Inverse Scattering Problems: To Overcome the Ill-posedness

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Acknowledgements

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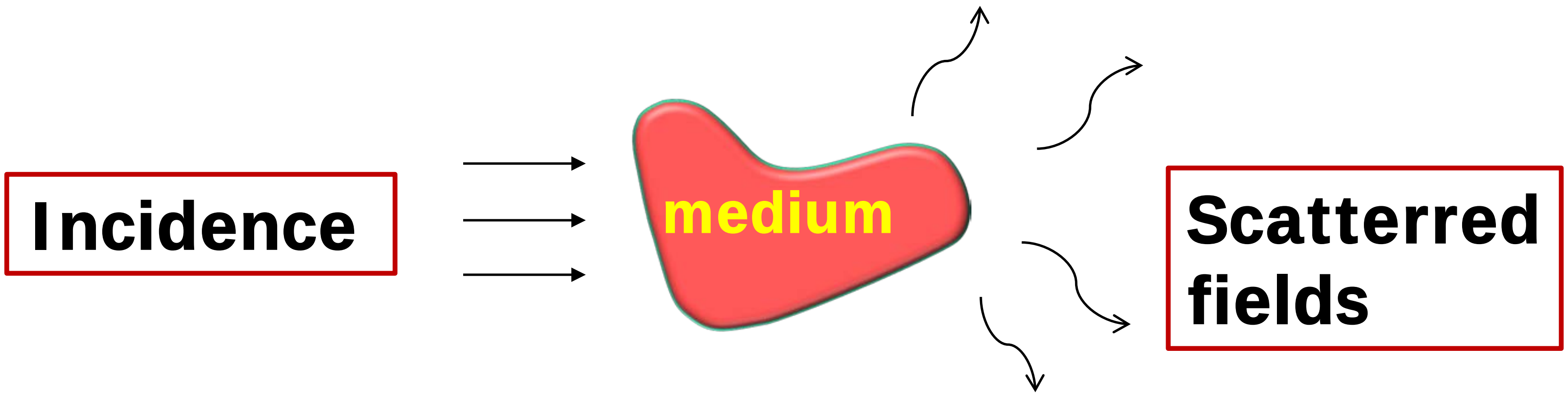
Outline

- **Introduction**
- **Inverse Scattering : Analysis and Computation**
- **Inverse Source Problems**
- **On-going Research**

Scattering and Inverse Scattering

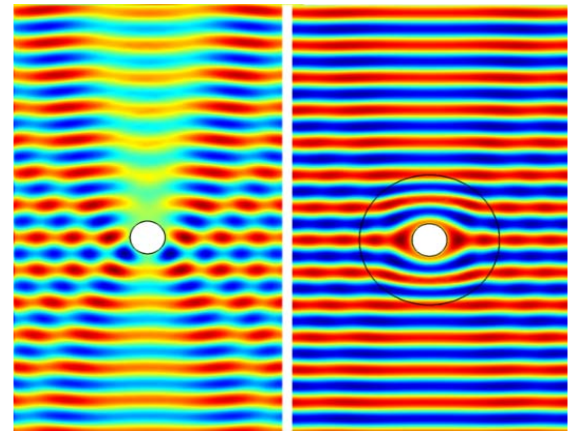


- **Scattering**
- **Inverse scattering**

Applications

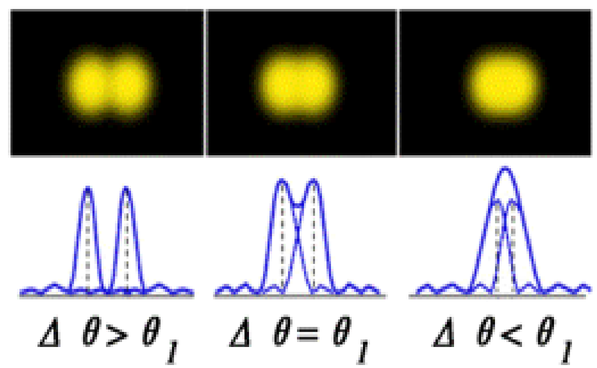
Stealth

Geophysical inspection



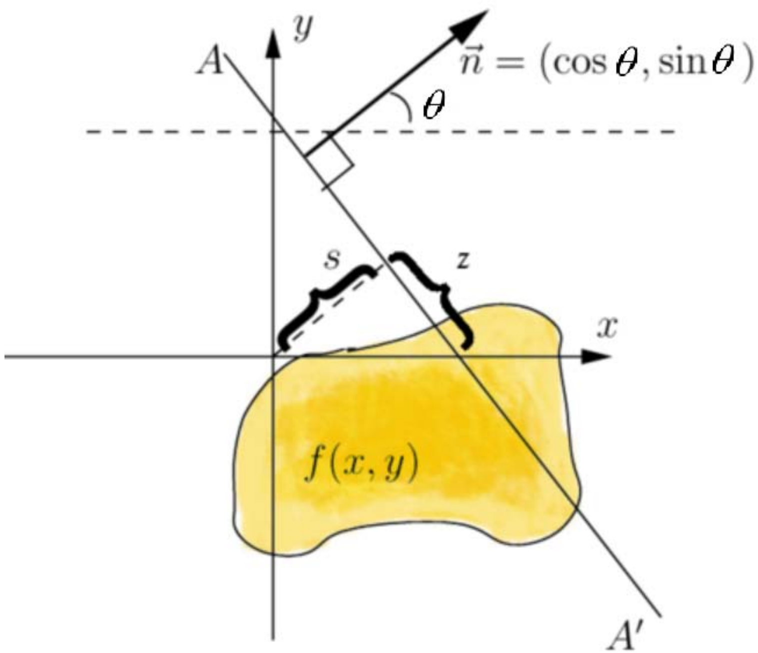
Cloaking

Medical imaging



Super-resolution

Application: Radon Transform



Radon transform（1917）

$$Rf(s,\theta) = g(s,\theta) = \int_{\langle x,\theta\rangle = s} f(x)dl = \int_L f$$

