Recent Studies on Scattering Problems for Elastic Waves

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Outline

- Motivation and model equations
- Random source problems
- Surface scattering problems
- Ongoing and future work

Scattering problems concerns the interaction between waves and media.

- Source scattering problems, e.g., medical imaging (MEG, ENG, EEG)
- Surface scattering problems, e.g., submarine detection

  obstacle scattering (bounded closed surface, exterior problems), diffractive optics (periodic structures), unbounded rough surface scattering (non-local perturbation of flat surfaces), cavity scattering (local perturbation of flat surfaces), interior cavity (bounded closed surface, interior problems)
- Medium scattering problems, e.g., geophysical exploration

- The Helmholtz equation - acoustic wave

$$\Delta u + \kappa^2 u = f \quad \text{in } \mathbb{R}^d.$$

- The Navier equation - elastic wave

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^d.$$

- The Maxwell equations - electromagnetic wave

$$\nabla \times \boldsymbol{E} - \mathrm{i}\omega\mu \boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} + \mathrm{i}\omega\varepsilon \boldsymbol{E} = \boldsymbol{J} \quad \text{in } \mathbb{R}^3.$$

Random source problems

- Random source scattering - white noise

  Joint with G. Bao and C. Chen

- Random source scattering - rough field

  Joint with T. Helin and J. Li

Related work

- Numerics: Marengo and Devaney ('99), Ammari, Bao, and Fleming ('02), Fokas, Kurylev, and Marinakis ('04), Devaney, Marengo, and Li ('07), Nara, Oohama, Hashimoto, Takeda, and Ando ('07), Eller and Valdivia ('09), Acosta, Chow, Taylor, and Villamizar ('12), Badia and Nara ('13), Bao, Lu, Rundell, and Xu ('15), Zhang and Guo ('15)
- Uniqueness: Bleistein and Cohen ('77), Devaney and Sherman ('82), He and Romanov ('98), Hauer, Kühn, and Potthast ('05), Albanese and Monk ('06), Badia and Nara ('11)
- Stability: Bao, Lin, and Triki ('10), Cheng, Isakov, and Lu ('16), L. and Yuan ('17)
- Random source: Devaney ('79), L. ('11), Bao and Xu ('13), Bao, Chow, L., and Zhou ('14), Bao, Chen, and L. ('16, '17), L. and Yuan ('17)
- Topic review: Bao, L., Lin, and Triki ('15)

The stochastic elastic wave equation

$$\mu\Delta \boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^2,$$

where the external force is driven by an additive white noise

$$\boldsymbol{f}(x) = \boldsymbol{g}(x) + \boldsymbol{\sigma}(x)\dot{W}_x.$$

More specifically,

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} + \begin{bmatrix} \sigma_1(x) & 0 \\ 0 & \sigma_2(x) \end{bmatrix} \begin{bmatrix} \dot{W}_1(x) \\ \dot{W}_2(x) \end{bmatrix}.$$

The Helmholtz decomposition

$$\boldsymbol{u} = \nabla\phi + \textbf{curl}\psi = \boldsymbol{u}_{\rm p} + \boldsymbol{u}_{\rm s} \quad \text{in } \mathbb{R}^2 \setminus \bar{\Omega},$$

where ϕ and ψ satisfy

$$\Delta\phi + \kappa_{\rm p}^2\phi = 0, \quad \Delta\psi + \kappa_{\rm s}^2\psi = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{\Omega}.$$

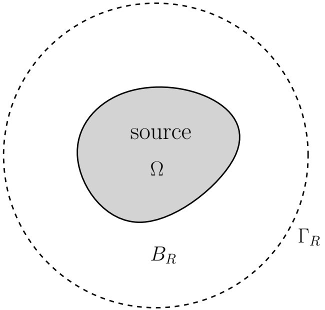
Here

$$\kappa_{\rm p} = \frac{\omega}{\sqrt{\lambda + 2\mu}}, \quad \kappa_{\rm s} = \frac{\omega}{\sqrt{\mu}}.$$

The Kupradze–Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}(\partial_r \boldsymbol{u}_{\rm p} - {\rm i}\kappa_{\rm p}\boldsymbol{u}_{\rm p}) = 0, \quad \lim_{r\to\infty} r^{1/2}(\partial_r \boldsymbol{u}_{\rm s} - {\rm i}\kappa_{\rm s}\boldsymbol{u}_{\rm s}) = 0.$$

Problem formulation



The direct problem: Given $\boldsymbol{g}$ and $\boldsymbol{\sigma}$, to determine the random wave field $\boldsymbol{u}$.

