CONVEXIFICATION OF COEFFICIENT INVERSE PROBLEMS

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PDE-based Numerical Method for X-Ray Tomography with Incomplete Data

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- The Radon transform works only with a Special Case of Incomplete Data
- The efforts of researchers were focused so far on the integral formulation
- Unlike these, our method is based on a PDE of the first order
- Works with a special case of incomplete data
- Carleman estimate is used to prove uniqueness, existence and convergence
- Applications are in checking luggage in airports and in checking walls

AN ILLUSTRATION OF A 3D TOMOGRAPHIC EXPERIMENT

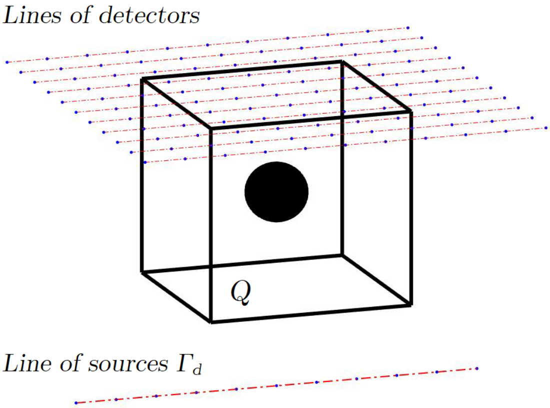


Figure 1: Source/detector configuration in 3D.

AN ILLUSTRATION OF A 3D TOMOGRAPHIC EXPERIMENT

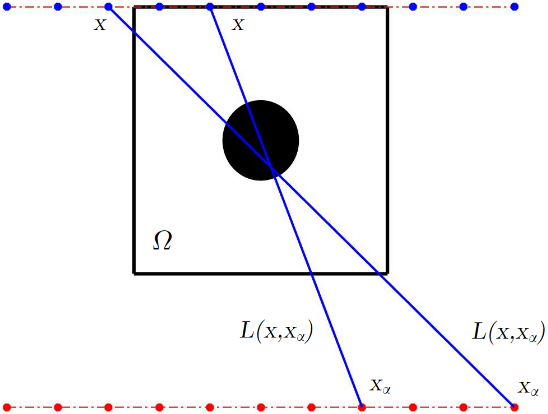


Figure 2: Source/detector configuration in any 2D cross section.

INVERSE PROBLEM STATEMENT

Let $a, b, R, d = const > 0, a < b$.

$$\Omega = (-R, R) \times (a, b), \quad \mathrm{x} = (x, y) \in \mathbb{R}^2$$

Let $f(\mathrm{x}) \in C^2(\mathbb{R}^2)$ be the unknown function,

$$f(\mathrm{x}) = 0 \text{ for } \mathrm{x} \in \mathbb{R}^2 \diagdown \Omega.$$

Consider sources

$$\mathrm{x}_\alpha = (\alpha, 0) \in \Gamma_d = \{\mathrm{x} = (\alpha, y) : \alpha \in (-d, d), y = 0\}.$$

We define the function $u(\mathrm{x}, \mathrm{x}_\alpha)$ as

$$u(\mathrm{x}, \mathrm{x}_\alpha) = \int\limits_{L(\mathrm{x}, \mathrm{x}_\alpha)} f(\xi) d\sigma,$$

where $L(\mathrm{x}, \mathrm{x}_\alpha)$ is the line segment connecting points x and x_α. x_α is the source and x is the detector.

RADON TRANSFORM PROBLEM

Determine the function f from the measurement of Rf

$$Rf = u(\mathsf{x}, \mathsf{x}_\alpha),$$

$$\forall \mathsf{x} = (x, y) \in \partial\Omega, \forall \mathsf{x}_\alpha = (\alpha, 0) \in \Gamma_d.$$

The function Rf is known as the Radon transform of the function f

A SPECIAL ORTHONORMAL BASIS IN $L_2(-d, d)$

(Klibanov, J. Inverse and Ill-Posed Problems, 25, 669-685,2017)

We need to construct such an orthonormal basis $\{\Psi_n(\alpha)\}_{n=0}^{\infty}$ in $L_2(-d, d)$ that:

1. $\Psi_n \in C^1[-d, d], \forall n = 0, 1, ...$
2. Let $(\cdot, \cdot)$ denotes the scalar product in $L_2(-d, d)$.

Let $a_{mn} = (\Psi_n', \Psi_m)$. Then the matrix $M_N = (a_{mn})_{m,n=0}^{N-1}$ is invertible for any $N = 1, 2, ...$

Neither standard orthonormal polynomials nor trigonometric functions $\left\{e^{[i(n\pi x)/d]}\right\}$ are not suitable since each such basis contains a constant function whose derivative is identical zero.

A SPECIAL ORTHONORMAL BASIS IN $L_2\,(-d,d)$

Functions $\{\alpha^k e^\alpha\}_{k=0}^\infty$ are linearly independent and form a complete set in $L_2(-d,d)$.

Gram-Schmidt orthonormalization procedure $\rightarrow \{\Psi_k\,(\alpha)\}_{k=0}^\infty$ in $L_2(-d,d)$, $\Psi_k(\alpha) = P_k(\alpha)e^\alpha$.

Lemma 1. *We have*

$$\phi_{mk} = \int_{-d}^{d} \Psi_k'(\alpha)\Psi_m(\alpha)d\alpha = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k < m. \end{cases}$$

Consequently, for any integer $N > 1$, the determinant of the matrix $M_N = (\phi_{mk})_{m,k=0}^N$ equals 1. Therefore, the matrix M_N is invertible.

[A. H. Hasanoğlu and V. G. Romanov, *Introduction to Inverse Problems for Differential Equations*, Springer, Cham, 2017]:

$$\frac{x-\alpha}{\sqrt{(x-\alpha)^2+y^2}}u_x+\frac{y}{\sqrt{(x-\alpha)^2+y^2}}u_y=f(x,y)\,,\quad (x,y)\in\Omega.$$

Differentiate with respect to α. Let $v(x,y,\alpha)=\partial_\alpha u(x,y,\alpha)\,.$

- Then the unknown function $f(x,y)$ is eliminated: as in the first step of the Bukhgeim-Klibanov method

$$v_y=-\frac{x-\alpha}{|\mathrm{x}-\mathrm{x}_\alpha|^2}u_y-\frac{x-\alpha}{y}v_x+\frac{y}{|\mathrm{x}-\mathrm{x}_\alpha|^2}u_x,\quad ,\alpha\in(-d,d). \tag{1}$$

PARTIAL DIFFERENTIAL EQUATION

- Approximate the function $u\,(x, y, \alpha)$ via a truncated Fourier series

$$u(x, y, \alpha) = \sum_{n=0}^{N-1} u_n(x, y)\Psi_n(\alpha), \;\; \mathsf{x} = (x, y) \in \Omega, \alpha \in (-d, d). \tag{2}$$

- The number N should be chosen numerically.

$$v(\mathsf{x}, \alpha) = \sum_{n=0}^{N-1} u_n(\mathsf{x})\Psi_n'(\alpha), \quad \mathsf{x} = (x, y) \in \Omega, \alpha \in (-d, d). \tag{3}$$

Substituting (2) and (3) in *(1)*, we obtain

$$\sum_{n=0}^{N-1} \partial_y u_n(\mathsf{x})\Psi_n'(\alpha) = -\frac{X - \alpha}{\left|\mathsf{x} - \mathsf{x}_\alpha\right|^2} \sum_{n=0}^{N-1} \partial_y u_n(\mathsf{x})\Psi_n(\alpha) - \tag{4}$$

$$-\frac{x-\alpha}{y}\sum_{n=0}^{N-1}\partial_x u_n(\mathrm{x})\Psi_n'(\alpha)+\frac{y}{|\mathrm{x}-\mathrm{x}_\alpha|^2}\sum_{n=0}^{N-1}\partial_x u_n(\mathrm{x})\Psi_n(\alpha).$$

