Unique continuation and inverse boundary problems for the Schrödinger equation with singular coefficients

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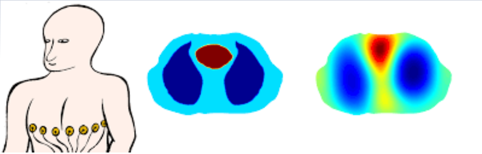
Outline

- Motivation: Calderón's problem (EIT)
- Connection to unique continuation property
- Quantitative uniqueness estimates and Landis' conjecture
- Inverse boundary value problem for the Schrödinger equation with singular potential in the plane

EIT



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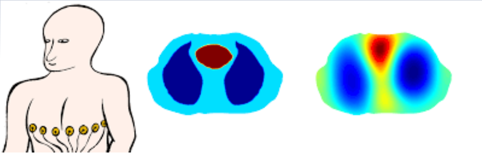

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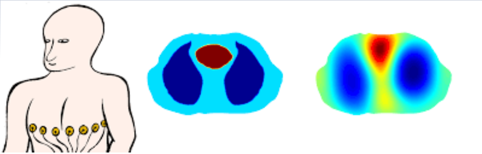

Problem: determination of the conductivity distribution from voltage-current measurements on the surface of the body. (Calderón's problem or Gel'fand-Calderón's problem).

Mathematical setup

Let u_f solve

$$\begin{cases} \operatorname{div}(\gamma \nabla u_f) = 0 & \text{in} \quad \Omega \\ u_f = f & \text{on} \quad \partial\Omega. \end{cases}$$

Dirichlet-Neumann map (Voltage-Current map):

$$\Lambda_\gamma : f \mapsto \gamma \frac{\partial u_f}{\partial \nu}|_{\partial\Omega}$$

IP: determine γ from Λ_γ.

Known uniqueness results

$$\boxed{\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2}$$

$(n \geq 3)$

- $\gamma_1, \gamma_2 \in C^\infty(\overline{\Omega})$, [Sylvester-Uhlmann, 87]
- $\gamma_1, \gamma_2 \in \mathrm{Lip}(\overline{\Omega})$ with $\|\nabla \gamma_i\|_{\mathrm{Lip}(\overline{\Omega})}$ small or $\gamma_1, \gamma_2 \in C^1(\overline{\Omega})$, [Haberman-Tataru, 11]
- $\gamma_1, \gamma_2 \in \mathrm{Lip}(\overline{\Omega})$, [Caro-Rogers, 14]
- $\gamma_1, \gamma_2 \in W^{1,n}(\Omega) \cap L^\infty(\Omega)$, $n = 3, 4$, [Haberman, 15]

2D

$(n = 2)$

- $\gamma_1, \gamma_2 \in W^{2,p}(\Omega)$, [Nachman, 96]; $\gamma_1, \gamma_2 \in W^{1,p}(\Omega), p > 2$, [Brown-Uhlmann, 97]
- $\gamma_1, \gamma_2 \in L^\infty(\Omega)$, [Astala-Päivärinta, 06]
- $\gamma_1, \gamma_2 \in W^{1,2}(\Omega)$ with $\|\nabla \log \gamma_i\|_{L^2(\Omega)}$ small (γ_1, γ_2 may be unbounded), [Carstea-W, 16]
- $\gamma_1, \gamma_2 \in W^{1,2}(\Omega)$, [Nachman-Regev-Tataru, 17]

Connection to UCP

The uniqueness question of the inverse boundary value problem is closely related to the unique continuation property. One expects the uniqueness of the inverse boundary value problem is true provided the unique continuation property holds.

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Consider the second order elliptic equation:

$$\Delta u + W \cdot \nabla u + Vu = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n.$$

In view of the scaling, the equation is invariant for $W \in L^n$ and $V \in L^{n/2}$ (or $W^{-1,n}$). One expects that UCP should hold for this class of coefficients.

