Instability of an inverse problem for the stationary radiative transport near the diffusion limit

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Joint work with Yimin Zhong

Radiative transport equation (RTE)

The radiative transport equation (RTE) is often used to model the propagation of radiative particles through a scattering medium.

$$
\begin{aligned}
v \cdot \nabla u(x,v) + \sigma_t(x,v)u(x,v) &= \int_\Omega k(x,v,v')u(x,v')d\mu(v'), && \text{in } D \times \Omega \\
u(x,v) &= f(x,v), && \text{on } \Gamma_-
\end{aligned}
$$

- $u(x,v)$ is particle density at x in direction v,
- $D \subset \mathbb{R}^d, d \geq 2$ is bounded with convex smooth boundary,
- $\Omega = \mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and $d\mu$ is the measure on Ω,
- $\sigma_t(x,v)$ and $k(x,v,v')$ are the total absorption coefficient and scattering coefficient respectively,
- $f(x,v)$ models an incident illumination density of particles,
- $\Gamma_\pm = \{(x,v) \in \partial D \times \Omega : \pm n_x \cdot v > 0\}$ is the outgoing and incoming boundary, where n_x is the outward unit normal vector at $x \in \partial D$.

The inverse problem

For many applications in medical imaging, remote sensing, nuclear engineering, astrophysics, etc., the task is to recover the optical parameters $\sigma_t(x)$ and $\sigma_s(x)$, where

$\sigma_t(x, v) = \sigma_t(x),\ k(x, v, v') = \sigma_s(x)p(v, v'),\ p(v, v')$ is the phase function,

from boundary measurements, e.g., the albedo operator

$$\Lambda : f(x, v) \to u|_{\Gamma_+}$$

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- In theory, angular resolved data make the inverse problem easier.
- In practice, only angular averaged data are available in many applications.

Stability for the inverse problem

Previous results:

- Stability for angular resolved data has been studied by (Bal, Jollivet, Wang, ...).
- Stability for angular averaged data for small optical parameters has been studied by Bal et al.
- Stability for angular resolved data near diffusion limit (Lai, Li and Uhlmann).

Our study

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- We consider the instability of reconstructing the absorption coefficient σ_a near the diffusion limit with angularly averaged measurement by constructive approach.
- Show the transition of stability from Hölder type to logarithmic type in terms of the the Knudsen number and the measurement perturbation.
- Explicitly characterize the balance between the transport and diffusion behaviors as well as the information content of the unknown coefficient and the measurements using Kolmogorov entropy and capacity. (Mandache, 2001)

The setup

Let $\sigma_a(x) := \sigma_t(x) - \sigma_s(x)$ and Knudsen number $0 < \varepsilon \ll 1$

$$
\begin{aligned}
v \cdot \nabla u(x,v) + \left(\varepsilon \sigma_a(x) + \frac{1}{\varepsilon}\sigma_s(x)\right) u(x,v) &= \frac{1}{\varepsilon}\sigma_s(x)\langle u \rangle, && \text{in } D \times \Omega \\
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$$

For simplicity, assume $p(v, v') \equiv 1$,

$$
\langle u \rangle(x) = \int_\Omega u(x,v) d\mu(v) (= \int_\Omega p(v,v') u(x,v') d\mu(v')),
$$

and define the scaled measurement by the averaged albedo operator

$$
\Lambda_{\sigma_a} : f(x) \in \mathcal{X} \mapsto \Lambda_{\sigma_a} f(x) = \mathcal{J}(x) \in \mathcal{Y}
$$

where

$$
\begin{aligned}
\mathcal{J}(x) &= \frac{1}{\varepsilon} \int_\Omega v \cdot n_x u(x,v) d\mu(v) = \mathcal{J}_+(x) + \mathcal{J}_-(x) \\
&= \frac{1}{\varepsilon} \left[\int_{n_x \cdot v > 0} v \cdot n_x u(x,v) d\mu(v) + \int_{n_x \cdot v < 0} v \cdot n_x f(x) d\mu(v) \right],
\end{aligned}
$$