Multiplying (4) by $\Psi_k(\alpha), k = 0, ..., N-1$ and integrating with respect to $\alpha \in (-d, d)$, we obtain

$$(M_N - D_1(\mathrm{x}))\,\mathrm{U}_y(\mathrm{x}) - D_2(\mathrm{x})\,\mathrm{U}_x(\mathrm{x}) = 0, \ \mathrm{x} = (x, y) \in \Omega, \tag{5}$$

$$\mathrm{U}(\mathrm{x}) = (u_0, ..., u_{N-1})^T(\mathrm{x}),$$

$$\max_{\mathrm{x}\in\overline{\Omega}} \|D_1(\mathrm{x})\| \le \frac{C}{a}.$$

Lemma 2. *For each $N > 1$ there exists a sufficiently large number $a_0 = a_0(N, R, d) > 1$ depending only on listed parameters such that for any $a \geq a_0$ the matrix $M_N\left(I - M_N^{-1}D_1(\mathrm{x})\right)$ is invertible. Denote $D(\mathrm{x}) = -\left[M_N\left(I - M_N^{-1}D_1(\mathrm{x})\right)\right]^{-1} D_2(\mathrm{x})$. Then there exists a constant $C^1 > 0$ such that*

$$\max_{\mathrm{x}\in\overline{\Omega}} \|D(\mathrm{x})\| \leq C_1$$

and equation (5) is equivalent with

$$\mathrm{U}_y(\mathrm{x}) + D(\mathrm{x})\,\mathrm{U}_x(\mathrm{x}) = 0,\ \mathrm{x} = (x, y) \in \Omega.$$

OVERDETERMINED BOUNDARY VALUE PROBLEM

Find the vector function $\mathbf{U}(\mathbf{x})$ satisfying the following conditions:

$$\mathbf{U}_y(\mathbf{x}) + D(\mathbf{x})\,\mathbf{U}_x(\mathbf{x}) = 0,\ \mathbf{x} = (x, y) \in \Omega, \tag{6}$$

$$\mathbf{U}(\mathbf{x}) = \mathbf{g}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{7}$$

- Semi-discrete formulation of the boundary value problem (6), (7).
- Consider the grid of the finite difference scheme with the step size h:

$$x_0 = -R < x_1 = h < ... < x_i = ih < ... < x_{K-1} = (K-1)\,h < x_K = R.$$

We define the domain Ω^h as

$$\Omega^h = \{\mathbf{x} = (x, y) : y \in [a, b]\,, x = ih, i = 1, ..., (K-1)\}\,.$$

$$\mathbf{U}_i^h(y) = \mathbf{U}(ih, y)\,, \mathbf{U}^h(y) = \left(\mathbf{U}_1^h, ..., \mathbf{U}_K^h\right)(y)\,,\ y \in [a, b]\,.$$

Since $\mathbf{U}$ is an N−D vector, then $\mathbf{U}^h$ is a $N \times (K+1)$ matrix.

OVERDETERMINED BOUNDARY VALUE PROBLEM

We approximate the derivative U_x at the point (ih, y) by the central finite difference as

$$\mathrm{U}_{ix}^h(ih, y) = \frac{\mathrm{U}_{i+1}^h(y) - \mathrm{U}_{i-1}^h(y)}{2h}, \; i = 1, ..., (K-1).$$

Semi finite difference formulation of the problem (6), (7):

$$\mathrm{U}_y^h(\mathrm{x}) + D^h(\mathrm{x})\,\mathrm{U}_x^h(\mathrm{x}) = 0, \; \mathrm{x} \in \Omega^h, \tag{8}$$

$$\mathrm{U}^h(\mathrm{x}) = \mathrm{g}^h(\mathrm{x}), \quad \mathrm{x} \in \partial\Omega. \tag{9}$$

Spaces of semi discrete functions:

$$L_2^h\left(\Omega^h\right) = \left\{\mathrm{U}^h : \left\|\mathrm{U}^h\right\|^2_{L_2^h(\Omega^h)} = \sum_{i=1}^{K-1} h \int_a^b \left[\mathrm{U}_i^h(y)\right]^2 dy < \infty\right\},$$

$$H^{1,h}\left(\Omega^h\right) = \left\{ \mathbf{U}^h : \left\|\mathbf{U}^h\right\|^2_{H^{1,h}\left(\Omega^h\right)} = \sum_{i=1}^{K-1} h \int_a^b \left[\left(\mathbf{U}^h_{ix}\left(ih, y\right)\right)^2 + \left(\mathbf{U}^h_{iy}\left(y\right)\right)^2 + \left(\mathbf{U}^h_{i}\left(y\right)\right)^2 \right] dy < \infty \right\},$$

$$H_0^{1,h}\left(\Omega^h\right) = \left\{ \mathbf{U}^h \in H^{1,h}\left(\Omega^h\right) : \mathbf{U}^h \mid_{\partial\Omega} = 0 \right\}.$$

There exists a constant $B_{h_0} = B_{h_0}\left(h_0\right) > 0$ depending only on h_0 such that

$$\left\|\mathbf{U}^h_x\left(y\right)\right\|^2_{L_2^h\left(\Omega^h\right)} \leq B_{h_0} \left\|\mathbf{U}^h\right\|^2_{L_2^h\left(\Omega^h\right)}, \ \forall h \in \left[h_0, 1\right].$$

Thus, the semi-discrete QUASI-REVERSIBILITY METHOD applied to problem (8), (9) is:

MINIMIZATION PROBLEM 1. *Let $\varepsilon \in [0,1)$ be the regularization parameter. Minimize the functional $J_{\alpha}^{h}\left(\mathbf{U}^{h}\right)$,*

$$J_{\varepsilon}^{h}\left(\mathbf{U}^{h}\right)=\left\|\mathbf{U}_{y}^{h}(y)+D^{h}(\mathbf{x})\,\mathbf{U}_{x}^{h}(y)\right\|_{L_{2}^{h}\left(\Omega^{h}\right)}^{2}+\epsilon\left\|\mathbf{U}^{h}(y)\right\|_{H^{1,h}\left(\Omega^{h}\right)}^{2}$$

on the space of matrices $\mathbf{U}^{h}\in H^{1,h}\left(\Omega^{h}\right)$ satisfying boundary condition $\mathbf{U}^{h}(\mathbf{x})=\mathbf{g}^{h}(\mathbf{x})$.