Some results

- For $n \geq 3$, $W \equiv 0$, $V \in L^{n/2}$, SUCP holds [Jerison-Kenig, 85]. The uniqueness of IP is true (Laplacian) [Lavine-Nachman, 91], (admissible manifolds) [Dos Santos Ferreira-Kenig-Salo, 13], and for $V \in W^{-1,n}(n = 3, 4)$ [Haberman, 15].
- For $n = 3, 4$, $W \in L^n$, $V \equiv 0$, SUCP holds [Wolff, 90] and fails for $n \geq 5$, but UCP holds for $W \in L^n$, $V \in L^{n/2}$ [Wolff, 92]. Partial result of the uniqueness of IP ($n = 3$) is proved [Haberman, 16].

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- For $n = 3, 4$, $W \in L^n$, $V \equiv 0$, SUCP holds [Wolff, 90] and fails for $n \geq 5$, but UCP holds for $W \in L^n$, $V \in L^{n/2}$ [Wolff, 92]. Partial result of the uniqueness of IP ($n = 3$) is proved [Haberman, 16]. General case is open.
- For $n = 2$, $W \in L^2$, $V \equiv 0$, SUCP holds [Kim, 89; Kenig-W, 15] and the local uniqueness of IP is true (restriction on $\|W\|_{L^2}$) [Carstea-W., 16]. Global uniqueness (without restriction on the size of $\|W\|_{L^2}$) was recently solved [Nachman-Regev-Tataru, 17]. Prior to these results, for $n = 2$, $W \in L^p (p > 2)$, the uniqueness of IP solved solved in [Cheng-Yamamoto, 04].

Some results

- For $n \geq 2$, counterexample to UCP for $\Delta u + Vu = 0$ with $V \in L^1_{loc}$ [Kenig-Nadirashvili, 00]. So SUCP fails also for $\Delta u + Vu = 0$ in $\mathbb{R}^2$ with $V \in L^1_{loc}$.

- For $n = 2$, $W \equiv 0$, $V \in L^p_{loc}$ with $p > 1$, UCP holds [Amrein-Berthier-Georgescu, 81].

- For $n = 2$, the global uniqueness of IP for $\Delta u + qu = 0$ is true when
  - $q \in W^{1,p}(p > 2)$ [Bukhgeim, 08]
  - $q \in L^p(p > 2)$ [Imanuvilo-Yamamoto, 12; Blasten-Imanuvilo-Yamamoto, 15]
  - $q \in L^p(4/3 < p)$ [Blasten-Tzou-W, 17]

  Open for $1 < p \leq 4/3$

Quantitative estimates

Local problem

Now assume that u is a nontrivial solution to

$$\Delta u + W \cdot \nabla u + V u = 0$$

and the SUCP holds, u can not vanishes to infinite order at any point. What is the maximal vanishing order of any nontrivial u and how it depends on the sizes of W and V?

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Continuation at the infinity

Assume that u is a solution of $\Delta u + W \cdot \nabla u + Vu = 0$ in $\mathbb{R}^n$. Then what decaying rate of u at ∞ will ensure that u is trivial? The decaying rate actually depends on the sizes of W and V.

Landis conjecture

Landis' conjecture ($\sim$'60)

If $\Delta u + Vu = 0$ in $\mathbb{R}^n$ with $\|V\|_{L^\infty(\mathbb{R}^n)} \leq 1$ and $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_0$ satisfying $|u(x)| \leq C\exp(-C|x|^{1+})$, then $u \equiv 0$.

Counterexample and quantitative estimate

[Meshkov, '92]

- constructed such V (*complex-valued*) and nontrivial u (*complex-valued*) satisfying $|u(x)| \leq C\exp(-C|x|^{\frac{4}{3}})$.
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[Bourgain-Kenig, '05]

Let $|V(x)| \leq 1$, $|u(x)| \leq C_0$, and $u(0) = 1$, then

$$\inf_{|x_0|=R} \sup_{|x-x_0|<1} |u(x)| \geq \exp(-CR^{4/3}\log R) \quad \text{for} \quad R \gg 1$$

where C depends on C_0.