The inverse problem: To recover $\boldsymbol{g}$ and $\boldsymbol{\sigma}$ from $\boldsymbol{u}|_{\Gamma_R}$ at $\omega_k, k = 1, \dots, m$.

The direct scattering problem

$$\begin{cases} \mu\Delta \boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = \boldsymbol{g} & \text{in } \mathbb{R}^2, \\ \partial_r \boldsymbol{u}_{\rm p} - {\rm i}\kappa_{\rm p}\boldsymbol{u}_{\rm p} = o(r^{-1/2}) & \text{as } r\to\infty, \\ \partial_r \boldsymbol{u}_{\rm s} - {\rm i}\kappa_{\rm s}\boldsymbol{u}_{\rm s} = o(r^{-1/2}) & \text{as } r\to\infty. \end{cases}$$

Given $\boldsymbol{g} \in L^2(\Omega)^2$, the scattering problem has a unique solution

$$\boldsymbol{u}(x,\omega) = \int_\Omega \boldsymbol{G}(x,y;\omega)\boldsymbol{g}(y){\rm d}y,$$

where the Green tensor function

$$\begin{aligned} \boldsymbol{G}(x,y;\omega) =& \frac{\rm i}{4\mu} H_0^{(1)}(\kappa_{\rm s}|x-y|)\boldsymbol{I}_{2\times 2} \\ &+ \frac{\rm i}{4\omega^2}\nabla_x\nabla_x^\top\Big[H_0^{(1)}(\kappa_{\rm s}|x-y|) - H_0^{(1)}(\kappa_{\rm p}|x-y|)\Big]. \end{aligned}$$

The stochastic direct scattering problem

$$\begin{cases} \mu\Delta \boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = \boldsymbol{g} + \boldsymbol{\sigma}\dot{W}_x & \text{in } \mathbb{R}^2, \\ \partial_r \boldsymbol{u}_{\rm p} - {\rm i}\kappa_{\rm p}\boldsymbol{u}_{\rm p} = o(r^{-1/2}) & \text{as } r\to\infty, \\ \partial_r \boldsymbol{u}_{\rm s} - {\rm i}\kappa_{\rm s}\boldsymbol{u}_{\rm s} = o(r^{-1/2}) & \text{as } r\to\infty. \end{cases}$$

Theorem (Bao-Chen-L., SINUM, '17)

Let $\boldsymbol{g} \in L^2(\Omega)^2, \sigma_j \in L^p(\Omega), p \in (2,\infty]$, and $\sigma_j \in C^{0,\eta}(\Omega), \eta \in (0,1]$. Then there exists a unique continuous stochastic process (mild solution) $\boldsymbol{u}$, which satifies

$$\boldsymbol{u}(x,\omega) = \int_\Omega \boldsymbol{G}(x,y;\omega)\boldsymbol{g}(y){\rm d}y + \int_\Omega \boldsymbol{G}(x,y;\omega)\boldsymbol{\sigma}(y){\rm d}W_y.$$

The mild solution at angular frequency ω_k:

$$\boldsymbol{u}(x,\omega_k) = \int_\Omega \boldsymbol{G}(x,y;\omega_k)\boldsymbol{g}(y)\mathrm{d}y + \int_\Omega \boldsymbol{G}(x,y;\omega_k)\boldsymbol{\sigma}(y)\mathrm{d}W_y.$$

Taking the expectation, we have

$$\mathbb{E}(\boldsymbol{u}(x,\omega_k)) = \int_\Omega \boldsymbol{G}(x,y;\omega_k)\boldsymbol{g}(y)\mathrm{d}y.$$

Taking the variance, we may obtain

$$\begin{aligned}
&\mathbb{V}(\mathrm{Re}u_1(x,\omega_k)) - \mathbb{V}(\mathrm{Im}u_1(x,\omega_k)) \\
&= \int_\Omega \Big[\big(G^{[11]}_{\mathrm{Re}}(x,y,\omega_k)\big)^2 - \big(G^{[11]}_{\mathrm{Im}}(x,y,\omega_k)\big)^2\Big]\sigma_1^2(y)\mathrm{d}y \\
&\quad + \int_\Omega \Big[\big(G^{[12]}_{\mathrm{Re}}(x,y,\omega_k)\big)^2 - \big(G^{[12]}_{\mathrm{Im}}(x,y,\omega_k)\big)^2\Big]\sigma_2^2(y)\mathrm{d}y.
\end{aligned}$$