Different regimes of the inverse problem

- If $\sigma_s \equiv 0$, the measurement is the Radon transform of $\sigma_a(x)$ (X-ray tomography) $\Rightarrow$ Hölder type of stability.
- As $\varepsilon \to 0$, the RTE approaches the diffusion limit

$$-\frac{1}{d}\nabla\left(\frac{1}{\sigma_s(x)}U(x)\right) + \sigma_a(x)U(x) = 0.$$

  The measurement corresponds to the DtN map and is related to the Calderón's problem (EIT) $\Rightarrow$ logarithmic stability.

Our main results

Let $D = B(0,1) \subset \mathbb{R}^d$, $K = B(0, r_0)$ $(0 < r_0 < 1)$, and assume $\sigma_s(x) \equiv \sigma_s > 0$. Define the admissible set of the absorption coefficient by

$$\mathcal{S} := \{\sigma_a \,|\, \sigma_a \geq 0, \operatorname{supp} \sigma_a \subset K, \sigma_a \in C^q(\overline{K}), q > 0\}$$

Theorem 1

For any $q > 0$ and any dimension $d \geq 2$, any $R > 0$, there is a constant $\beta > 0$ such that for any $\theta \in (0, \frac{R}{2})$ and $\sigma_{a,0} \in L^\infty(D)$ with $\|\sigma_{a,0}\|_\infty \leq \frac{R}{2}$, $\operatorname{supp} \sigma_{a,0} \subset K$, there are absorption coefficients $\sigma_{a,1}, \sigma_{a,2} \in \mathcal{S}$, such that

$$\begin{aligned}
&\|\Lambda_1 - \Lambda_2\|_{H^s(\partial D) \to H^{-s}(\partial D)} \leq 8\sqrt{2}\omega(\theta^{-\frac{d}{(2d+1)q}}), \\
&\|\sigma_{a,1} - \sigma_{a,2}\|_\infty \geq \theta, \\
&\|\sigma_{a,i} - \sigma_{a,0}\|_{C^q} \leq \beta, \quad i = 1, 2 \\
&\|\sigma_{a,i} - \sigma_{a,0}\|_\infty \leq \theta, \quad i = 1, 2
\end{aligned}$$

where Λ_1, Λ_2 are the averaged albedo operators for $\sigma_{a,1}, \sigma_{a,2}$ respectively, $\omega(t)$ is the solution to the following equation

$$t = \log(\omega^{-1}) + \frac{\varepsilon}{\omega} + \left(\frac{\varepsilon}{\omega}\right)^{-1/\tau}, \quad \tau = \frac{d+4}{2} - s < 0.$$

Two regimes

Let $s = \frac{d+4}{2} + 1$. Depending on the relation of ε and θ, we have the following diffusion and transport regimes under the same assumptions of Theorem 1.

Corollary 2 (Diffusion)

When Knudsen number ε is small, $\varepsilon \le \frac{1}{3}\theta^{-\frac{d}{(2d+1)q}} \exp\left(-\frac{1}{3}\theta^{-\frac{d}{(2d+1)q}}\right)$,

$$\|\Lambda_1 - \Lambda_2\|_{H^s(\partial D)\to H^{-s}(\partial D)} \le 8\sqrt{2}\exp\left(-\frac{1}{3}\theta^{-\frac{d}{(2d+1)q}}\right),$$

$$\|\sigma_{a,1} - \sigma_{a,2}\|_\infty \ge \theta.$$

Corollary 3 (Transport)

When Knudsen number ε satisfies $1 \gg \varepsilon > \frac{1}{3}\theta^{-\frac{d}{(2d+1)q}} \exp\left(-\frac{1}{3}\theta^{-\frac{d}{(2d+1)q}}\right)$

$$\|\Lambda_1 - \Lambda_2\|_{H^s(\partial D)\to H^{-s}(\partial D)} \le 24\sqrt{2}\varepsilon\theta^{\frac{d}{(2d+1)q}},$$

$$\|\sigma_{a,1} - \sigma_{a,2}\|_\infty \ge \theta.$$

A few basic lemmas

Lemma 4

If the boundary source $f \in L^p(\partial D)$ for $p \geq 1$, then $u(x,v) \in L^p(D \times \Omega)$ and $\langle u \rangle \in L^p(D)$.