Maximal vanishing order

Landis' problem is equivalent to the estimate of the maximal vanishing order of any nontrivial solution.

Assume that $v \not\equiv 0$ is a solution of

$$\Delta v + qv = 0 \quad \text{in} \quad B_2.$$

Let $\|q\|_{L^\infty(B_2)} \leq M$. Assume that $\|v\|_{L^\infty(B_2)} \leq C_0$ and $\|v\|_{L^\infty(B_1)} \geq 1$. Find $\beta = \beta(C_0, M)$ such that

$$\|v\|_{L^\infty(B_r)} \geq r^\beta.$$

[Bourgain-Kenig, '05]: $\beta = CM^{2/3}$. The proof is based on the Carleman estimate.

General second order equations

By the same arguments, one can also show that if v satisfies

$$\Delta v + W \cdot \nabla v + Vv = 0 \quad \text{in} \quad B_2$$

and $\|v\|_{L^\infty(B_2)} \le C_0$, $\|v\|_{L^\infty(B_1)} \ge 1$,

$$\|W\|_{L^\infty(B_2)} \le K, \quad \|V\|_{L^\infty(B_2)} \le M,$$

then

$$\|v\|_{L^\infty(B_r)} \ge r^{C(K^2+M^{2/3})}.$$

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So the asymptotic quantitative uniqueness estimate behaves like $\exp(-R^2 \log R)$ at $R \gg 1$ [Davey, '14, Lin-W., 14']

Complex-valued solutions

For complex V (and W), and the complex solution u, the vanishing rate $CM^{2/3}$ (and $C(K^2 + M^{2/3})$) and the decay rate $\exp(-R^{4/3}\log R)$ (and $\exp(-R^2\log R)$) are sharp.

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Similar question: under the same setting and u solves $\Delta u + W\cdot\nabla u + Vu = 0$ with real W and $\|W\|_{L^\infty(B_2)} \leq K$, show that

$$\|u\|_{L^\infty(B_r)} \geq r^{C(M^{1/2}+K)}.$$

Related results

Related results

- Δ: Laplace-Beltrami operator on a compact smooth manifold and $V = \lambda$: eigenvalue, the sharp vanishing rate $C\lambda^{1/2}$ is true. [Donnelly-Fefferman, '93]
- $C(1 + \|V_-\|_{L^\infty}^{1/2} + (\text{osc } V)^2)$. [Kukavica, '98]
- $C(1 + \|V\|_{C^1}^{1/2})$, where $\|V\|_{C^1} = \|V\|_{L^\infty} + \|\nabla V\|_{L^\infty}$. [Bakri, '11]

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- $C(1 + \|V\|_{C^1}^{1/2} + \|W\|_{C^1})$. [Bakri, '13]

Simple equation

[Kenig-W., '15]

Consider the Schrödinger operator with an bounded drift term

$$\Delta u + W \cdot \nabla u = 0 \quad \text{in} \quad \mathbb{R}^2,$$

where $W = (W_1, W_2)$ is a real vector-valued functions with $\|W\|_{L^\infty} \leq 1$.

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where $W = (W_1, W_2)$ is a real vector-valued functions with $\|W\|_{L^\infty} \leq 1$.

Quantitative Landis: assume that $\|u\|_{L^\infty(\mathbb{R}^2)} \leq C_0$ and satisfies certain a priori assumption at 0. Then u (real solution) satisfies the following asymptotic estimates at $R \gg 1$

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq \exp(-C_1 R \log R)$$

Ideas

As above, the problem at ∞ is reduced to the estimate of the maximal vanishing order for the local problem.