Consider the operator equations

$$A_k q = p_k, \quad k = 1, \ldots, K.$$

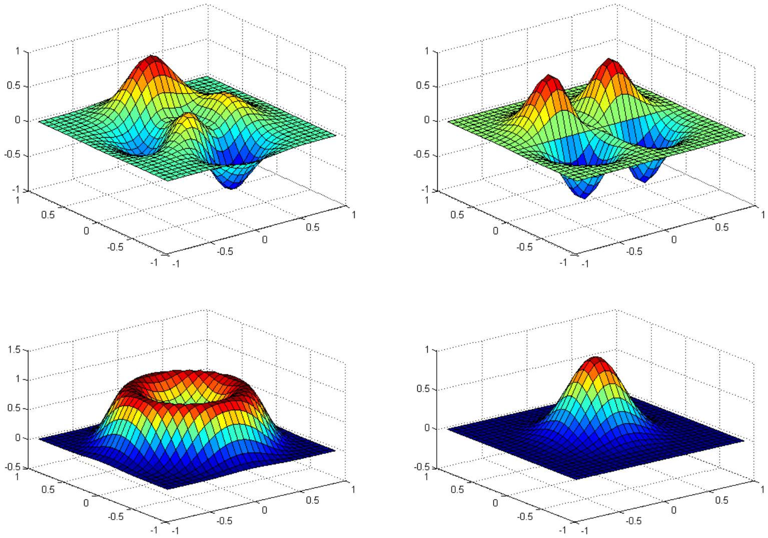
Regularized block Kaczmarz method: Given q_γ^0, for $l = 0, 1, \ldots,$

$$\begin{cases} q_0 = q_\gamma^l, \\ q_k = q_{k-1} + A_k^\top (\gamma I + A_k A_k^\top)^{-1}(p_k - A_k q_{k-1}), \quad k = 1, \ldots, K, \\ q_\gamma^{l+1} = q_K, \end{cases}$$

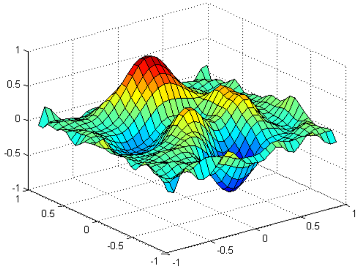
Theorem (Bao-Chen-L., SINUM, '17)

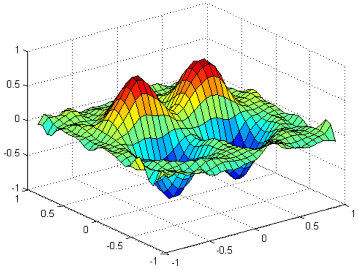
Assume that the system is consistent with the minimal norm solution $q^\dagger$ and the initial guess $q_\gamma^0 \in \mathcal{R}(A^\top)$, e.g., $q_\gamma^0 = 0$. Then q_γ^l converges, as $l \to \infty$, to $q_\gamma^\dagger$, which converges to $q^\dagger$ as $\gamma \to 0$.

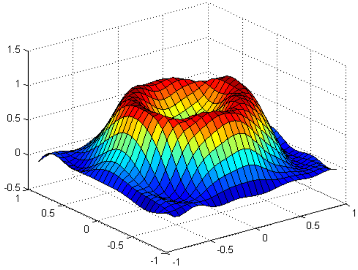
Numerical experiments: exact mean and variance

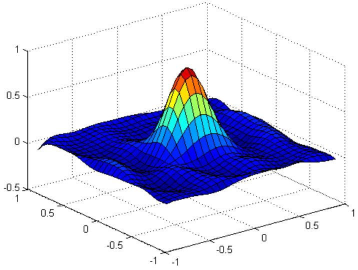


Numerical experiments: reconstructed mean and variance









The elastic wave equation

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^d,$$

where $\boldsymbol{f}$ is a generalized Gaussian random function (rough field).

The Kupradze–Sommerfeld radiation condition:

$$\lim_{r\to\infty} r^{(d-1)/2}(\partial_r\boldsymbol{u}_{\rm p} - {\rm i}\kappa_{\rm p}\boldsymbol{u}_{\rm p}) = 0, \quad \lim_{r\to\infty} r^{(d-1)/2}(\partial_r\boldsymbol{u}_{\rm s} - {\rm i}\kappa_{\rm s}\boldsymbol{u}_{\rm s}) = 0.$$

Generalized Gaussian random function

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A generalized Gaussian random function $f : \Omega \to \mathcal{D}'(\mathbb{R}^d)$ is a measurable map such that

$$\langle f(\hat{\omega}), \psi \rangle, \quad \forall \psi \in C_0^\infty(\mathbb{R}^d),$$

is a Gaussian random variable.

