Complex Geometrical Optics and Calderón's Problem

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0 Introduction

Inverse problems arise in practical situations such as medical imaging, exploration geophysics, and non-destructive evaluation where measurements made in the exterior of a medium are used to deduce properties of the hidden interior. Since the 1980's there have been substantial developments in the mathematical theory of inverse problems in the multidimensional case. The purpose of these lectures is to describe some of these developments. We concentrate on giving some examples on

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how complex geometrical optics (CGO) solutions to partial differential equations, which were motivated by the work of Calderón [3], are used in inverse problems.

Calderón considered the problem of determining the electrical conductivity of a medium by making voltage and current measurements at the boundary. This inverse method is known as *Electrical Impedance Tomography* (EIT). Calderón's motivation was geophysical prospection; he worked as an engineer for YPF (Yacimientos Petroleros Fiscales) in Argentina in the 1940's. EIT is also called in geophysics *resistivity imaging.* See [48]. Other applications include detection of leaks, see for instance [17]. More recently EIT has received considerable attention for the potential medical applications running from early detection of breast cancer, to continuous monitoring of brain and lung functions. See the proceedings [16], [14], and [13], [50] for more details. We give the mathematical formulation of EIT below.

Let Ω be a bounded region in $\mathbf{R}^n$ ($n \geq 2$) which models a conducting medium. Let $u(x)$ represent a voltage potential (*i.e.*, $u(x) - u(y)$ is the voltage difference measured by a voltmeter with electrodes attached at the points x and y). The current is now represented by a vector which we denote by $i(x)$, and Ohm's law becomes

$$i(x) = -\gamma(x)\nabla u(x) \ \ .$$

The current is no longer independent of position in Ω, however, since we are considering steady state conduction, charge cannot accumulate in any subset $\tilde{\Omega} \subset \Omega$. This means that the net flow of current across $\partial\tilde{\Omega}$ is zero, *i.e.*,

$$\int_{\partial\tilde{\Omega}} i(x) \cdot \nu(x) \ dS(x) = 0 \ \ ,$$

where ν denotes the unit outer normal to $\partial\tilde{\Omega}$. The divergence theorem implies that

$$\int_{\tilde{\Omega}} \nabla \cdot i(x) \ dx = 0 \ \ .$$

As $\tilde{\Omega}$ is arbitrary, we have, for every x

$$\nabla \cdot i(x) = 0 \ \ .$$

Substituting 0 into this identity we arrive at

$$\nabla \cdot (\gamma(x)\nabla u(x)) = 0 \quad \text{in } \Omega \ \ .$$

The coefficient $\gamma(x)$ is in general a positive definite symmetric $n \times n$ matrix; if $\gamma(x)$ is a scalar valued function we say that the medium is isotropic, in all other cases we refer to it as anisotropic.

The central aim of EIT is to infer as much as we can about $\gamma(x)$ from multiple boundary measurements of voltages and currents. This is therefore an example of a nondestructive testing situation: it is forbidden to penetrate the interior of Ω with

a probe, electrodes may only be attached to the boundary. If Ω is a smooth domain, then the set of all possible smooth measurements consists of

$$\{(f,g) \in C^\infty(\partial\Omega) \times C^\infty(\partial\Omega) \ : f = u|_{\partial\Omega}, \ g = \gamma\frac{\partial u}{\partial\nu}|_{\partial\Omega} \ \text{ and } u \text{ satisfies } 0\} \ .$$

A mathematically (and practically) somewhat more satisfactory approach is to consider the set $\mathcal{C}_\gamma$ of all Cauchy data associated to the equation 0. The set $\mathcal{C}_\gamma$ is larger than the set 0 – if we restrict attention to solutions of 0 with finite energy, then $\mathcal{C}_\gamma$ is the closure of the set 0 in the $H^{1/2} \times H^{-1/2}$ norm. Whereas it is natural to think of all the information contained in the set 0 (or $\mathcal{C}_\gamma$) as emerging from a special type of experiment – fix voltage pattern and measure current flux across the boundary (or vice versa)– it does also encode the information related to all other possible experiments, such as fixing voltage pattern on part of the boundary, $\partial\Omega_1$, fixing current flux on the remainder of the boundary, $\partial\Omega_2$, and then measuring current flux and voltage pattern on $\partial\Omega_1$ and $\partial\Omega_2$ respectively.

To elaborate a little more on the natural interpretation of 0 we mentioned above, consider the Dirichlet problem

$$\nabla \cdot (\gamma\nabla u) = 0 \quad \text{in } \Omega \ , \quad u = f \quad \text{on } \partial\Omega \ .$$

This problem is well posed, and therefore the first component of an element of 0 can be any function in $C^\infty(\partial\Omega)$. For any such f there is exactly one pair (f,g) contained in 0, namely the pair $(f, \gamma\frac{\partial u}{\partial\nu}|_{\partial\Omega})$. We define the map Λ_γ

$$\Lambda_\gamma(f) = \gamma\frac{\partial u}{\partial\nu}|_{\partial\Omega} \quad \text{where } u \text{ solves } 0 \ .$$

The map Λ_γ is referred to as the Dirichlet-to-Neumann (DN) map. The set 0 is the graph of this map (over $C^\infty(\partial\Omega)$). Our mathematical formulation of EIT is to infer information about γ from the Dirichlet- to-Neumann map, Λ_γ.

In section 1 of these notes we define precisely the Dirichlet-to-Neumann map and give some of its properties. In section 2 we describe how to find the Taylor series of a smooth conductivity is determined by the DN map. We also describe the layer stripping algorithm based on the boundary determination.

For dimensions $n \geq 2$ it is known that Λ_γ does indeed provide sufficient information to determine an isotropic conductivity (assuming for instance that the conductivity is twice differentiable.) This was proven by Sylvester and Uhlmann [40] in dimension $n \geq 3$ by using CGO solutions solutions. The construction of CGO solutions for the Schrösinger equation is done in section 3. The proof of the result of Sylvester and Uhlmann mentioned above is in section 4. We also give in this section an application of CGO solutions to determine cavities.

In section 5 we construct CGO solutions for first order perturbations of the Laplacian. The intertwining property formulated in this section is the main ingredient in such construction. Examples of equations or systems that can be reduced to first order perturbations of the Laplacian are the magnetic Schrödinger equation, the Dirac system and the elasticity system.

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1   The Dirichlet-to-Neumann Map

In this section we state the basic properties of the DN map as defined in the introduction. We recall first the basic existence and uniqueness results for the solution to the Dirichlet Problem

(1.1)
$$L_\gamma u = \nabla \cdot \gamma \nabla u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \gamma_{ij} \frac{\partial}{\partial x_j} u = F \quad \text{in } \Omega$$
$$u = f \quad \text{on } \partial\Omega.$$

where γ is bounded, symmetric and strictly positive definite matrix, *i.e.*

$$\gamma_{ij} \in L^\infty(\Omega), \quad \gamma_{ij} = \gamma_{ji} \quad \text{and}$$

$$0 < c|\xi|^2 \le \sum_{i,j=1}^{n} \gamma_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbf{R}^n \ \text{ a.e. } x \in \Omega,$$

the domain Ω is always assumed to be smooth (C^∞).

Here $H^s(\Omega)$ and $H^s(\partial\Omega)$ denote the standard L^2 based Sobolev spaces. $H_0^1(\Omega)$ denotes the closed subspace of $H^1(\Omega)$ with zero trace at the boundary. We denote by R the restriction (or trace) map $R : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$.

Before we are able to formulate the theorem that asserts the unique solvability of the Dirichlet problem 1.1 we note that $L_\gamma = \sum_{i=j=1}^{n} \frac{\partial}{\partial x_i} \gamma_{ij} \frac{\partial}{\partial x_j}$ may be conveniently viewed as an operator from $H^1(\Omega)$ to the dual of $H_0^1(\Omega)$, denoted $H^{-1}(\Omega)$, by means of the following identity

(1.2)
$$-\langle L_\gamma u, v \rangle := \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \ dx \quad \forall v \in H_0^1(\Omega).$$

The notation $\langle \cdot, \cdot \rangle$ is used to signify the standard duality pairing between the Hilbert space $H_0^1(\Omega)$ and its dual, $H^{-1}(\Omega)$. This duality pairing is the extension of the L^2 inner product. If γ is in C^∞, then the above definition is consistent with the natural way in which a differential operator acts on distributions (or in this case functions in $H^1(\Omega)$).

Theorem 1.1. *The mapping*

$$\mathcal{F} \ : \ H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \ ,$$

defined by

(1.3)
$$\mathcal{F}u := \begin{pmatrix} L_\gamma u \\ Ru \end{pmatrix}$$

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is an isomorphism.

That is, for any $F \in H^{-1}(\Omega)$ and $f \in H^{\frac{1}{2}}(\partial\Omega)$ there exists a unique $u \in H^1(\Omega)$ such that

$$\mathcal{F}u = \begin{pmatrix} F \\ f \end{pmatrix} \quad.$$

This solution u satisfies the estimate

$$(1.4) \qquad \|u\|_{H^1(\Omega)} \leq C(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}) \quad.$$

The above theorem guarantees the existence of a unique solution to the boundary value problem in a very specific sense. Because of the definition 1.2 the u which solves $\mathcal{F}u = (F, f)^t$ is also the unique function in $H^1(\Omega)$ which satisfies

$$(1.5) \qquad \int_\Omega \sum_{i,j=1}^n \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \; dx = -\langle F, v \rangle \quad \forall v \in H_0^1(\Omega) \; ,$$

$$\text{and } Ru = f \quad.$$

This formulation is quite standard; u is frequently referred to as the weak solution of the boundary value problem 1.1.

Theorem 1.2. *Let F and f be elements of $H^{-1}(\Omega)$ and $H^{\frac{1}{2}}(\Omega)$ respectively. The weak solution to the boundary value problem*

$$L_\gamma u = F \quad in \quad \Omega, \quad u = f \quad on \quad \partial\Omega \; ,$$

introduced by Theorem 1.4 may also be characterized as the unique minimizer to the Dirichlet integral

$$D_F(w) = \frac{1}{2} \int_\Omega \gamma_{ij} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} \; dx + \langle F, Pw \rangle \quad,$$

in the set $\{w \in H^1(\Omega) \; : \; w|_{\partial\Omega} = f \}$.

If the coefficient γ is assumed to be infinitely often differentiable then the weak solution, u, is as regular as the data F and f naturally permit. If the data are sufficiently regular then it follows immediately that the weak solution is also the unique strong solution (by this we mean a function in $C^2(\overline{\Omega})$ which satisfies 1.1 in the classical sense).

Corollary 1.3. *If γ, in addition to being positive definite, is in $C^\infty(\overline{\Omega})$, then the map $\mathcal{F}$ defined by 1.3 is an isomorphism*

$$\mathcal{F} \; : \; H^t(\Omega) \to H^{t-2}(\Omega) \times H^{t-\frac{1}{2}}(\partial\Omega)$$

for any value $t \geq 1$.

The fact that the weak solution to 1.1 satisfies an estimate of the form

$$(1.6) \qquad \|u\|_{H^k(\Omega)} \le C(\|F\|_{H^{k-2}(\Omega)} + \|f\|_{H^{k-\frac{1}{2}}(\partial\Omega)})$$

for any integer $k \ge 1$ follows from the regularity theory concerning elliptic boundary value problems. We shall not here give a proof of 1.6, but instead refer the reader to chapter 7 of [8]. Corollary 1.4 now follows for arbitrary $t \ge 1$ from interpolation between Sobolev spaces and the estimates 1.6 corresponding to integer k. We note that the interior regularity of u is fairly easy to assert; the regularity up to the boundary is slightly more tricky and in particular requires that the boundary of the domain Ω is C^∞. We also note that the estimate 1.6 for a fixed k does not really require that the conductivity γ be in $C^\infty(\overline{\Omega})$, it suffices that γ be in $C^{k-1}(\overline{\Omega})$.

We are now finally in a position to define the DN map. Consider the boundary value problem 1.1 with F equal to zero

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega \ .$$

If the boundary data f is in $H^{\frac{3}{2}}(\partial\Omega)$, and γ is in $C^1(\overline{\Omega})$, then the unique solution to this problem, as we have just seen, belongs to $H^2(\Omega)$. Therefore ∇u is in $H^1(\Omega)$ and as a consequence $\nabla u|_{\partial\Omega} = R(\nabla u)$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$. We may now define

$$(1.7) \qquad \Lambda_\gamma f := (\gamma \nabla u) \cdot \nu|_{\partial\Omega)} = \sum_{i,j=1}^{n} \gamma_{ij} \frac{\partial u}{\partial x_j} \nu_i|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega) \ ,$$

where ν denotes the outward unit normal to $\partial\Omega$. As we shall see below Λ_γ is defined for $f \in H^{1/2}(\partial\Omega)$ (and $\gamma \in L^\infty(\Omega)$) even though the classical formula above, in terms of the restriction map, does not make sense. The classical formula fails to make sense for $f \in H^{1/2}(\partial\Omega)$ because in that case ∇u is generally only in $L^2(\Omega)$ and therefore there is no appropriate notion of a restriction to the boundary. Similarly if γ is only in $L^\infty(\Omega)$ then we would generally only know that $\gamma_{ij} \frac{\partial u}{\partial x_j}$ is in $L^2(\Omega)$ and there would again not be an appropriate notion of its restriction to $\partial\Omega$. In order to define Λ_γ on all of $H^{\frac{1}{2}}(\partial\Omega)$ we shall need its dual space $H^{-\frac{1}{2}}(\partial\Omega)$. The duality pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ is the extension of the $L^2(\partial\Omega)$ inner product; we shall also use the notation $\langle \cdot , \cdot \rangle$ for this duality pairing.