Consider

$$\Delta v + W \cdot \nabla v = 0 \quad \text{in} \quad B_8$$

with

$$\|W\|_{L^\infty(B_8)} \le K.$$

If $g = v_x - iv_y$, then g satisfies

$$\bar{\partial} g = \alpha g \quad \text{in} \quad B_8, \tag{1}$$

where

$$\|\alpha\|_{L^\infty(B_8)} \le K.$$

Representation of solutions

Any solution of (1) is represented by

$$g = \exp(w)h \quad \text{in} \quad B_8,$$

where h is holomorphic in B_8 and

$$w(z) = -\frac{1}{\pi}\int_{B_8} \frac{\alpha(\xi)}{\xi - z} d\xi = (T\alpha)(z).$$

Cauchy transform

For $p > 2$, the Cauchy transform T maps L^p to L^∞, i.e.

$$\|W\|_{L^\infty(B_8)} = \|T\alpha\|_{L^\infty(B_8)} \leq CK.$$

By Hadamard's 3-circle theorem, we can prove:

If v satisfies $|v(z)| \leq C_0$ for all $z \in B_8$ and $\sup_{B_1} |\nabla v(z)| \geq 1$, then

$$\|v\|_{L^\infty(B_r)} \geq r^{C_1+C_2K}.$$

Nonpositive potentials

[Kenig-Silvestre-W., '15]

Assume that $W_1(x,y)$, $W_2(x,y)$ and $V(x,y)$ are real-valued and $V(x,y) \geq 0$ *a.e.* in $\mathbb{R}^2$, furthermore,

$$\|W\|_{L^\infty(\mathbb{R}^2)} \leq 1, \quad \|V\|_{L^\infty(\mathbb{R}^2)} \leq 1.$$

Let u be a real solution to

$$\Delta u - \nabla(Wu) - Vu = 0 \quad \text{in} \quad \mathbb{R}^2.$$

Assume that $|u(z)| \leq \exp(C_0|z|)$ and $u(0) = 1$. Then we have that

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq \exp(-CR\log R) \quad \text{for} \quad R \gg 1$$

where C depends on C_0.

Maximal vanishing order

Assume that $W_1(x, y)$, $W_2(x, y)$ and $V(x, y)$ are real-valued, and $V(x, y) \geq 0$ *a.e.* in B_2, moreover,

$$\|W\|_{L^\infty(B_2)} \leq K, \quad \|V\|_{L^\infty(B_2)} \leq M.$$

Let u be a real solution to

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Assume that $\|u\|_{L^\infty(B_2)} \leq \exp(C_0(\sqrt{M} + K))$ and $\|u\|_{L^\infty(B_1)} \geq 1$. Then

$$\|u\|_{L^\infty(B_r)} \geq r^{C(\sqrt{M}+K)}$$

for all sufficiently small r, where C depends on C_0.

Ideas

- Reduce to the Beltrami equation. Here we need the sign condition of V to construct a global positive multiplier.
- Explicit representation of solutions to the Beltrami equation.
- Hadamard's three-circle theorem.

Unbounded drift

[Kenig-W., '15]

Consider the Schrödinger operator with an unbounded drift term

$$\Delta u + W \cdot \nabla u = 0 \quad \text{in} \quad \mathbb{R}^2,$$

where $W = (W_1, W_2)$ is a real vector-valued functions with $\|W\|_{L^p} \leq 1$ for $2 \leq p < \infty$.

We are also interested in the lower bound of the decay rate for any nontrivial solution u.

$p = \infty$: discussed above.

Results

$2 < p < \infty$: $\|W\|_{L^p} \leq 1$

Assume that $\|u\|_{L^\infty(\mathbb{R}^2)} \leq C_0$ and satisfies certain a priori assumption at 0. Then u satisfies the following asymptotic estimates at $R \gg 1$

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Note that L^2 drift term W is scale invariant in $\mathbb{R}^2$

More singular lower order terms

[Davey-W., '17]

Consider real-valued solutions to

$$-\mathrm{div}(A\nabla u + W_1 u) + W_2 \cdot \nabla u + Vu = 0 \text{ in } \mathbb{R}^2,$$

where

$$V \in L^p, \qquad W_i \in L^{q_i} \quad p \in (1, \infty],\ q_i \in (2, \infty]$$

and satisfy suitable sign conditions and size conditions.