Lemma 5

If the boundary source $f \in L^2(\partial D)$, then $\Lambda_{\sigma_a} f \in H^{-1/2}(\partial D) \subset H^{-s}(\partial D)$.

Lemma 6

If u is the solution to the RTE and w satisfies the following adjoint RTE with outgoing boundary condition,

$$\begin{aligned} -v \cdot \nabla w + (\varepsilon \sigma_a + \frac{1}{\varepsilon}\sigma_s) w = \frac{1}{\varepsilon}\sigma_s \langle w \rangle \qquad & \text{in } D \times \Omega, \\ w(x,v) = g(x), \qquad & \text{on } \Gamma_+. \end{aligned}$$

then u and w satisfy the following relation,

$$\int_{\partial D} \left(\int_\Omega n_x \cdot v u(x,v) d\mu(v) \right) g(x) dS(x) = -\int_{\partial D} \left(\int_\Omega n_x \cdot v w(x,v) d\mu(v) \right) f(x) dS(x).$$

A few relations

Let $u(x,v)$ be the solution to the RTE and $u_0(x,v)$ be the solution to the RTE without absorption ($\sigma_a \equiv 0$). Then $\phi = u - u_0$ satisfies

$$\begin{aligned} v \cdot \nabla\phi(x,v) + \left(\varepsilon\sigma_a(x) + \frac{1}{\varepsilon}\sigma_s(x)\right)\phi(x,v) &= \frac{1}{\varepsilon}\sigma_s(x)\langle\phi\rangle - \varepsilon\sigma_a(x)u_0 \quad \text{in } D \times \Omega \\ \phi(x,v) &= 0 \quad \text{on } \Gamma_- \end{aligned}$$

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Define the linear operator $\Gamma(\sigma_a) := \Lambda_{\sigma_a} - \Lambda_0$. $\forall f, g \in H^s(\partial D)$, we have

$$\begin{aligned} \langle\Gamma(\sigma_a)f, g\rangle &= -\int_K \sigma_a(x)\overline{\hat{g}(x)}\langle u\rangle(x)dx + \frac{1}{\varepsilon}\int_D \nabla\overline{\hat{g}(x)}\cdot\left(\int_\Omega v\phi(x,v)d\mu(v)\right)dx \\ &= \int_K \sigma_a(x)\hat{f}(x)\overline{\langle w\rangle(x)}dx - \frac{1}{\varepsilon}\int_D \nabla\hat{f}(x)\cdot\int_\Omega v\overline{\varphi(x,v)}d\mu(v)dx \end{aligned}$$

where $\hat{g}(x), \hat{f}(x) \in H^1(D)$ are arbitrary extensions of $g(x), f(x)$ in D, $w(x,v), w_0(x,v)$ are the solutions to the corresponding adjoint RTE, and $\varphi = w - w_0$.

The key estimate

Introduce d-dimensional spherical harmonics on the unit sphere $\mathbb{S}^{d-1}$

$$\mathbb{H}^d := \{Y_{mj} \mid m \geq 0, 1 \leq j \leq p_m\}$$

where Y_{mj} is a spherical harmonic of order m and

$$p_m = \binom{m+d-1}{d-1} - \binom{m+d-3}{d-1} \leq 2(1+m)^{d-2}.$$

$\mathbb{H}^d$ forms a Schauder basis for $\mathcal{X} = H^s(\partial D)$ and $\mathcal{Y} = H^{-s}(\partial D)$.