Theorem 1.4. *Assume that $\gamma \in C^1(\overline{\Omega})$. The DN map, Λ_γ, defined by 1.7, extends as a bounded map*

$$\Lambda_\gamma \ : \ H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \ .$$

Proof. We shall use from now on the summation convention. If u, v and γ are arbitrary but smooth functions then Green's formula immediately gives

$$(1.8) \qquad \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \ dx = - \int_\Omega \frac{\partial}{\partial x_i} \left(\gamma_{ij} \frac{\partial u}{\partial x_j} \right) \ v \ dx + \int_{\partial\Omega} \gamma_{ij} \frac{\partial u}{\partial x_j} \nu_i \ v \ dS(x) \ .$$

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From the continuity of all the terms involved it is clear that this formula holds for $u \in H^2(\Omega)$, $v \in H^1(\Omega)$ and $\gamma_{ij} \in C^1(\overline{\Omega})$. If u is the solution to $\nabla \cdot \gamma \nabla u = 0$ in Ω, $u = f$ on $\partial\Omega$, defined in the sense of Theorem 1.3, then we know that

$$(1.9) \qquad \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx = 0 \quad \forall v \in H^1_0(\Omega) \ .$$

As noted above the solution, u, is in $H^2(\Omega)$ if f is in $H^{\frac{3}{2}}(\partial\Omega)$; it follows from a combination of 1.8 and 1.9 that

$$\int_\Omega \frac{\partial}{\partial x_i} \left(\gamma_{ij} \frac{\partial u}{\partial x_j} \right) v \, dx = 0 \quad \forall v \in H^1_0(\Omega) \ .$$

From the above identity it follows immediately that the L^2 function $\frac{\partial}{\partial x_i} \gamma_{ij} \frac{\partial u}{\partial x_j}$ vanishes. Inserting this fact and the definition of Λ_γ, 1.7, into 1.8 we get

$$(1.10) \qquad \int_{\partial\Omega} \Lambda_\gamma f \, v \, dS(x) = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \quad \forall v \in H^1(\Omega) \ .$$

Given any $g \in H^{\frac{1}{2}}(\partial\Omega)$ Theorem 1.1 guarantees the existence of a $v \in H^1(\Omega)$ such that

$$v|_{\partial\Omega} = Rv = g \quad \text{and} \quad \|v\|_{H^1(\Omega)} \le C\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \ .$$

Insertion of this v into 1.10 yields

$$\int_{\partial\Omega} \Lambda_\gamma f \, g \, dS(x) = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx$$
$$\le C\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$
$$\le C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \ .$$

Here we have also used the estimate of the $H^1(\Omega)$ norm of u in terms of the $H^{\frac{1}{2}}(\partial\Omega)$ norm of f, as given in Theorem 1.4. Since the duality between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ is the extension of the $L^2(\partial\Omega)$ inner product we get, by taking the maximum over all g with $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \le 1$,

$$\|\Lambda_\gamma f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \ .$$

The above estimate was proven under the assumption that $f \in H^{\frac{3}{2}}(\partial\Omega)$. This estimate, however, is sufficient to insure the existence of a unique extension of Λ_γ from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$. $\qquad\square$

The above discussion leads to a quite general definition of the DN map for any positive definite $\gamma \in L^\infty(\Omega)$. Given f and g in $H^{\frac{1}{2}}(\partial\Omega)$ we let u denote the weak solution to $L_\gamma u = 0$, $u|_{\partial\Omega} = f$ and we let v be any function in $H^1(\Omega)$, with the property that $v|_{\partial\Omega} = g$. We then define $\Lambda_\gamma f \in H^{-\frac{1}{2}}(\partial\Omega)$ by the requirement that

$$(1.11) \qquad \langle \Lambda_\gamma f, g \rangle = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \ .$$

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It is easy to see that the right hand side of 1.11 is independent of which v satisfying $v|_{\partial\Omega} = g$ we take. This follows immediately from the fact that

$$\int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx = 0 \ \ \forall w \in H_0^1(\Omega) \ \ .$$

Furthermore, since there exists $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$ and $\|v\|_{H^1(\Omega)} \le C\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$, the right hand side of 1.11 for fixed u (*i.e.*, for fixed f) defines a bounded linear functional on $H^{\frac{1}{2}}(\partial\Omega)$. This ensures the existence (and uniqueness) of $\Lambda_\gamma f \in H^{-\frac{1}{2}}(\partial\Omega)$ satisfying 1.11. From the inequalities

$$\int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx \le C\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \le C\|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \ \ ,$$

and the definition 1.11 it follows immediately that the "generalized" map Λ_γ is bounded from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$ It is obvious that this general definition yields exactly the extension of the map defined by 1.7 for $\gamma \in C^1(\overline{\Omega})$.

Returning to the variational characterization of the weak solution to

$$L_\gamma u = 0 \ \ \text{in } \Omega, \ \ u = f \ \ \text{on } \partial\Omega \ \ ,$$

we recall that it was shown in Theorem 1.5 that this u is also the unique minimizer to

$$D_0(w) = \frac{1}{2} \int_\Omega \gamma_{ij} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx$$

in the set $\{w \ : \ w \in H^1(\Omega), \ w|_{\partial\Omega} = f \ \}$. The functional Q_γ defined by

$$Q_\gamma(f) = \int_\Omega \gamma_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \, dx = \min_{w \in H^1(\Omega) \ w|_{\partial\Omega}=f} \int_\Omega \gamma_{ij} \frac{\partial w}{\partial x_j} \frac{\partial w}{\partial x_i} \, dx \ \ ,$$

is a quadratic functional on $H^{\frac{1}{2}}(\partial\Omega)$. From the formula 1.11 it follows immediately that Λ_γ is the selfadjoint linear map associated to this quadratic functional, *i.e.*,

$$\langle \Lambda_\gamma f, f \rangle = Q_\gamma(f) \ \ \forall f \in H^{\frac{1}{2}}(\partial\Omega) \ \ .$$

Knowledge of $Q_\gamma(f)$ is therefore the same as knowledge of $\langle \Lambda_\gamma f, f \rangle$. Since knowledge of the two quadratic expressions $\langle \Lambda_\gamma (f + g), (f + g) \rangle$ and $\langle \Lambda_\gamma (f - g), (f - g) \rangle$ by means of the formula

$$4\langle \Lambda_\gamma f, g \rangle = \langle \Lambda_\gamma (f + g), (f + g) \rangle - \langle \Lambda_\gamma (f - g), (f - g) \rangle$$

leads to knowledge of the expression $\langle \Lambda_\gamma f, g \rangle$, it follows that

Proposition 1.1. *Knowledge of $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ leads to knowledge of $\langle \Lambda_\gamma f, g \rangle$ for all $f, g \in H^{\frac{1}{2}}(\partial\Omega)$ and therefore leads to knowledge of the map Λ_γ. Conversely knowledge of Λ_γ also leads to knowledge of $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$.*

In the later chapters we shall see that many of the developments for the inverse conductivity problem have parallels when the boundary value problem $\nabla \cdot \gamma \nabla u = 0$ in Ω $u = f$ on $\partial\Omega$ is replaced by the Schrödinger equation

(1.12)
$$\Delta u + qu = 0 \quad \text{in } \Omega$$
$$u = f \quad \text{on } \partial\Omega \ .$$

We assume that $q \in L^\infty(\Omega)$. The operator $\Delta + q$ with Dirichlet boundary conditions is self adjoint from $\mathcal{D}(\Delta + q) \subset L^2(\Omega)$ to $L^2(\Omega)$. The domain of definition, $\mathcal{D}(\Delta + q)$, equals $H^2(\Omega) \cap H_0^1(\Omega)$, and the operator has a compact resolvent [20]. Given $f \in H^{\frac{3}{2}}(\partial\Omega)$ the boundary value problem, 1.12, therefore has a unique solution, $u \in H^2(\Omega)$, exactly when zero is not an eigenvalue for $\Delta + q$. In this case it is also possible to show that 1.12 has a unique weak solution for any $f \in H^{\frac{1}{2}}(\partial\Omega)$; this weakly defined solution is in $H^1(\Omega)$. If zero is not an eigenvalue we may define the DN map

(1.13)
$$\Lambda_q f = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

as a map from $H^{\frac{3}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$. Based on what we have seen earlier it is not surprising that this map extends as bounded map from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$. We summarize the previous discussion with

Theorem 1.5. *Suppose that $q \in L^\infty(\Omega)$ and that zero is not an eigenvalue of $\Delta + q$ with Dirichlet boundary conditions, then the boundary value problem 1.12 has a unique solution satisfying*

$$\|u\|_{H^t(\Omega)} \leq C\|f\|_{H^{t-\frac{1}{2}}(\partial\Omega)} \quad t \geq 1 \ .$$

The map Λ_q defined by 1.13 has a unique extension as a bounded map

$$\Lambda_q \ : \ H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \ .$$

There is a very direct relation between the isotropic conductivity equation and the Schrödinger equation. Suppose that γ is in $C^2(\overline{\Omega})$ and that $u \in H^2(\Omega)$ is a solution to

$$L_\gamma u = \nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial\Omega \ ,$$

for some $f \in H^{\frac{3}{2}}(\partial\Omega)$. If we define

$$w = \gamma^{\frac{1}{2}} u$$

then we find that

$$\Delta w + qw = 0 \quad \text{in } \Omega \ ,$$

with

$$q = -\frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \ .$$

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At the same time

$$w = \gamma^{\frac{1}{2}} f \quad \text{on } \partial\Omega \ ,$$

and we therefore easily calculate that

$$\Lambda_q(\gamma^{\frac{1}{2}} f) = \frac{\partial}{\partial\nu} w|_{\partial\Omega} = \frac{\partial}{\partial\nu}(\gamma^{\frac{1}{2}} w)|_{\partial\Omega}$$
$$= \frac{1}{2}\gamma^{-\frac{1}{2}}\frac{\partial\gamma}{\partial\nu} f + \gamma^{-\frac{1}{2}}\Lambda_\gamma f \ .$$

Through substitution of $g = \gamma^{\frac{1}{2}} f$ this yields

$$\Lambda_q g = \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu} g + \gamma^{-\frac{1}{2}}\Lambda_\gamma(\gamma^{-\frac{1}{2}} g) \ .$$

We have therefore proven

Theorem 1.6. *Let γ be a conductivity in $C^2(\overline{\Omega})$ and define*

$$q = -\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \ ,$$

then

$$\Lambda_q = \gamma^{-\frac{1}{2}}\Lambda_\gamma(\gamma^{-\frac{1}{2}} \ \cdot \ ) + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}I \ .$$

We conclude this chapter with a remark owing to the fact that the DN map is not always well defined for a Schrödinger operator. It is therefore often convenient to work with the set of Cauchy data

$$\mathcal{C}_q = \left\{ (f,g) \in H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \ : \ f = w|_{\partial\Omega}, \ g = \frac{\partial w}{\partial\nu}|_{\partial\Omega}, \ \text{with } \Delta w + qw = 0 \right\} \ .$$

The normal derivative $\frac{\partial w}{\partial\nu}|_{\partial\Omega}$ is here defined by Green's formula:

$$\langle \frac{\partial w}{\partial\nu}|_{\partial\Omega}, \phi \rangle = \int_\Omega \left(\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i} - qwv \right) \ dx \ ,$$

where $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = \phi$. We note that that when Λ_q exists, $\mathcal{C}_q$ is just its graph

$$\mathcal{C}_q = \{ (f, \Lambda_q f) \ : \ f \in H^{\frac{1}{2}}(\partial\Omega) \} \ .$$

2 Boundary Determination and Layer Stripping

The goal of this section is to show that if two conductivities γ_1 and γ_2 are in $C^\infty(\overline{\Omega})$ and give rise to the same boundary measurements (i.e., $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$) on the entire boundary, then the conductivities, and their normal derivatives of all orders agree on $\partial\Omega$. This was first proven in [22]. The result is

Theorem 2.1. *Suppose that γ_1 and γ_2 are in $C^\infty(\bar{\Omega})$ and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then, for any integer $\ell \geq 0$

$$(2.1) \qquad \left(\frac{\partial}{\partial\nu}\right)^\ell \gamma_1 = \left(\frac{\partial}{\partial\nu}\right)^\ell \gamma_2 \qquad on \ \partial\Omega \ .$$

The assertion of Theorem 2.1 immediately guarantees that all derivatives of γ_1 and γ_2 agree on $\partial\Omega$. As a consequence it follows that Λ_γ uniquely determines γ within the class of real-analytic γ. There is also a local version of Theorem 2.1 which guarantees the coincidence of all the derivatives of γ_1 and γ_2 near a point p solely based on the coincidence of $\Lambda_{\gamma_1}(f)$ and $\Lambda_{\gamma_2}(f)$ near p for any f with support near p.

A slightly more careful argument can be used to prove a stability estimate for the inverse problem at the boundary. For that purpose we define the operator norm

$$\|A\|_{1/2,-1/2} = \sup_{\|\phi\|_{H^{1/2}(\partial\Omega)}=1} \|A\phi\|_{H^{-1/2}(\partial\Omega)}.$$

If the operator A is an unbounded self adjoint operator on $L^2(\partial\Omega)$, then

$$\|A\|_{1/2,-1/2} = \sup_{\|\phi\|_{H^{1/2}(\partial\Omega)}=1} (\phi, A\phi)_{L^2(\partial\Omega)}|$$

$$= \sup_{\|\phi\|_{H^{1/2}(\partial\Omega)}=1} |Q_A(\phi)|$$

Where Q_A denotes the unique quadratic from associated to A [8].

Theorem 2.2. *Suppose that γ_0 and γ_1 are isotropic C^∞ conductivities on $\bar{\Omega} \subset \mathbf{R}^n$ satisfying:*

i) $1/E \leq \gamma_i \leq E$

ii) $\|\gamma_i\|_{C^2(\bar{\Omega})} \leq E$,

Given any $0 < \sigma < 1/(n+1)$ there exists $C = C(\Omega, E, n, \sigma)$ such that

$$(2.2) \qquad \|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{1/2,-1/2}$$

and

$$(2.3) \qquad \|\frac{\partial\gamma_1}{\partial\nu} - \frac{\partial\gamma_2}{\partial\nu}\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^\sigma_{1/2,-1/2} \ .$$

In this section we sketch an alternative approach to the proof of Theorem 2.1 developed first in [41] which uses the fact that Λ_γ is a pseudodifferential operator of order 1. Then one computes its full symbol in appropriate coordinates. In [28] this approach was further simplified by using a "factorization" method. We will follow this approach here because it also leads to a Riccati type equation satisfied by the DN map. This, and the boundary determination of the conductivity, are the key elements of the layer stripping algorithm developed in [39]. Also the factorization method leads to boundary determination results for more general equations and systems ([28], [30], [33]). Furthermore it provides an easy way to show that the DN map is a pseudodifferential operator. For related approaches using the singularities of the Green's kernel instead see [1] and [29].

We start with a very simple example. Let $\Omega = \mathbb{R}^n_+ = \{x = (x', x_n), x_n > 0\}$. Then $\partial\Omega = \mathbb{R}^{n-1}$. Let $f \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$. Let us consider the unique solution, $u \in H^1(\mathbb{R}^n_+)$, of

(2.4)
$$\Delta u = 0 \text{ in } \mathbb{R}^n_+$$
$$u|_{\partial\Omega} = f$$

Then the DN map is

(2.5)
$$f \to -\frac{\partial u}{\partial x_n}|_{\mathbb{R}^{n-1}}$$

where u solves (2.4).