Inverse Radon transform

$$f(x) = \frac{1}{4\pi^2}\, p.v. \int_{S^1} d\theta \int \frac{\frac{d}{ds} g(s,\theta) ds}{\langle x,\theta\rangle - s}$$

Radon, J.: ***Uber die Bestimmung von Funktionen durch ihre Integralwerte Langs Gewisser Mannigfaltigkeiten.*** **Ber. Saechsische Akad. Wiss. 29, 262 (1917)**

Application: Radon Transform

1979 Nobel Prize in Medicine
Computed Tomography（CT）

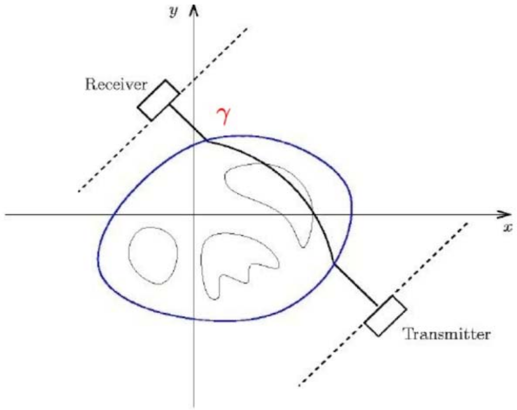
A. Cormack

G. Hounsfield

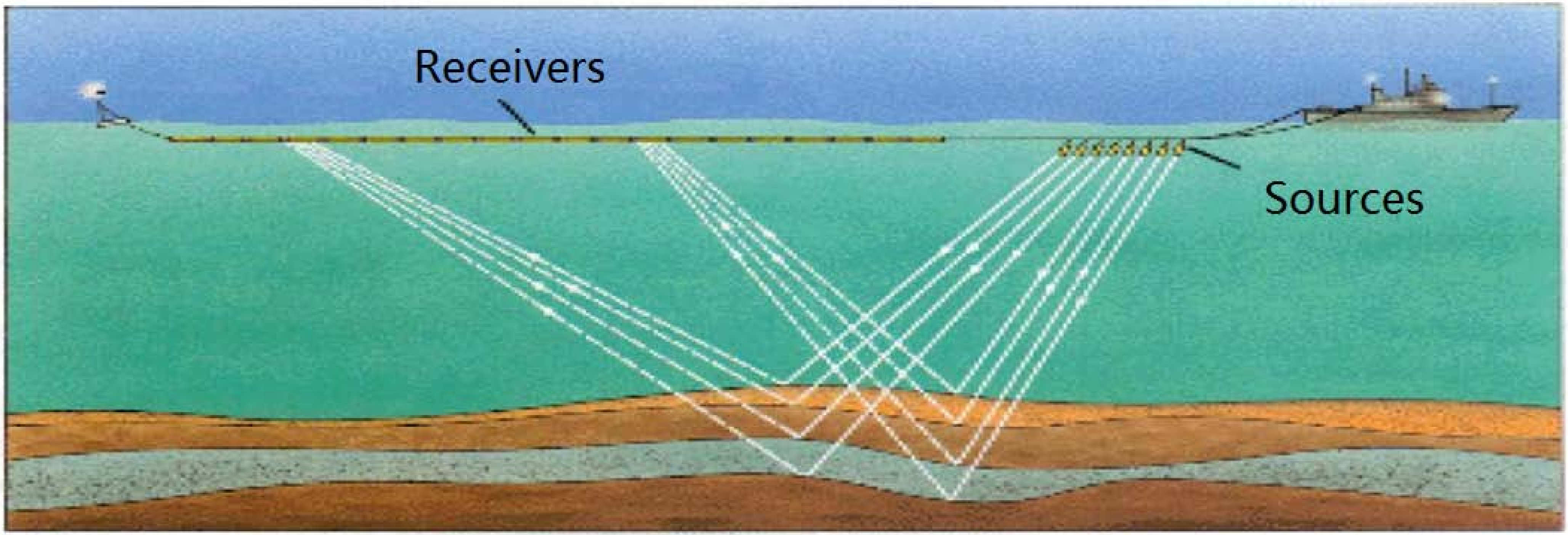
(From Wikipedia)

Integral geometry

$$T = \int_{\gamma} \frac{1}{c(x)} ds = \text{Travel time}$$



Application: Seismic Inversion



$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) p(x, y, z, t) = 0$$

Data: $p(x, y, z, t)|_{z=0}$

To determine $c(x, y, z)$?

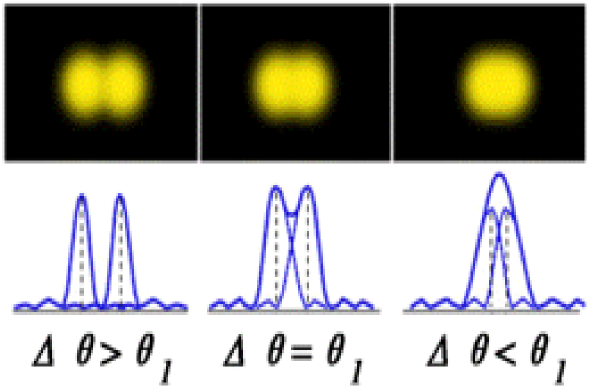
Boundary rigidity problem:

Michel 81', Gromov, 83', …

Lens rigidity problem:

Largely open!

