -\varepsilon^2 u''(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x}) = \lambda u(\mathbf{x}), & \mathbf{x} \in \Omega = [-1, 1], \\ u(-1) = u(1) = 0, \\ \int_{-1}^{1} |u(\mathbf{x})|^2 d\mathbf{x} = 1. \end{cases} \tag{21}$$

We get a partition $\mathcal{T}_h$ of Ω by

$$-1 = \mathbf{x}_0 < \mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_{M-2} < \mathbf{x}_{M-1} < \mathbf{x}_M = 1,$$

such that $V(\mathbf{x})$ is smooth enough on $(\mathbf{x}_{i-1}, \mathbf{x}_i)$, and then the mesh size is taken to be $h_i = \mathbf{x}_i - \mathbf{x}_{i-1}$. Besides, U_i is the numerical approximation of $u(\mathbf{x}_i)$. On each subinterval $I_i = \{\mathbf{x} | \mathbf{x}_{i-1} \le \mathbf{x} \le \mathbf{x}_{i+1}\}$, we will construct a three-point discrete scheme at $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, *i.e.*

$$\alpha_i(\lambda)U_i + \beta_i(\lambda)U_{i-1} + \gamma_i(\lambda)U_{i+1} = 0, \tag{22}$$

where the formula of $\alpha_i(\lambda)$, $\beta_i(\lambda)$ and $\gamma_i(\lambda)$ depended on λ will be explained in detail below.

Smooth potential function case

Case 1

If $V(\mathbf{x})$ is differentiable at $\mathbf{x}_i$, we can linearize $V(\mathbf{x})$ locally on I_i by Taylor expansion:

$$V_h(\mathbf{x}) = a_i\mathbf{x} + b_i, \quad \mathbf{x} \in I_i. \tag{23}$$

Hence, we obtain an approximated differential equation of (21):

$$-\varepsilon^2 u''(\mathbf{x}) + (V_h(\mathbf{x}) - \lambda)u(\mathbf{x}) = 0, \quad \mathbf{x} \in I_i. \tag{24}$$

If $a_i \neq 0$, the general solutions of equation (24) have the following form:

$$u(\mathbf{x}) = \alpha Ai\big(z_i(\mathbf{x})\big) + \beta Bi\big(z_i(\mathbf{x})\big),$$

where $Ai(\mathbf{y})$, $Bi(\mathbf{y})$ are Airy functions and $z_i(\mathbf{x}) = \frac{a_i\mathbf{x}+b_i-\lambda}{\sqrt[3]{\varepsilon^2 a_i^2}}$, α and β are some constants.

Smooth potential function case

If $a_i = 0$ and $b_i > \lambda$, the equation (24) is elliptic and its general solutions are given by:

$$u(\mathbf{x}) = \alpha e^{\tau_i \mathbf{x}} + \beta e^{-\tau_i \mathbf{x}}, \tag{25}$$

where $\tau_i = \frac{\sqrt{|b_i - \lambda|}}{\varepsilon}$.

Similarly , when $a_i = 0$ and $b_i < \lambda$, the equation (24) is hyperbolic and its general solutions are given by:

$$u(\mathbf{x}) = \alpha \sin(\tau_i \mathbf{x}) + \beta \cos(\tau_i \mathbf{x}). \tag{26}$$

Moreover, when $a_i = 0$ and $b_i = \lambda$, the equation (24) has the following form:

$$-u''(\mathbf{x}) = 0.$$

Its solutions have the following form:

$$u(\mathbf{x}) = \alpha + \beta \mathbf{x}. \tag{27}$$

TFPM scheme for SPEP

According to the principle of the tailored finite point method, we can determine the coefficients of (22) by

$$\alpha_i(\lambda)w(\mathbf{x}_i,\lambda)+\beta_i(\lambda)w(\mathbf{x}_{i-1},\lambda)+\gamma_i(\lambda)w(\mathbf{x}_{i+1},\lambda)=0, \quad \forall w(\mathbf{x},\lambda)\in\mathcal{W}_{i,\lambda}, \quad (28)$$

for the solution space $\mathcal{W}_{i,\lambda}$ of the equation (24). Namely, we can choose the two dimensional subspace

$$\mathcal{W}_{i,\lambda}=\{w_i^1(\mathbf{x},\lambda),w_i^2(\mathbf{x},\lambda)\},$$