If $|u(z)| \leq \exp(C_0 |z|^\beta)$ and $u(0) = 1$, then for any R sufficiently large and any arbitrarily small $\varepsilon > 0$,

$$\inf_{|z_0|=R} \|u\|_{L^\infty(B_1(z_0))} \geq \exp(-CR^{\beta(1+\varepsilon)} \log R),$$

where $\beta = \max\{1 - \frac{1}{p}, 1 - \frac{2}{q_1}, 1 - \frac{2}{q_2}\}$.

Sign-changing potentials

[Davey-Kenig-W., '18]

$$\Delta u - Vu = 0 \quad \text{in} \quad \mathbb{R}^2.$$

We assume that $V = V_+ - V_-$ where $V_\pm \geq 0$ satisfies

$$\|V_+\|_{L^\infty(\mathbb{R}^2)} \leq 1 \tag{2}$$

$$V_-(z) \leq \exp\left(-c_0|z|^{1+\varepsilon_0}\right) \quad \forall z \in \mathbb{R}^2, \tag{3}$$

for some $\varepsilon_0 > 0$.

Theorem

Assume that $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies (2) and (3). Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a solution for which

$$|u(z)| \leq \exp(C_0|z|)$$
$$|u(0)| \geq 1.$$

Then for any $\varepsilon > 0$ and any $R \geq R_0(C_0, c_0, \varepsilon_0, \varepsilon)$, we have

$$\inf_{|z_0|=R} \|u\|_{L^\infty(B_1(z_0))} \geq \exp\left(-R^{1+\varepsilon}\right).$$

IP for L^p potentials, $p > 4/3$

Assume that $q_1, q_2 \in L^p(\Omega)$ and $\Lambda_{q_1} = \Lambda_{q_2}$. Then Alessandrini's identity holds

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0,$$

where u_j solves $\Delta u_j + q_j u_j = 0$ in Ω.

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where u_j solves $\Delta u_j + q_j u_j = 0$ in Ω.

Look for solutions u of $\Delta u + qu = 0$ having the following form

$$u = e^{i\tau\Phi} f \quad \text{or} \quad u = e^{i\tau\bar{\Phi}} f,$$

where

$$\Phi(z) = (z - z_0)^2, \quad z \in \mathbb{C}.$$

Why Φ?

Previously, we choose a linear phase function $\phi(x) = \rho \cdot x$ with $\rho \cdot \rho = 0$. Such phase function satisfies

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In $\mathbb{R}^2 = \mathbb{C}$, if we denote

$$\bar\partial = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}) \quad \partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}),$$

then $\Delta = 4\partial\bar\partial = 4\bar\partial\partial$ and so

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Φ is holomorphic and $\bar{\Phi}$ is anti-holomorphic.

Quadratic phases

The quadratic phase Φ was first introduced by [Uhlmann-W., '08]. We applied these solutions to do the reconstruction of discontinuity.

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In fact, we used polynomial phase functions $(z - z_0)^k$ with $k \in \mathbb{N}$. We put the critical point z_0 outside of the domain. But, Bukhgeim placed the critical point inside the domain Ω and took advantage of the fact that z_0 is a non-degenerate critical point. This is the setup of the method of the stationary phase.

Non-stationary point

Let f be a nice function, e.g., in $C_0^\infty(\mathbb{R}^n)$. Then consider the integral

$$I(\tau) = \int e^{i\tau\phi} f(x)dx.$$

If ϕ does not have critical points in the support of f, i.e., $\nabla\phi \neq 0$, then, by integration by parts,

$$I(\tau) = O(\tau^{-N})$$

for any $N \in \mathbb{N}$.