The covariance operator $C_f : C_0^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$:

$$\langle C_f \psi_1, \psi_2 \rangle = \mathrm{cov}(\langle f, \psi_1 \rangle, \langle f, \psi_2 \rangle) = \mathbb{E}(\langle f - \mathbb{E}f, \psi_1 \rangle \langle f - \mathbb{E}f, \psi_2 \rangle),$$

which is a pseudo-differential operator with principle symbol

$$\phi(x)|\xi|^{-(2\epsilon+2)}, \quad \epsilon \in [0, \frac{1}{2}),$$

where $\phi \in C_0^\infty(\mathbb{R}^d)$, $\mathrm{supp}\phi \subset \Omega$, $\phi \geq 0$.

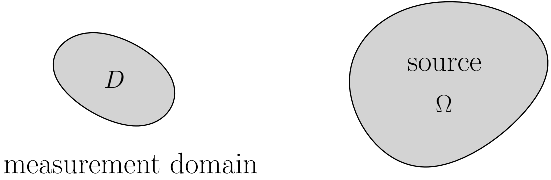
Lemma

The generalized Gaussian random function $f(\hat{\omega})$ belongs with probability one to the Sobolev space $W^{-\varepsilon,p}(\Omega)$ for all $\varepsilon > 0$ and $1 < p < \infty$.

Theorem (Helin-Li-L.)

For some fixed $s \in (0, 1 - \frac{d}{6})$, let $1 < p < \frac{2d}{d+2(1-s)}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Given $\boldsymbol{f} \in W_0^{-1,p'}(\Omega)^d$, then the scattering problem attains a unique solution $\boldsymbol{u} \in H^s(\mathbb{R}^d)^d$ satisfying

$$\boldsymbol{u}(x, \omega) = \int_\Omega \boldsymbol{G}(x, y; \omega) \boldsymbol{f}(y) \mathrm{d}y.$$



Theorem (Helin-Li-L.)

Let $\boldsymbol{f}$ *be a Gaussian random vector field with*

$$\textit{principle symbol } C_{f_j} = \phi(x)|\xi|^{-(2\epsilon+2)}.$$

Then for all $x \in D$*, it holds almost surely that*

$$\lim_{K\to\infty} \int_1^K \omega^{2\epsilon+2} |\boldsymbol{u}(x,\omega)|^2 \mathrm{d}\omega = C(\lambda,\mu) \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} \phi(y) \mathrm{d}y.$$

Moreover the micro-correlation strength ϕ *can be uniquely determined.*

- Time-harmonic and time-domain obstacle scattering

  Joint with J. Lai, Y. Wang, Z. Wang, X. Yuan, and Y. Zhao

- Time-domain periodic surface scattering

  Joint with J. Wang, and L. Zhang

- Time-domain coupled model problems

  Joint with G. Bao, Y. Gao, J. Wang, and L. Zhang

- Obstacle scattering: Hähner and Hsiao ('93), Alves and Ammari ('01), Elschner and Yamamoto ('10), Hu, Li, Liu, and Sun ('14), Hu, Kirsch, and Sini ('13), Kar and Sini ('15), Louër ('12), Chen, Xiang, and Zhang ('16), L., Wang, Wang, and Zhao ('16)
- Periodic structures: Arens ('99), Charalambopoulos, Gintides, and Kiriaki ('01), Hu, Lu, and Zhang ('13), L., Wang, and Zhao ('15, '16), Jiang, L., Lv, and Zheng ('18)
- Unbounded surfaces: Abubakar ('62), Arens ('01, '02), Elschner and Hu ('12), Liu, Zhang, and Zhang ('18)
- Acoustic-elastic interaction: Luke and Martin ('95), Hsiao, Kleinman, and Schuetz ('89), Hsiao ('94), Yin, Hu, Xu, and Zhang, ('16), Hsiao, Sánchez-Vizuet, and Sayas ('17), L. and Jiang ('17), Gao, L., and Zhang ('17), Bao, Gao, and L. ('18)