Lemma 3. (Carleman estimate). *Let the parameter $\lambda>0$. Let $H_{0}^{1}(a,b)$ be the subspace of functions $w\in H^{1}(a,b)$ satisfying the boundary condition $w(b)=0$. Then the following Carleman estimate holds*

$$\int_{a}^{b}\left(w'\right)^{2}e^{2\lambda y}dy\geq\frac{1}{2}\int_{a}^{b}\left(w'\right)^{2}e^{2\lambda y}dy+\frac{1}{2}\lambda^{2}\int_{a}^{b}w^{2}e^{2\lambda y}dy,$$

$\forall w\in H_{0}^{1}(a,b),\forall\lambda>0.$

Theorem 1. *Assume that* $a \geq a_0 = a_0(N, R, d) > 1$, *where* $a_0(N, R, d)$ *is the number of Lemma 2. Also assume that there exists a* $N \times (K+1)$ *matrix* $\mathrm{F}^h \in H^{1,h}\left(\Omega^h\right)$ *such that* $\mathrm{F}^h \mid_{\partial\Omega} = \mathbf{g}^h(\mathrm{x})$. *Then for each number* $\epsilon \in [0,1)$ *and for each* $h \in [h_0, 1)$ *there exists unique solution* $\mathrm{U}^h \in H^{1,h}\left(\Omega^h\right)$ *of the Minimization Problem 1. Furthermore, there exists a constant* $C_{h_0} = C_{h_0}(N, \Omega, d, h_0) > 0$ *depending only on listed parameters such that for all* $h \in (0, h_0)$ *the following estimate holds:*

$$\left\|\mathrm{U}^h\right\|_{H^{1,h}\left(\Omega^h\right)} \leq C_{h_0} \left\|\mathrm{F}^h\right\|_{H^{1,h}\left(\Omega^h\right)}.$$

Let $\delta \in (0,1)$ be the level of noise in the data. Let $\mathsf{U}^{*,h}(\mathsf{x})$ be the exact solution of problem (8), (9) with noiseless data $\mathsf{g}^{*,h}(\mathsf{x})$ in (9). Suppose that there exists a matrix $\mathsf{F}^{*,h} \in H^{1,h}\left(\Omega^h\right)$ such that $\mathsf{F}^{*,h}\mid_{\partial\Omega} = \mathsf{g}^{*,h}(\mathsf{x})$.

Also, let $\mathsf{g}_\delta^h(\mathsf{x})$ be the noisy data in (9) and assume that there exists a matrix $\mathsf{F}_\delta^h \in H^{1,h}\left(\Omega^h\right)$ such that $\mathsf{F}_\delta^h\mid_{\partial\Omega} = \mathsf{g}_\delta^h(\mathsf{x})$. We assume the following error estimate:

$$\left\|\mathsf{F}_\delta^h - \mathsf{F}^{*,h}\right\|_{H^{1,h}\left(\Omega^h\right)} \leq \delta. \tag{10}$$

CONVERGENCE RATE THEOREM

Theorem 2. (convergence rate) *Let* $\mathbf{U}^{*,h}(\mathbf{x})$ *be the exact solution of equation (8) with noiseless data* $\mathbf{g}^{*,h}(\mathbf{x})$ *in (9). Let* $\mathbf{U}_{\delta}^{h}(\mathbf{x})$ *be the solution of equation (8) with noisy data* $\mathbf{g}_{\delta}^{h}(\mathbf{x})$ *in (9), which was constructed in Theorem 1. Assume that conditions of Theorem 1 hold and also that error estimate (10) is valid. Then for all* $h \in [h_0, 1)$ *the following convergence rate is valid:*

$$\left\|\mathbf{U}_{\delta}^{h} - \mathbf{U}^{*,h}\right\|_{H^{1,h}(\Omega^h)} \leq C_{h_0}\left(\delta + \sqrt{\epsilon}\left\|\mathbf{U}^{*,h}\right\|_{H^{1,h}(\Omega^h)}\right).$$

Equation:

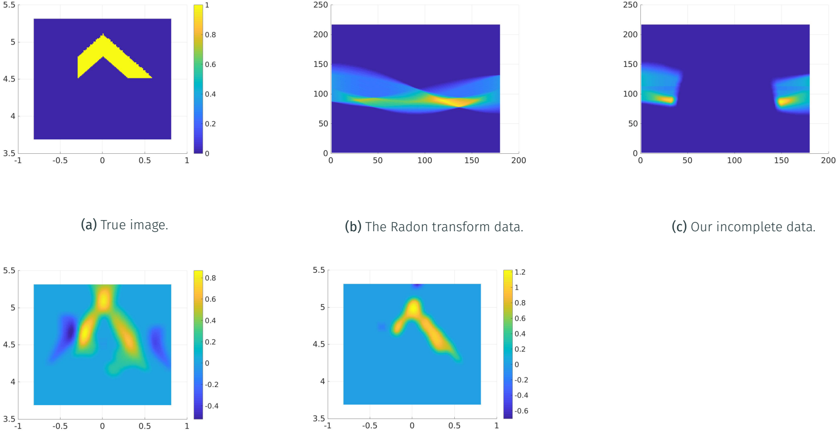
$$\frac{x-\alpha}{\sqrt{(x-\alpha)^2+y^2}}u_x+\frac{y}{\sqrt{(x-\alpha)^2+y^2}}u_y+c(x,y)\,u=f(x,y), \quad (x,y)\in\Omega.$$

The only difference with the original equation is in the term $c(x,y)\,u$.

However, since Carleman estimates are insentive to lower terms of PDE operators, then our technique works for this case.

Inversion formula was derived by Novikov in 2002.

NUMERICAL STUDIES: COMPARISON WITH FBP



(a) True image.

(b) The Radon transform data.

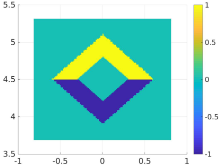
(c) Our incomplete data.

(d) FBP, artifacts removed.

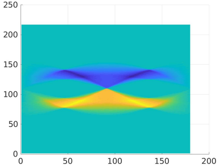
(e) Our method, artifact removed

Figure 3: The reconstructions of the function with an L-shape image.

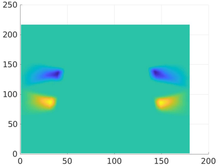
NUMERICAL STUDIES: COMPARISON WITH FBP



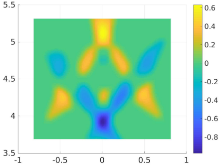
(a) True image.



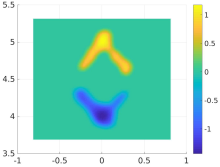
(b) The Radon transform data.



(c) Our incomplete data.



(d) FBP, artifacts removed.



(e) Our method, artifact removed

Figure 4: The reconstructions of the function with an L-shape image.

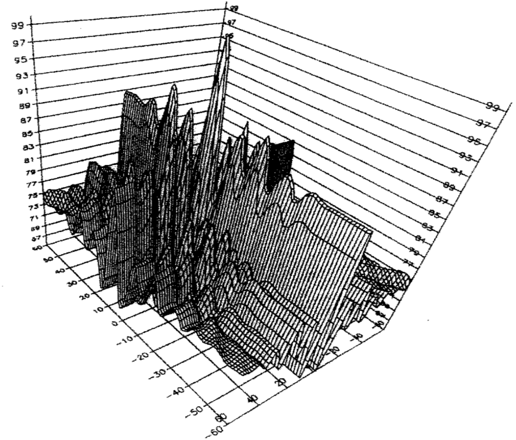


Figure 5: An example of multiple local minima which are depicted a maxima for convenience. J. A. Scales, M. L. Smith, and T. L. Fischer, J. of Computational Physics, 103, 258-268, 1992.

CONVEXIFICATION

- Coefficient inverse problems for PDEs are nonlinear
- Therefore, conventional least squares functionals for them usually have many local minima and ravines
- Hence, only LOCALLY convergent numerical methods can work in the conventional case, i.e. small perturbation approaches, e.g. gradient-like methods and Newton-like methods
- Convexification constructs globally strictly convex cost functionals for CIPs
- The main ingredient: CARLEMAN WEIGHT FUNCTION
- GLOBALLY convergent numerical methods are constructed and tested numerically

DEFINITION. We call a numerical method for a CIP *globally convergent* if a theorem is proved, which claims that this method delivers at least one point in a sufficiently small neighborhood of the exact solution without any advanced knowledge of this neighborhood.

A. CIPs with single measurement data.

- Two types of globally convergent numerical methods: Klibanov and his group, 1995-2018.
- Carleman Weight Functions Method.
- The tail function method.
- Both methods are fully verified on experimental backscattering data.