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Lemma 7

There is a constant $C_0 = C_0(r_0, d, s)$ such that for any 4-tuple (m, j, n, k) with $m, n \geq 0$ and $j \leq p_m$, $k \leq p_n$,

$$|\langle \Gamma(\sigma_a) Y_{mj}, Y_{nk} \rangle| \leq C_0 \|\sigma_a\|_\infty (1+l) \left(r_0^l + \varepsilon(1+l) \right). \qquad (1)$$

where $l = \max(m, n)$.

The proof of the estimate

$$\langle \Gamma(\sigma_a) Y_{mj}, Y_{nk} \rangle = -\int_K \sigma_a(x) \overline{\hat{Y}_{nk}(x)} \langle u_{mj} \rangle (x) dx + \frac{1}{\varepsilon} \int_D \nabla \overline{\hat{Y}_{nk}(x)} \cdot \int_\Omega v \phi_{mj}(x,v) d\mu(v) dx$$
$$= I_{1,mjnk} + I_{2,mjnk}.$$

$\hat{Y}_{nk}(x) = |x|^n Y_{nk}(x/|x|)$ is the Harmonic extensions of Y_{nk}
u_{mj} is the solution to the RTE with $f = Y_{mj}$
$u_{0,mj}$ is the solution to the RTE with $\sigma_a = 0$ and $f = Y_{mj}$
$\phi_{mj} = u_{mj} - u_{0,mj}$ satisfies

$$v \cdot \nabla \phi_{mj} + \left(\varepsilon \sigma_a(x) + \frac{1}{\varepsilon} \sigma_s(x) \right) \phi_{mj} = \frac{1}{\varepsilon} \sigma_s(x) \langle \phi_{mj} \rangle - \varepsilon \sigma_a(x) u_{0,mj}, \qquad \text{in } D \times \Omega$$
$$\phi_{mj}(x, v) = 0, \qquad \text{on } \Gamma_-.$$

The proof of the estimate

$$
\begin{aligned}
\langle \Gamma(\sigma_a) Y_{mj}, Y_{nk} \rangle &= -\int_K \sigma_a(x) \overline{\hat{Y}_{nk}(x)} \langle u_{mj} \rangle (x) dx + \frac{1}{\varepsilon} \int_D \nabla \overline{\hat{Y}_{nk}(x)} \cdot \int_\Omega v \phi_{mj}(x,v) d\mu(v) dx \\
&= I_{1,mjnk} + I_{2,mjnk}.
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$$
\begin{aligned}
v \cdot \nabla \phi_{mj} + \left(\varepsilon \sigma_a(x) + \frac{1}{\varepsilon} \sigma_s(x) \right) \phi_{mj} &= \frac{1}{\varepsilon} \sigma_s(x) \langle \phi_{mj} \rangle - \varepsilon \sigma_a(x) u_{0,mj}, \qquad && \text{in } D \times \Omega \\
\phi_{mj}(x,v) &= 0, && \text{on } \Gamma_-.
\end{aligned}
$$

We want to show, when $n \geq m$

1. $|I_{1,mjnk}| \leq C_0 \|\sigma_a\|_\infty (1+n) r_0^n$.
2. $|I_{2,mjnk}| \leq C_0 \|\sigma_a\|_\infty \varepsilon (1+n)^2$.