The time-harmonic elastic wave equation

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}.$$

Elastically rigid obstacle

$$\boldsymbol{u} = 0 \quad \text{on } \partial D.$$

The Helmholtz decomposition for the scattered field

$$\boldsymbol{v} = \nabla\phi_1 + \mathbf{curl}\phi_2,$$

where ϕ_j satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\partial_r\phi_j - \mathrm{i}\kappa_j\phi_j\right) = 0, \quad r = |\boldsymbol{x}|.$$

The boundary value problem for $\boldsymbol{u}$:

$$\begin{cases} \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = 0 & \text{in } \Omega = B\setminus\bar{D}, \\ \boldsymbol{u} = 0 & \text{on } \partial D, \\ \mathscr{B}\boldsymbol{u} = \mathscr{T}\boldsymbol{u} + \boldsymbol{g} & \text{on } \partial B. \end{cases}$$

The coupled boundary value problem for ϕ_j:

$$\begin{cases} \Delta\phi_j + \kappa_j^2\phi_j = 0 & \text{in } \Omega, \\ \partial_\nu\phi_1 - \partial_\tau\phi_2 = u & \text{on } \partial D, \\ \partial_\nu\phi_2 + \partial_\tau\phi_1 = v & \text{on } \partial D, \\ \partial_r\phi_j - \mathscr{T}_j\phi_j = 0 & \text{on } \partial B. \end{cases}$$

Theorem (L.-Wang-Wang-Zhao, IP, '16)

The direct problem admits a unique solution $\boldsymbol{u} \in H^1_{\partial D}(\Omega)$.

Theorem (L.-Wang-Wang-Zhao, IP, '16)

Let $\boldsymbol{u}$ *be the solution of the direct problem. Given* $\boldsymbol{p} \in C^2(\partial D)$*, the domain derivative of the scattering operator* $\mathscr{S}$ *is* $\mathscr{S}'(\partial D; \boldsymbol{p}) = \gamma \boldsymbol{u}'$*, where* $\boldsymbol{u}'$ *is the unique weak solution of the boundary value problem:*

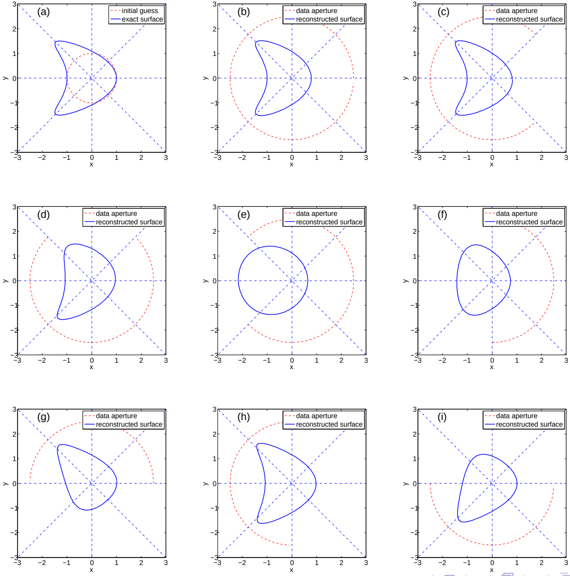
$$\begin{cases} \mu\Delta\boldsymbol{u}' + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}' + \omega^2\boldsymbol{u}' = 0 & \text{in } \Omega, \\ \boldsymbol{u}' = -(\boldsymbol{p}\cdot\nu)\partial_\nu\boldsymbol{u} & \text{on } \partial D, \\ \mathscr{B}\boldsymbol{u}' = \mathscr{T}\boldsymbol{u}' & \text{on } \partial B. \end{cases}$$

Theorem (L.-Yuan)

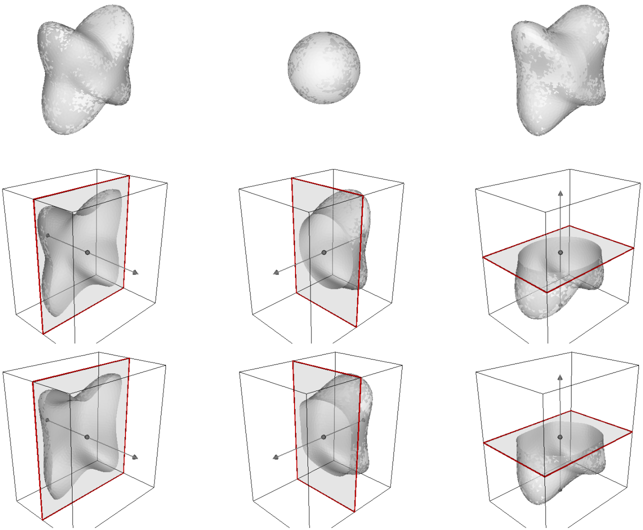
Let ϕ_j be solution of the direct problem. Given $\boldsymbol{p} \in C^2(\partial D)$, the domain derivative of the scattering operator $\mathscr{T}_j'(\partial D, \boldsymbol{p}) = \gamma\phi_j$, where ϕ_j' is the unique weak solution of the boundary value problem:

$$\begin{cases} \Delta\phi_j' + k_j^2\phi_j' = 0 & \text{in } \Omega, \\ \partial_\nu\phi_1' + \partial_\tau\phi_2' = \kappa_1^2(\boldsymbol{p}\cdot\nu)\phi_1 & \text{on } \partial D, \\ \partial_\nu\phi_2' - \partial_\tau\phi_1' = \kappa_2^2(\boldsymbol{p}\cdot\nu)\phi_2 & \text{on } \partial D, \\ \partial_\rho\phi_1' - \mathscr{T}_1\phi_1' = 0 & \text{on } \partial B, \\ \partial_\rho\phi_2' - \mathscr{T}_2\phi_2' = 0 & \text{on } \partial B. \end{cases}$$

Numerical experiments - 2D



Numerical experiments - 3D



Integral representation

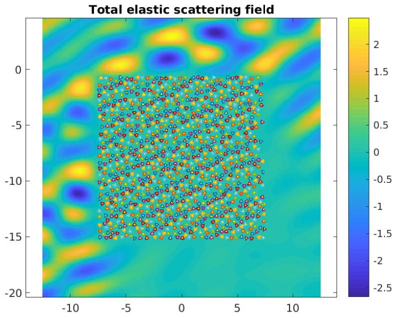
$$\phi_1(x) = (\mathcal{D}_{\kappa_\mathrm{p}} - \mathrm{i}\mathcal{S}_{\kappa_\mathrm{p}})\alpha(x), \quad \phi_2(x) = (\mathcal{D}_{\kappa_\mathrm{p}} - \mathrm{i}\mathcal{S}_{\kappa_\mathrm{p}})\beta(x)$$

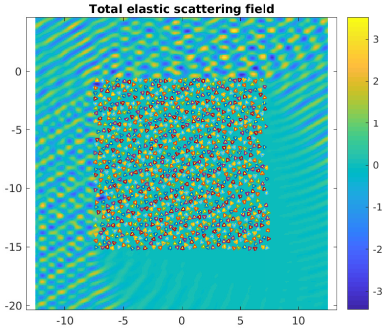
Boundary integral equation

$$\left(\begin{bmatrix} \partial_\nu D_{\kappa_\mathrm{p}} & \partial_\tau(\frac{1}{2}I + D_{\kappa_\mathrm{s}}) \\ \partial_\tau(\frac{1}{2}I + D_{\kappa_\mathrm{p}}) & -\partial_\nu D_{\kappa_\mathrm{s}} \end{bmatrix} - \mathrm{i}\begin{bmatrix} -\frac{1}{2}I + D'_{\kappa_\mathrm{p}} & H'_{\kappa_\mathrm{s}} \\ H'_{\kappa_\mathrm{p}} & \frac{1}{2}I - D'_{\kappa_\mathrm{s}} \end{bmatrix}\right)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Theorem (Lai-L.)

For any $\omega > 0$, the boundary integral equation admits a unique solution.

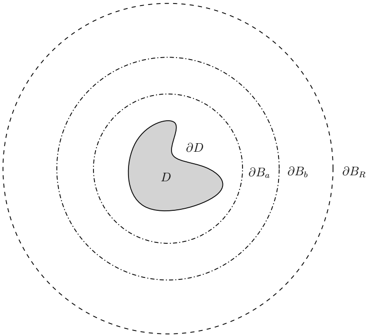

Total elastic scattering field
0
-5
-10
-15
-20
-10
-5
0
5
10
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5


Total elastic scattering field
0
-5
-10
-15
-20
-10
-5
0
5
10
3
2
1
0
-1
-2
-3

Time-domain obstacle scattering problem

The time-domain elastic wave equation

$$\partial_t^2 \boldsymbol{u}(\boldsymbol{x},t) - \mu\Delta\boldsymbol{u}(\boldsymbol{x},t) - (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in\mathbb{R}^d\setminus\bar{D},\ t>0.$$