with the linear independent basic functions:

$$w_i^1(\mathbf{x},\lambda)=\begin{cases}Ai\big(z_i(\mathbf{x})\big),\\ e^{-\tau_i\mathbf{x}},\\ \sin(\tau_i\mathbf{x}),\\ 1.\end{cases} \qquad w_i^2(\mathbf{x},\lambda)=\begin{cases}Bi\big(z_i(\mathbf{x})\big), & a_i\neq 0, \quad b_i\in\mathbb{R},\\ e^{\tau_i\mathbf{x}}, & a_i=0, \quad b_i>\lambda,\\ \cos(\tau_i\mathbf{x}), & a_i=0, \quad b_i<\lambda,\\ \mathbf{x}, & a_i=0, \quad b_i=\lambda;\end{cases}$$

TFPM scheme for SPEP

As $h_i \le \dfrac{\pi\varepsilon}{\sqrt{\lambda}}$, the relation among coefficients in (22) can be uniquely determined by (28):

$$\begin{cases} \beta_i(\lambda) = \dfrac{w_i^1(\mathbf{x}_i,\lambda)w_i^2(\mathbf{x}_{i+1},\lambda)-w_i^2(\mathbf{x}_i,\lambda)w_i^1(\mathbf{x}_{i+1},\lambda)}{w_i^2(\mathbf{x}_{i-1},\lambda)w_i^1(\mathbf{x}_{i+1},\lambda)-w_i^1(\mathbf{x}_{i-1},\lambda)w_i^2(\mathbf{x}_{i+1},\lambda)}\alpha_i(\lambda), \\ \gamma_i(\lambda) = \dfrac{w_i^1(\mathbf{x}_i,\lambda)w_i^2(\mathbf{x}_{i-1},\lambda)-w_i^2(\mathbf{x}_i,\lambda)w_i^1(\mathbf{x}_{i-1},\lambda)}{w_i^2(\mathbf{x}_{i+1},\lambda)w_i^1(\mathbf{x}_{i-1},\lambda)-w_i^1(\mathbf{x}_{i+1},\lambda)w_i^2(\mathbf{x}_{i-1},\lambda)}\alpha_i(\lambda). \end{cases} \tag{29}$$

Especially, when $a_i = 0$ and $h_i = h_{i+1} = h$, we have

$$\frac{\alpha_i(\lambda)}{\beta_i(\lambda)} = \frac{\alpha_i(\lambda)}{\gamma_i(\lambda)} = \begin{cases} -e^{\tau_i h} - e^{-\tau_i h}, & b_i > \lambda, \\ -2\cos\tau_i h, & b_i < \lambda, \\ -2, & b_i = \lambda. \end{cases} \tag{30}$$

Non-smooth potential case

Case 2

If $V(\mathbf{x})$ is non-differentiable at $\mathbf{x}_i$, we can approximate $V(\mathbf{x})$ by piecewise linear function

$$\tilde{V}_h(\mathbf{x}) = \begin{cases} \tilde{a}_{i-1}\mathbf{x} + \tilde{b}_{i-1}, & \mathbf{x} \in [\mathbf{x}_{i-1}, \mathbf{x}_i], \\ \tilde{a}_i\mathbf{x} + \tilde{b}_i, & \mathbf{x} \in [\mathbf{x}_i, \mathbf{x}_{i+1}]. \end{cases} \tag{31}$$

Then, let us consider the following two boundary-value problems:

$$\begin{cases} -\varepsilon^2 u''(\mathbf{x}) + \big(\tilde{V}_h(\mathbf{x}) - \lambda\big)u(\mathbf{x}) = 0, & \mathbf{x} \in [\mathbf{x}_{i-1}, \mathbf{x}_i], \\ u(\mathbf{x}_{i-1}) = U_{i-1}, \quad u(\mathbf{x}_i) = U_i, \end{cases} \tag{32}$$

and

$$\begin{cases} -\varepsilon^2 u''(\mathbf{x}) + \big(\tilde{V}_h(\mathbf{x}) - \lambda\big)u(\mathbf{x}) = 0, & \mathbf{x} \in [\mathbf{x}_i, \mathbf{x}_{i+1}], \\ u(\mathbf{x}_{i+1}) = U_{i+1}, \quad u(\mathbf{x}_i) = U_i, \end{cases} \tag{33}$$