We factorize

(2.6)
$$-\Delta = (D_{x_n} + i\sqrt{-\Delta'})(D_{x_n} - i\sqrt{-\Delta'})$$

where $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$ and $-\Delta' = -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}$. $\sqrt{-\Delta'}$ is the pseudodifferential operator given by

$$\sqrt{-\Delta'}f = \frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}|\xi'|\widehat{f}(\xi')d\xi'.$$

The point is that we can solve

(2.7)
$$(D_{x_n} + i\sqrt{-\Delta'})u = 0 \text{ in } \mathbb{R}^n_+$$
$$u|_{x^n=0} = f.$$

We simply take for $x_n > 0$

(2.8)
$$u(x', x_n) = \int e^{ix'\xi'}e^{-x_n|\xi'|}\widehat{f}(\xi')d\xi'.$$

From (2.7) we then deduce that

$$-\frac{\partial u}{\partial x_n}\bigg|_{x_n=0} = \sqrt{-\Delta'}f.$$

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So the DN map in this case is just $\sqrt{-\Delta'}$ whose full symbol is $|\xi'|$. Note that the term $(D_{x_n} - i\sqrt{-\Delta'})$ behaves like a heat equation in $\mathbb{R}^n_+$ and $(D_{x_n} + i\sqrt{-\Delta'})$ behaves like a backwards heat equation.

Now we try a similar idea for $(-\Delta + q)$, $q \in C^\infty(\overline{\Omega})$ where Ω is a bounded domain with smooth boundary. First we take coordinates near a point $x_0 \in \partial\Omega$ so that locally $\Omega = \{(x', x_n), x_n > 0\}$ and $-\frac{\partial}{\partial x_n}\big|_{\partial\Omega} = \frac{\partial}{\partial\nu}\big|_{\partial\Omega}$ with ν the unit outer normal to $\partial\Omega$.

In these coordinates

$$(2.9) \qquad -\Delta + q = (D_{x_n}^2 + iE(x)D_{x_n} + Q(x, D_{x'}) + q)$$

with $E \in C^\infty(\overline{\Omega})$ real-valued and $Q(x, D_{x'})$ a differential operator of order 2 in x', with no zero order term, depending smoothly on x_n, with full symbol $g_2(x, \xi') + g_1(x, \xi')$ with $g_2 > 0$ and g_i homogeneous of degree i in ξ', $i = 1, 2$.

We try to find an operator $B(x, D_{x'})$ so that we have the factorization

$$(2.10) \qquad (-\Delta + q) = (D_{x_n} + iE(x) + iB(x, D_{x'}))(D_{x_n} - iB(x, D_{x'}).$$

Using (2.9) and (2.10) $B(x, D_{x'})$ must solve

$$(2.11) \qquad i[D_{x_n}, B(x, D_{x'})] + EB(x, D_{x'}) + B^2 - Q - q = 0$$

where $[A, B] = AB - BA$ denotes the commutator. Notice that (2.11) is a Riccati type equation for B. We solve (2.11) using the calculus of pseudodifferential operators. If $b(x, \xi')$ denotes the full symbol of $B(x, D_{x'})$ a pseudodifferential operator of order 1 then the full symbol of $i[D_{x_n}, B(x, D_{x'}]$ is $\frac{\partial}{\partial x_n} b(x, \xi')$. The full symbol of B^2 is $\sum_\alpha \frac{1}{\alpha!} \partial_{\xi'}^\alpha b(x, \xi') D_{x'}^\alpha b(x, \xi')$ where the $\sum$ is interpreted asymptotically (as usual).

The full symbol of EB is $b(x, \xi')E(x')$. Therefore the equation we must solve for $b(x, \xi')$ is

(2.12)
$$\partial_{x_n} b(x, \xi') + b(x, \xi')E(x) + \sum_\alpha \frac{1}{\alpha!} \partial_{\xi'}^\alpha b(x, \xi') D_{x'}^\alpha b(x, \xi') - g_2(x, \xi') - g_1(x, \xi') - q = 0.$$

Now we write

$$(2.13) \qquad b(x, \xi') \sim \sum_{j \leq 1} b_j(x, \xi')$$

with b_j homogeneous of degree j in ξ'.

Now we compare terms of the same homogeneity in (2.12). The term homogeneous of degree 2 in (2.12) is

$$b_1^2(x, \xi') - g_2(x, \xi') = 0.$$

Therefore we choose

$$(2.14) \qquad b_1(x, \xi') = \sqrt{g_2(x, \xi')}.$$

We choose the positive sign in the square root in (2.14) since we want the term $D_{x_n} - iB(x, D_{x'})$ to behave like a heat equation in Ω. The term homogeneous of degree 1 in (2.12) is

$$\partial_{x_n} b_1(x, \xi') + b_1(x, \xi')E(x) + \sum_{j=1}^{n-1} \partial_{\xi_j} b_1 D_{x'_j} b_1 + 2b_0 b_1 - g_1 = 0.$$

Therefore we choose

$$(2.15) \qquad b_0 = \frac{1}{2b_1}\left\{-\partial_{x_n} b_1(x, \xi') - b_1(x, \xi')E(x) - \sum_{j=1}^{n-1} \partial_{\xi_j} b_1 D_{x'_j} b_1 + g_1\right\}.$$

We will do one more step. The term homogeneous of degree 0 in (2.12) is

$$\partial_{x_n} b_0(x, \xi') + b_0(x, \xi')E(x) + 2b_{-1}b_1 + b_0^2$$

$$+ \sum_{|\alpha|=1} \partial_{\xi'}^{\alpha} b_0 D_{x'}^{\alpha} b_1 + \sum_{|\alpha|=1} \partial_{\xi'}^{\alpha} b_1 D_{x'}^{\alpha} b_0 + \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_{\xi'}^{\alpha} b_1 D_{x'}^{\alpha} b_1 - q = 0$$

We then choose

$$b_{-1} = \frac{1}{2b_1}\left\{-\partial_{x_n} b_0(x, \xi') - b_0(x, \xi')E(x) - \sum_{|\alpha|=1} \partial_{\xi'}^{\alpha} b_0 D_{x'}^{\alpha} b_1\right.$$

$$(2.16)$$

$$\left. - \sum_{|\alpha|=1} \partial_{\xi'}^{\alpha} b_1 D_{x'}^{\alpha} b_0 - \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_{\xi'}^{\alpha} b_1 D_{x'}^{\alpha} b_1 + q\right\}.$$

Now the inductive procedure is clear. For any $j < -2$, collecting terms homogeneous of degree $j + 1$, we obtain

$$(2.17) \quad b_j = \frac{1}{2b_1}\left\{-\partial_{x_n} b_{j+1} - b_{j+1}E - 2 \sum_{\substack{l+k=j+1 \\ l,k \geq j+1}} b_l b_k - \sum_{\substack{|\alpha|\geq 1 \\ l+k-|\alpha|=j+1 \\ l,k \leq 1}} \frac{1}{\alpha!} \partial_{\xi'}^{\alpha}(b_l) D_{x'}^{\alpha}(b_k)\right\}$$

Note, that this forces $l, k \geq j + |\alpha| \geq j + 1$, i.e. the procedure is recursive. Then we have proven

Theorem 2.3. *In local coordinates (x', x_n) as chosen above, there exists a pseudodifferential operator $B(x, D_{x'})$ of order 1 depending smoothly on x_n such that*

$$(2.18) \qquad -\Delta + q = (D_{x_n} + iE(x) + iB(x, D_{x'}))(D_{x_n} - iB(x, D_{x'}))$$

modulo a smoothing operator.

Remark. The equation is solved modulo smoothing since we have only compared the full symbol of both sides of (2.18).

Now we can solve the pseudodifferential equation

(2.19)
$$(D_{x_n} - iB(x, D_{x'}))u = 0 \qquad \text{(mod smoothing)}$$
$$u|_{\partial\Omega} = f$$

in the form

(2.20)
$$u(x', x_n) = \int e^{ix'\xi'} e^{-x_n|\xi'|} a(x, \xi') \widehat{f}(\xi') d\xi'$$

with $a \sim \sum_{j \le 0} a_j$, $a_j(x, \xi')$ homogeneous of degree j in ξ'. (See [42].)

Now the other term in the factorization (2.18) is a smoothing operator (see [42]). Therefore we conclude that if u is the solution of

(2.21)
$$(-\Delta + q)u = 0$$
$$u|_{\partial\Omega} = f.$$

Then in local coordinates (x', x^n)

$$D_{x_n} u = iB(x, D_{x'})u \qquad \text{mod smoothing.}$$

Therefore

(2.22)
$$\Lambda_q = B(x', 0, D_{x'}) \qquad \text{mod smoothing.}$$

proving that the DN map is a pseudodifferential operator of order 1 on $\partial\Omega$. Now we prove

Theorem 2.4. *From the full symbol of Λ_q we can recover $\partial^\alpha q|_{\partial\Omega} \, \forall \, \alpha$.*

Proof. Using (2.22) we need only to compute the full symbol of $B(x', 0, D_{x'})$ i.e. $b(x', 0, \xi') \sim \sum_{i \le 1} b_j(x', 0, \xi')$.

The terms b_1, b_0 don't give any information on q (see (2.14) and (2.15)). Now from (2.16) we conclude that if we know $b_{-1}(x', 0, \xi')$ we can determine $q(x', 0)$ since all of the other terms in the RHS of (2.16) are known.

Proceeding inductively: if we know that from b_{-j+1} we can determine $\frac{\partial^j q}{\partial x_n^j}(x', 0)$, and if we know b_k, $k \ge -j + 1$, then from (2.17) we conclude that we can recover from $b_{-j}(x', 0, \xi')$, $\frac{\partial^{j+1} q}{\partial x_n^{j+1}}(x', 0)$ finishing the proof. $\qquad \square$

We now use Theorem 2.4 to prove the Kohn-Vogelius result.

Theorem 2.5. *Let $\gamma_i \in C^\infty(\overline{\Omega})$, $\gamma_i \ge \epsilon > 0$, so that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\partial^\alpha \, \gamma_1|_{\partial\Omega} = \partial^\alpha \, \gamma_2|_{\partial\Omega} \qquad \forall \, \alpha$$

Remark. This result is also local, i.e., one only needs to take $x_0 \in \partial\Omega$ and a neighborhood $U(x_0)$ of x_0 in Ω so that $\Lambda_{\gamma_1}(f)|_{U(x_0)} = \Lambda_{\gamma_2}(f)|_{U(x_0)} \forall f$ supported in $U(x_0)$ to conclude that $\partial^\alpha \gamma_1(x_0) = \partial^\alpha \gamma_2(x_0) \forall \alpha$.

We recall Theorem 1.9 relating Λ_{q_j} and Λ_{γ_j} if $q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}}$, $j = 1, 2$

$$(2.22) \qquad \Lambda_{q_j} f = \gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega} \Lambda_{\gamma_j}\left(\gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega} f\right) + \frac{1}{2}\left(\gamma_j^{-1}\frac{\partial\gamma_j}{\partial\nu}\right)\Big|_{\partial\Omega} f, j = 1, 2.$$

Now we know that

$$\sigma_1(\Lambda_{q_j}) = \sqrt{g_2(x', 0, \xi')} = \gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega} \sigma_1(\Lambda_{\gamma_j})\, \gamma_j^{-\frac{1}{2}}\Big|_{\partial\Omega}$$

where $\sigma_m(A)$ denotes the principal symbol of a pseudodifferential operator of order m. So we deduce that

$$\gamma_1|_{\partial\Omega}\sqrt{g_2(x', 0, \xi')} = \sigma_1(\Lambda_{\gamma_1}) = \sigma_1(\Lambda_{\gamma_2}) = \gamma_2|_{\partial\Omega}\sqrt{g_2(x', 0, \xi')}$$

We then have that $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$. Therefore under the hypotheses of Theorem 2.5, from (2.22) we conclude that

$$\Lambda_{q_1} - \Lambda_{q_2} - \frac{1}{2}\left(\gamma_1^{-1}\frac{\partial\gamma_1}{\partial\nu}\right)\Big|_{\partial\Omega} - \frac{1}{2}\left(\gamma_2^{-1}\frac{\partial\gamma_2}{\partial\nu}\right)\Big|_{\partial\Omega} = 0.$$

Now we take the principal symbol of order zero in the above equation. We have that $\Lambda_{q_1} - \Lambda_{q_2}$ is a pseudodifferential operator of order zero and $\sigma_0(\Lambda_{q_1} - \Lambda_{q_2}) = 0$. We then obtain that

$$\frac{\partial\gamma_1}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial\gamma_2}{\partial\nu}\Big|_{\partial\Omega}.$$

Therefore we have that $\Lambda_{q_1} = \Lambda_{q_2}$ and by Theorem 2 $\qquad \partial^\alpha q_1|_{\partial\Omega} = \partial^\alpha q_2|_{\partial\Omega} \forall \alpha$. Since $q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}}$ we arrive to the conclusion of the theorem. $\qquad\square$

In [?] it was shown that knowing the principal symbol of Λ_γ $\sigma_1(\Lambda_\gamma) = \gamma|_{\partial\Omega}|\xi'|$ implies the stability estimate (2.2) for just continuous conductivities. The computation of $\sigma_0(\Lambda_\gamma - \gamma|_{\partial\Omega}\Lambda_1)$ where Λ_1 is the DN associated to the conductivity 1 leads to the following result in [?].

Theorem 2.6. *Let γ_i be measurable functions such that*

$$0 < \frac{1}{\lambda} \leq \gamma_i \leq \lambda.$$

If γ_i are Lipschitz continuous in $\overline{\Omega}$ and for some β

$$\sup_{x\in\overline{\Omega}} |\nabla\gamma_i| \leq \beta$$

Then we have that the bounded linear map

$$B_i : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

where

$$B_i = \Lambda_{\gamma_i} - \gamma_i|_{\partial\Omega} \Lambda_1,$$

satisfies

$$\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} \leq C(\lambda,\beta)\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\Omega)}$$

and

$$\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\partial\Omega)} \leq C\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$$

where $\| \quad \|_{\frac{1}{2},\frac{1}{2}}$ *and* $\| \quad \|_{\frac{1}{2},-\frac{1}{2}}$ *denotes the corresponding operator norms.*

This method has been generalized to the anisotropic case in [28].

(a) **(Anisotropic conductivities)** Let $g(x) = g_{ij}(x)$ be a smooth Riemannian metric in $\overline{\Omega}$, i.e. g_{ij} is assumed to be a smooth, symmetric, positive definite matrix in $\overline{\Omega}$. Let Δ_g be the Laplace-Beltrami operator associated to g, i.e.

$$(2.22) \qquad \Delta_g = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j} \right)$$

where $(g^{ij}) = (g_{ij})^{-1}$.

Let $f \in H^{\frac{1}{2}}(\partial\Omega)$. Let $u \in H^1(\Omega)$ be the unique solution of

$$(2.23) \qquad \begin{aligned} -\Delta_g u &= 0 \\ u|_{\partial\Omega} &= f \end{aligned}$$

The DN map is defined by

$$(2.25) \qquad \Lambda_g(f) = \sum_{i,j=1}^{n} g^{ij} \nu_j \left. \frac{\partial u}{\partial x_i} \right|_{\partial\Omega}$$

where ν_j denotes the components of ν. In section 1 of [28] the following result was proven: Let (x', x^n) denote boundary normal coordinates with respect to the metric g. Then the full symbol of Λ_g determines $\partial_{x_n}^{\alpha} g|_{\partial\Omega} \,\forall\, \alpha$.