Transformed elastic wave equation

Initial boundary value problem

$$\begin{cases} \beta\partial_t^2 \boldsymbol{u} - \mu\nabla\cdot(M\nabla\boldsymbol{u}) - (\lambda+\mu)K\nabla(\beta^{-1}\nabla\cdot(K\boldsymbol{u})) = \boldsymbol{f} & \text{in } \Omega\times(0,T], \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega\times[0,T], \\ \boldsymbol{u}|_{t=0} = \boldsymbol{g}, \quad \partial_t\boldsymbol{u}|_{t=0} = \boldsymbol{h} & \text{in } \Omega. \end{cases}$$

Theorem (L.-Yuan)

The initial boundary value problem admits a unique weak solution.

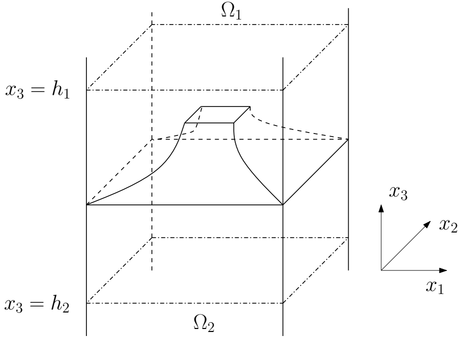
Theorem (L.-Yuan)

Let $\boldsymbol{u}$ be the unique weak solution of the initial boundary value problem. Given $\boldsymbol{g}\in H_0^1(\Omega)^d, \boldsymbol{h}\in L^2(\Omega)^d, \boldsymbol{f}\in L^1(0,T;L^2(\Omega))^d$, there exists a positive constant C such that

$$\max_{t\in[0,T]}\left(\|\partial_t\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^d} + \|\nabla\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^{d\times d}}\right) \leq C\left(\|\boldsymbol{h}\|_{L^2(\Omega)^d} + \|\boldsymbol{g}\|_{H^1(\Omega)^d} + \|\boldsymbol{f}\|_{L^1(0,T;L^2(\Omega))^d}\right).$$

The time-domain elastic wave equation

$$\rho\partial_t^2 \boldsymbol{u} - \nabla\cdot[\mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)] - \nabla(\lambda\nabla\cdot\boldsymbol{u}) = 0 \quad \text{in } [0, \boldsymbol{\Lambda}] \times \mathbb{R} \times \mathbb{R}^+.$$



Incident plane wave

$$\boldsymbol{u}^{\mathrm{inc}}(\boldsymbol{x}, t) = \boldsymbol{p} f(c_1 t - \boldsymbol{p} \cdot \boldsymbol{x}),$$

where

$$c_1 = \sqrt{(\lambda_1 + 2\mu_1)/\rho_1}, \quad \boldsymbol{p} = (\alpha, p_3)^\top.$$

It is easy to verify that

$$\boldsymbol{u}^{\mathrm{inc}}(\hat{\boldsymbol{x}} + \boldsymbol{\Lambda}, x_3, t) = \boldsymbol{u}^{\mathrm{inc}}(\hat{\boldsymbol{x}}, x_3, t - c_1^{-1} \alpha \cdot \boldsymbol{\Lambda}).$$

Change of variables

$$\boldsymbol{U}(\hat{\boldsymbol{x}}, x_3, t) = \boldsymbol{u}(\hat{\boldsymbol{x}}, x_3, t + c_1^{-1} \alpha \cdot (\hat{\boldsymbol{x}} - \boldsymbol{\Lambda})).$$

The compressed coordinate transformation: $x_3 = \psi(\tilde{x}_3)$.

Consider the initial boundary value problem in $\tilde{\Omega} \times (0, T]$:

$$\rho \partial_t^2 \boldsymbol{V} - \tilde{\nabla} \cdot [\mu(\tilde{\nabla} \boldsymbol{V} + \tilde{\nabla} \boldsymbol{V}^\top)] - \tilde{\nabla}(\lambda \tilde{\nabla} \cdot \boldsymbol{V}) = 0$$

where

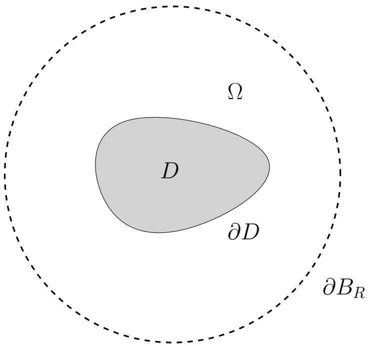
$$\tilde{\nabla} = \nabla_\psi - c_1^{-1} \boldsymbol{p} \partial_t.$$