Application: Near-Field Optics



IOP: **Betzig 93'**

Diffraction limit: $\theta_1 = \lambda/2$

2014 Noble Prize in Chemistry

Eric Betzig (92', 93') evanescent $\theta_1 = \lambda/10$,

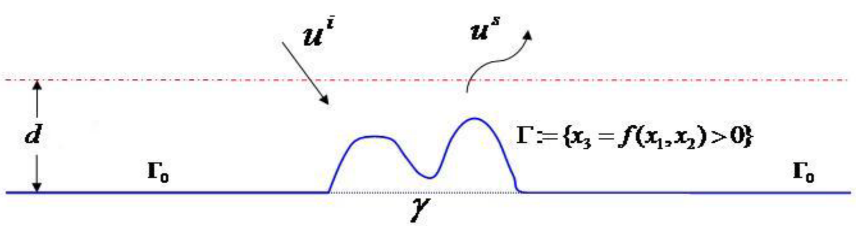
(95') PSF

$$\Delta_{\min} = \frac{\Delta}{\sqrt{N}} = \frac{1}{\sqrt{N}} \frac{\lambda}{2n \sin\alpha}$$

Stefan W. Hell (94', 95') STED

$$\Delta_{\min} \approx \frac{\lambda}{2n \sin\alpha(\sqrt{1 + I_0 / I_{sat}})}$$

William E. Moerner experiment



$$\begin{cases} \Delta u + k^2 u = 0 \\ \dfrac{\partial u}{\partial r} - ikr = o(\dfrac{1}{r}) \quad \text{as } r \to \infty \\ u + u^i + u^r = 0 \end{cases}$$

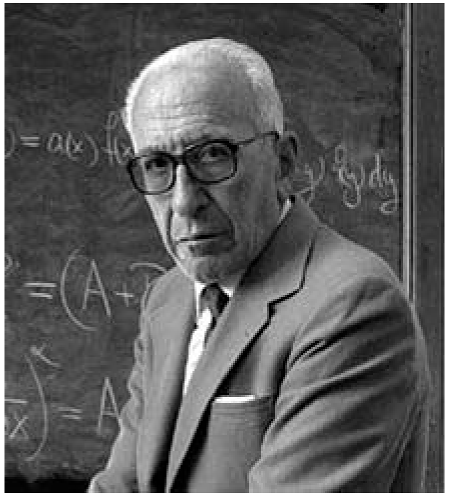
$$\text{Measurements}: u(x_1, x_2, d), \quad d - \max f < \lambda,$$

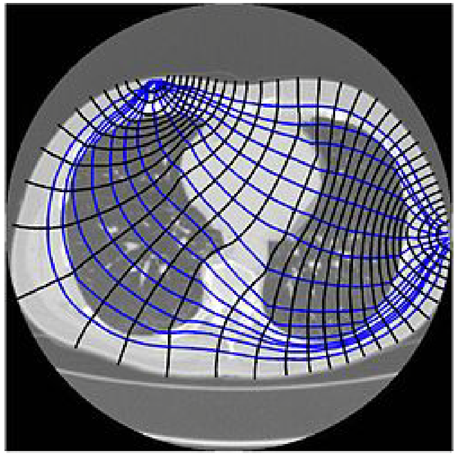
$$\text{to determine:} \quad f(x_1, x_2)$$

Challenges for Inverse Scattering

- **Ill-posedness**
- **Nonlinearity**
- **Computation**
- **Uncertainty**

Calderón Problem





To determine γ from Λ

(voltage to current map)

$$\nabla \cdot (\gamma(x)\nabla \mathbf{u}) = 0$$

$$\Lambda\colon \mathbf{u}\Big|_{\Gamma} \to \gamma \frac{\partial u}{\partial \nu}\Big|_{\Gamma}$$

- **Calderón A. P.: On an inverse boundary value problem, *Seminar on Numerical Analysis and its Applications to Continuum Physics* (Rio de Janeiro, 1980) Soc. Brasil Mater. 65-73 Rio de Janeiro**

Calderón Problem: Progress

- **Kohn & Vogelius 1984, CPAM**
- **Sylvester & Uhlmann 1987, Ann. Math.**
- **Nachman 1988, Ann. Math.**
- **Nachman 1995, Ann. Math.**
- **Pestov & Uhlmann 2005, Ann. Math.**
- **Astala & Päivärinta 2006, Ann. Math.**
- **Kenig, Sjöstrand & Uhlmann 2007, Ann. Math.**
- **Imanuvilov, Uhlmann & Yamamoto 2010, JAMS**
- **Uhlmann & Vasy 2016, Invent. Math.**

Calderón Problem: Challenges

- **Unstable! Ill-posedness…**

  Alessandrini 1987,

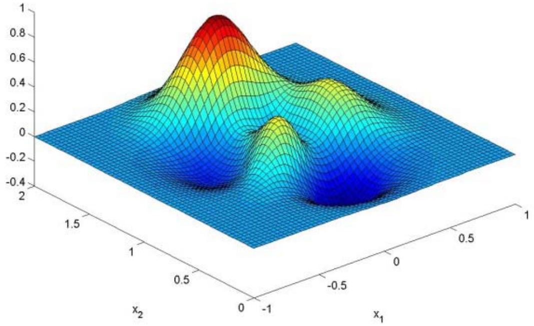
  Mandache 2001

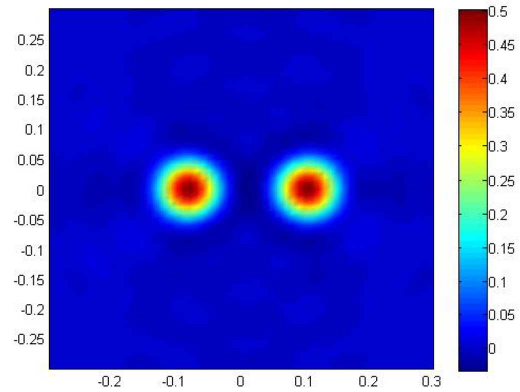
Same ill-posedness for Helmholtz/Maxwell eqns at a fixed frequency! No lucky break…

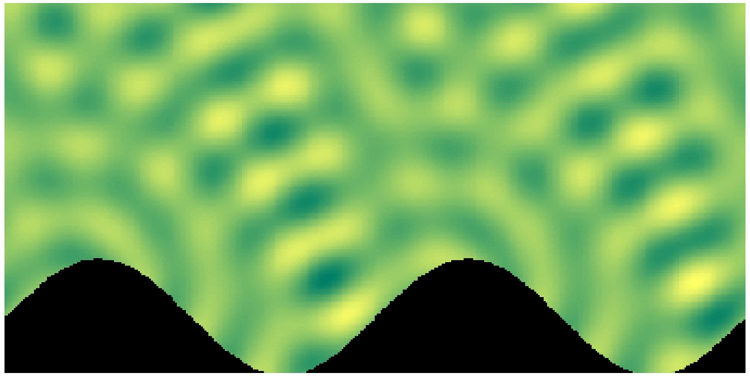
- **Remedy:**

  Hybrid, Multiple frequency data

Inverse Scattering Problems







IMP **ISP** **IOP**

- **Models: Electromagnetic, Optics, Acoustics, Elasticity**
- **Measurements: boundary**
- **Multiple Frequency**