We recall the definition of boundary normal coordinates. For each $r \in \partial\Omega$, let $\alpha_q : [0,\epsilon) \to \overline{\Omega}$ denote the limit-speed geodesic starting at r and normal to $\partial\Omega$. If $\{x^1, \ldots, x^n\}$ are local coordinates for $\partial\Omega$ near $p \in \partial\Omega$, we can extend them smoothly to functions on a neighborhood of p in $\overline{\Omega}$ by letting them to be constant along each normal geodesic. If we define x^n to be the parameter along each α_r, then $\{x^1, \ldots, x^n\}$ are coordinates called boundary normal coordinates.

The method of proof proceeds as before. Namely we find $B(x, D_{x'})$ a pseudodifferential operator of order 1 in x', depending smoothly on x_n such that we have the factorization

$$\begin{aligned} -\Delta_g &= D_{x^n}^2 + iE(x)D_{x^n} + Q(x, D_{x'}) \\ &= (D_{x^n} + iE(x) + iB(x, D_{x'}))(D_{x^n} - iB(x, D_{x'})) \end{aligned}$$

modulo smoothing.

Then we prove

$$\Lambda_g = B(x', 0, D_{x'}) \qquad \text{mod smoothing.}$$

Finally from the full symbol of $B(x', 0, D_{x'})$ we recover $\left.\frac{\partial^\alpha g}{(\partial x^n)^\alpha}\right|_{\partial\Omega}$. Notice that this statement depends on the coordinates (x', x^n).

B Layer stripping algorithm

Layer stripping algorithms have been developed for several inverse problems. For the Electrical Impedance Tomography the corresponding algorithm was developed in [39]. The idea is quite simple: We embed the domain $\partial\Omega$ in domains Ω_a $a \geq 0$ small with $\Omega_0 = \Omega$. In the case of the Schrödinger equation $-\Delta + q$, Ω_a is given by

$$\Omega_a = \{(x', x_n); x_n = a\}$$

so we have a family of DN maps $\Lambda_q^{(a)} = \Lambda_q|_{x_n=a}$. We know that we can determine $q|_{\partial\Omega_0}$ from Λ_q. Then we use the Riccati equation (2.11) (note that $B(x', a, D_{x'})$ is the DN map) to compute $\left.\frac{d\Lambda_q}{dx_n}\right|_{x_n=0}$. We now use the approximation

$$\Lambda_q|_{x_n=a} \approx \Lambda_q|_{x_n=0} + a \left.\frac{d\Lambda_q}{dx_n}\right|_{x_n=0}$$

In this way we can determine $\Lambda_q|_{x_n=a}$. We can then determine $q|_{x_n=a}$ and therefore we can use (2.12) again to write

$$\Lambda_q|_{x_n=a+\Delta a} \approx \Lambda_q|_{x_n=a} + \Delta a \left.\frac{d\Lambda_q}{dx_n}\right|_{x_n=a}.$$

Of course, the problem is that the Riccati equation is non-linear and there will be "blow-up" as x_n increases. Some regularization of this is needed. In [39] this was investigated for a ball $\mathbb{R}^2$ by dropping the high frequency modes. / We finish this section by showing another derivation of the Riccati equation (2.11') is satisfied exactly, not just up to a smoothing operator as shown earlier. We follow [39].

We solve the family of Dirichlet problems

$$(2.26) \qquad L_q u = (D_{x_n}^2 + iE(x)D_{x_n} + Q + q)u(x, a) = 0 \text{ in } \Omega_a$$

$$(2.27) \qquad u(x, a)|_{x_n=a} = f(x')$$

We define the family of DN maps

$$(2.28) \qquad \Lambda_q^{(a)}(f) = -\left.\frac{\partial u}{\partial x_n}\right|_{x_n=a}.$$

We differentiate (2.26) and (2.27) with respect to a to obtain

$$L_q \frac{\partial u}{\partial a} = 0$$

$$\left.\frac{\partial u}{\partial a}\right|_{x_n=a} = \left.\frac{\partial u}{\partial x_n}\right|_{x_n=a} = -\Lambda_q^{(a)}(f)$$

Therefore

$$(2.29) \qquad (\Lambda_q^{(a)})^2(f) = \left.\frac{\partial^2 u}{\partial x_n \partial a}\right|_{x_n=a}$$

Now we differentiate (2.28) with respect to a. We get

$$
\begin{aligned}
(2.30) \qquad \left(\frac{d\Lambda_q^{(a)}}{da}\right)(f) &= \frac{d}{da}(\Lambda_q^{(a)}f) \\
&= \frac{d}{da}\left(-\left.\frac{\partial u}{\partial x_n}\right|_{x_n=a}\right) \\
&= -\left(\left(\frac{\partial}{\partial x_n}+\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial x_n}\right)\right)(u)|_{x_n=a} \\
&= -iE(x)\Lambda_q^a f + Qf + qf - (\Lambda_q^{(a)})^2 f
\end{aligned}
$$

by using (2.26), (2.27), (2.28) and (2.29). So we get

$$(2.31) \qquad \frac{d\Lambda_q^{(a)}}{da} + E\Lambda_q^a + (\Lambda_q^{(a)})^2 - Q - q = 0$$

which is equation (2.11) since $i[D_{x_n}, B(x, D'_x)]f(x') = \partial_{x_n}B(x, D_{x'})f(x')$.

3 Complex Geometrical Optics Solutions

If we look for "plane wave" exponential solutions to Laplace's equation, i.e., if we seek

$$u = e^{x\cdot\zeta} \qquad \zeta \in \mathbf{C}^n$$

which satisfy

$$\Delta u = 0,$$

then we must necessarily have

$$\zeta \cdot \zeta = 0 \; ;$$

conversely 3 solves Laplace's equation whenever 3 is satisfied. As any nontrivial solutions to 3 will have non-zero real part, the corresponding solution 3 will grow exponentially in most directions. The search for solutions analogous to 3 with the Laplacian replaced by $\Delta + q$ will be the main subject of this section. The utility of complex geometrical optics solutions in solving the inverse conductivity problem was first observed by Calderón in [3]. We begin by exhibiting Calderón's proof of injectivity of the linearized inverse boundary value problem.

We recall that the mapping Λ defined by

$$\gamma \xmapsto{\Lambda} \Lambda_\gamma$$

is an analytic map from $L^\infty(\Omega)$ to $\mathcal{BL}_{1/2,-1/2}$, the vector space of bounded linear maps from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ endowed with the operator norm. We denote by $D\Lambda_\gamma[\delta\gamma]$ the Frechet derivative of Λ at γ acting on the perturbation $\delta\gamma$. Calderón proved the following result.

Theorem 3.1. *The Frechet derivative of Λ at $\gamma = 1$, $D\Lambda_1[\ \cdot\ ]$, is injective. That is, if*

$$D\Lambda_1[\delta\gamma] = 0 \text{ for some } \delta\gamma \in L^\infty(\Omega)$$

then

$$\delta\gamma = 0$$

Proof. Let $\gamma = \gamma(t)$ be a smooth curve in $L^\infty(\Omega)$ and let $u(t)$ and $v(t)$ satisfy, for each t

$$\left(\begin{array}{cc} L_\gamma u = 0, & L_\gamma v = 0 \\ u|_{\partial\Omega} = f, & v|_{\partial\Omega} = g \\ \gamma\frac{\partial u}{\partial\nu}|_{\partial\Omega} = \alpha, & \gamma\frac{\partial v}{\partial\nu}|_{\partial\Omega} = \beta, \end{array}\right)$$

then integration by parts gives the identity

$$\int_{\partial\Omega} (f\beta(t) - g\alpha(0))\ dS(x) = \int_\Omega (\nabla u(0)^T \gamma(t)\nabla v(t) - \nabla u(0)^T \gamma(0)\nabla v(t))\ dx.$$

Differentiation with respect to t at $t = 0$ gives

$$\int_{\partial\Omega} f\,\overset{\bullet}{\beta}(0)\ dS(x) = \int_\Omega \nabla u(0)^T \overset{\bullet}{\gamma}(0)\nabla v(0)\ dx\ ,$$

where $\bullet$ denotes $\frac{d}{dt}$ and u and v satisfy 3. Since $\overset{\bullet}{\beta}(0)$ equals $D\Lambda_{\gamma(0)}[\overset{\bullet}{\gamma}(0)]g$ this identity may also be written

$$\langle f, D\Lambda_{\gamma(0)}[\overset{\bullet}{\gamma}(0)]g\rangle = \int_\Omega \nabla u(0)^T \overset{\bullet}{\gamma}(0)\nabla v(0)\ dx\ ,$$

for every f and $g \in H^{1/2}(\partial\Omega)$. By taking $\overset{\bullet}{\gamma}(0) = \delta\gamma$ it now follows that he equation

$$D\Lambda_{\gamma(0)}[\delta\gamma] = 0$$

is equivalent to

$$\int_\Omega \nabla u^T \delta\gamma \nabla v\ dx = 0$$

for every u and $v \in H^1(\Omega)$ which satisfy $L_{\gamma(0)}u = L_{\gamma(0)}v = 0$. If we further restrict to $\gamma(0) = 1$ then 3 must hold for every pair of harmonic functions u and v. A very natural set of choices for u and v are those exponentials 3 which satisfy 3, i.e., let

$$u = e^{x\cdot\zeta_1}, \qquad v = e^{x\cdot\zeta_2}$$

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with $\zeta_j \cdot \zeta_j = 0$. Then it follows from 3 that

$$\zeta_1 \cdot \zeta_2 \int_\Omega e^{x \cdot (\zeta_1 + \zeta_2)} \delta\gamma \ dx = 0$$

or

$$\frac{(\zeta_1 + \zeta_2) \cdot (\zeta_1 + \zeta_2) - (\zeta_1 - \zeta_2) \cdot (\zeta_1 - \zeta_2)}{4} \int_\Omega e^{x \cdot (\zeta_1 + \zeta_2)} \delta\gamma \ dx = 0 \ .$$

We now require that

$$\zeta_1 + \zeta_2 = ik, \quad k \in \mathbf{R}^n,$$

and note that since $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$

$$-(\zeta_1 - \zeta_2) \cdot (\zeta_1 - \zeta_2) = (\zeta_1 + \zeta_2) \cdot (\zeta_1 + \zeta_2) = -k \cdot k \ \ .$$

Hence $D\Lambda_1[\delta\gamma] = 0$ implies that

$$k \cdot k \int_\Omega e^{ix \cdot k} \delta\gamma \ dx = 0 \ \ ,$$

which again implies that

$$\text{supp} \ \widehat{(\chi_\Omega \delta\gamma)} \subset \{0\} \ \ ,$$

with χ_Ω denoting the characteristic function of the set Ω. However, $\chi_\Omega \delta\gamma$ is an element of $L^2(\mathbf{R}^n)$, so that $\widehat{\chi_\Omega \delta\gamma}$ is in $L^2(\mathbf{R}^n)$ and therefore cannot be supported at a single point. As a consequence

$$\chi_\Omega \delta\gamma = 0 \ ,$$

which proves that $D\Lambda_1[\ \cdot \ ]$ is injective. $\qquad\qquad\square$

Before proceeding, we formulate the analog of Theorem 3.1 for Schrödinger operators. Let $\mathcal{C}_q$ denote the Cauchy data for $\Delta + q$, defined by

$$\mathcal{C}_q = \{(f, g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) :$$

$$\exists v \in H^1(\Omega) \text{ with } \Delta v + qv = 0 \text{ in } \Omega, \text{ and } v|_{\partial\Omega} = f, \ \frac{\partial v}{\partial \nu}|_{\partial\Omega} = g\} \ .$$

The map

$$L^\infty(\Omega) \ni q \overset{\mathcal{C}}{\mapsto} \mathcal{C}_q \in \{ \text{ linear subspaces of } H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)\}$$

is also real analytic, and we denote by $D\mathcal{C}_q$, its Frechet derivative at q.

Theorem 3.2. *The Frechet derivative of $\mathcal{C}$ at $q = 0$, $D\mathcal{C}_0[\ \cdot \ ]$, is injective.*

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Remark. For any fixed Ω and q sufficiently small ($\|q\|_{L^\infty} <$ smallest eigenvalue of $-\Delta$ with Dirichlet boundary conditions) $\mathcal{C}_q$ is the graph of the Dirichlet to Neumann map, Λ_q, corresponding to the operator $\Delta + q$. Hence, the Frechet derivative acting on the perturbation δq is given by

$$DC_q[\delta q] = \{(0, D\Lambda_q[\delta q]f) : \ f \in H^{1/2}(\partial\Omega)\} \ ,$$

and the statement that $DC_0[\ \cdot\ ]$ is injective is equivalent to the statement $D\Lambda_q[\ \cdot\ ]$ is injective at $q = 0$. $\qquad\square$

The approach that we will use to prove identifiability later in § 5 is based on exponential solutions which are constructed in a way that naturally extends the previous construction for the Laplacian. To construct these solutions we shall make use of the following norms, defined for any $u \in C_0^\infty(\mathbf{R}^n)$ and any $-\infty < \delta < \infty$:

$$\|u\|_{L_\delta^2} = \left(\int_{\mathbf{R}^n} \left(1 + |x|^2\right)^\delta |u|^2 \ dx \right)^{1/2} .$$

The space L_δ^2 is defined as the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\|\cdot\|_{L_\delta^2}$.

The main theorem in this section is:

Theorem 3.3. *Let $-1 < \delta < 0$. There exists $\epsilon = \epsilon(\delta)$ and $C = C(\delta)$ such that, for every $q \in L_{\delta+1}^2 \cap L^\infty$ and every $\zeta \in \mathbf{C}^n$ satisfying*

$$\zeta \cdot \zeta = 0 \quad and$$

$$\frac{\|(1 + |x|^2)^{1/2} q\|_{L^\infty} + 1}{|\zeta|} \leq \varepsilon \ ,$$

there exists a unique solution to

$$\Delta u + qu = 0 \quad in \ \mathbf{R}^n$$

of the form

$$u = e^{x \cdot \zeta} \left(1 + \psi(x, \zeta)\right)$$

with $\psi(x, \zeta) \in L_\delta^2$. Furthermore,

$$\|\psi\|_{L_\delta^2} \leq \frac{C}{|\zeta|} \|q\|_{L_{\delta+1}^2}.$$

This theorem has a counterpart for the conductivity problem, which is obtained by invoking the correspondence in Theorem 0.6 between the Schrödinger equation and the conductivity equation. The statement is

Theorem 3.4. *Let* $-1 < \delta < 0$. *There exists* $\epsilon = \epsilon(\delta)$ *and* $C = C(\delta)$ *such that, for every positive* γ *with* $\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \in L_{\delta+1}^2 \cap L^\infty$ *and every* $\zeta \in \mathbf{C}^n$ *satisfying*

$$\zeta \cdot \zeta = 0 \quad and$$

$$\frac{\|(1 + |x|^2)^{1/2}\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}\|_{L^\infty} + 1}{|\zeta|} \leq \epsilon \; ,$$

there exists a unique solution to

$$L_\gamma u = 0$$

of the form

$$u = \gamma^{-1/2}e^{x \cdot \zeta}\left(1 + \psi(x, \zeta)\right)$$

with $\psi(x, \zeta) \in L_\delta^2$. *Furthermore,*

$$\|\psi(x, \zeta)\|_{L_\delta^2} \leq \frac{C}{|\zeta|}\|\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}\|_{L_{\delta+1}^2}.$$

Most of the work necessary for the proof of Theorem 3.3 is associated with establishing the following proposition.