The proof of the estimate

$$|I_{1,mjnk}| = \left| \int_K \sigma_a(x)|x|^n \overline{Y_{nk}(x/|x|)} \langle u_{mj} \rangle (x) dx \right|$$
$$\leq \|\sigma_a\|_\infty \frac{1}{\sqrt{2n+d}} r_0^{n+d/2} \|\langle u_{mj} \rangle\|_K \overset{?}{\leq} C_0 \|\sigma_a\|_\infty (1+n) r_0^n$$

$$\langle u_{mj} \rangle = (I - \mathcal{K}_1)^{-1} \mathcal{K}_2 Y_{mj}.$$

where

$$\mathcal{K}_1 f(x) = \int_D \mathcal{K}(x,y) \frac{\sigma_s}{\varepsilon} f(y) dy, \;\; \mathcal{K}_2 f(x) = \int_{\partial D} \mathcal{K}(x,y) \frac{y-x}{|y-x|} \cdot n_y f(y) dS(y)$$

$$\mathcal{K}(x,y) = \frac{1}{\nu_d} \frac{E(x,y)}{|x-y|^{d-1}}, \;\; E(x,y) = \exp\left(-\frac{|x-y|}{\varepsilon} \int_0^1 (\varepsilon^2 \sigma_a + \sigma_s)(x + t(y-x)) dt \right)$$

The proof of the estimate

$$|I_{1,mjnk}| = \left| \int_K \sigma_a(x)|x|^n \overline{Y_{nk}(x/|x|)} \langle u_{mj} \rangle (x) dx \right|$$

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Lemma 8

$$\|(I - \mathcal{K}_1)^{-1} \mathcal{K}_2 f\|_{L^2(D)} < C \|f\|_{H^{3/2}(\partial D)}$$

where C is a constant independent of ε.

The proof of the estimate

$$|I_{2,mjnk}| = \left| \frac{1}{\varepsilon} \int_D \nabla \overline{\hat{Y}_{nk}(x)} \cdot \int_\Omega v \phi_{mj}(x,v) d\mu(v) dx \right| \overset{?}{\leq} C_0 \|\sigma_a\|_\infty \varepsilon (1+n)^2$$

$\phi_{mj} = u_{mj} - u_{0,mj}$ satisfies

$$v \cdot \nabla \phi_{mj} + \left(\varepsilon \sigma_a(x) + \frac{1}{\varepsilon} \sigma_s(x) \right) \phi_{mj} = \frac{1}{\varepsilon} \sigma_s(x) \langle \phi_{mj} \rangle - \varepsilon \sigma_a(x) u_{0,mj} \quad \text{in } D \times \Omega,$$

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$u_{0,mj} = u^I_{0,mj} + u^B_{0,mj}$: $|u^B_{0,mj}(x,v)| \leq C \exp(-\frac{\ell}{\varepsilon}(1-|x|))$ is the boundary layer solution and $u^I_{0,mj}$ is the interior solution.

$$u^I_{0,mj}(x,v) = U_{0,mj} - \frac{\varepsilon}{\sigma_s} v \cdot \nabla U_{0,mj} + O(\varepsilon^2),$$

$U_{0,m,j}(x)$ is the solution to the diffusion equation, and

$$\phi_{mj} = \Phi_{mj}(x) - \frac{\varepsilon}{\sigma_s} v \cdot \nabla \Phi_{mj}(x) + \varepsilon^2 R_{mj},$$

where Φ_{mj} satisfies the diffusion equation with zero Dirichlet BC,

$$-\frac{1}{d} \nabla \cdot \left(\frac{1}{\sigma_s} \nabla \Phi_{mj} \right) + \sigma_a \Phi_{mj} = -\sigma_a U_{0,mj},$$

and $\|R_{mj}\|_{L^2(D\times\Omega)} \leq C \|\sigma_a u_{0,mj}\|_{L^2(D\times\Omega)} \leq C \|\sigma_a\|_\infty \|u_{0,mj}\|_{L^2(D\times\Omega)}$