Non-smooth potential case

By simple calculation, we have:

$$u(\mathbf{x}) = \begin{cases} p_{i-1}(\mathbf{x},\lambda)U_i + q_{i-1}(\mathbf{x},\lambda)U_{i-1}, & \mathbf{x} \in [\mathbf{x}_{i-1}, \mathbf{x}_i], \\ p_i(\mathbf{x},\lambda)U_{i+1} + q_i(\mathbf{x},\lambda)U_i, & \mathbf{x} \in [\mathbf{x}_i, \mathbf{x}_{i+1}], \end{cases} \quad (34)$$

where

$$p_i(\mathbf{x},\lambda) = \frac{\tilde{w}_i^2(\mathbf{x}_i,\lambda)\tilde{w}_i^1(\mathbf{x},\lambda) - \tilde{w}_i^1(\mathbf{x}_i,\lambda)\tilde{w}_i^2(\mathbf{x},\lambda)}{\tilde{w}_i^2(\mathbf{x}_i,\lambda)\tilde{w}_i^1(\mathbf{x}_{i+1},\lambda) - \tilde{w}_i^1(\mathbf{x}_i,\lambda)\tilde{w}_i^2(\mathbf{x}_{i+1},\lambda)},$$

and

$$q_i(\mathbf{x},\lambda) = \frac{\tilde{w}_i^1(\mathbf{x}_{i+1},\lambda)\tilde{w}_i^2(\mathbf{x},\lambda) - \tilde{w}_i^2(\mathbf{x}_{i+1},\lambda)\tilde{w}_i^1(\mathbf{x},\lambda)}{\tilde{w}_i^2(\mathbf{x}_i,\lambda)\tilde{w}_i^1(\mathbf{x}_{i+1},\lambda) - \tilde{w}_i^1(\mathbf{x}_i,\lambda)\tilde{w}_i^2(\mathbf{x}_{i+1},\lambda)}.$$

The above functions $\{\tilde{w}_i^k(\mathbf{x},\lambda), k = 1, 2\}$ are defined in $[\mathbf{x}_i, \mathbf{x}_{i+1}]$ by the same way as in **Case 1**.

In fact, $u(\mathbf{x})$ defined in (34) is continuous at $\mathbf{x}_i$. To ensure the regularity at $\mathbf{x}_i$, $u(\mathbf{x})$ needs to be differentiable at $\mathbf{x}_i$, that is

$$p_{i-1,\mathbf{x}}(\mathbf{x}_i,\lambda)U_i + q_{i-1,\mathbf{x}}(\mathbf{x}_i,\lambda)U_{i-1} = p_{i,\mathbf{x}}(\mathbf{x}_i,\lambda)U_{i+1} + q_{i,\mathbf{x}}(\mathbf{x}_i,\lambda)U_i.$$

Nonlinear algebra eigenvalue problem

Hence, we can get the coefficients of discrete scheme (22):

$$\begin{cases} \beta_i(\lambda) = \frac{q_{i-1,\mathbf{x}}(\mathbf{x}_i,\lambda)}{p_{i-1,\mathbf{x}}(\mathbf{x}_i,\lambda)-q_{i,\mathbf{x}}(\mathbf{x}_i,\lambda)}\alpha_i(\lambda), \\ \gamma_i(\lambda) = \frac{p_{i,\mathbf{x}}(\mathbf{x}_i,\lambda)}{q_{i,\mathbf{x}}(\mathbf{x}_i,\lambda)-p_{i-1,\mathbf{x}}(\mathbf{x}_i,\lambda)}\alpha_i(\lambda). \end{cases} \tag{35}$$

Let the number of total interior points be M, and denote the vector

$$\mathbf{u} = [U_1, U_2, \cdots, U_M]^T.$$

Then the discrete scheme (22) leads to a nonlinear eigenvalue problem (NLEP)

$$\begin{cases} \mathbf{A}(\lambda)\mathbf{u} = 0, \\ \|\mathbf{u}\|_2 = 1, \end{cases} \tag{36}$$