Theorem (L.-Wang-Zhang)

The initial boundary value problem admits a unique weak solution which satisfies the energy estimate

$$\max_{t \in [0,T]} \left(\|\partial_t \boldsymbol{V}(\cdot, t)\|^2_{L^2(\tilde{\Omega})^3} + \|\nabla_\psi \boldsymbol{V}(\cdot, t)\|^2_{L^2(\tilde{\Omega})^{3\times 3}} + \|\nabla_\psi \cdot \boldsymbol{V}(\cdot, t)\|^2_{L^2(\tilde{\Omega})} \right)$$
$$\lesssim \|\boldsymbol{g}_1\|^2_{L^1(0,T;L^2(\tilde{\Omega})^3)} + \|\boldsymbol{g}_2\|^2_{H^1(\tilde{\Omega})^3} + \|\boldsymbol{g}_3\|^2_{L^2(\tilde{\Omega})^3}.$$

Time-domain acoustic-elastic interaction problem



Acoustic wave equation

$$\Delta p(\boldsymbol{x}, t) - \frac{1}{c^2}\partial_t^2 p(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in D^c, \; t > 0.$$

Elastic wave equation

$$\mu \Delta \boldsymbol{u}(\boldsymbol{x}, t) + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t) - \rho_2 \partial_t^2 \boldsymbol{u}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in D, \; t > 0.$$

Interface conditions on ∂D for $t > 0$:

$$\partial_{\boldsymbol{n}_D} p(\boldsymbol{x}, t) = -\rho_1 \boldsymbol{n}_D \cdot \partial_t^2 \boldsymbol{u}(\boldsymbol{x}, t),$$
$$-p(\boldsymbol{r}, t)\boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D} \boldsymbol{u}(\boldsymbol{r}, t) + (\lambda + \mu)(\nabla \cdot \boldsymbol{u}(\boldsymbol{r}, t))\boldsymbol{n}_D.$$

Theorem (Bao-Gao-L., ARMA, '18)

The initial-boundary value problem has a unique solution which satisfies

$$p(\boldsymbol{x},t) \in L^2\left(0,T;\ H^1(\Omega)\right) \cap H^1\left(0,T;\ L^2(\Omega)\right),$$
$$\boldsymbol{u}(\boldsymbol{x},t) \in L^2\left(0,T;\ H^1(D)^2\right) \cap H^1\left(0,T;\ L^2(D)^2\right),$$

and the stability estimates

$$\max_{t\in[0,T]} \left(\|\partial_t p\|_{L^2(\Omega)} + \|\nabla\partial_t p\|_{L^2(\Omega)^2}\right)$$
$$\lesssim \|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + \max_{t\in[0,T]} \|\partial_t\rho\|_{H^{-1/2}(\partial B_R)} + \|\partial_t^2\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))},$$
$$\max_{t\in[0,T]} \left(\|\partial_t \boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(D)} + \|\nabla\boldsymbol{u}\|^2_{L^2(D)^{2\times 2}}\right)$$
$$\lesssim \|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + \max_{t\in[0,T]} \|\partial_t\rho\|_{H^{-1/2}(\partial B_R)} + \|\partial_t^2\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}.$$

Transformed acoustic-elastic interaction problem

Consider the initial boundary value problem

$$\begin{cases} \frac{\beta}{c^2}\partial_t^2 p - \nabla\cdot(M\nabla p) = f & \text{in } \Omega\times(0,T], \\ p = 0 & \text{on } \partial B_b\times(0,T], \\ p|_{t=0} = g, \quad \partial_t p|_{t=0} = h & \text{in } \Omega \\ \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} - \rho_2\partial_t^2\boldsymbol{u} = 0, & \text{in } D\times(0,T], \\ \boldsymbol{u}|_{t=0} = \partial_t\boldsymbol{u}|_{t=0} = 0 & \text{in } D, \\ \partial_{\boldsymbol{n}_D} p = -\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u} & \text{in } \partial D\times(0,T] \\ -p\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}\boldsymbol{u} + (\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{n}_D & \text{in } \partial D\times(0,T]. \end{cases}$$

- Medium scattering problems for elastic waves
- Random medium scattering problem - rough field
- Compressed coordinate transformation & Time-domain PML

Thank You !