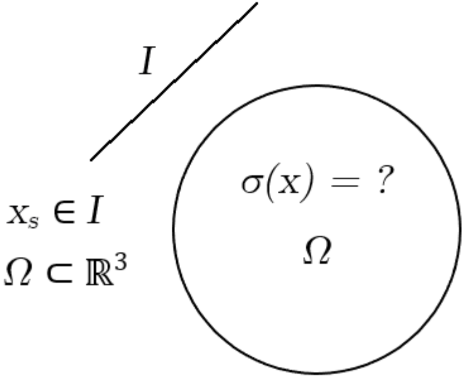
B. CIPs with restricted Dirichlet-to-Neumann map data: convexification

C. CIPs with Dirichlet-to-Neumann map data

- Methods of M.I. Belishev and S.I. Kabanikhin.

ELECTRICAL IMPEDANCE TOMOGRAPHY WITH RESTRICTED DIRICHLET-TO-NEUMANN DATA

Michael V. Klibanov, Jingzhi Li and Wenlong Zhang



Let $\Omega, G \subset \mathbb{R}^n, n = 2, 3$ be two bounded domains with piecewise smooth boundaries, $\Omega \subset G, \partial\Omega \cap \partial G = \varnothing$.

Let $\overline{x} \in \mathbb{R}^{n-1}$ be a fixed point. For $s \in [0,1]$ denote $x_s = (x_{1s}, \overline{x})$ the position of the point source. Let

$$I = \{x_s = (x_{1s}, \overline{x}) : s \in [0,1]\} \subset G,$$

$$I \cap \overline{\Omega} = \varnothing.$$

be the interval of the straight line of sources.

Consider a small neighborhood I_ε of the interval I,

$$I_\varepsilon = \{x \in \mathbb{R}^n : dist(x, I) < \varepsilon\} \subset \left(G \diagdown \overline{\Omega}\right),$$

$$f(x - x_s) = \begin{cases} \frac{1}{\varepsilon} \exp\left(-\frac{1}{1 - |x - x_s|^2/\varepsilon}\right), \text{ if } (x - x_s)^2 < \varepsilon, \\ 0, \text{ otherwise.} \end{cases}$$

$$\begin{cases} \nabla \cdot (\sigma(x) \nabla u(x,s)) & = -f(x - x_s), & x \in G, \forall x_s \in I, \\ u(x,s)|_{x \in \partial G} & = 0, & \forall x_s \in I. \end{cases}$$

COEFFICIENT INVERSE PROBLEM (CIP)

Let the conductivity function

$$\sigma(x) \in C^{2+\alpha}(\overline{G}), \sigma(x) = 1 \text{ for } x \in G \setminus \Omega \text{ and } \sigma(x) \geq const. > 0.$$

We measure both Dirichlet and Neumann boundary conditions of the function u on a part $\Gamma \subseteq \partial\Omega$ of the boundary $\partial\Omega$,

$$u(x,s)|_{x\in\Gamma, x_s\in I} = g_0(x,s) \text{ and } \partial_n u(x,s)|_{x\in\Gamma, x_s\in I} = g_1(x,s). \tag{11}$$

- We call (11) Restricted Dirichlet-to-Neumann Data
- These data are non overdetermined in both 2D and 3D cases

CIP. Assume that the function $\sigma(x)$ *is unknown for* $x \in \Omega$. *Also, assume that functions* $g_0(x,s)$ *and* $g_1(x,s)$ *in (11) are known for all* $x \in \Gamma, x_s \in I$. *Determine the function* $\sigma(x)$.

First, we introduce the well known change of variables $u_1 = \sqrt{\sigma} u$.

$$a_0(x) = -\frac{\Delta\left(\sqrt{\sigma(x)}\right)}{\sqrt{\sigma(x)}}$$

Then

$$\begin{cases} \Delta u_1(x,s) + a_0(x) u_1(x,s) & = -f(x - x_s), \quad x \in G, \qquad \forall x_s \in \bar{I}, \\ u_1(x,s)|_{x \in \partial G} & = 0, \qquad\qquad\qquad\qquad \forall x_s \in \bar{I}, \end{cases} \tag{12}$$

$$u_1(x,s)|_{x \in \Gamma, s \in (0,1)} = g_0(x,s) \text{ and } \partial_\nu u_1(x,s)|_{x \in \Gamma, s \in (0,1)} = g_1(x,s). \tag{13}$$

If we would recover the function $a_0(x)$ for $x \in G$ from conditions (12), (13), then we would recover the function $\sigma(x)$.

We have $u_1(x,s) > 0$ for all $x \in G$ and all $s \in [0,1]$. Hence, we can consider the function $v(x,s)$,

$$v(x,s) = \ln u_1(x,s).$$

Then (12) implies that

$$\Delta v(x,s) + (\nabla v(x,s))^2 = -a_0(x), \qquad x \in \Omega, \forall s \in [0,1]. \tag{14}$$

In addition, using (13), we obtain

$$v(x,s)|_{x\in\Gamma, s\in[0,1]} = \tilde{g}_0(x,s) \text{ and } \partial_\nu v(x,s)|_{x\in\Gamma, s\in[0,1]} = \tilde{g}_1(x,s). \tag{15}$$

Differentiate equation (14) with respect to s,

$$\Delta v_s + 2\nabla v_s \cdot \nabla v = 0, \qquad x \in \Omega, \forall s \in [0,1]. \tag{16}$$

Now the CIP1 is reduced to the following problem:

Reduced Problem. *Recover the function $v(x,s)$ from equation (16), given the boundary measurements $\tilde{g}_0(x,s)$ and $\tilde{g}_1(x,s)$ in (15).*

Let $\{\Psi_n(s)\}_{n=0}^{\infty} \subset L_2(0,1)$ be the orthonormal basis as above.

APPROXIMATE MATHEMATICAL MODEL. We assume that the function $v(x,s)$ is represented via a truncated Fourier series with respect to the basis $\{\Psi_n(s)\}_{n=0}^{\infty}$,

$$v(x,s) = \sum_{n=0}^{N-1} v_n(x)\,\Psi_n(s),\ x \in \Omega, s \in (0,1),$$

where the vector function $V(x) = (v_0(x), ..., v_{N-1}(x))^T$ of coefficients is unknown.

As above, the number N should be chosen numerically.

Similarly with the tomography case

$$M_N \Delta V - \widetilde{F}(\nabla V) = 0,\ x \in \Omega, V \in C^3\left(\overline{\Omega}\right),$$

$$\phi_{mk} = \int_0^1 \Psi_k'(s)\Psi_m(s)ds,$$

$M_N = (\phi_{mk})_{m,k=0}^{N-1}$. M_N^{-1} exists. Let $F(\nabla V) = M_N^{-1}\widetilde{F}(\nabla V)$. Then

$$\Delta V - F(\nabla V) = 0, x \in \Omega, V \in C^3\left(\overline{\Omega}\right), \tag{17}$$

$$V(x)\mid_\Gamma = p_0(x), \partial_\nu V(x)\mid_\Gamma = p_1(x). \tag{18}$$

Let $\mu, \rho > 0$ and $\mu < \rho$. Denote $r = |x|$. Let

$$\Omega = \left\{ x \in \mathbb{R}^3 : |x| < \rho \right\},$$

$$\Omega_\mu = \left\{ x \in \mathbb{R}^3 : \mu < |x| < \rho \right\} \subset \Omega,$$

$$H_0^2 \left(\Omega_\mu \right) = \left\{ w \in H^2 \left(\Omega_\mu \right) : w \mid_{r=R} = w_r \mid_{r=R} = 0 \right\}.$$