Nonlinear algebra eigenvalue problem

where

$$\mathbf{A}(\lambda) = \begin{pmatrix} \alpha_1(\lambda) & \gamma_1(\lambda) & & & \\ \beta_2(\lambda) & \alpha_2(\lambda) & \gamma_2(\lambda) & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{M-1}(\lambda) & \alpha_{M-1}(\lambda) & \gamma_{M-1}(\lambda) \\ & & & \beta_M(\lambda) & \alpha_M(\lambda) \end{pmatrix}. \tag{37}$$

and

$$\|\mathbf{u}\|_2^2 = \sum_{i=1}^{M} \frac{(\mathbf{x}_{i+1} - \mathbf{x}_{i-1})|U_i|^2}{2}. \tag{38}$$

Nonlinear algebra eigenvalue problem

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and

$$\|\mathbf{u}\|_2^2 = \sum_{i=1}^{M} \frac{(\mathbf{x}_{i+1} - \mathbf{x}_{i-1})|U_i|^2}{2}. \quad (38)$$

In general, it is not easy to solve the nonlinear eigenvalue problem (36) with high accuracy. Next, we introduce a linear TFPM scheme.

Linear TFPM scheme for SPEP

Consider the following elliptic equation:

$$-\varepsilon^2 u''(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}),$$

and then we can approximate locally $V(\mathbf{x})$ on I_i by $V_i(\mathbf{x}) = a_i\mathbf{x} + b_i$ and $f(\mathbf{x})$ by $\bar{f}_i = \frac{1}{|\mathbf{x}_{i+1} - \mathbf{x}_{i-1}|}\int_{\mathbf{x}_{i+1}}^{\mathbf{x}_{i-1}} f(\mathbf{x})d\mathbf{x}$. Assume that $u(\mathbf{x})$ is a solution of the following equation:

$$-\varepsilon^2 u''(\mathbf{x}) + V_i(\mathbf{x})u(\mathbf{x}) = \bar{f}_i.$$

If we denote $w(\mathbf{x}) = u(\mathbf{x}) - \bar{f}\mu(\mathbf{x})$ where $\mu(\mathbf{x})$ is a particular solution of the following equation:

$$-\varepsilon^2 \mu''(\mathbf{x}) + V(\mathbf{x})\mu(\mathbf{x}) = 1,$$

and we can choose

$$\mu(\mathbf{x}) = \varepsilon^{-2}\int_0^{\mathbf{x}} \frac{\omega_i^2(s,0)\omega_i^1(\mathbf{x},0) - \omega_i^1(s,0)\omega_i^2(\mathbf{x},0)}{\dot{\omega}_i^2(s,0)\omega_i^1(s,0) - \dot{\omega}_i^1(s,0)\omega_i^2(s,0)}ds.$$

Linear TFPM scheme for SPEP

Then it is straightforward to show

$$-\varepsilon^2 w''(\mathbf{x}) + V_i(\mathbf{x})w(\mathbf{x}) = 0. \quad (39)$$

Then, we can construct a TFPM scheme to solve equation (39) in I_i:

$$\alpha_i(0)W_i + \beta_i(0)W_{i-1} + \gamma_i(0)W_{i+1} = 0, \quad (40)$$

where W_i is the numerical approximation of $w(\mathbf{x}_i)$. According to the definition of $w(\mathbf{x})$, we have

$$W_i = U_i - \bar{f}_i\mu_i. \quad (41)$$

Substituting (41) into (40), we get a new TFPM scheme:

$$\alpha_i U_i + \beta_i U_{i-1} + \gamma_i U_{i+1} = \bar{f}_i.$$

where

$$\begin{cases} \alpha_i = \frac{\alpha_i(0)}{\alpha_i(0)\mu_i+\beta_i(0)\mu_{i-1}+\gamma_i(0)\mu_{i+1}}, \\ \beta_i = \frac{\beta_i(0)}{\alpha_i(0)\mu_i+\beta_i(0)\mu_{i-1}+\gamma_i(0)\mu_{i+1}}, \\ \gamma_i = \frac{\gamma_i(0)}{\alpha_i(0)\mu_i+\beta_i(0)\mu_{i-1}+\gamma_i(0)\mu_{i+1}}, \end{cases} \quad (42)$$

Linear TFPM scheme for SPEP

The above TFPM scheme leads to a linear eigenvalue problem (LEP):

$$\begin{cases} \mathbf{A}\mathbf{u} = \lambda \mathbf{u}, \\ \|\mathbf{u}\|_2 = 1, \end{cases} \tag{43}$$

where the vector $\mathbf{u}$, as well as the norm $\|\cdot\|_2$ are defined as (36)–(38) and

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \gamma_1 & & & \\ \beta_2 & \alpha_2 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{M-1} & \alpha_{M-1} & \gamma_{M-1} \\ & & & \beta_M & \alpha_M \end{pmatrix}. \tag{44}$$

We can solve the LEP (43) by inverse power method. Furthermore, the LEP (43) is a linear approximation to NLEP (36), and then we can set its solutions as the initial value when we use Newton's method to solve the NLEP (36).