Stability for the IP of Wave Equation

$$\left(\frac{1}{c^2(x)}\partial_t^2 - \Delta\right)u = 0$$

$$\Lambda : u\big|_\Gamma \to \frac{\partial u}{\partial \upsilon}\bigg|_\Gamma, \text{ to find } c(x)$$

Uniqueness:

Boundary control theory, Belishev & Kurylev 92'

Stability (partial results): Uhlmann, Lassas, Vasy et.al, 98' --now

GO, No Caustics!

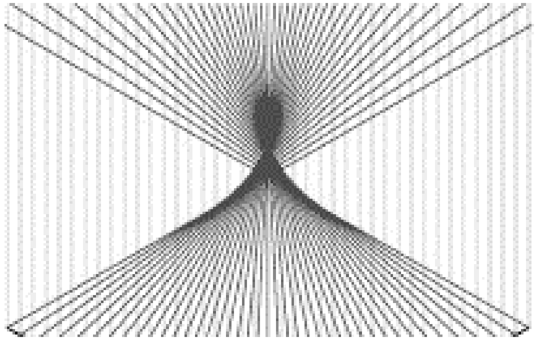
Geometric Optics

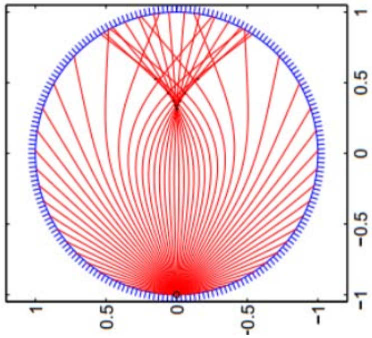
For wave equation in time domain, WKB expansion for GO is:

$$u(x,t) = A(x,t)e^{i\omega\varphi(x,t)}, \quad x \in \Omega, t \in [0,T],$$

where the frequency $\omega \gg 1$,and the phase function $\varphi(x,t)$ is real. The solution is defined on a ray which follows the characteristic of the eikonal equation.

However, the GO solution blows up at the caustics!





Since the GO solution is local, at the caustics, the rays are intersecting and $A(x,t) \to \infty$

Gaussian Beams

Gaussian Beam solution is also based on WKB form. Instead of being a local solution, each GB is a global solution.

In particular, near the ray, the phase function admits the following expansion

$$\varphi(x,t) = p(t)\cdot\big(x - x(t)\big) + \frac{1}{2}\big(x - x(t)\big)^T \cdot M(t)\cdot\big(x - x(t)\big) + O(\|x - x(t)\|^3)$$

where the Hessian matrix $M(t)$ has a positive definite imaginary part.

Since GB solution conserve the energy during the propagation and is linearly independent of the other GB solutions, it will not blow up at the caustics.

New Stability Result

G. Bao and H. Zhang: *Sensitivity analysis of an inverse problem for the wave equations in the presence of caustics*, J. of AMS, 2014

New Method:

Gaussian beam and microlocal analysis

Key Ingredients:

Linearized Hamiltonian with respect to the velocity,
Stability analysis of the X-ray transform,
Gaussian beams/microlocal analysis

New Stability Result

- **First stability result for the inverse problem with caustics ; general well-posedness result**

- **Lens rigidity, first stability result for the non-simple metric case, ARMA, 2017**

Inverse Scattering Algorithms

Objective: Stable reconstruction methods:

- Ill-posedness
- Nonlinarity
- Systematic initial guesses
- Uncertainty principle, limitation of resolution

Scope

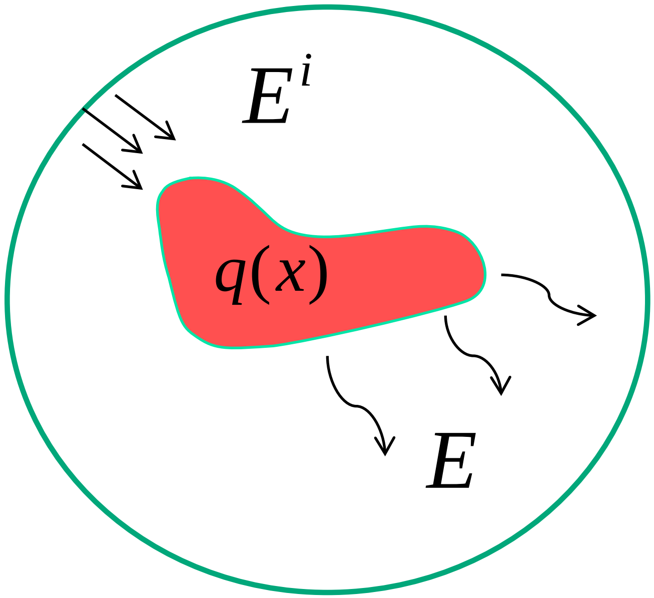
- Related direct imaging approaches: Linear sampling, factorization methods, transmission eigenvalues, Colton, Kirsch, Monk, Cakoni, et al
- Many issues not addressed!

Inverse Medium Problem

Maxwell's Eq: $\nabla\times(\nabla\times E^t)-k^2(1+q(x))E^t=0$

k : wavenumber

$q(x)>-1.$ supported in $\Omega\subset R^3$



$$E^t=E^i+E$$

E^i : incidence

$$\nabla\times(\nabla\times E^i)-k^2E^i=0$$

E: scattered field

$$\nabla\times(\nabla\times E)-k^2(1+q(x))E=k^2q(x)E^i$$

$$\lim_{r\to\infty} r\left[\nabla\times E\times\frac{x}{r}-ikE\right]=0,\quad r=|x|$$

Inverse Medium Problem

Abstract Setting:

$$M(q,k) = \mathrm{Data}(k)$$

Previous work:

- Dorn, Bertete-Aguirre, Berryman, Papanicolaou (1999), Haddar and Monk (2002), Hohage (2003), Colton and Kress, W. Chew et al (2003, 2004), Sini and Thanh (2012), de Hoop, Scherzer, Qiu (2014)...