Proposition 3.1. *Suppose that* $\zeta \cdot \zeta = 0$, $|\zeta| \geq c > 0$, $f \in L_{\delta+1}^2$ *and* $-1 < \delta < 0$. *There exists a unique* $\varphi \in L_\delta^2$ *such that*

$$(\Delta + 2\zeta \cdot \nabla)\varphi = f.$$

Moreover,

$$\|\varphi\|_{L_\delta^2} \leq \frac{C(\delta, c)}{|\zeta|}\|f\|_{L_{\delta+1}^2}.$$

We postpone the proof of this proposition to the end of this chapter, instead we first show how it may be applied for the

Proof of Theorem 3.3. We seek u of the form

$$u = e^{x \cdot \zeta}(1 + \psi)$$

satisfying

$$(\Delta + q)\{e^{x \cdot \zeta}(1 + \psi)\} = 0$$

or

$$\Delta\psi + 2\zeta \cdot \nabla\psi = -q - q\psi.$$

To solve 3, we define

$$\psi_{-1} = 1$$

and we recursively define ψ_j by

$$(\Delta + 2\zeta \cdot \nabla)\psi_j = -q\psi_{j-1} \quad \text{for } j \geq 0.$$

We claim that

$$\psi := \sum_{j=0}^{\infty} \psi_j$$

is the desired solution. It needs to be proved that the functions ψ_j, $j \geq 0$, are well defined, and that the series 3 is convergent. We may without loss of generality restrict our attention to $\epsilon < 1$, so that we only consider ζ for which $|\zeta| \geq 1$. Since $q \in L_{\delta+1}^2$ and $\psi_{-1} = 1$ it follows from Proposition 3.1 that there exists a unique $\psi_0 \in L_\delta^2$ that solves 3 with $j = 0$. This ψ_0 furthermore satisfies

$$\|\psi_0\|_{L_\delta^2} \leq \frac{C(\delta)}{|\zeta|} \|q\|_{L_{\delta+1}^2} .$$

If v is an element in L_δ^2, then the fact that $(1 + |x|^2)^{1/2}q$ is in L^∞ immediately implies that qv is in $L_{\delta+1}^2$ with the estimate

$$\|qv\|_{L_{\delta+1}^2} \leq \|(1 + |x|^2)^{1/2}q\|_{L^\infty}\|v\|_{L_\delta^2} .$$

Using this observation in connection with Proposition 3.1 we conclude that if ψ_{j-1} is in L_δ^2 then there exists a unique solution to 3 in L_δ^2. This solution ψ_j furthermore satisfies

(3.16)
$$\|\psi_j\|_{L_\delta^2} \leq \frac{C(\delta)}{|\zeta|}\|q\psi_{j-1}\|_{L_{\delta+1}^2}$$
$$\leq \left(\frac{C(\delta)\|(1 + |x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|}\right)\|\psi_{j-1}\|_{L_\delta^2} .$$

An induction argument based on the estimates 3 and 3.16 now gives that ψ_j, $j \geq 0$, are all elements of L_δ^2 and satisfy the estimates

$$\|\psi_j\|_{L_\delta^2} \leq \frac{C(\delta)}{|\zeta|}\theta^j\|q\|_{L_{\delta+1}^2} \quad \text{with} \quad \theta = \frac{C(\delta)\|(1 + |x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|} .$$

By selecting ϵ sufficiently small, say that $\theta < 1/2$, we now obtain that the series 3 is convergent, with the bound

$$\|\psi\|_{L_\delta^2} \leq 2\frac{C(\delta)}{|\zeta|}\|q\|_{L_{\delta+1}^2} .$$

This completes the proof of the existence part of Theorem 3.3. $\qquad \square$

To verify the uniqueness of the solution ψ (and therefore of u), suppose that

$$\Delta\psi + 2\zeta \cdot \nabla\psi = -q - q\psi$$

and

$$\Delta\tilde{\psi} + 2\zeta \cdot \nabla\tilde{\psi} = -q - q\tilde{\psi} \ ,$$

with ψ and $\tilde{\psi} \in L^2_\delta$. Then

$$\Delta(\tilde{\psi} - \psi) + 2\zeta \cdot \nabla(\tilde{\psi} - \psi) = q(\psi - \tilde{\psi}) \ ,$$

so that according to Proposition 3.1

$$\|\tilde{\psi} - \psi\|_{L^2_\delta} \leq \frac{C\|(1 + |x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|}\|\tilde{\psi} - \psi\|_{L^2_\delta}$$

$$\leq \frac{1}{2}\|\tilde{\psi} - \psi\|_{L^2_\delta} \ ,$$

which can only occur if

$$\|\tilde{\psi} - \psi\|_{L^2_\delta} = 0 \ .$$

$\square$

It is not exactly Theorem 3.3 we use later on in our proof of identifiability, rather it is the following version for a bounded domain.

Corollary 3.5. *Let Ω be a bounded domain in $\mathbf{R}^n$. There exist constants ϵ and C such that for every $q \in L^\infty(\Omega)$ and every $\zeta \in \mathbf{C}^n$ satisfying*

$$\zeta \cdot \zeta = 0 \quad and$$

$$\frac{\|q\|_{L^\infty} + 1}{|\zeta|} \leq \epsilon \ ,$$

there exists a solution $u \in H^1(\Omega)$ to

$$\Delta u + qu = 0 \quad in \ \Omega$$

of the form

$$u = e^{x \cdot \zeta}\big(1 + \psi(x, \zeta)\big)$$

with

$$\|\psi\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}\|q\|_{L^2(\Omega)} \quad and$$

$$\|\psi\|_{H^1(\Omega)} \leq C\|q\|_{L^2(\Omega)} \ .$$

Proof. Define

$$\tilde{q} = \begin{cases} q & \text{in } \Omega \\ 0 & \text{in } \mathbf{R}^n \setminus \Omega \ . \end{cases}$$

We may apply Theorem 3.3 to $\tilde{q}$, say with $\delta = 1/2$. This way we obtain the existence of a solution to $\Delta u + \tilde{q}u = 0$ in $\mathbf{R}^n$ (and therefore a solution to $\Delta u + qu = 0$ in Ω) of the form $u = e^{x \cdot \zeta}(1 + \psi(x, \zeta))$ with

$$\|\psi\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|} \|q\|_{L^2(\Omega)} \ .$$

By interior elliptic regularity estimates it follows that $u \in H^1(\Omega)$. It only remains to prove the estimate concerning the H^1 norm of ψ. As a means to obtain this estimate we establish a particular interior estimate for solutions to

$$\Delta v = -F \quad \text{in } \mathbf{R}^n \ ,$$

namely

$$\|v\|_{H^1(\Omega)} \leq C\big(\|F\|_{H^{-1}(\Omega')} + \|v\|_{L^2(\Omega')}\big) \ ,$$

provided $\Omega \subset\subset \Omega'$. Let $\chi \in C_0^1(\mathbf{R}^n)$, $0 \leq \chi \leq 1$ be such that $\chi \equiv 1$ on Ω and supp $\chi \subset \Omega'$, then integration by parts and use of Hölder's inequality yields

$$\begin{aligned}
\int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx &= \int_{\mathbf{R}^n} F\chi^2 v \, dx + 2 \int_{\mathbf{R}^n} \chi \nabla\chi \cdot \nabla v \, v \, dx \\
&\leq C\|F\|^2_{H^{-1}(\Omega')} + \frac{1}{8}\|\chi^2 v\|^2_{H^1(\Omega')} \\
&\quad + \frac{1}{4}\int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx + C \int_{\mathbf{R}^n} v^2 |\nabla\chi|^2 \, dx \ .
\end{aligned}$$

(3.19)

On the other hand

$$\begin{aligned}
\|\chi^2 v\|^2_{H^1(\Omega')} &= \int_{\mathbf{R}^n} |\nabla(\chi^2 v)|^2 \, dx + \int_{\mathbf{R}^n} |\chi^2 v|^2 \, dx \\
&\leq 2\int_{\mathbf{R}^n} \chi^4 |\nabla v|^2 \, dx + 2\int_{\mathbf{R}^n} v^2 |\nabla\chi^2|^2 \, dx \\
&\quad + \int_{\mathbf{R}^n} |\chi^2 v|^2 \, dx \\
&\leq 2\int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx + C\|v\|^2_{L^2(\Omega')} \ .
\end{aligned}$$

(3.20)

A combination of 3.19 and 3.20 gives

$$\begin{aligned}
\int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx &\leq C\big(\|F\|^2_{H^{-1}(\Omega')} + \|v\|^2_{L^2(\Omega')}\big) \\
&\quad + \frac{1}{2}\int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx \ ,
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla v|^2 \, dx &\leq \frac{1}{2}\int_{\mathbf{R}^n} \chi^2 |\nabla v|^2 \, dx \\
&\leq C\big(\|F\|^2_{H^{-1}(\Omega')} + \|v\|^2_{L^2(\Omega')}\big) \ .
\end{aligned}$$

This immediately leads to the estimate 3.

Going back to the equation 3 we get that

$$\Delta\psi = -2\zeta \cdot \nabla\psi - \tilde{q} - \tilde{q}\psi \quad \text{in } \mathbf{R}^n ,$$

and the estimate 3 thus gives

$$\|\psi\|_{H^1(\Omega)} \leq C \left(\|2\zeta \cdot \nabla\psi + \tilde{q} + \tilde{q}\psi\|_{H^{-1}(\Omega')} + \|\psi\|_{L^2(\Omega')} \right) ,$$

with $\Omega \subset\subset \Omega'$. On the other hand we also have

$$(3.22) \quad \begin{aligned} \|2\zeta \cdot \nabla\psi + \tilde{q} + \tilde{q}\psi\|_{H^{-1}(\Omega')} &\leq 2|\zeta| \|\psi\|_{L^2(\Omega')} + \|\tilde{q}\|_{L^2(\Omega')} + \|\tilde{q}\|_{L^2(\Omega')} \|\psi\|_{L^2(\Omega')} \\ &= 2|\zeta| \|\psi\|_{L^2(\Omega')} + \|q\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega')} , \end{aligned}$$

and

$$\|\psi\|_{L^2(\Omega')} \leq \frac{C}{|\zeta|} \|\tilde{q}\|_{L^2(\Omega')} = \frac{C}{|\zeta|} \|q\|_{L^2(\Omega)} .$$

The estimate 3 is obtained by replacing Ω by Ω' in the estimate 3 (the constant C changes). A combination of 3-3 yields

$$\|\psi\|_{H^1(\Omega)} \leq C \left(\|q\|_{L^2(\Omega)} + \frac{\|q\|_{L^2(\Omega)}^2}{|\zeta|} + \frac{\|q\|_{L^2(\Omega)}}{|\zeta|} \right) ,$$

and since the assumption about $|\zeta|$ implies that $1/|\zeta| \leq C \min(1, \|q\|_{L^2(\Omega)}^{-1})$, we immediately get

$$\|\psi\|_{H^1(\Omega)} \leq C \|q\|_{L^2(\Omega)} ,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now return to the

Proof of Proposition 3.1. We first prove uniqueness. Suppose that $w \in L^2_\delta$ and

$$\Delta w + 2\zeta \cdot \nabla w = 0.$$

Fourier transformation gives

$$(-|\xi|^2 + 2\zeta \cdot i\xi)\hat{w} = 0.$$

As this equation is invariant under rotations, we may without loss of generality assume that

$$\zeta = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \end{pmatrix} + is \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \end{pmatrix} = se_1 + ise_2, \quad s = \frac{|\zeta|}{\sqrt{2}} ,$$

in which case 3 is equivalent to

$$\left[\left(\xi_1^2 + (\xi_2 - s)^2 + \xi_3^2 \cdots + \xi_n^2 - s^2 \right) + 2is\xi_1 \right] \cdot \hat{w} = 0.$$

The content of 3 is that the tempered distribution $\hat{w}$ is supported on the manifold $\mathcal{M}(s)$, where $\mathcal{M}(s)$ denotes the codimension 2 sphere which arises as the intersection of the plane $\xi_1 = 0$ and the n-1 dimensional sphere with center se_2 and radius s. Whenever misunderstandings are excluded we shall for brevity use the notation $\mathcal{M}$ in place of $\mathcal{M}(s)$.