Asymptotical preserving property

In a sense, our TFPM scheme can preserve some asymptotic properties of the eigen-states of the SPE problem as $\varepsilon \to 0^+$.

According to Martinez (1988) and Reyes (2016), for the case $V(\mathbf{x}) \in C^\infty(\Omega)$ has finite isolated zeros, the eigenfunctions $u_k^\varepsilon(\mathbf{x})$ will exponentially decay as $\varepsilon \to 0^+$ for any $\mathbf{x} \in \{\mathbf{x} \in \Omega \subset \mathbb{R} | V(\mathbf{x}) - \lambda_k^\varepsilon > 0\}$, *i.e.*

$$|u_k^\varepsilon(\mathbf{x})| \le Ce^{-\frac{\int_{c(\varepsilon)}^{\mathbf{x}} \sqrt{V(t)-\lambda_k^\varepsilon}\,dt}{\varepsilon}}, \qquad \varepsilon \to 0^+, \tag{45}$$

with some selected functions $c(\varepsilon)$ and a positive constant C. In fact, the numerical solution U_i^{FDM} by typical finite difference method (FDM) satisfies that for any $\mathbf{x} \in \{\mathbf{x} \in \Omega \subset \mathbb{R} | V_h(\mathbf{x}) - \lambda_h^\varepsilon > 0\}$,

$$\frac{|U_i^{FDM}|}{\max_{k=0}^{M} |U_k^{FDM}|} \le \frac{2\varepsilon^2}{h_i h_{i+1} \left(V_h(\mathbf{x}_i) - \lambda_h^\varepsilon\right)}, \qquad \varepsilon \to 0^+.$$

Hence, the U_i^{FDM} will decay at a rate of ε^2 as $\varepsilon \to 0^+$, which is much slower than theoretical result.

Asymptotical preserving property

Consider our three-point discrete scheme (22). According to the asymptotic formula for $Ai(\mathbf{y})$ and $Bi(\mathbf{y})$ as $\mathbf{y} \to +\infty$, we have that for any $\mathbf{x} \in \{\mathbf{x} \in \Omega | V_h(\mathbf{x}) - \lambda_h^\varepsilon > 0\}$,

$$w_i^k(\mathbf{x}, \lambda_h^\varepsilon) \sim \frac{\sqrt[6]{|a_i \varepsilon|} e^{(-1)^k \cdot \frac{2\sqrt{[V_h(\mathbf{x}) - \lambda_h^\varepsilon]^3}}{3|a_i \varepsilon|}}}{2\sqrt[4]{\pi^2 V_h(\mathbf{x}) - \pi^2 \lambda_h^\varepsilon}}, \quad k = 1, 2, \qquad \varepsilon \to 0^+.$$

Then a routine computation gives rise to

$$\frac{\beta_i(\lambda_h^\varepsilon)}{\alpha_i(\lambda_h^\varepsilon)} \approx -e^{\frac{-\sqrt{V_h(\mathbf{x}_i) - \lambda_h^\varepsilon}}{\varepsilon} h_i}, \quad \frac{\gamma_i(\lambda_h^\varepsilon)}{\alpha_i(\lambda_h^\varepsilon)} \approx -e^{\frac{-\sqrt{V_h(\mathbf{x}_i) - \lambda_h^\varepsilon}}{\varepsilon} h_{i+1}}, \qquad \varepsilon \to 0^+.$$

Hence, we obtain that

$$\frac{|U_i|}{\max_{k=0}^M |U_k|} \le C e^{\frac{-\sqrt{V_h(\mathbf{x}_i) - \lambda_h^\varepsilon}}{\varepsilon} h}, \qquad \varepsilon \to 0^+,$$

where $h = \min\{h_i, h_{i+1}\}$. That means, the numerical solution U_i by our TFPM scheme preserves the asymptotic property (45).

Outline

Numerical Examples

In this section, we present some numerical examples to verify our theory obtained in Section 2 and show the efficiency of our TFPM given in Section 3.