- Optimization, initial guess/stability

Low Frequency Approximation

Born

$$\nabla \times (\nabla \times E) - k^2 E = k^2 q(E^i + E)$$

$$\nu \times (\nabla \times E) + ik\nu \times (\nu \times E) = 0$$

Important identity:

$$\int_\Omega q(x) p_1 \cdot p_2 e^{ikx\cdot(n_1+n_2)} dx = \frac{i}{k} \int_\Gamma (\nu \times E) \cdot ((n_2 + \nu) \times p_2) e^{ikx\cdot n_2} ds - \int_\Omega q(x) p_2 \cdot E e^{ikx\cdot n_2} dx.$$

Linearization:

$$\int_\Omega q(x) e^{ikx\cdot(n_1+n_2)} dx = \frac{i}{(p_1 \cdot p_2) k} \int_\Gamma (\nu \times E) \cdot ((n_2 + \nu) \times p_2) e^{ikx\cdot n_2} ds$$

$$\hat{q}(\xi) = \int_\Omega q(x) e^{ikx\cdot(n_1+n_2)} dx \qquad \xi = k(n_1 + n_2) \qquad |\xi| \le 2k$$

Algorithm beyond Born

Born + Recursive linearization

Born q_{k_0}

Do $i = 1, 2, \ldots$ （wave number）

$$q_{k_i}^{0} = q_{k_{i-1}}$$

Do $j = 1, \ldots, m$ （incidence）

$$\delta q_j = \frac{1}{\beta_k} DM_j^*(q_{k_i}^{j-1}) R_j(q_{k_i}^{j-1})$$

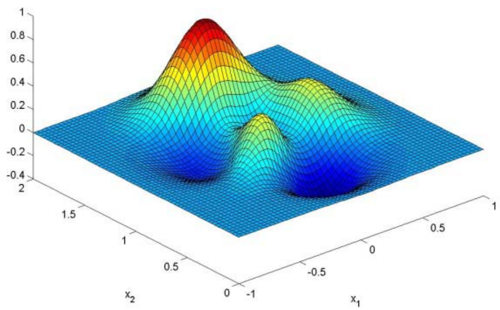
$$q_{k_i}^{j} = q_{k_i}^{j-1} + \delta q_j$$