Lemma 2. (Carleman estimate). *There exists a number $\lambda_0 = \lambda_0 \left(\Omega_\mu \right) \geq 1$ and a number $C = C \left(\Omega_\mu \right) > 0$, both depending only on the domain Ω_μ, such that for any function $w \in H_0^2 \left(\Omega_\mu \right)$ and for all $\lambda \geq \lambda_0$ the following Carleman estimate with the CWF $e^{2\lambda r}$ holds:*

$$\int\limits_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx \geq \frac{1}{2} \int\limits_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx + C\lambda \int\limits_{\Omega_\mu} (\nabla w)^2 e^{2\lambda r} dx + C\lambda^3 \int\limits_{\Omega_\mu} w^2 e^{2\lambda r} dx$$

$$-C\lambda^3 e^{2\lambda \mu} \left\| w \right\|^2_{H^2(\Omega_\mu)}.$$

Let

$$\Gamma = \partial\Omega = \{r = R\}$$

Arrange zero Dirichlet and Neumann boundary conditions at Γ,

$$P(r,\varphi,\theta) = p_0(r,\varphi,\theta) + (r-\rho)\,p_1(r,\varphi,\theta),$$

$$W(r,\varphi,\theta) = V(r,\varphi,\theta) - P(r,\varphi,\theta);\; W(r,\varphi,\theta) = (W_0,...,W_{N-1})^T(r,\varphi,\theta).$$

Hence, (17) and (18) imply that

$$\Delta W + \Delta P - F(\nabla W + \nabla P) = 0,$$

$$W \in H_0^3(\Omega_\mu).$$

Weighted Tikhonov-like functional:

$$J_{\lambda,\beta}(W) = \tag{19}$$

$$= e^{-2\lambda(\mu+\eta)} \int_\Omega [\Delta W + \Delta P - F(\nabla W + \nabla P)]^2 e^{2\lambda r} dx + \beta \|W + P\|^2_{H^3(\Omega_\mu)}.$$

Let $R > 0$ be an arbitrary number,

$$B(R) = \left\{ W \in H_0^3(\Omega_\mu) : \|W\|_{H^3(\Omega_\mu)} < R \right\}.$$

Minimization Problem 2. *Minimize the functional $J_{\lambda,\beta}(W)$ on the closed ball $\overline{B(R)}$.*

Theorem 3. (global strict convexity) *Let $\eta > 0$ be such that $\mu + \eta < \rho$. The functional $J_{\lambda,\beta}(W)$ has the Frechét derivative $J'_{\lambda,\beta}(W)$ at every point $W \in H_0^3(\Omega_\mu)$. Furthermore, there exists numbers*

$\lambda_2 = \lambda_2(\mu, \eta, F, N, P, R) \geq \lambda_0 > 0$ and $C_2 = C_2(\mu, \eta, F, N, P, R) > 0$ depending only on listed parameters if $2e^{-\lambda_2 \eta} < 1$ then for all $\lambda \geq \lambda_2$ the functional $J_{\lambda,\beta}(W)$ is strictly convex on $\overline{B(R)}$ for the choice of $\beta \in \left(2e^{-\lambda\eta}, 1\right)$. More precisely, the following inequality holds:

$$J_{\lambda,\beta}(W_2) - J_{\lambda,\beta}(W_1) - J'_{\lambda,\beta}(W_1)(W_2 - W_1)$$

$$\geq C_2 \|\Delta(W_2 - W_1)\|_{L_2(\Omega_{\mu+\eta})} + C_2 \|W_2 - W_1\|^2_{H^1(\Omega_{\mu+\eta})} + \frac{\beta}{2} \|W_2 - W_1\|^2_{H^3(\Omega_\mu)},$$

$$\forall W_1, W_2 \in \overline{B(R)}.$$

Let $P_B : H_0^3(\Omega_\mu) \to \overline{B(R)}$ be the projection operator of the space $H_0^3(\Omega_\mu)$ on the closed ball $\overline{B(R)}$.

Let $W_0 \in B(R)$ be an arbitrary point. The gradient projection method:

$$W_n = P_B\left(W_{n-1} - \zeta J'_{\lambda,\beta}(W_{n-1})\right),\ n = 1, 2, ..., \tag{20}$$

where $\zeta \in (0,1)$ is a sufficiently small number. $[,]$ denotes the scalar product in the space of real valued $N-$D vector functions $H^3(\Omega_\mu)$.

Theorem 4. *Let $\lambda_2 = \lambda_2(\mu, \eta, F, N, P, R) \geq \lambda_0 > 0$ be the number of Theorem 3 and let the regularization parameter $\beta \in \left(2e^{-\lambda\eta}, 1\right)$. Then for every $\lambda \geq \lambda_2$ there exists unique minimizer $W_{\min,\lambda,\beta} \in \overline{B(R)}$ of the functional $J_{\lambda,\beta}(W)$ on the closed ball $\overline{B(R)}$. Furthermore, the following inequality holds*

$$\left[J'_{\lambda,\beta}\left(W_{\min,\lambda,\beta}\right), W - W_{\min,\lambda,\beta}\right] \geq 0,\ \forall W \in \overline{B(R)}.$$

In addition, there exists a sufficiently small number $\zeta_0 = \zeta_0(\mu, \eta, F, N, P, R, \lambda, \beta) \in (0,1)$ depending only on listed parameters such that for every $\zeta \in (0, \zeta_0)$ the sequence (20) converges to the minimizer $W_{\min,\lambda,\beta}$ and the following estimate of the convergence rate holds:

$$\left\|W_n - W_{\min,\lambda,\beta}\right\|_{H^3(\Omega)} \leq \omega^n \left\|W_0 - W_{\min,\lambda,\beta}\right\|_{H^3(\Omega)}, \ n = 1, 2, ...,$$

where $\omega = \omega(\zeta) \in (0,1)$ depends only on the parameter ζ.

Let $\delta \in (0,1)$ be the level of the noise in the data.

Let W^* be the exact solution with the noiseless data $P^* \in C^3\left(\overline{\Omega}_\mu\right)$,

$$\Delta W^* + \Delta P^* - F(\nabla W^* + \nabla P^*) = 0,$$

$$W^* \in H_0^3(\Omega_\mu).$$

Let $P \in C^3\left(\overline{\Omega}_\mu\right)$ be the noisy data. Denote $\widetilde{P} = P - P^*$. We assume that

$$\left\|\widetilde{P}\right\|_{H^3\left(\overline{\Omega}_\mu\right)} \leq \delta.$$

Theorem 5. *Let* $\lambda_2 \geq \lambda_0 > 0$ *and* $C_2 > 0$ *be numbers of Theorem 3. Choose the number* $\delta_1 > 0$ *so small that* $\delta_1 < \min\left(e^{-4\rho\lambda_2}, 3^{-4\rho/\eta}\right)$ *and let* $\delta \in (0, \delta_1)$ *. Set* $\lambda = \lambda(\delta) = \ln \delta^{-1/(4\rho)}$, $\beta = \beta(\delta) = 3\delta^{\eta/(4\rho)}$. *Also, assume that the vector function* $W^* \in B(R)$. *Let* $W_{\min,\lambda(\delta),\beta(\delta)} \in \overline{B(R)}$ *be the minimizer of the functional* $J_{\lambda,\beta}(W)$, *which is guaranteed by Theorem 4. Also, let the number* $\zeta \in (0, \zeta_0)$ *in (20) be the same as in Theorem 4, so as the number* $\omega \in (0, 1)$. *Then the following estimates hold:*

$$\left\|W^* - W_{\min,\lambda(\delta),\beta(\delta)}\right\|_{H^1(\Omega_{\mu+\eta})} \leq C_2\delta^{\eta/(8\rho)}, \tag{21}$$