The proof of the estimate

Let J_{mj} denote the velocity averaged vector field,

$$J_{mj}(x) = \int_\Omega v\phi_{mj}(x,v)d\mu(v) = -\frac{\varepsilon}{d\sigma_s}\nabla\Phi_{mj}(x) + \varepsilon^2\tilde{R}_{mj}.$$

$$\begin{aligned} |I_{2,mjnk}| &= \left|\frac{1}{\varepsilon}\int_D \nabla(|x|^n\overline{Y_{nk}(x/|x|)})\cdot J_{mj}(x)dx\right| \\ &\le \varepsilon\left|\int_D \nabla(|x|^n\overline{Y_{nk}(x/|x|)})\cdot \tilde{R}_{mj}dx\right| \\ &\le C\varepsilon\|\sigma_a\|_\infty\|u_{0,mj}\|_{L^2(D\times\Omega)}\|\nabla(|x|^n\overline{Y_{nk}(x/|x|)})\|_D \end{aligned}$$

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Since $\|\nabla(|x|^n\overline{Y_{nk}(x/|x|)})\|_D \leq C\sqrt{1+n}$ and $\|u_{0,mj}\|_{L^2(D\times\Omega)} \leq (1+m)^{\frac{3}{2}}$ by

Lemma 9

Suppose u is the solution to the RTE, then

$$\|u\|_{L^2(D\times\Omega)} \leq C\|f\|_{H^{3/2}(\partial D)}$$

where C is independent of ε when $\varepsilon \ll 1$.

$$\Rightarrow |I_{2,mjnk}| \leq C_0\|\sigma_a\|_\infty\varepsilon(1+n)^2 \text{ when } n > m.$$

Kolmogorov entropy and capacity for function space

Definition 10

Let (X, d) be a metric space and $\delta > 0$, then we say

1. A set $Y \subset X$ is a δ-net for $X_1 \subset X$ if for any $x \in X_1$ there exists $y \in Y$ such that $\mathrm{d}(x, y) \leq \delta$.
2. A set $Z \subset X$ is θ-distinguishable if for any distinct $z_1, z_2 \in Z$, we have $\mathrm{d}(z_1, z_2) \geq \theta$.

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Lemma 11 (Kolmogorov)

Let $d \geq 2$ and $q > 0$. For $\theta, \beta > 0$, consider the metric (induced by L^∞ norm) space

$$X_{q\theta\beta} = \{f \in C_0^q(K) : \|f\|_\infty \leq \theta, \|f\|_{C^q} \leq \beta \text{ and } f \geq 0\}.$$

There is a constant $\mu > 0$ such that for any $\beta > 0$ and $\theta \in (0, \mu\beta)$, $X_{q\theta\beta}$ has a θ-distinguishable set Z, its cardinality satisfies the lower bound,

$$|Z| \geq \exp\left(2^{-d-1}(\mu\beta/\theta)^{d/q}\right).$$

Kolmogorov entropy and capacity for function space

For any bounded linear operator $\mathcal{A} : \mathcal{X} = H^s(\partial D) \to \mathcal{Y} = H^{-s}(\partial D)$ with matrix representation $a_{mjnk} = \langle \mathcal{A} Y_{mj}, Y_{nk} \rangle$, then

$$\|\mathcal{A}\|^2_{\mathcal{X}\to\mathcal{Y}} \leq \sum_{m,n\geq 0, 1\leq j\leq p_m, 1\leq k\leq p_n} (1+m)^{-2s}(1+n)^{-2s}|a_{mjnk}|^2.$$

Lemma 12

$\|\mathcal{A}\|^2_{\mathcal{X}\to\mathcal{Y}} \leq 32 \sup_{m,j,n,k}(1+\max(m,n))^{d-2s}|a_{mjnk}|^2.$

Let $X_s := \{(a_{mjnk}) \big| \|(a_{mjnk})\|_{X_s} := \sup_{m,j,n,k}(1+\max(m,n))^{\frac{d}{2}-s}|a_{mjnk}| < \infty\}$ then $\|\mathcal{A}\|_{X\to Y} \leq 4\sqrt{2}\,\|(a_{mjnk})\|_{X_s}$.