End

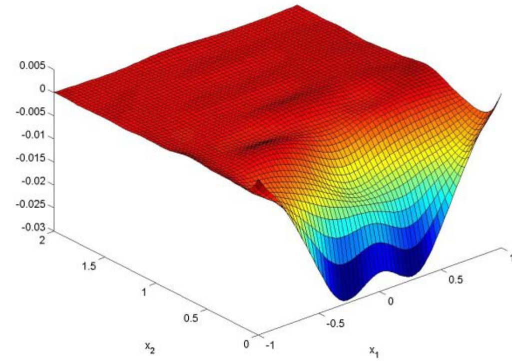
End

Numerical Results

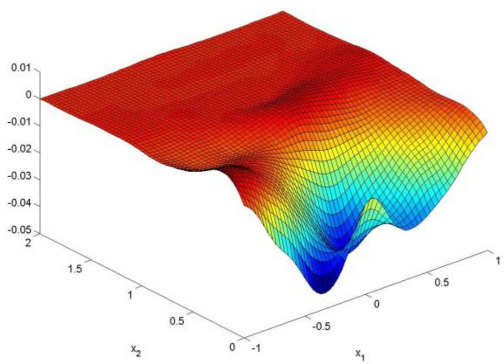
2D smooth medium



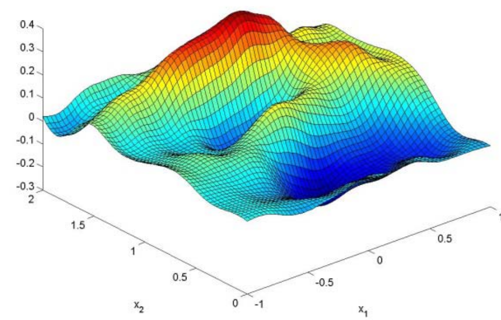
$q(x)$



q_{k_0}

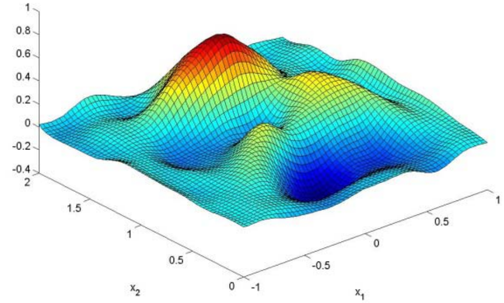


$\eta = 10.2$

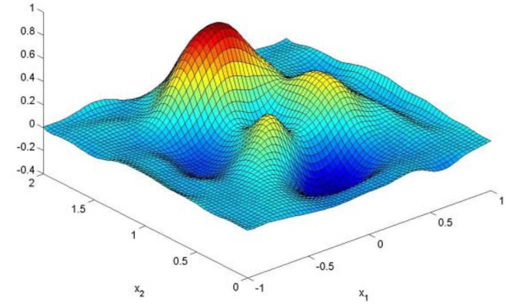


$\eta = 8.4$

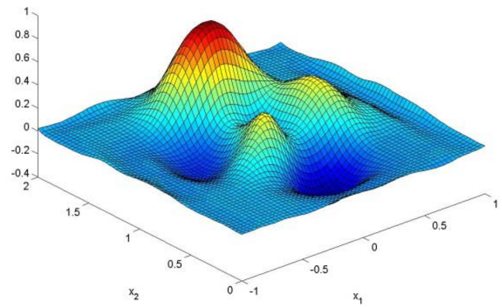
Numerical Results



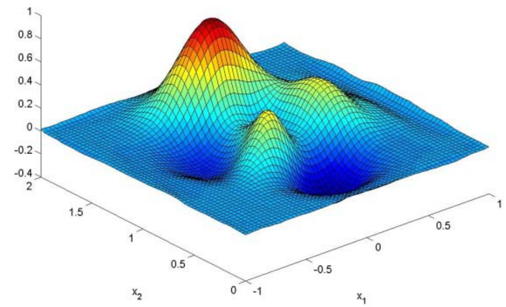
$\eta = 6.6$



$\eta = 4.8$



$\eta = 3.0$



$\eta = 0$

Features on the Algorithm I

- Multiple frequency data is crucial, spectral information
- Systematic selection of initial guesses
- Convergence, analysis is related to the stability analysis
- Other types of inverse problems in wave propagation
- Uncertainty based continuation method

Features on the Algorithm II

- Not distorted Born approximation (W. Chew et al)
- Not direct frequency hopping,...
- It is a continuation method!

At each frequency, optimization, regularization are involved, the problem needs not be solved precisely.

Uncertainty Based Continuation Method

- Inverse medium scattering:

2-D Full aperture, Chen & Rokhlin 97', Chen 97', B., Chen, & Ma 00'; Limited aperture, B. & Liu 03', B. & Li 07'; Fixed frequency, B. & Li 05', 06', 07'; Convergence, B. & Triki 08'; 3-D, B. & Li 04', 05', 07', 08', 09'

- Inverse source problems: B., Lin, &Triki 10', 11', B., Lu, Rundell, Xu, 15'

- Inverse obstacle scattering:

Coifman, Goldberg, Hrycak, & Rokhlin 99', B., Hou, & Li 07', B., Lin 10', 11', B. & Li 12', B., Li, & Lv 13'

Inverse scattering problems with multi-frequency data

B., Li, Lin, Triki, Topical Review, Inverse Problems, 15'

Inverse Source Problems

Model problem: 2D Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = S & \text{in } R^2 \\ \dfrac{\partial u}{\partial r} - iku = o(r^{-1/2}) & \text{as } r \to +\infty \end{cases}$$

(1) S compact supported

(2) Boundary measurements $\left\{ u_k, \dfrac{\partial u_k}{\partial n} \right\}$ for $k \in [k_{\min}, k_{\max}]$

ISP:

From $\left\{ u_k, \dfrac{\partial u_k}{\partial n} \right\}$ for $k \in [k_{\min}, k_{\max}]$

to determine S

Earlier Work

Bleistein & Cohen 77’
He & Romanov 98’,
Ammari, Gang Bao, Fleming 02’
Albanese & Monk 06’
Davaney et al 04’, 07’, ...

Difficulties：
Nonuniqueness/ill-posedness at fixed frequency

ISP: Our Results

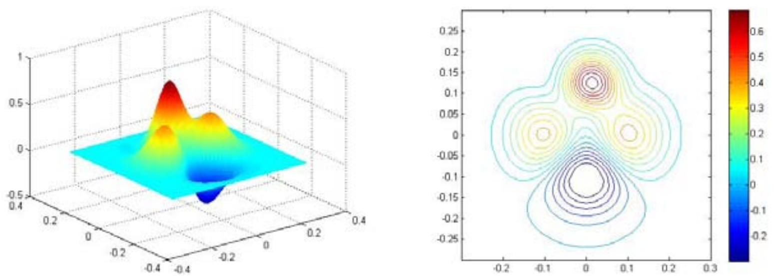
Uniqueness: **B., Lin, Triki, JDE, 10'**

Let $\{k_j\}_{j=1}^{\infty}$ be a bounded and strictly monotone sequence, then $\left\{u_k, \frac{\partial u_k}{\partial n}\right\}_{k=1}^{\infty}$ uniquely determine S.

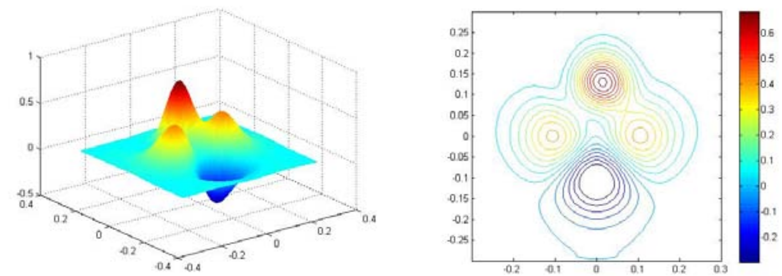
Stability:

There is a critical number, when the highest frequency exceeds the number. The stability is Lipschitz; otherwise it is logrithmic.