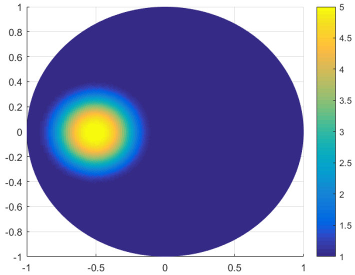
$$\|W^* - W_n\|_{H^1(\Omega_{\mu+\eta})} \leq C_2\delta^{\eta/(8\rho)} + \omega^n \left\|W_0 - W_{\min,\lambda(\delta),\beta(\delta)}\right\|_{H^3(\Omega)}, \ n = 1, 2, ..., \tag{22}$$

$$\|\Delta W^* - \Delta W_n\|_{L^2(\Omega_{\mu+\eta})} \leq C_2\delta^{\eta/(8\rho)} + \omega^n \left\|W_0 - W_{\min,\lambda(\delta),\beta(\delta)}\right\|_{H^3(\Omega)}, \ n = 1, 2, ... \tag{23}$$

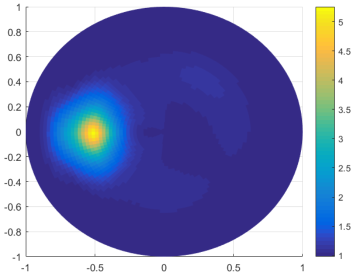
In the case of noiseless data with $\delta = 0$ *one should replace in (21)-(23)* $\delta^{\eta/(8\rho)}$ *with* $\sqrt{\beta}$, *where* $\beta = 3e^{-\lambda\eta}$ *and* $\lambda \geq \lambda_2$.

CONCLUSION

Theorem 5. ensures the GLOBAL convergence of the gradient projection method to the exact solution.

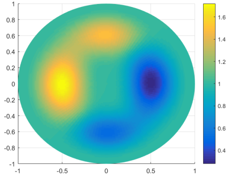


(a) Example 1, true solution.

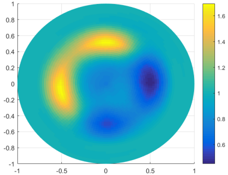


(b) Computed for (a). $N = 8$.

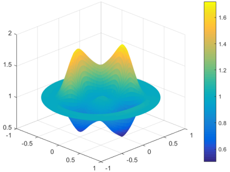
Figure 6: The first example of the reconstruction.



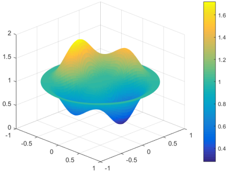
(a) True solution.



(c) Computed for (a).



(b) 3D visualization of (a).



(d) 3D visualization of (c). $N = 8$.

Figure 7: The second example of the reconstruction..

COEFFICIENT INVERSE PROBLEM IN FREQUENCY DOMAIN.

$$\Delta u + k^2 c(\mathbf{x}) u = 0, \quad \mathbf{x} \in \mathbb{R}^3,$$

$$u = e^{ikz} + u_{sc},$$

$$\partial_r u_{sc} - iku_{sc} = o\left(\frac{1}{r}\right), r \to \infty, r = |\mathbf{x}|,$$

$$c \in C^{15}(\mathbb{R}^3), \quad c(\mathbf{x}) \geq 1,$$

$$c(\mathbf{x}) = 1 \quad \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \Omega.$$

Regularity Assumption. *We assume that geodesic lines of the metric (8) satisfy the regularity condition, i.e. for each two points* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ *there exists a single geodesic line* $\Gamma(x, y)$ *connecting these points.*

A sufficient condition for the validity of Regularity Assumption is (V.G. Romanov, 2014):

$$\sum_{i,j=1}^{3} \frac{\partial^2 \ln c(\mathbf{x})}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0,\ \forall \xi \in \mathbb{R}^3, \forall \mathbf{x} \in \overline{\Omega}.$$

-

$$\Omega = \{\mathbf{x} = (x, y, z) : -b < x, y < b, z \in (-\xi, a)\},$$

$$\Gamma = \{\mathbf{x} = (x, y, z) : -b < x, y < b, z = -\xi\}.$$

COEFFICIENT INVERSE SCATTERING PROBLEM (CISP)

Let the backscattering data be given:

$$u(\mathrm{x},k) = g_1(\mathrm{x},k),\, \partial_z u(\mathrm{x},k) = g_2(\mathrm{x},k),\, x \in \Gamma,\, k \in [\underline{k}, \overline{k}].$$

$$u(\mathrm{x},k) = e^{ikz},\, \mathrm{x} \in \partial\Omega \diagdown \Gamma,\, k \in [\underline{k}, \overline{k}].$$

Find the coefficient $c(\mathrm{x})$ *for* $\mathrm{x} \in \Omega$.

One can prove that $u(\mathrm{x},k) \neq 0$ for sufficiently large k.

Furthermore, using asymptotic behaviour of $u(\mathrm{x},k)$ at $k \to \infty$ (Vainberg, 1975; Klibanov and Romanov, 2016), one can uniquely define $\log u(\mathrm{x},k)$ for sufficiently large k.

$$v(\mathrm{x},k) = \frac{\log\left(u(\mathrm{x},k)\, e^{-ikz}\right)}{k^2},\, \mathrm{x} \in \Omega,\, k \in [\underline{k}, \overline{k}],$$

$$q(\mathrm{x},k) = \partial_k v(\mathrm{x},k),$$

$$v(\mathrm{x},k) = -\int\limits_{k}^{\overline{k}} q(\mathrm{x},\tau)\,d\tau + v\left(\mathrm{x},\overline{k}\right).$$

Denote

$$V(\mathrm{x}) = v\left(\mathrm{x},\overline{k}\right).$$

$V(x)$ is the tail function.

To use the SIMPLEST POSSIBLE Carleman Weight Function

$$\varphi(z) = e^{-2\lambda z},$$

we consider the semi-discrete form: finite differences with respect to x, y with the step size $h \geq h_0 > 0$. But we do not let $h \to 0$.

$$L(q) = \Delta^h q + 2k\left(\nabla^h V - \int_k^{\overline{k}} \nabla^h q(\kappa)d\kappa\right)\cdot\left(k\nabla^h(q+V) - \int_k^{\overline{k}} \nabla^h q(\kappa)\,d\kappa\right) + \quad (24)$$

$$+2i\left(kq_z + V_z - \int_{\underline{k}}^{\overline{k}} q_z(\kappa)\, d\kappa\right) = 0.$$

$$q(\mathrm{x}, k) = \phi_0(\mathrm{x}, k), \partial_z q(\mathrm{x}, k) = \phi_1(\mathrm{x}, k), \mathrm{x} \in \Gamma, k \in [\underline{k}, \overline{k}],$$

$$q(\mathrm{x}, k) = 0, \mathrm{x} \in \partial\Omega \diagdown \Gamma, k \in [\underline{k}, \overline{k}]. \tag{25}$$

In (25) we heuristically complement the data on Γ by the data on $\partial\Omega \diagdown \Gamma$.