We will apply Plancherel's theorem to $\hat{w}$, but, in order to do so, we first smooth the distribution $\hat{w}$ by introducing

$$\hat{w}_\varepsilon(\cdot) = \varepsilon^{-n} \widehat{\beta(|x|)} \left(\frac{\cdot}{\epsilon} \right) * \hat{w}(\cdot)$$

where

$$\beta(|x|) = \left(\frac{1}{2\pi} \right)^n \int_{\mathbf{R}^n} \chi(|\xi|^2) e^{-i\xi x} \, d\xi \quad \text{with}$$
$$\chi \in C^\infty([0, \infty)) \quad \text{and} \quad \text{supp } \chi \subset [0, 1] \ .$$

From these definitions it follows immediately that

$$\widehat{\beta(|x|)}(\xi) = \chi(|\xi|^2) \ .$$

We furthermore normalize β by the requirement that

$$\int_{\mathbf{R}^n} \widehat{\beta(|x|)}(\xi) \, d\xi = \int_{\mathbf{R}^n} \chi(|\xi|^2) \, d\xi = 1 \ .$$

It straightforward to see that

$$\lim_{\epsilon \downarrow 0} \hat{w}_\epsilon = \hat{w} \ ,$$

$$\text{supp } \hat{w}_\varepsilon \subset N_\varepsilon \left(\mathcal{M}(s) \right) = \{ \xi \mid \text{dist}(\xi, \mathcal{M}(s)) \leq \varepsilon \} \ ,$$

$$\left(\varepsilon^{-n} \widehat{\beta(|\cdot|)} \left(\frac{\xi}{\epsilon} \right) \right)^\vee (x) = \beta(\epsilon|x|) \ .$$

For any $\varphi \in \mathcal{S}(\mathbf{R}^n)$

$$\langle w, \varphi \rangle = \langle \hat{w}, \check{\varphi} \rangle$$
$$= \lim_{\epsilon \downarrow 0} \int_{\mathbf{R}^n} \hat{w}_\epsilon \check{\varphi} \, dx$$

so that

$$|\langle w, \varphi \rangle| \leq \overline{\lim_{\epsilon \downarrow 0}} \ \epsilon \left(\int_{N_\varepsilon} |\hat{w}_\epsilon|^2 d\xi \right)^{1/2} \cdot \left(\frac{1}{\varepsilon^2} \int_{N_\epsilon} |\check{\varphi}(\xi)|^2 d\xi \right)^{1/2}$$

As $\check{\varphi}$ is smooth and the (volume of N_ϵ) $/\epsilon^2$ converges to a constant times the surface area of $\mathcal{M}$,

$$|\langle w, \varphi \rangle| \leq C \left(\overline{\lim_{\epsilon \downarrow 0}} \; \epsilon \|\hat{w}_\epsilon\|_{L^2} \right) \left(\int_{\mathcal{M}(s)} |\check{\varphi}(\xi)|^2 dS(\xi) \right)^{1/2}.$$

Moreover,

$$\left(\frac{1}{2\pi} \right)^n \|\hat{w}_\epsilon\|_{L^2}^2 = \|w_\epsilon\|_{L^2}^2$$

$$= \int_{\mathbf{R}^n} |\beta(\epsilon|x|)|^2 \, |w(x)|^2 \, dx$$

$$\leq \sup(|\beta^2(\epsilon|x|)(1+|x|^2)^{-\delta}|) \cdot \|w\|_{L_\delta^2}^2.$$

As $\beta \in \mathcal{S}(\mathbf{R}^n)$, and $\delta < 0$, it therefore follows that

$$\|\hat{w}_\epsilon\|_{L^2}^2 \leq C \sup(|(1+\epsilon^2|x|^2)^\delta| \cdot |(1+|x|^2)^{-\delta}|) \cdot \|w\|_{L_\delta^2}^2$$

$$\leq C\epsilon^{2\delta} \|w\|_{L_\delta^2}^2.$$

Returning to 3

$$|\langle w, \varphi \rangle| \leq C \; \overline{\lim_{\epsilon \downarrow 0}} (\epsilon \cdot \epsilon^\delta) \|w\|_{L_\delta^2} \left(\int_{\mathcal{M}(s)} |\check{\varphi}(\xi)|^2 dS(\xi) \right)^{1/2}.$$

Since $\delta > -1$, it therefore follows that

$$\langle w, \varphi \rangle = 0$$

for every $\varphi \in \mathcal{S}$, so that $w = 0$.

We turn to prove existence of a solution to 3.1. Suppose for now that $f \in \mathcal{S}(\mathbf{R}^n)$ and define

$$\hat{w}(\xi) = \frac{\hat{f}(\xi)}{-|\xi|^2 + 2i\zeta \cdot \xi}.$$

We shall prove that w is well defined and satisfies the estimate

$$\|w\|_{L_\delta^2} \leq \frac{C}{|\zeta|} \|f\|_{L_{\delta+1}^2}.$$

Once this estimate is established we can dispense with the assumption that $f \in \mathcal{S}(\mathbf{R}^n)$ by continuity. As we did in the uniqueness proof, we may assume that

$$\zeta = s(e_1 + ie_2) \,, \quad s = \frac{|\zeta|}{\sqrt{2}} \,,$$

and therefore

$$-|\xi|^2 + 2i\zeta \cdot \xi = \xi_1^2 + (\xi_2 - s)^2 + \xi_3^2 \cdots + \xi_n^2 - s^2 + 2is\xi_1 = P(\xi, s).$$

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With this definition of the polynomial P it is easy to see that

$$P(\xi, s) = s^2 P(\xi/s, 1) \ .$$

As before we denote

$$N_r(\mathcal{M}(s)) = \{\xi \in \mathbf{R}^n | \ \text{dist} \ (\xi, \mathcal{M}(s)) \leq r\}$$

and we define an open cover of $\mathbf{R}^n$ by

$$\mathcal{O}_1(s) = \mathbf{R}^n \backslash N_{s/2n}(\mathcal{M}(s))$$

$$\mathcal{O}_2(s) = \{|\xi_2 - s| \geq s/2n\} \cap \overset{\circ}{N}_s(\mathcal{M}(s))$$

$$\mathcal{O}_j(s) = \{|\xi_j| \geq s/2n\} \cap \overset{\circ}{N}_s(\mathcal{M}(s)) \ \text{for} \ j > 2.$$

It is useful to note that $\mathcal{M}(s) = s\mathcal{M}(1)$ and that $\mathcal{O}_j(s) = s\mathcal{O}_j(1)$.
Let $\chi_j(\xi)$ be a partition of unity subordinate to this open cover, so that

$$\hat{w}(\xi) = \sum_{j=1}^{n} \hat{w}_j(\xi) = \sum_{j=1}^{n} \frac{\chi_j(\xi)\hat{f}(\xi)}{P(\xi, s)}$$

Since $\mathcal{O}_1(1)$ is disjoint from $\mathcal{M}(1)$ and since $P(\xi, 1) \to \infty$ as $|\xi| \to \infty$ there exists a constant c such that

$$P(\xi, 1) \geq c > 0 \ \ \forall \xi \in \mathcal{O}_1(1) \ .$$

For $\xi \in \mathcal{O}_1(s)$ this leads to the estimate

$$P(\xi, s) = s^2 P(\xi/s, 1) \geq cs^2$$

so that

$$\|w_1\|_{L^2_\delta} \leq \|w_1\|_{L^2} \leq \frac{1}{cs^2}\|f\|_{L^2} \leq \frac{1}{cs^2}\|f\|_{L^2_{\delta+1}} \ .$$

Here we are use the facts that $\delta < 0$ and $\delta + 1 > 0$. Since our hypothesis guarantees that $|\zeta| = \sqrt{2}s$ is greater than some $c > 0$, 3 gives the desired estimate for w_1.
To estimate each of the $w_j, j = 2, ..., n$ we first introduce new coordinates in $\mathcal{O}_j(s)$ by

$$\eta_1 = 2\xi_1$$

$$\eta_\ell = \xi_\ell \ \text{for} \ \ell \neq 1, \ j$$

$$\eta_j = \frac{\xi_1^2 + (\xi_2 - s)^2 + \xi_3^2 \cdots + \xi_n^2 - s^2}{s}$$

In terms of these new coordinates

$$\hat{w}_j(\eta) = \frac{\chi_j(\xi)\hat{f}(\xi)}{s(\eta_j + i\eta_1)} \ .$$

The Jacobian of this coordinate transformation on $\mathcal{O}_j(s)$ is easily calculated to be

$$\text{Det } \left(\frac{\partial \eta}{\partial \xi}\right) = \frac{4\xi_j}{s} \quad \text{for } j \neq 2 \ ,$$

and

$$\text{Det } \left(\frac{\partial \eta}{\partial \xi}\right) = \frac{4(\xi_2 - s)}{s} \quad \text{for } j = 2 \ .$$

These expressions are in all cases bounded from above and below on $\mathcal{O}_j(s)$, $j = 2, ..., n$ independently of s. At this point we shall make use of the following three results, the proofs of which will be given later.

Lemma 3.1. *The maps Z_j defined by*

$$(Z_j f)(\xi) = \left(\frac{\hat{f}}{\xi_j + i\xi_1}\right)^{\vee} \quad f \in \mathcal{S}(\mathbf{R}^n)$$

are bounded from $L^2_{\delta+1}$ to L^2_δ.

Lemma 3.2. *For any $\chi \in C_0^\infty(\mathbf{R}^n)$ and any $f \in \mathcal{S}(\mathbf{R}^n)$*

$$\|\left(\chi(\xi)\hat{f}(\xi)\right)^{\vee}\|_{L^2_{\delta+1}} \leq C\|f\|_{L^2_{\delta+1}} \ ,$$

where the constant C depends on χ, but is independent of f.

Lemma 3.3. *Let $\mathcal{O}$ and $\mathcal{O}'$ be open subsets of $\mathbf{R}^n$ Let $\hat{f}$ be in $C_0^\infty(\mathcal{O}')$ and let Ψ be a diffeomorphism from $\mathcal{O}$ to $\mathcal{O}'$ such that the Jacobians $D\Psi$ and $D\Psi^{-1}$ are bounded on $\mathcal{O}$ and $\mathcal{O}'$ respectively, then*

$$\|(\hat{f} \circ \Psi)^{\vee}\|_{L^2_{\delta+1}} \leq C\|f\|_{L^2_{\delta+1}} \ .$$

The constant C depends on Ψ, but is independent of f.

The proof of Proposition 3.1 now proceeds as follows. Let

$$g_j(x) = [(\chi_j \hat{f})(\xi(\eta))]^{\vee}(x) \ ,$$

then it follows immediately from the formula for the w_j that

$$w_j(x) = \frac{1}{s}\left[\frac{\chi_j \hat{f}(\xi(\eta))}{\eta_j + i\eta_1}\right]^{\vee}(x) = \frac{1}{s}\left[\frac{\hat{g}_j(\eta)}{\eta_j + i\eta_1}\right]^{\vee}(x) \ .$$

Using Lemma 3.1 we obtain that

(3.30) $$\|w_j\|_{L^2_\delta} \leq \frac{C}{s}\|g_j\|_{L^2_{\delta+1}} \ .$$

At the same time, if we define

$$h_j = \left[\chi_j \hat{f}\right]^{\vee}(x) \quad \text{and} \quad \Psi(\eta) = \xi(\eta) \ ,$$

then we get
$$g_j = \left[\chi_j \hat{f} \circ \Psi\right]^{\vee} = \left[\hat{h}_j \circ \Psi\right]^{\vee} \quad,$$

so that according to Lemma 3.2 and Lemma 3.3

(3.31)
$$\|g_j\|_{L^2_{\delta+1}} \le C\|h_j\|_{L^2_{\delta+1}} = C\|(\chi_j \hat{f})^{\vee}\|_{L^2_{\delta+1}}$$
$$\le C\|f\|_{L^2_{\delta+1}} \quad.$$

A combination of 3.30 and 3.31 gives estimate

$$\|w_j\|_{L^2_\delta} \le \frac{C}{s}\|f\|_{L^2_{\delta+1}} \qquad s = \frac{|\zeta|}{\sqrt{2}} \quad.$$

Invoking the formula $w = \sum_{j=1}^{n} w_j$ this completes the proof of Proposition 3.1. $\quad\square$

It still remains to prove the three auxiliary lemmas 3.1–3.3. If we note that $\|\hat{f}\|_{H^{\delta+1}} = \|f\|_{L^2_{\delta+1}}$ then lemmas 3.2 and 3.3 merely state that multiplication by smooth, compactly supported functions and composition with smooth diffeomorphisms are bounded operators on $H^s(\mathbf{R}^n)$; two facts that are well known. It thus only remains to give the

Proof of Lemma 3.1. The map

$$f \mapsto \left(\frac{\hat{f}}{\xi_j + i\xi_1}\right)^{\vee}$$

may also be written

$$f \mapsto f * \left(\frac{1}{\xi_j + i\xi_1}\right)^{\vee} \quad f \in \mathcal{S}(\mathbf{R}^n) \quad.$$

Furthermore it is well known that for an appropriate constant C

$$\left(\frac{1}{\xi_j + i\xi_1}\right)^{\vee} = \frac{C}{x_j + ix_1}\delta_0(\tilde{x}) \quad,$$

where $\tilde{x} = (x_2, \ldots x_{j-1}, x_{j+1}, \ldots x_n)$ and δ_0 denotes a Dirac delta function at the origin. Therefore Z_j is given by

$$f \mapsto f * \frac{C}{x_j + ix_1}\delta_0(\tilde{x}) \quad,$$

which is to say that Z_j is proportional to the solution operator for the inhomogeneous equation

$$(\partial_{x_1} - i\partial_{x_j})v = f \quad \text{in } \mathbf{R}^n.$$

To prove Lemma 3.1 it clearly suffices to consider a single value of the index j, for instance $j = 2$. We furthermore claim that it suffices to prove the estimate $\|Z_2 f\|_{L^2_\delta} \leq C\|f\|_{L^2_{\delta+1}}$ in $\mathbf{R}^2$. To see this we note that

$$\|u\|^2_{L^2_\delta(\mathbf{R}^n)} = \int_{\mathbf{R}^n} (1 + |x|^2)^\delta |u(x)|^2 \, dx$$
$$\leq \int_{\mathbf{R}^n} (1 + x_1^2 + x_2^2)^\delta |u(x)|^2 \, dx$$

since $\delta < 0$. Therefore

$$\|Z_2 f\|^2_{L^2_\delta(\mathbf{R}^n)} \leq \int dx_3 \ldots dx_n \left[\|Z_2 f(\cdot, \cdot, x_3, \ldots x_n)\|^2_{L^2_\delta(\mathbf{R}^2)} \right] \quad .$$

Here we use the fact that $(Z_2 f)(x_1, x_2, \ldots x_n) = [Z_2 f(\cdot, \tilde{x})](x_1, x_2)$, i.e., we use that $\tilde{x} = (x_3, \ldots x_n)$ may be treated as parameters untouched by Z_2. At the same time

$$\|f\|^2_{L^2_{1+\delta}(\mathbf{R}^n)} = \int_{\mathbf{R}^n} (1 + |x|^2)^{1+\delta} |f(x)|^2 \, dx$$
$$\geq \int_{\mathbf{R}^n} (1 + x_1^2 + x_2^2)^{1+\delta} |f(x)|^2 \, dx \quad ,$$

since $1 + \delta > 0$. Therefore

$$\|f\|^2_{L^2_{\delta+1}(\mathbf{R}^n)} \geq \int dx_3 \ldots dx_n \left[\|f(\cdot, \cdot, x_3, \ldots x_n)\|^2_{L^2_\delta(\mathbf{R}^2)} \right].$$

The estimates 3 and 3 immediately imply that it suffices to prove the estimate $\|Z_2 f\|_{L^2_\delta} \leq C\|f\|_{L^2_{\delta+1}}$ in two dimensions. This latter estimate follows from the following lemma with $p = 2$. $\qquad \square$

Lemma 3.4. *If Z is defined by*

$$Z f := \int_{\mathbf{R}^2} \frac{1}{(u_2 - v_2) + i(u_1 - v_1)} f(v_1, v_2) \, dv \qquad f \in \mathcal{S}(\mathbf{R}^2) \quad ,$$

then Zf is bounded from $L^p_{\delta+1}(\mathbf{R}^2)$ to $L^p_\delta(\mathbf{R}^2)$ provided $p > 1$ and $-2/p < \delta < 1 - 2/p$