Stationary point

Now consider the oscillatory integral

$$I(\tau) = \int_{-\infty}^{\infty} e^{i\tau\phi(x)} f(x)dx$$

with $\phi'(x_0) = 0$ and $\phi''(x_0) \neq 0$ (non-degenerate). Then

$$I(\tau) \approx e^{i\tau\phi(x_0)} \int_{-\infty}^{\infty} e^{i\tau\phi''(x_0)(x-x_0)^2/2} f(x)dx$$

$$\approx f(x_0)e^{i\tau\phi(x_0)} e^{i\pi \text{sign}\phi''(x_0)/4} \sqrt{\frac{2\pi}{\tau|\phi''(x_0)|}}$$

The same computation can be extended to higher dimensions.

Existence of Bukhgeim's CGO

We look for solutions to $\Delta u_j + q_j u_j = 0$ in X, $\Omega \Subset \Omega$, of the form

$$u_1 = e^{i\tau\Phi} f_1, \quad u_2 = e^{i\tau\bar{\Phi}} f_2.$$

Here we show that such solutions exist as long as $q_j \in L^p(\Omega)$ with $p > 1$.

Existence of Bukhgeim's CGO

We look for solutions to $\Delta u_j + q_j u_j = 0$ in X, $\Omega \Subset \Omega$, of the form

$$u_1 = e^{i\tau\Phi} f_1, \quad u_2 = e^{i\tau\bar{\Phi}} f_2.$$

Here we show that such solutions exist as long as $q_j \in L^p(\Omega)$ with $p > 1$. Define two operators

$$D_1 f = -4e^{-i\tau(\Phi+\overline{\Phi})}\partial(e^{i\tau(\Phi+\overline{\Phi})}\bar{\partial} f),$$
$$D_2 f = -4e^{-i\tau(\Phi+\overline{\Phi})}\bar{\partial}(e^{i\tau(\Phi+\overline{\Phi})}\partial f).$$

$$(\Delta + q_1)(e^{i\tau\Phi} f_1) = 0 \quad \Leftrightarrow \quad D_1 f_1 = q_1 f_1,$$
$$(\Delta + q_2)(e^{i\tau\overline{\Phi}} f_2) = 0 \quad \Leftrightarrow \quad D_2 f_2 = q_2 f_2.$$

Cauchy operators

For inverting the operators D_1 and D_2 we will have to use conjugated versions of the *Cauchy operators* ∂^{-1} and $\bar{\partial}^{-1}$, where

$$\partial^{-1} u = \frac{1}{\pi z} * u, \quad \bar{\partial}^{-1} u = \frac{1}{\pi \bar{z}} * u.$$

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- If $1 < p < 2$, then $W^{1,p}(X) \hookrightarrow L^{p*}(X)$ with $\frac{1}{p} = \frac{1}{2} + \frac{1}{p*}$. So ∂^{-1} and $\bar{\partial}^{-1}$ are bounded $L^p(X) \to L^{p*}(X)$.

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- If $p > 2$, then $W^{1,p}(X) \hookrightarrow L^{\infty}(X)$ and so ∂^{-1} and $\bar{\partial}^{-1}$ are bounded $L^p(X) \to L^{\infty}(X)$.

Some operators

Let χ be a cut-off function in X with $\chi = 1$ on Ω. Define

$$S_1 f = -\frac{1}{4}\bar{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\chi\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}q_1 f)\right),$$

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Then $D_j S_j f = q_j f$ in Ω.

For $z_0 \in \mathbb{C}$, let β_j be a bounded function on X. We define

$$\varphi_1 = \frac{1}{4}\bar{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\chi(\beta_1(z_0) - \partial^{-1}q_1)\right),$$

$$\varphi_2 = \frac{1}{4}\partial^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\chi(\beta_2(z_0) - \bar{\partial}^{-1}q_2)\right).$$

Neumann series

Let f_j be defined by

$$f_j = e^{-i\tau(\Phi+\overline{\Phi})} + \sum_{m=0}^{\infty} S_j^m \varphi_j,$$

then $D_j f_j = q_j f_j$ and hence $u_1 = e^{i\tau\Phi} f_1$ and $u_2 = e^{i\tau\bar{\Phi}} f_2$ are solutions we are look for.