Example 4.1

Consider the following 1D/2D harmonic potential case:

$$\begin{cases} -\varepsilon^2 u''(\mathbf{x}) + \mathbf{x}^2 u(\mathbf{x}) = \lambda u(\mathbf{x}), \quad \mathbf{x} \in \Omega = [-1,1]^n,\ n = 1,2, \\ u|_{\partial\Omega} = 0, \\ \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x} = 1. \end{cases} \tag{46}$$

Numerical Examples

Table 1: The 1st, 3rd and 5th eigenvalue for SPEP with harmonic potential in 1-D compared with the asymptotic analysis result for different ε, with $h = 1/1000$.

ε	0.2	0.1	0.05	0.025
$\lambda_{0,h}$	0.2061	0.1000	0.0500	0.0250
$\lvert\lambda_{0,h} - \varepsilon\rvert$	6.1e-03	3.2e-05	9.6e-07	9.5e-07
$\lambda_{2,h}$	1.1991	0.5041	0.2500	0.1250
$\lvert\lambda_{2,h} - 5\varepsilon\rvert$	2.0e-01	4.1e-03	6.4e-07	4.2e-08
$\lambda_{4,h}$	2.7985	0.9571	0.4500	0.2250
$\lvert\lambda_{4,h} - 9\varepsilon\rvert$	9.9e-01	5.7e-02	5.6e-05	2.3e-06

Numerical Examples

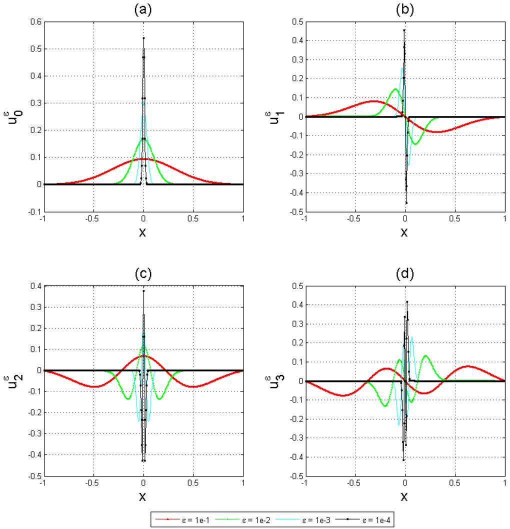


Figure 1: First four eigenfunctions with harmonic potential in 1-D, h = 1/200.

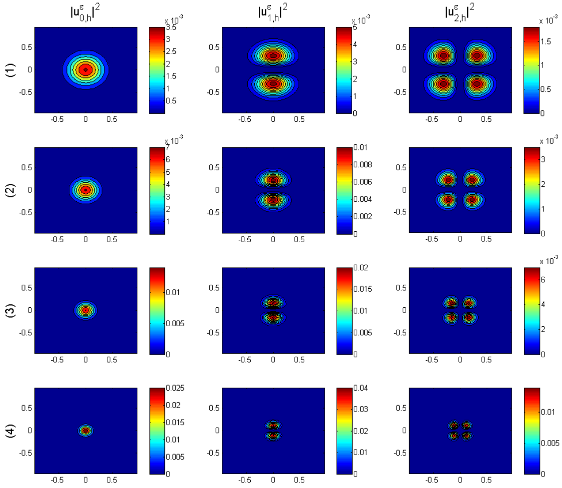
Numerical Examples

Table 2: Errors of the first three eigenvalues and eigenfunctions for SPEP with harmonic potential as $\varepsilon = 0.001$

h	1/20	1/40	1/80	1/160
$\|\lambda_0 - \lambda_{0,h}\|$	2.1145e-03	5.2262e-04	1.3034e-04	3.2562e-05
$\|\|u_0^\varepsilon - u_{0,h}^\varepsilon\|\|_2$	4.6305e-02	1.0431e-03	5.3166e-05	4.1527e-06
$\|\lambda_1 - \lambda_{1,h}\|$	2.0383e-03	5.2790e-04	1.3063e-04	3.2585e-05
$\|\|u_1^\varepsilon - u_{1,h}^\varepsilon\|\|_2$	3.9557e-03	2.0981e-03	9.6228e-05	8.4877e-06
$\|\lambda_2 - \lambda_{2,h}\|$	1.4479e-03	5.3455e-04	1.3094e-04	3.2615e-05
$\|\|u_2^\varepsilon - u_{2,h}^\varepsilon\|\|_2$	6.9979e-02	3.7266e-03	1.5043e-04	1.4425e-05

Numerical Examples

Figure 2: The plot for the square of the absolute value of the first three eigenfunctions with 2D harmonic potential for different ε as $h = 1/20$: (1) $\varepsilon = 0.1$, (2) $\varepsilon = 0.05$, (3) $\varepsilon = 0.025$, (4) $\varepsilon = 0.0125$.