Proof The space L^p_δ consists of the functions

$$\{u : \ (1 + |x|^2)^{\delta/2} u \in L^p\} \, ,$$

equipped with the norm $\|u\|_{L^p_\delta} = \|(1 + |x|^2)^{\delta/2} u\|_{L^p}$. It is well known that the spaces $L^q_{-\delta}$ and L^p_δ are dual, provided $1/p + 1/q = 1$. Due to this fact, it suffices to verify the estimate $|\langle g, Zf \rangle| \leq C\|f\|_{L^p_{\delta+1}} \|g\|_{L^q_{-\delta}}$ for any $g \in L^q_{-\delta}$. We have

$$|\langle g, Zf \rangle| = \left| \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \frac{g(u)f(v)}{(u_2 - v_2) + i(u_1 - v_1)} \, dudv \right|$$
$$\leq \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \frac{\left(|g(u)|(1 + |u|)^\beta (1 + |v|)^{-\alpha} \right) \cdot \left(|f(v)|(1 + |u|)^{-\beta} (1 + |v|)^\alpha \right)}{|u - v|} \, dudv \quad ,$$

where α and β will be chosen later. Employing Hölder's inequality,

$$|\langle g, Zf \rangle| \le \left(\int_{\mathbf{R}^2} \left\{ \int_{\mathbf{R}^2} \frac{(1+|u|)^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)}}{|u-v|} du \right\} (1+|v|)^{p(\delta+1)} |f(v)|^p dv \right)^{1/p}$$

$$\times \left(\int_{\mathbf{R}^2} \left\{ \int_{\mathbf{R}^2} \frac{(1+|u|)^{q(\beta+\delta)}(1+|v|)^{-q\alpha}}{|u-v|} dv \right\} (1+|u|)^{-q\delta} |g(u)|^q du \right)^{1/q}$$

$$\le C \|f\|_{L^p_{\delta+1}} \cdot \|g\|_{L^q_{-\delta}}$$

with the constant C given by

$$C = \left(\sup_v \int_{\mathbf{R}^2} \frac{(1+|u|)^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)}}{|u-v|} du \right)^{1/p}$$

$$\times \left(\sup_u \int_{\mathbf{R}^2} \frac{(1+|u|)^{q(\beta+\delta)}(1+|v|)^{-q\alpha}}{|u-v|} dv \right)^{1/q} .$$

In order to guarantee that C is finite, it suffices to require that

$$1/p < \beta < 2/p \quad \text{and} \quad 1/q < \alpha < 2/q$$

with

$$\delta = \alpha - \beta - 1/q .$$

On the other hand, if $p > 1$ and δ satisfies

$$-2/p < \delta < 1 - 2/p ,$$

then it is not difficult to check that it is always possible to select α and β such that 3 and 3 are satisfied. This completes the proof of Lemma 3.5 and consequently the proof of Lemma 3.1. $\qquad\square$

See the notes of Mikko Salo [34] for another construction, due to Peter Hähner, [11], of CGO solutions for the Schrödinger equation.

4 Applications of Complex Geometrical Optics Solutions

We give two applications of CGO solutions. One is to prove unique identifiability of a C^2 conductivity a result due to [40] and to determine cavities using the enclosure method [15].

4.1 Uniqueness for Calderón's Problem

For any $\zeta \in \mathbf{C}^3$ satisfying $\zeta \cdot \zeta := {}^t\zeta\zeta = 0$, $e^{x \cdot \zeta}$ satisfies $\Delta e^{x \cdot \zeta} = 0$. This harmonic function is called the complex plane wave solution. There are two important properties for the complex plane wave solution. The one is that the complex plane wave solution is exponentially decaying and growing for each side of the surface $x \cdot \mathrm{Re}\zeta = 0$ and oscillating on this surface as $|\zeta| \to \infty$. The other is that the linear combinations of their products of two complex plane wave solutions are complete in $L^2(\Omega)$. We will refer this second property by the completeness of product.

The CGO solutions is a generalization of complex plane wave solution for more general equation than the Laplace equation . It still has the two properties. For example, let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, then the CGO solutions u of the Schrödinger equation $(\Delta + q(x))u = 0$ in Ω with potential $q \in L^\infty(\Omega)$ has the form

$$(4.1) \qquad u = e^{x \cdot \zeta}(1 + O(|\zeta|^{-1})) \quad (|\zeta| \to \infty).$$

These CGO solutions u can be used to prove the uniqueness for identifying q from the Dirichlet to Neumann map Λ_q defined by

$$(4.2) \qquad \Lambda_q(f) := \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{1/2}(\partial\Omega),$$

where ν is the outer unit normal vector of $\partial\Omega$ and $u = u(f)$ is the solution to

$$(4.3) \qquad \begin{cases} (\Delta + q)u = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = f \in H^{3/2}(\Omega) \end{cases}$$

The outline of the proof is as follow. Let $q_j \in L^\infty(\Omega)$ and Λ_{q_j} be its Dirichlet to Neumann map. We have to prove that $\Lambda_{q_1} = \Lambda_{q_2}$ implies $q_1 = q_2$. By Green's formula, we have the identity:

$$(4.4) \qquad \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2}) f_1 f_2 d\sigma = \int_\Omega (q_1 - q_2) u_1(f_1) u_2(f_2) \, dx,$$

where $d\sigma$ is the surface element of $\partial\Omega$. For any $k \in \mathbf{R}^3 \setminus \{0\}$ and $r > 0$, let $\zeta^{(j)} = \zeta^{(j)}(k, r)$ $(j = 1, 2)$ satisfy

$$(4.5) \qquad \begin{cases} (\zeta^{(j)})^2 = 0, \quad |\zeta^{(j)}|^2 = 2r^2 + \frac{1}{2}|k|^2, \quad |\zeta^{(j)}| = O(r) \ (r \to \infty) \ (j = 1, 2) \\ \zeta^{(1)} + \zeta^{(2)} = ik \end{cases}$$

Substitute the CGO solutions

$$(4.6) \qquad u_j = e^{x \cdot \zeta^{(j)}}(1 + \psi_j(x, \zeta^{(j)})), \quad \|\psi^{(j)}\|_{L^2(\Omega)} = O(r^{-1}) \ (r \to \infty)$$

to the identity recalling that $\Lambda_{q_1} = \Lambda_{q_2}$. Then, we have

$$(4.7) \qquad \int_\Omega e^{ix \cdot k}(q_1 - q_2) \, dx = -\int_\Omega e^{ix \cdot k}(q_1 - q_2)(\psi_1 + \psi_2 + \psi_1\psi_2) \, dx.$$

The right-hand side of (4.7) is $O(r^{-1})$ $(r \to \infty)$ and hence

$$(4.8) \qquad \int_{\Omega} e^{ix \cdot k}(q_1 - q_2) \, dx = 0$$

which implies $q_1 = q_2$ in Ω. This completes the proof.

By the relation:

$$(4.9) \qquad u = \gamma^{-1/2} v, \quad q = -\frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}},$$

u is the solution of the conductivity equation:

$$(4.10) \qquad \nabla \cdot \gamma \nabla u = 0 \quad \text{in} \quad \Omega$$

if v is the solution to the Schrödinger equation with potential q. Here $\gamma > 0$ in Ω is the conductivity in the Sobolev space $W^{2,\infty}(\Omega)$. Hence, (4.10) has the CGO solution u of the form:

$$(4.11) \qquad u = e^{x \cdot \zeta} \gamma^{-1/2}(1 + O(|\zeta|^{-1})) \quad (|\zeta| \to \infty).$$

Also, we have the uniqueness result for identifying the conductivity γ from the Dirichlet to Neumann map Λ_γ defined by

$$(4.12) \qquad \Lambda_\gamma(f) := \gamma \frac{\partial u(f)}{\partial \nu}|_{\partial\Omega},$$

where $u = u(f) \in H^2(\Omega)$ is the solution of the conductivity equation satisfying the boundary condition $u|_{\partial\Omega} = f \in H^{2/3}(\Omega)$.

Further Developments There has been many other developments in EIT using CGO solutions. See the survey papers [45], [46] and the special volume [12]. For recent improvements on the regularity assumed in the unique identifiability of the conductivity from the DN map see [10] and [5]. A subsequent review paper applying CGO solutions to hybrid inverse problems is [21]. We also recommend the excellent notes of Mikko Salo [35].

4.2 Determining Cavities

So far we have only used the completeness of the product of the CGO solutions. Combining this property with the exponential decay and growth of the CGO solutions, we can give the reconstruction of a cavity D with strongly convex C^2 boundary ∂D inside a conductive medium Ω with conductivity 1 such that $\Omega \setminus \overline{D}$ is connected. This result is in [15] and its proof is as follows. Define the Dirichlet to Neumann map Λ_D by

$$(4.13) \qquad \Lambda_D(f) := \frac{\partial u(f)}{\partial \nu}|_{\partial\Omega},$$

where $u = u(f) \in H^2(\Omega)$ is the solution to

(4.14)
$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \frac{\partial u}{\partial \nu}|_{\partial D} = 0, \\ u|_{\partial \Omega} = f \in H^{3/2}(\Omega) \end{cases}$$

and ν is the unit normal of ∂D directed outside D. If $D = \emptyset$, we denote Λ_D by Λ_0. Let $\omega, \omega^\perp$ be unit 3 dimensional real vectors perpendicular to each other. For $\tau > 0$, consider the CGO solutions $v = v(x, \tau, \omega, \omega^\perp)$ of the Laplacian in Ω :

(4.15)
$$v(x, \tau, \omega, \omega^\perp) = e^{\tau x \cdot (\omega + i\omega\perp)}.$$

For $t \in \mathbf{R}$, define the indicator function $I_{\omega,\omega^\perp}(\tau, t)$ by

(4.16)
$$I_{\omega,\omega^\perp}(\tau, t) := e^{-2\tau t} \int_{\partial \Omega} ((\Lambda_D - \Lambda_0)v|_{\partial \Omega})\overline{v|_{\partial \Omega}} \, d\sigma.$$

Define the support function $h_D(\omega)$ of D by

(4.17)
$$h_D(\omega) := \sup_{x \in D} x \cdot \omega.$$

Then, we can characterize the support function in terms of the indicator function. That is we have

(4.18)
$$(h_D(\omega), \infty) = \{t \in \mathbf{R} \; ; \; \lim_{\tau \to \infty} I_{\omega,\omega^\perp}(\tau, t) = 0\}.$$

Hence, by taking many ω, we can recover the shape of D. The outline of characterizing the support function is as follow. Let w be the reflected solution of $e^{-\tau t}v$ which is defined as the solution to

(4.19)
$$\begin{cases} \Delta w = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \frac{\partial w}{\partial \nu}|_{\partial D} = -\frac{\partial e^{-\tau t}v}{\partial \nu}|_{\partial D}, \\ w|_{\partial \Omega} = 0. \end{cases}$$

Then, by integration by parts, we have

(4.20)
$$-I_{\omega,\omega^\perp}(\tau, t) = \int_{\Omega \setminus \overline{D}} |\nabla w|^2 \, dx + \int_D |\nabla(e^{-\tau t}v)|^2 \, dx.$$

By reminding the well known inequality:

(4.21)
$$\|w\|_{H^1(\Omega \setminus \overline{D})} \le C \|e^{-\tau t}v\|_{H^1(D)}$$

for some constant $C > 0$ and analyzing the asymptotic behaviour of $\int_D |\nabla(e^{-\tau t}v)|^2 \, dx$ as $\tau \to \infty$, we can prove (4.18).

Further Developments The method indicated above, called enclosure method, has been applied to determine cavities , inclusions and other defects in several other situations. See the review paper [47] and the references given there. A recent development is the paper [9] where inclusions within inclusions and the surrounding medium can be determined using CGO solutions.

5 Complex Geometrical Optics Solutions for First Order Perturbations of the Laplacian

Several partial differential equations arising in applications can be transformed to $\Delta + \text{l.o.t}$ and $\Delta I + \text{l.o.t.}$, where I is the identity matrix and "l.o.t" denotes lower order terms. We mention the magnetic Schrödinger equation, the linear elasticity system and the Dirac system among many. It is interesting and important to find a systematic way of constructing CGO solutions and this is what we do in this section.

Let

$$(5.1) \qquad M = \Delta I + N(x, D),$$

where $N(x, D)$ is a $\ell \times \ell$ system of differential operator of the first order whose coefficients are C^∞ and compactly supported in $\mathbf{R}^3$. Of course when $\ell = 1$, we have the differential operator $\Delta + \text{l.o.t.}$ By conjugating M with $e^{x \cdot \zeta}$, we have

$$(5.2) \qquad M_\zeta \cdot = (\Delta_\zeta + N_\zeta) \cdot$$

with

$$(5.3) \qquad \Delta_\zeta = \Delta + 2\zeta \cdot \nabla$$

and a system of differential operator N_ζ of the first order.

The idea to construct solutions of 5.2 is to find invertible operators A_ζ, B_ζ so that

$$(5.4) \qquad M_\zeta A_\zeta = B_\zeta(\Delta_z eta + C_\zeta)$$

where C_ζ is an operator of order zero in an appropriate sense. This operator plays the same role as a potential term in section 3. One can then, using the same approach as in section 3, construct solutions v_ζ of $\Delta_\zeta + C_\zeta$. The CGO solutions of M will then be $u_\zeta = A_\zeta v_\zeta$.

Equation 5.4 is referred to as the intertwining property. The remaining of this section is devoted to the construction of A_ζ, B_ζ. The material in this section is in [32].

5.1 Intertwining property (part 1)

The operators A_ζ, B_ζ will be constructed in a class of pseudodifferential operators depending on a parameter studied, for instance, in Shubin's book [38] and formulate precisely the intertwining property for M_ζ defined by (5.2).

Here we quote here the definitions and theorems from Chapter 2, section 9 in [38].

Definition 5.1. Let $l \in \mathbf{R}$ and

$$(5.5) \qquad Z := \{\zeta \in \mathbf{C}^3; \ |\zeta| \geq 1, \ \zeta \cdot \zeta = {}^t\zeta\zeta = 0\}.$$

(i) Let $V \subset \mathbf{R}^m$ ($m = 3$ or 6) be an open set. For a matrix $a_\zeta = a_\zeta(x, \xi)$, $a_\zeta \in S^l(V, Z)$ if and only if the following (i–1), (i–2) are satisfied.

(i–1) For any fixed $\zeta \in Z$, $a_\zeta(x, \xi) \in C^\infty(V \times \mathbf{R}^3)$.

(i–2) For any $\alpha \in \mathbf{Z}^3_+$, $\beta \in \mathbf{Z}^m_+ := \{\beta = (\beta_1, \dots, \beta_m); \ \beta_j \in \mathbf{Z}_+ \ (1 \leq j \leq m)\}$ and compact set $K \subset V$, there exist a constant $C_{\alpha, \beta, K} > 0$ such that the estimate

$$(5.6) \quad |\partial_\xi^\alpha \partial_x^\beta a_\zeta(x, \xi)| \leq C_{\alpha, \beta, K}(|\xi| + |\zeta|)^{l - |\alpha|} \quad (x \in K, \ \xi \in \mathbf{R}^3, \ \zeta \in Z)$$

holds.

(ii) Define $S^{-\infty}(V, Z) := \bigcap_{l \in \mathbf{R}} S^l(V, Z)$.