We can show that the Neumann series converges in $L^\infty(X)$.

Estimates of terms

Substitute $u_1 = e^{i\tau\Phi} f_1$ and $u_2 = e^{i\tau\bar{\Phi}} f_2$ into Alessandrini's identity:

$$\frac{2\tau}{\pi}\int (q_1 - q_2) u_1 u_2 dx = \sum_{k+l=0}^{\infty} \frac{2\tau}{\pi}\int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx$$
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We now estimate each term in the above identity.

Higher order terms

Let $q_1, q_2 \in L^p(\Omega)$ with $4/3 < p < 2$. Then if $q_1 - q_2 \in L^2(\Omega)$ we have

$$\left|\tau \int (q_1 - q_2) e^{i\tau(\Phi+\bar{\Phi})} F_{1,k} F_{2,l} dx\right| \leq C^{k+l} \tau^{-(k+l-2)\alpha}.$$

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We only know that $q_1, q_2 \in L^p(\Omega)$ with $p < 2$. How do we guarantee $q_1 - q_2 \in L^2(\Omega)$?

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It turns out $q_1 - q_2 \in L^2(\Omega)$ if both $q_1, q_2 \in L^p(\Omega)$ with $p > 4/3$ [Serov-Päivärinta, '05]. This is, Λ_q contains the information of the lower integrability part.

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So it is not artificial to assume that $q_1 - q_2 \in L^2(\Omega)$.

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Let $q_1, q_2 \in L^p(\Omega)$ with $4/3 < p < 2$. Assume that $q_1 - q_2 \in L^2(\Omega)$. For $k+l=2$, we have

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Thus, this term also decays in τ provided $p > 4/3$.

Next to the principal term

Note that we can treat

$$\left(\frac{2\tau}{\pi}\int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx\right)(z_0)$$

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Then taking the $L^2(X)$ with respect to z_0 implies

$$\lim_{\tau\to\infty}\left\|\frac{2\tau}{\pi}\int (q_1 - q_2)e^{i\tau(\Phi+\overline{\Phi})}F_{1,1}F_{2,0}dx\right\|_2$$
$$\leq C\|\beta_2 - \bar{\partial}^{-1}q_2\|_{p*}\|q_1 - q_2\|_p$$

Next to the principal term

Likewise, we have

$$\lim_{\tau\to\infty}\left\|\frac{2\tau}{\pi}\int(q_1-q_2)e^{i\tau(\Phi+\overline{\Phi})}F_{1,0}F_{2,1}\,dx\right\|_2$$
$$\leq C\|\beta_1-\partial^{-1}q_1\|_{p*}\|q_1-q_2\|_p$$

Since $\bar{\partial}^{-1},\partial^{-1}:L^p(X)\to L^{p*}(X)$, by the density argument, for any $\varepsilon>0$, we can choose $\beta_1,\beta_2\in C_0^\infty(X)$ such that

$$\|\beta_2-\bar{\partial}^{-1}q_2\|_{p*}<\varepsilon,\quad \|\beta_1-\partial^{-1}q_1\|_{p*}<\varepsilon.$$

Principal term

$$\lim_{\tau\to\infty}\left\|\frac{2\tau}{\pi}\int(q_1-q_2)e^{i\tau(\Phi+\bar{\Phi})}F_{1,0}F_{2,0}dx-(q_1-q_2)\right\|_2=0,$$

Putting everything together,

$$\|q_1-q_2\|_2<C\varepsilon$$

for any $\varepsilon>0$. Thus $q_1=q_2$.

Summary

- We gave some quantitative uniqueness estimates for solutions of the Schrödinger equation in terms of the sizes of the coefficients.
- We proved the global uniqueness of identifying q in $\Delta u + qu = 0$ on the plane from the boundary measurements for $q \in L^p$ with $p > 4/3$.
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THANK YOU