$$H^{n,h}(\Omega_h) = \{f(x_j, y_s, z) : \|f\|^2_{H^{n,h}(\Omega_h)} = \sum_{j,s=1}^{N_h} \sum_{r=0}^{n} h^2 \int_{-\xi}^{d} \left|\partial_z^r f(x_j, y_s, z)\right|^2 dz < \infty\}, n = 2, 3.$$

$$L_2^h(\Omega_h) = \{f(x_j, y_s, z) : \|f\|^2_{L_2^h(\Omega_h)} = \sum_{j,s=1}^{N_h} h^2 \int_{-\xi}^{d} \left|f(x_j, y_s, z)\right|^2 dz < \infty\},$$

$$H_n^h = \{f(x_j, y_s, z, k) : \|f\|^2_{H_n^h} = \int\limits_{\underline{k}}^{\overline{k}} \|f(\mathbf{x}, k)\|^2_{H^{n,h}(\Omega_h)} \, dk < \infty\}, \quad n = 2, 3.$$

$$H_0^{2,h}(\Omega_h) = \{f(x_j, y_s, z, k) \in H_2^h(\Omega_h) : f \mid_{\partial\Omega_h} = f_z \mid_{\Gamma} = 0\},$$

$$L_2^h(\Omega_h) = \{f(x_j, y_s, z) : \|f\|^2_{L_2^h(\Omega_h)} = \sum_{j,s=1}^{N_h} h^2 \int\limits_{-\xi}^{d} \left| f\left(x_j, y_s, z\right)\right|^2 dz < \infty\},$$

$$H_n^h = \{f(x_j, y_s, z, k) : \|f\|^2_{H_n^h} = \int\limits_{\underline{k}}^{\overline{k}} \|f(\mathbf{x}, k)\|^2_{H^{n,h}(\Omega_h)} \, dk < \infty\}, \quad n = 2, 3.$$

Theorem 6. (Carleman estimate). *For $\lambda > 0$ let*

$$B_h(u,\lambda) = \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} |\Delta^h u(x_j, y_s, z)|^2 \varphi_\lambda(z) dz.$$

Then there exists a sufficiently large number $\lambda_0 = \lambda_0(\xi, d) > 1$ such that for all $\lambda \geq \lambda_0$ the following estimate is valid for all functions $u \in H_0^{2,h}(\Omega_h)$

$$B_h(u,\lambda) \geq C \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left| u_{zz}\left(x_j, y_s, z\right)\right|^2 \varphi_\lambda\left(z\right) dz +$$

$$+ C\lambda \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left| u_{z}\left(x_j, y_s, z\right)\right|^2 \varphi_\lambda(z) dz +$$

$$+ C\lambda^3 \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left| u\left(x_j, y_s, z\right)\right|^2 \varphi_\lambda(z) dz.$$

$$V(\mathrm{x}) = v\left(\mathrm{x}, \overline{k}\right) = \frac{p(\mathrm{x})}{\overline{k}} + O\left(\frac{1}{\overline{k}^2}\right), \quad \overline{k} \to \infty, x \in \Omega,$$

$$q\left(\mathrm{x}, \overline{k}\right) = -\frac{p(\mathrm{x})}{\overline{k}^2} + O\left(\frac{1}{\overline{k}^3}\right), \quad \overline{k} \to \infty, x \in \Omega.$$

Drop the terms with $O\left(\overline{k}^{-2}\right)$ and $O\left(\overline{k}^{-3}\right)$ and substitute in (24) at $k = \overline{k}$,

$$\Delta^h V = 0, \quad \mathrm{x} \in \Omega,$$

$$V(\mathrm{x}) = V^{(0)}(\mathrm{x}),\ V_z(\mathrm{x}) = V^{(1)}(\mathrm{x}),\ \mathrm{x} \in \Gamma;\ V(\mathrm{x}) = 0,\ x \in \partial\Omega \setminus \Gamma. \tag{26}$$

Minimization Problem 3. *For $V \in H^{2,h}(\Omega_h)$, minimize the functional $I_\mu(V)$,*

$$I_\mu(V) = e^{2\mu d} \sum_{j,s=1}^{N_h} h^2 \int_{-\xi}^{d} \left|(\Delta^h V)(x_j, y_s, z)\right|^2 \varphi_\mu(z) dz, \tag{27}$$

subject to boundary conditions (26). The multiplier $e^{2\mu d}$ is introduced here to ensure that $e^{2\mu d} \min_{[-\xi,d]} \varphi_\mu(z) = 1$.

One can prove existence and uniqueness of the solution of this problem as well as convergence of minimizers to the exact solution as long as the level of noise in the data tends to zero.

Suppose that there exists a function $F(\mathrm{x}, k) \in H_3^h$ such that

$$F(\mathrm{x}, k) = \phi_0(\mathrm{x}, k),\ F_z(\mathrm{x}, k) = \phi_1(\mathrm{x}, k), \quad \mathrm{x} \in \Gamma, \quad F(\mathrm{x}, k) = 0, \quad \mathrm{x} \in \partial\Omega \setminus \Gamma. \tag{28}$$

$$p(\mathrm{x}, k) = q(\mathrm{x}, k) - F(\mathrm{x}, k).$$

Let an arbitrary number $R > 0$ and

$$B(R) = \{r \in H_0^{2,h}(\Omega_h) : \|r\|_{H^{2,h}} < R\}.$$

CONSTRUCTION OF WEIGHTED TIKHONOV-LIKE FUNCTIONAL

Construct the weighted Tikhonov-like functional,

$$J_\lambda(p) = e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_{\underline{k}}^{\overline{k}} \int_{-\xi}^{d} |L^h(p+F)(x_j, y_s, z, \kappa)|^2 \varphi_\lambda(z) dz d\kappa, \quad p \in \overline{B(R)}, \tag{29}$$

where the tail function in L^h is the solution of the Minimization Problem 4:

Minimization Problem 4. *Minimize the functional $J_{\lambda,\rho}(p)$ on the set $\overline{B(R)}$.*

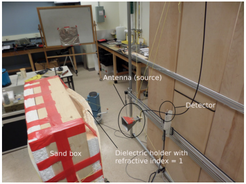
ANALOGS OF THEOREMS 3-5 ARE VALID FOR $J_\lambda(p)$

PERFORMANCE OF THE CONVEXIFICATION FOR THE MOST CHALLENGING CASE:

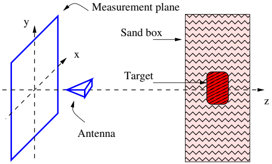
EXPERIMENTAL MICROWAVE BACKSCATTERING DATA FOR TARGETS BURIED IN A SANDBOX

- Frequencies $\omega \in [1, 10]$ GHz.
- Useful frequencies only $\omega \in [2.8, 3.2]$ GHz. The rest of frequencies are too noisy.
- Dry sand. It would be harder to work out a moistured sand: signal does not go through well
- A simple data preprocessing: we have subtracted from the data with a target in the data without that target.
- Only backscattering data were measured.

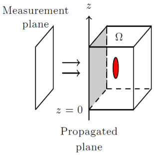
$$\lambda = 3.$$



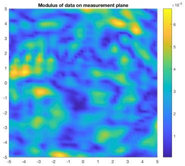
(a) A photo of the experimental device.



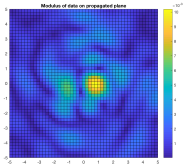
(b) Schematic diagram of measurements.



(c) Schematic diagram of data propagation.



(d) Original data.



(e) Propagated data.

Figure 8: The reconstructions of the function with an *L*-shape image.