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Lemma 12

$\|\mathcal{A}\|^2_{\mathcal{X}\to\mathcal{Y}} \le 32 \sup_{m,j,n,k}(1+\max(m,n))^{d-2s}|a_{mjnk}|^2.$

Let $X_s := \{(a_{mjnk}) \big| \|(a_{mjnk})\|_{X_s} := \sup_{m,j,n,k}(1+\max(m,n))^{\frac{d}{2}-s}|a_{mjnk}| < \infty\}$ then $\|\mathcal{A}\|_{X\to Y} \le 4\sqrt{2}\,\|(a_{mjnk})\|_{X_s}$.

$$\text{Define} \quad B^\infty_{+,R} = \{f \in L^\infty(K) | \|f\|_\infty \le R, f \ge 0\},$$

Lemma 13

The operator Γ *maps* $B^\infty_{+,R}$ *into* X_s *for any* $s > \frac{d+4}{2}$*. There exists a constant* $0 < \eta = \eta(R,s,d)$ *that for every* $\delta \in (0, e^{-1})$*, there is a* δ*-net* Y *for* $\Gamma(B^\infty_{+,R})$ *in* X_s*, with at most*

$\exp\left(\eta\left(\log\delta^{-1} + \frac{\varepsilon}{\delta} + \left(\frac{\varepsilon}{\delta}\right)^{-1/\tau}\right)^{2d+1}\right)$ *elements, where* $\tau = \frac{d+4}{2} - s < 0$.

Proof of the main result

- Lemma 11 shows $\sigma_{a,0}+X_{q\theta\beta}$ has a θ-distinguishable set $\sigma_{a,0}+Z$, pair wise distance of which are at least θ in L^∞ and $\sigma_{a,0}+X_{q\theta\beta}\subset B^\infty_{+,R}$.
- Lemma 13 constructs a δ-net Y for $\Gamma(\sigma_{a,0}+X_{q\theta\beta})$.
- When $|\sigma_{a,0}+X_{q\theta\beta}|>|Y|$, there are two absorption coefficients $\sigma_{a,1},\sigma_{a,2}\in\sigma_{a,0}+X_{q\theta\beta}$ that are θ apart, their images under Γ are in the same X_s-ball of radius δ.

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Let δ be the unique solution to the following equation

$$\theta^{-\frac{d}{(2d+1)q}} = \log\delta^{-1} + \left(\frac{\varepsilon}{\delta}\right) + \left(\frac{\varepsilon}{\delta}\right)^{-1/\tau}, \quad \tau = \frac{d+4}{2} - s < 0,$$

and choose $\beta > \mu^{-1}\max\left(\frac{R}{2}, \left(2^{(d+1)}\eta\right)^{q/d}\right)$. Then $\mu\beta \geq \frac{R}{2} > \theta$ and

$$|\sigma_{a,0} + X_{q\theta\beta}| = |X_{q\theta\beta}| \geq \exp(2^{-d-1}(\mu\beta/\theta)^{d/q}) > \exp(\eta\theta^{-d/q}) \geq |Y|.$$

Special cases

Assume $s - \frac{d+4}{2} = 1$, δ solves the equation

$$\theta^{-\frac{d}{(2d+1)q}} = \log \delta^{-1} + 2\left(\frac{\varepsilon}{\delta}\right).$$

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- Diffusion regime (Corollary 2): if $\varepsilon \leq \delta \log \delta^{-1}$,

$$\left(\frac{\varepsilon}{\delta}\right) \leq \log \delta^{-1} \Rightarrow \delta \leq \exp\left(-\frac{1}{3}\theta^{-\frac{d}{(2d+1)q}}\right).$$

- Transport regime (Corollary 3): if $\varepsilon \geq \delta \log \delta^{-1}$,

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Remarks:

1. Our result shows sharpness of the stability estimate by Lai, Li and Uhlmann, and that angular resolved data do not provide better stability near the diffusion limit.
2. Our constructive approach using Kolmogorov entropy characterizes explicitly how regularity of the unknown coefficient affects the (in)stability in terms of information content.

Thank you!