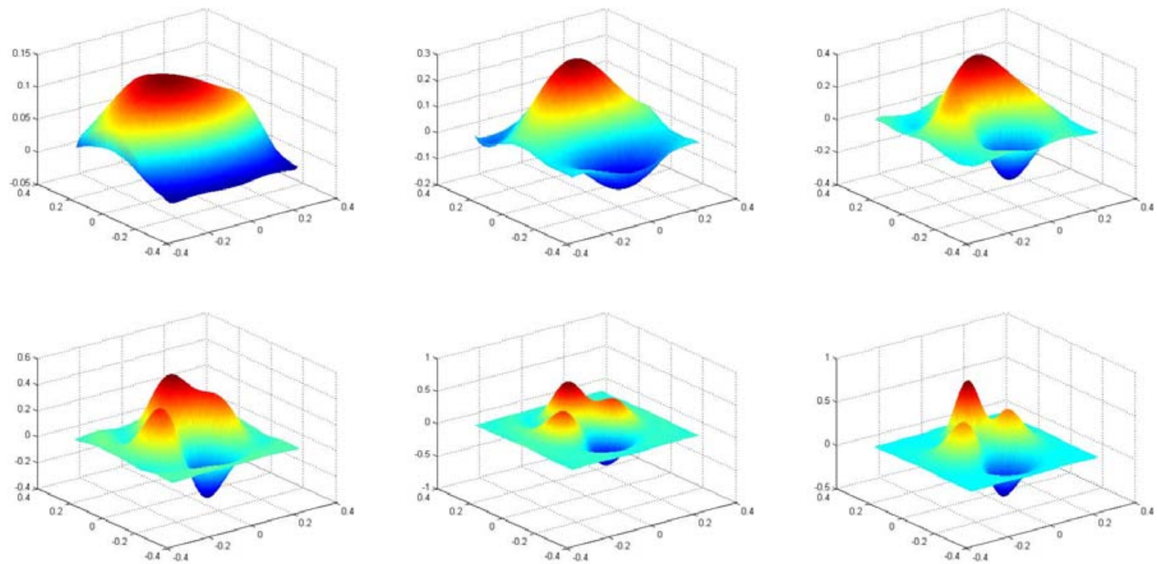
Numerical Results for ISP



(a) The source function $S(x)$



(b) Reconstruction of $S(x)$



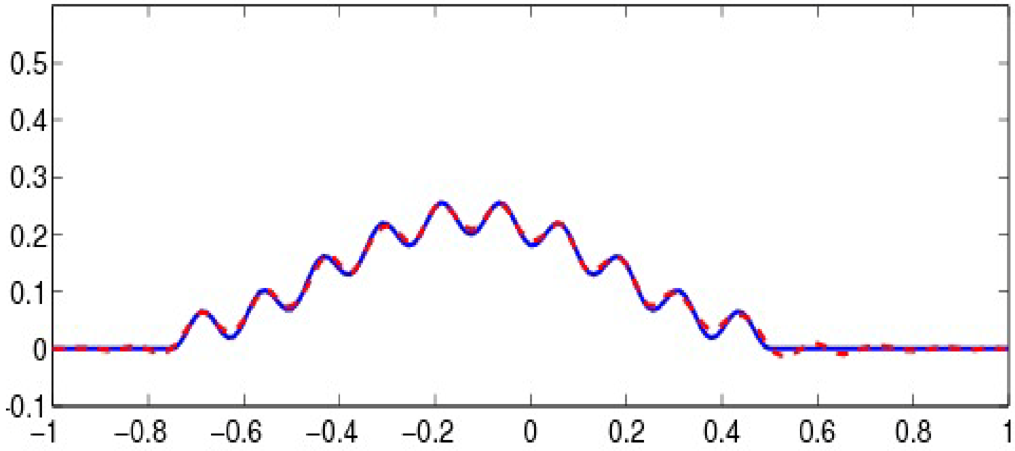
(c) Reconstruction of $S(x)$

$k = 9,17,25$

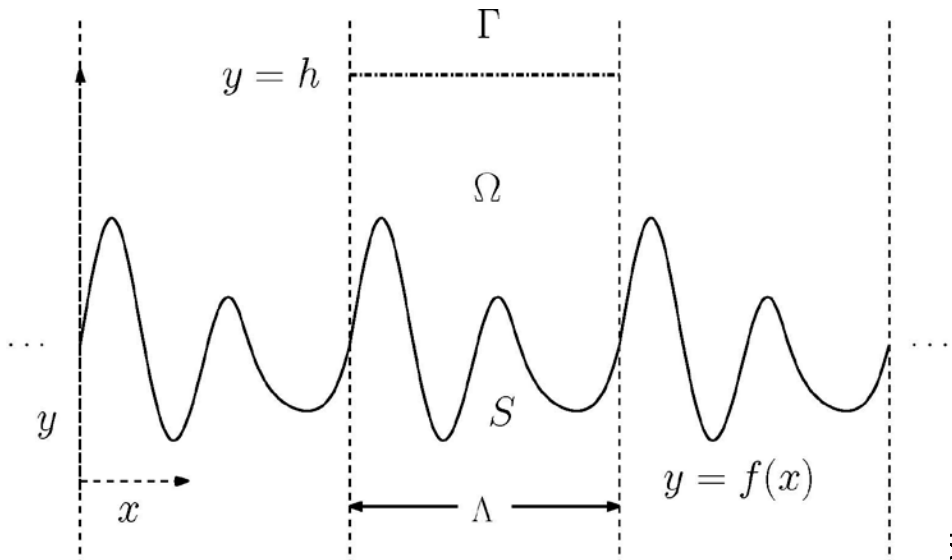
$k = 33, 41, 61$

Inverse Obstacle Problems

- **Multiscale**
  B., J. Lin, SIAP, 11'



- **Rough surface**
  B., P. Li, et al , 13', 14'...



Ongoing and Future Research

- Inverse medium problems: Stability
- Inverse problems with uncertainty: Inverse random source problems, rough surface scattering
- Inverse scattering for elastic waves
- Inverse problems via AI

Stability for Multi-frequency IMP 1-D

1-D Case:

$$\varphi''(x,k)+k^2(1+q(x))\varphi(x,k)=0,\quad x\in(0,1)$$

IP： Given the reflection coefficients $k\in[0,k_0]$
to determine $q(x)$

Theorem(B., Triki , 17’)

Let $q,\tilde{q}\in C_0^m(0,1),\quad \|q\|,\|\tilde{q}\|\le M,\quad q,\tilde{q}>q_0>-1.$

Then for any $k_0\ge k_{M,q_0}$,

$$\|q-\tilde{q}\|_{L^\infty}\le C_{M,q_0}\left(\|d(k)-\tilde{d}(k)\|_{L^1(-k_0,k_0)}+\frac{1}{k_0^{m-1}}\right)$$