Numerical Examples

Table 3: The first six eigenvalues for SPEP with 2D harmonic potential as $h = 1/20$

ε	$\lambda_{0,h}$	2ε	$\lambda_{1,h}$	$\lambda_{2,h}$	4ε	$\lambda_{3,h}$	$\lambda_{4,h}$	$\lambda_{5,h}$	6ε
0.1	0.1999	0.2000	0.4004	0.3997	0.4000	0.6003	0.6021	0.6025	0.6000
0.05	0.0986	0.1000	0.2000	0.1997	0.2000	0.3005	0.2990	0.2970	0.3000
0.025	0.0500	0.0500	0.1000	0.1000	0.1000	0.1498	0.1497	0.1500	0.1500
0.00625	0.0125	0.0125	0.0250	0.0250	0.0250	0.0375	0.0375	0.0375	0.0375

Table 4: The approximate eigenvalues for SPEP with 2D harmonic potential obtained by FDM and TFPM for $\varepsilon = 0.01$

$h = 1/20$	FDM	TFPM
$\lambda_0 = 0.0200$	0.0197	0.0200
$\lambda_2 = 0.0400$	0.0390	0.0400
$\lambda_4 = 0.0600$	0.0577	0.0599
$\lambda_6 = 0.0800$	0.0757	0.0800
$\lambda_{12} = 0.1000$	0.0929	0.0999
$\lambda_{16} = 0.1200$	0.1092	0.1199

Numerical Examples

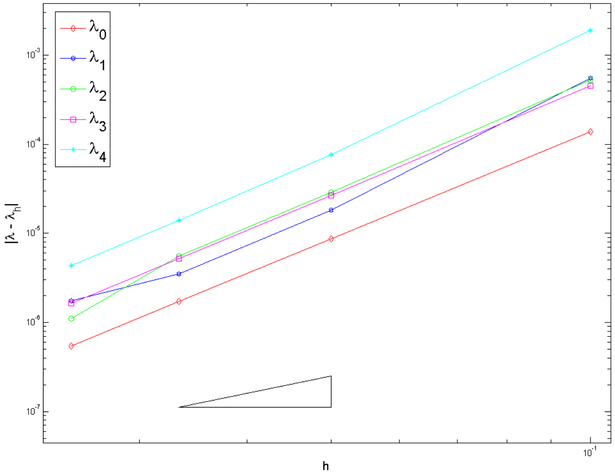


Figure 3: Errors of eigenvalues for different h as $\varepsilon = 0.01$ in logarithm scale where the slope of the hypotenuse is '2'.

Numerical Examples

Example 4.2

We finally consider a piecewise constant potential model:

$$\begin{cases} -\varepsilon^2 u''(\mathbf{x}) + V(\mathbf{x})u(\mathbf{x}) = \lambda u(\mathbf{x}), & \mathbf{x} \in \Omega \\ u|_{\partial\Omega} = 0, \\ \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x} = 1. \end{cases} \quad (47)$$

Here

$$\Omega = [-1, 1], \quad V(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in [-1, -0.5] \\ 0, & \mathbf{x} \in [-0.5, 0.5] \\ 1, & \mathbf{x} \in [0.5, 1] \end{cases}$$

or

$$\Omega = [-1, 1]^2, \quad V(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in [-0.5, 0.5] \times [-0.5, 0.5] \\ 1, & \text{otherwise} \end{cases}$$