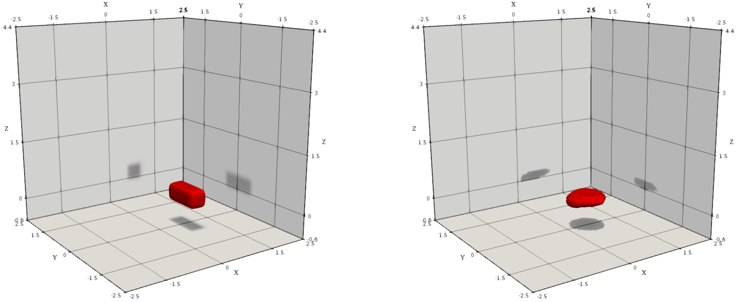
BURIED TARGETS IN EXPERIMENTS

Table 1: Buried targets tested in experiments

Target number	Description	Size in $x \times y \times z$ directions, cm
1	Bamboo	$3.8 \times 11.6 \times 3.8$
2	Geode	$8.8 \times 8.8 \times 8.8$
3	Rock	$10.5 \times 7.5 \times 4.0$
4	Sycamore	$3.8 \times 9.9 \times 3.8$
5	Wet wood	$9.1 \times 5.7 \times 5.8$
6	Yellow pine	$9.0 \times 8.3 \times 5.8$

Table 2: Directly measured c_{meas} and computed c_{comp} dielectric constants of objects of Table 1.

Target number	c_{meas}, error	c_{comp}, error
1	4.50, 5.99%	4.69, 4.22%
2	5.45, 1.13%	5.28, 3.12%
3	5.61, 21.3%	5.07, 9.63%
4	4.89, 2.89%	4.95, 1.23%
5	7.58, 4.69%	8.06, 6.33%
6	4.89, 1.54%	5.22, 8.75%



(a) Correct image of target no. 4.

(b) Computed image of target no. 4.

Figure 9: True and computed images of target no. 4. Other images are similar.

CONCLUSION

The convexification has demonstrated its EXCELLENT performance accuracy for the MOST CHALLENGING case of experimental data for buried targets.

INVERSION OF THE WAVETHE TIME DEPENDENT FRONT IN A HETEROGENEOUS MEDIUM

Michael V. Klibanov, Jingzhi Li and Wenlong Zhang

The forward problem. *Find the solution* $u(\mathrm{x}, \mathrm{x}_0, t)$ *of the following Cauchy problem*

$$c(\mathrm{x})\, u_{tt} = \Delta u, \mathrm{x} \in \mathbb{R}^3, t > 0,$$

$$u(\mathrm{x}, 0) = 0, u_t(\mathrm{x}, 0) = \delta(\mathrm{x} - \mathrm{x}_0).$$

The travel time function $\tau(\mathrm{x}, \mathrm{x}_0)$ is the solution of the eikonal equation

$$|\nabla_{\mathrm{x}} \tau(\mathrm{x}, \mathrm{x}_0)|^2 = c(\mathrm{x}),$$

$$\tau(\mathrm{x}, \mathrm{x}_0) = O(|\mathrm{x} - \mathrm{x}_0|) \text{ as } |\mathrm{x} - \mathrm{x}_0| \to 0,$$

$$\tau\left(\mathbf{x},\mathbf{x}_0\right)=\int\limits_{\Gamma(\mathbf{x},\mathbf{x}_0)}\sqrt{c\left(\mathbf{x}\right)}d\sigma.$$

Fix the source $\mathbf{x}_0=0$,

$$\Omega=\left\{\mathbf{x}=(x,y,z):-A<x,y<A,z\in(0,B)\right\}.$$

Coefficient Inverse Problem. *Let $S_T=\partial\Omega\times(0,T)$. Let $\mathbf{x}_0=0$. Suppose that the following two functions are given:*

$$u\left(\mathbf{x},t\right)\mid_{(\mathbf{x},t)\in S_T}=g_0\left(\mathbf{x},t\right),\partial_n u\left(x,y,B,t\right)=g_1\left(x,y,t\right),t\in(0,T).$$

Find the function $c\left(\mathbf{x}\right)$ for $\mathbf{x}\in\Omega$ assuming that $c\left(\mathbf{x}\right)=1$ outside of Ω.

Denote $\tau^0\left(\mathbf{x}\right)=\tau\left(\mathbf{x},0\right)$. Consider the function p,

$$p\left(\mathbf{x},t\right)=\int\limits_0^t dy\int\limits_0^y u\left(\mathbf{x},s\right)ds.$$

$$w\left(\mathbf{x},t\right)=p\left(\mathbf{x},t+\tau^0\left(\mathbf{x}\right)\right),\text{ for }\left(\mathbf{x},t\right)\in\Omega\times(0,T).$$

We obtain

$$\Delta w - 2\sum_{i=1}^{3} w_{x_i t}\tau_{x_i}^0 - w_t\Delta\tau^0 = 0, \mathrm{x}\in\Omega, t\in(0,T).$$

Let $\{P_n(t)\}_{n=1}^{\infty}$ be the set of polynomials, forming an orthonormal basis in $L_2(0,T)$ and such that

$$P_n(0) = 0, \forall n = 1, 2, ...$$

Approximate the function $w(\mathrm{x}, t)$ as

$$w(\mathrm{x}, t) = \sum_{n=1}^{N} w_n(\mathrm{x}) P_n(t).$$

Then

$$\Delta\tau^0(\mathrm{x}) + 2\left[\sum_{i=1}^{3}\tau_{x_i}^0\sum_{n=1}^{N} P_n'(0)\,\partial_{x_i} w_n(\mathrm{x})\right]\left[\sum_{n=1}^{N} P_n'(0)\, w_n(\mathrm{x})\right]^{-1} = 0, \mathrm{x}\in\Omega.$$

$$\Delta W = F\left(\nabla\tau^0, \nabla W, W\right), \mathrm{x}\in\Omega,$$

$$W(x) = (w_1(\mathrm{x}), ..., w_N(\mathrm{x}))^T,\ Q(\mathrm{x}) = \left(\tau^0(\mathrm{x}), W(\mathrm{x})\right).$$

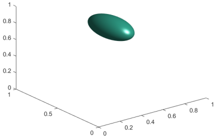
$$\Delta Q + F(\nabla Q, Q) = 0, \mathrm{x} \in \Omega,$$

$$Q\mid_{\partial\Omega} = q_0(\mathrm{x}), \partial_z Q\mid_{z=B} = q_1(\mathrm{x}).$$

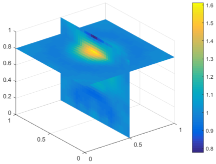
The Strictly Convex Tikhonov-like Functional

$$J_{\lambda,\beta}(Q) = \int\limits_{\Omega} \left(\Delta Q + F(\nabla Q, Q)\right)^2 e^{2\lambda z^2} d\mathrm{x} + \beta \|Q\|^2_{H^3(\Omega)}.$$

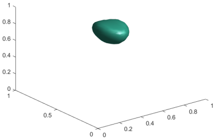
Analogs of Theorem 3-5 Are Valid in This Case: Global Convergence.



(a) True image.



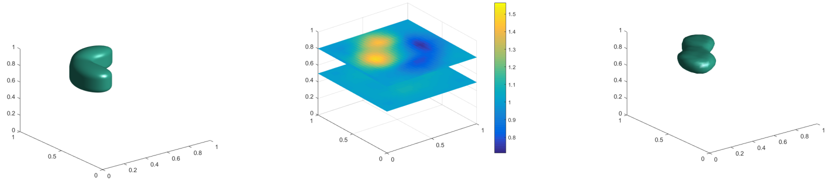
(b) A 2d slice of the computed image.



(c) 3D visualization of the computed image.

Figure 10: A preliminary numerical example. The reconstruction of a target of a complicated shape.

A PRELIMINARY NUMERICAL EXAMPLE



(a) True image.

(b) A 2d slice of the computed image.

(c) 3D visualization of the computed image.

Figure 11: A preliminary numerical example. The reconstruction of a target of a complicated shape.