Key：Trace formula

Inverse Scattering for Elastic Waves

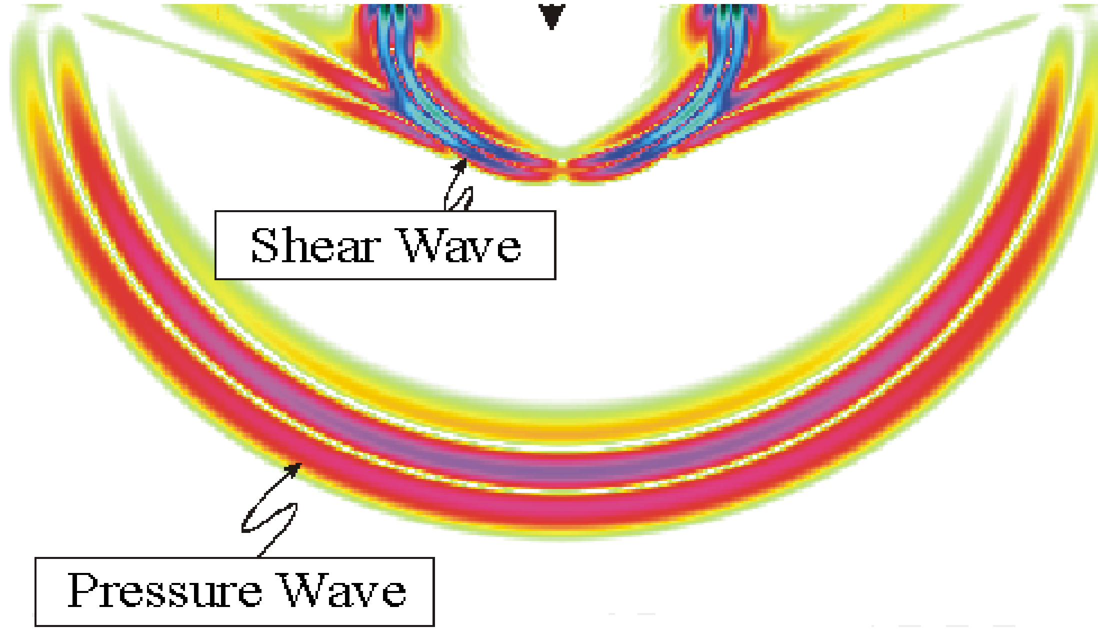
Model

$$\nabla \cdot \sigma(u) + \rho\omega^2 u = 0$$

$$\sigma(u) = C : \varepsilon(u)$$

Kupadze radiation condition

Lame parameters: λ, μ



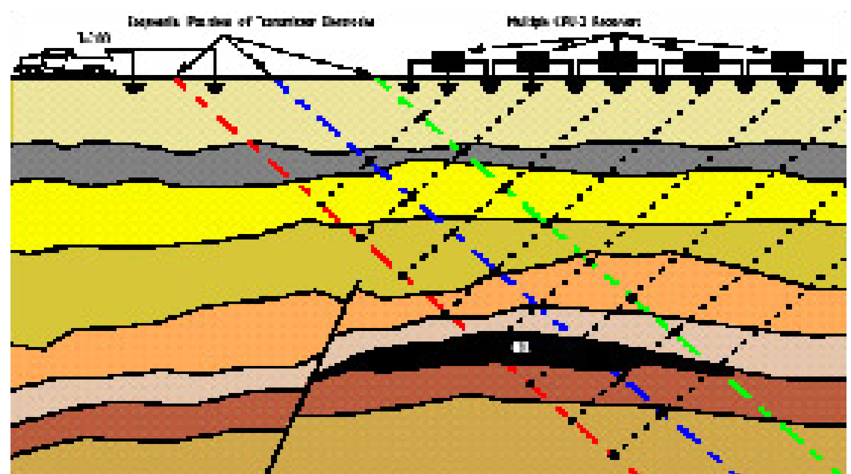
Shear + pressure

- Forward problem

Solution $u^s = u - u^i$

- Inverse Problem

Reconstruct medium/obstacle/source from boundary measurements



Research for Scattering in Elasticity

- **Scattering:**

Finite element method, transparent BC

B., G. Hu, J. Sun and T. Yin, JMPA, 18'.

Boundary integral equation method

B., L. Xu and T. Yin, JCP, 17'

Time-domain

B., Y. Gao, P. Li, ARMA, 18'

- **Inverse scattering:**

Uniqueness/stability: Beretta, Yamamoto, Uhlmann, De Hoop...

Object: Recursive linearization algorithm for inverse elastic scattering problems with multi-frequency data.

Inverse source problems: stability analysis, B., P. Li et al.

Inverse Source Problem for Elasticity

Model problem: elasticity equation

$$\begin{cases} \Delta^* u + \omega^2 u = f(x) & \text{in } B_R, \\ Tu = \mathcal{T} u & \text{on } \Gamma_R. \end{cases}$$

$\Delta^* = \mu\Delta + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}$ is the Lamé operator

ISP. Let f be a complex function with a compact support $\Omega \subset B_R$. The ISP is to determine f from the data $u(x, \omega), x \in \Gamma_R, \omega \in (0, K)$, where $K > 1$ is a constant.

Stability for ISP

$$\|u(\cdot,\omega)\|_{\Gamma_R}^2 = \int_{\Gamma_R} \left(|\mathcal{T} u(x,\omega)|^2 + \omega^2 |u(x,\omega)|^2\right) ds_x$$

$$\mathbb{F}_M = \{f \in H^m(\Omega)^3 : \|f\|_{H^m(\Omega)^3} \le M,\ \operatorname{supp} f = \Omega \subset B_R\}$$

Theorem (B.-Li-Zhao)

Let $f \in \mathbb{F}_M$ and u be the solution of the scattering problem corresponding to f. Then

$$\|f\|_{L^2(B_R)^3}^2 \lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}}}{(R+1)(6m-15)^3}\right)^{2m-5}},$$

where

$$\epsilon = \left(\int_0^K \omega^2 \|u(\cdot,\omega)\|_{\Gamma_R}^2 \mathrm{d}\omega\right)^{\frac{1}{2}}.$$

Concluding Remarks

- Inverse scattering problems are an exciting and fast growing area of mathematics driven by interdisciplinary applications.
- Many basic math questions to be addressed; effective computational methods are in demand.
- Interactions of crossed-disciplinary efforts, Prof. Bolomey...
- New problems are emerging; AI, deep learning, big data,...

Thanks !