Numerical Examples

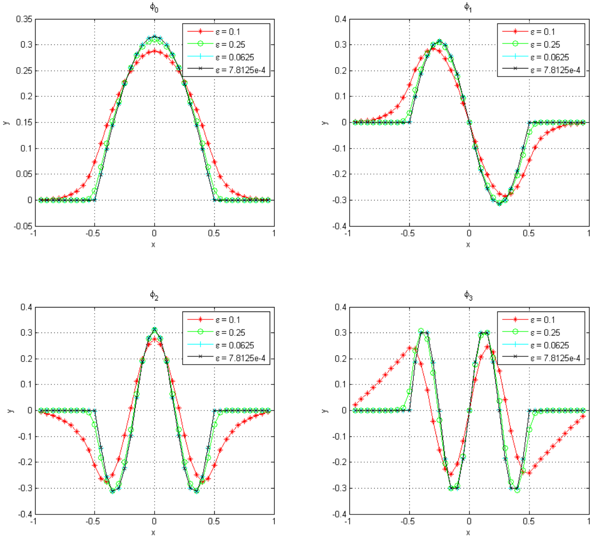


Figure 4: First four eigenfunctions with piecewise constant potential in 1-D, h = 1/20.

Numerical Examples

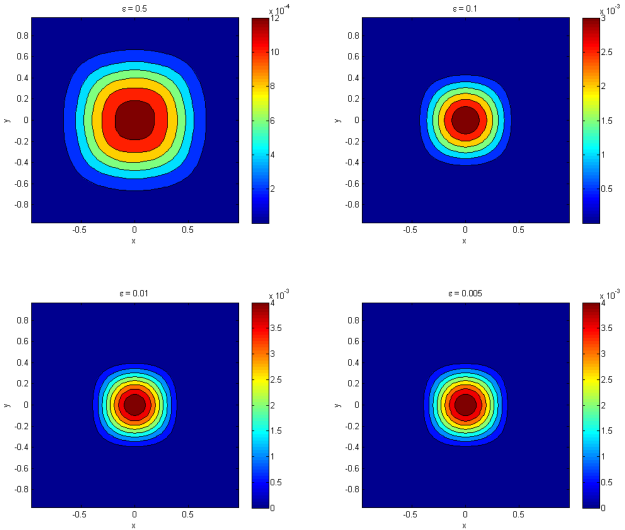


Figure 5: Ground-state with piecewise constant potential in 2-D, h = 1/20.

Numerical Examples

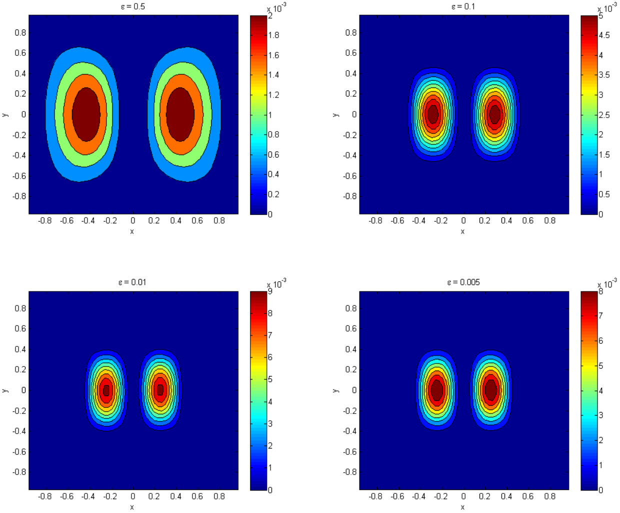


Figure 6: First excited-state with piecewise constant potential in 2-D, h = 1/20.

Outline

1. Asymptotic analysis of the singularly perturbed eigenvalue problem
2. TFPM for the singularly perturbed eigenvalue problem (SPEP)
   - Efficient TFPM for SPEP
   - Asymptotical preserving property
3. Numerical Implementation
4. Conclusion

Conclusion

In this talk, we study the asymptotic behavior for the solutions of the singularly perturbed eigenvalue problem (SPEP) (1). For a wide kind of potential functions, we prove that, as $\varepsilon \to 0^+$,

- the eigenvalues converges to the minimum value of the potential $V(\mathbf{x})$
- and the eigenfunctions are concentrated in the immediate vicinity of the minimal points of the potential.

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To get the high accurate approximation of the eigenvalues and the fine structure of the eigenfunctions, we introduce the linear/nonlinear tailored finite point method (TFPM) to solve SPEP (1) numerically.

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Certainly, the construction of the TFPM schemes for more complicated (non)linear SPEP and the corresponding convergence analysis are still open.

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Certainly, the construction of the TFPM schemes for more complicated (non)linear SPEP and the corresponding convergence analysis are still open.

Thank you for your attention!