(iii) Let $U \subset \mathbf{R}^3$ be an open set and $a_\zeta(x, y, \xi) \in S^l(U \times U, Z)$. Then, define $A_\zeta = O_p(a_\zeta)$ by

$$(5.7) \quad A_\zeta f(x) = (2\pi)^{-3} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} e^{i(x-y) \cdot \xi} a_\zeta(x, y, \xi) f(y) dy d\xi$$

for any $f \in C_0^\infty(U)$. Here, the integral is the oscillatory integral and i denotes the unit of the pure imaginary number if there isn't any confusion. Furthermore, define

$$L^l(U, Z) := \{O_p(a_\zeta); \ a_\zeta \in S^l(U \times U, Z)\}.$$

(iv) Define $L^{-\infty}(U, Z) := \cap_{l \in \mathbf{R}} L^l(U, Z)$.

(v) Denote the Schwartz kernel of $A_\zeta \in L^l(U, Z)$ by K_ζ. We call A_ζ is properly supported if there exists a proper closed set $H \subset U \times U$ such that $\operatorname{supp} K_{A_\zeta} \subset H$ for any $\zeta \in Z$. Here H is proper if

$$(\pi_j)^{-1}(K) \cap H \subset \mathbf{R}^6 \quad (j = 1, 2)$$

is relatively compact for any compact set $K \subset U$, where $\pi_1(x, y) = x$, $\pi_2(x, y) = y$ $((x, y) \in U \times U)$.

(vi) For a properly supported $A_\zeta = O_p(a_\zeta) \in L^l(U, Z)$, define

$$\widetilde{\sigma}(A_\zeta)(x, \xi) := e^{-ix \cdot \xi} A_\zeta(e^{ix \cdot \xi})$$

and call it the total symbol of A_ζ.

Theorem 5.2.

(i) The formula for the full symbols of the composition, adjoint etc. of pseudodifferential operators are the same as those of the usual classical pseudodifferential operators without parameter.

(ii) For any $A_\zeta \in L^l(U, Z)$, there exist a properly supported $A'_\zeta \in L^l(U, Z)$ such that $A'_\zeta - A_\zeta \in L^{-\infty}(U, Z)$.

(iii) For any properly supported $A_\zeta = O_p(a_\zeta) \in L^l(U, Z)$, we have

(5.8)
$$\widetilde{\sigma}(A_\zeta)(x, \xi) - \sum_{|\alpha| \leq N-1} (\alpha!)^{-1} \partial_\xi^\alpha D_y^\alpha a_\zeta(x, y, \xi)|_{y=x} \in S^{l-N}(U, Z) \quad (N \in \mathbf{N}).$$

We denote (5.8) by

(5.9)
$$\widetilde{\sigma}(A_\zeta)(x, \xi) \sim \sum (\alpha!)^{-1} \partial_\xi^\alpha D_y^\alpha a_\zeta(x, y, \xi)|_{y=x}.$$

(iv) Let $A_\zeta \in L^l(\mathbf{R}^3, Z)$, $r \geq l$. Assume that there exist a compact set $K \subset \mathbf{R}^6$ such that $\mathrm{supp}\, K_{A_\zeta} \subset K$ for any $\zeta \in Z$. Then, for any $k \in \mathbf{R}$, there exists a constant $C_{k,r} \geq 0$ depending on k, r such that

(5.10)
$$\|A_\zeta\|_{k,k-r} \leq C_{k,r} \begin{cases} |\zeta|^l & (r \geq 0) \\ |\zeta|^{l-r} & (r \leq 0) \end{cases} \quad (\zeta \in Z),$$

where $\|A_\zeta\|_{k,k-r}$ is the operator norm of $A_\zeta : H^k(\mathbf{R}^3) \to H^{k-r}(\mathbf{R}^3)$.

(v) For any partial differential operator $P(x, D)$ of order l with $C^\infty(U)$ coefficients,

$$P_\zeta(x, D) \cdot := e^{-x \cdot \zeta} P(x, D)(e^{x \cdot \zeta} \cdot)$$

is properly supported and $P_\zeta(x, D) \in L^l(U, Z)$, where $D = D_x$.

(vi) For $\chi \in B^\infty(\mathbf{R}^3) := \{x \in C^\infty(\mathbf{R}^3); \ \forall \alpha \in \mathbf{Z}_+^3, \ \sup_{x \in \mathbf{R}^3} |D^\alpha \chi(x)| \leq \infty\}$ satisfying $\nabla \chi \in C_0^\infty(\mathbf{R}^3)$, we have $\chi(|\zeta|^{-1}\xi) \in S^0(\mathbf{R}^3 \times \mathbf{R}^3, Z)$.

Proof. The proofs are given in Chapter 2, Section 9 in [38] except (v), (vi). (v) is clear if we note $P_\zeta(x, \xi) = P(x, \xi - i\zeta)$.

As for (vi), we only have to note

$$\partial_{\xi_j} \chi(|\zeta|^{-1}\xi) = (\partial_{x_j} \chi)(|\zeta|^{-1}\xi)|\zeta|^{-1}$$

and

$$|\zeta|^{-1} \leq (1 + r^2)^{\frac{1}{2}} (|\xi|^2 + |\zeta|^2)^{-\frac{1}{2}} \quad (|\xi| \leq r|\zeta|).$$

Here we have assumed $\mathrm{supp}\, \nabla \chi \subset \{|x| \leq r\}$. $\qquad\square$

Definition 5.3.

(i) Let $\chi \in C_0^\infty(\mathbf{R}^3)$, $\chi(x) = 1(|x| \ll 1)$. Then, for $l \in \mathbf{R}$, define

(5.11)
$$\lambda_\zeta^l(x, y, \xi) := \chi(x - y)(|\xi|^2 + |\zeta|^2)^{\frac{l}{2}} \in S^l(\mathbf{R}^3 \times \mathbf{R}^3, Z)$$

(5.12)
$$\Lambda_\zeta^l := O_p(\lambda_\zeta^l) \in L^l(\mathbf{R}^3, Z).$$

(ii) Let a properly supported $A_\zeta \in L^l(U, Z)$ admit an asymptotic expansion :

$$\widetilde{\sigma}(A_\zeta)(x,\xi) \sim \sum_{j=0}^{\infty} a_{l-j}(x,\xi,\zeta),$$

where for each $\in \mathbf{Z}_+$,

$$\begin{cases} a_{l-j}(x,\xi,\zeta) \in S^{l-j}(U,Z) \\ a_{l-j}(x,t\xi,t\zeta) = t^{l-j} a_{l-j}(x,\xi,\zeta) \end{cases} \quad (t > 0, \ \zeta \in Z, \ \xi \in \mathbf{R}^3).$$

Then we call such an A_ζ classical pseudodifferential operator and $\sigma(A_\zeta)(x,\xi) := a_0(x,\xi,\zeta)$ is called the principal symbol of A_ζ.

(iii) We call a properly supported pseudodifferential operator $A_\zeta \in L^l(U, Z)$ elliptic if

$$\det\sigma(A_\zeta)(x,\xi) \neq 0 \qquad (x \in U, \ \xi \in \mathbf{R}^3, \ \zeta \in Z)$$

holds.

Theorem 5.4. Let $A_\zeta \in L^l(U, Z)$ be a properly supported pseudodifferential operator with principal symbol $a_\zeta(x,\xi)$.

(i) If A_ζ is elliptic, then there exist properly supported elliptic classical pseudodifferential operators B_ζ, $C_\zeta \in L^{-\infty}(U, Z)$ such that

$$A_\zeta B_\zeta - I, \ C_\zeta A_\zeta - I \in L^{-\infty}(U, Z).$$

B_ζ and C_ζ are called the right and left parametrix of A_ζ, respectively.

(ii) Let $A_\zeta \in L^m(\mathbf{R}^3, Z)$ be a classical properly supported with the principal symbol $a_\zeta^{(m)}(x,\xi)$. Let $a_\zeta^{(m-1)}(x,\xi)$ be the $m - 1$ th order symbol of A_ζ, $\phi_1, \phi_2 \in C_0^\infty(\mathbf{R}^3)$ and $f \in C^\infty(\mathbf{R}^3)$. Then, we have the followings.

(ii-1) $\{|\zeta|^{-m+1}\phi_1(A_\zeta - a_\zeta^{(m)}(x,0))\phi_2 f\}_{\zeta \in Z}$ is bounded in $B^\infty(\mathbf{R}^3) := \{f \in C^\infty(\mathbf{R}^3) \ ; \ f$ and its derivatives are bounded$\}$.

(ii-2)

$$\{|\zeta|^{-m+2}\phi_1(A_\zeta - a_\zeta^{(m)}(x,0) - a_\zeta^{(m-1)}(x,0) - (\partial_\xi a_\zeta^{(m)}(x,0) \cdot D_x)\phi_2 f\}_{\zeta \in Z}$$

is bounded in $B^\infty(\mathbf{R}^3)$.

Proof. (i) is proven in Chapter 2, Section 9 in [38]. We first prove (ii-1).

$$\begin{aligned} \widetilde{\sigma}(A_\zeta)(x,\xi) - a_\zeta^{(m)}(x,0) \ &\equiv a_\zeta^{(m)}(x,\xi) - a_\zeta^{(m)}(x,0) \\ &= \int_0^1 (\partial_\xi a_\zeta^{(m)})(x,\theta\xi) \ d\theta \cdot \xi \quad \mod \quad S^{(m-1)}(\mathbf{R}^3, Z) \end{aligned}$$

Put

$$b_\zeta^{(m-1)}(x,\xi) := \int_0^1 (\partial_\xi a_\zeta^{(m)}(x,\theta\xi) \ d\theta.$$

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$$|\zeta|^{-m+1}(\phi_1(A_\zeta - a_\zeta(x,0))\phi_2 f)(x) = |\zeta|^{-m+1} \int e^{ix\cdot\xi}\phi_1(x)b_\zeta^{(m-1)}(x,\xi)\widehat{\phi_2 f}(\xi)\tilde{d}\xi,$$

$$||\zeta|^{-m+1}\phi_1(x)b_\zeta^{(m-1)}\hat{\phi_2 f}(\xi)| \le C < \xi >^{-4} \begin{cases} |\zeta|^{-m+1}(|\zeta| + |\xi|)^{m-1} < \xi >^{-m+1} & (m \ge 1) \\ |\zeta|^{-m+1}|\zeta|^{m-1} & (m < 1), \end{cases}$$

where C depends on ϕ_2, f and $\widetilde{d\xi} := (2\pi)^{-3} d\xi$. For the "mod term" ($=r_\zeta \in S^{m-1}(\mathbf{R}^3, Z)$), use the estimate

$$\|\phi_1 \mathrm{Op}(r_\zeta)(\phi_2 f)\|_{k,k} \le C_k|\zeta|^{m-1}\|\phi_2 f\|_k.$$

Similar estimate hold for the derivatives. Hence we have proven (i).

Next we prove (ii).

$$\tilde{\sigma}(A_\zeta)(x,\xi) - a_\zeta^{(m)}(x,0) - a_\zeta^{(m-1)}(x,0) - (\partial_\xi a_\zeta^{(m)})(x,0) \cdot \xi$$
$$\equiv b_\zeta^{(m-2)}(x,\xi) \cdot \xi + \sum_{j,k=1}^3 b_{j,k,\zeta}(x,\xi)\xi_j\xi_k \quad \mathrm{mod} \quad S^{m-2}(\mathbf{R}^3, Z),$$

where

$$b_\zeta^{m-2}(x,\xi) := \int_0^1 (\partial_\xi a_\zeta^{m-1})(x,\theta\xi) \, d\theta,$$

$$b_{j,k,\zeta}^{(m-2)}(x,\xi) := \int_0^1 (1 - \theta)(\partial_{\xi_j}\partial_{\xi_k} a_\zeta^{(m)})(\xi,\theta\xi) \, d\theta.$$

Hence, likewise (i) before, we have (ii).

$\square$

From now on we only consider properly supported classical pseudodifferential operators.

Now, for $\zeta \in Z$, we define

$$(5.13) \qquad\qquad M_\zeta \cdot := e^{-x\cdot\zeta}M(e^{x\cdot\zeta}\cdot) \in L^2(\mathbf{R}^2, Z)$$
$$(5.14) \qquad\qquad \Delta_\zeta \cdot := e^{-x\cdot\zeta}\Delta(e^{x\cdot\zeta}\cdot) \in L^2(\mathbf{R}^2, Z).$$

We also define

$$(5.15) \qquad\qquad q_\zeta(\xi) := \sigma(\Delta_\zeta) = -|\xi|^2 + 2i\zeta \cdot \xi.$$

Then, we have the following.

Lemma 5.1.

(i) By putting $q_\zeta^{-1}(0) := \{\xi \in \mathbf{R}^3; \; q_\zeta(\xi) = 0\}$ we have

$$(5.16) \qquad q_\zeta^{-1}(0) = \{\xi \in \mathbf{R}^3; \; (\Re\zeta) \cdot \xi = 0, \; |\xi + \Im\zeta|^2 = |\Im\zeta|^2\},$$

where $\Re\zeta_j, \; \Im\zeta_j \; (1 \le j \le 3)$ are the real and imaginary parts of ζ_j, and $\Re\zeta = (\Re\zeta_1, \Re\zeta_2, \Re\zeta_3), \Im\zeta = (\Im\zeta_1, \Im\zeta_2, \Im\zeta_3)$ for $\zeta = (\zeta_1, \zeta_2, \zeta_3)$.

(ii) $\nabla_\xi \Re q_\zeta$, $\nabla_\xi \Im q_\zeta$ are linearly independent on $q_\zeta^{-1}(0)$, where $\nabla_\xi \Re q_\zeta := {}^t(\partial_{\xi_1} \Re q_\zeta, \partial_{\xi_2} \Re q_\zeta, \partial_{\xi_3} \Re q_\zeta)$ etc.

Proof. Since (i) is easy, we only prove (ii).

It is enough to show

(5.17) $$\nabla_\xi \Re(-|\xi|^2 + 2i\zeta \cdot \xi) = -2(\xi + \Im\zeta)$$
(5.18) $$\nabla_\xi \Im(-|\xi|^2 + 2i\zeta \cdot \xi) = 2\Re\zeta$$

are linearly independent on $q_\zeta^{-1}(0)$.

Let $\alpha, \beta \in \mathbf{R}$ satisfy

(5.19) $$\alpha(\xi + \Im\zeta) + \beta \, \Re\zeta = 0 \qquad \text{on} \qquad q_\zeta^{-1}(0).$$

First note that we have

(5.20) $$|\Re\zeta| = |\Im\zeta| > 0, \quad \Re\zeta \cdot \Im\zeta = 0$$

from $\zeta \in Z$. Then, using this and taking the inner product of the both hand sides of (5.19) with $\Re\zeta$, we have $\beta = 0$. Hence,

(5.21) $$\alpha(\xi + \Im\zeta) = 0 \qquad \text{on} \qquad q_\zeta^{-1}(0).$$

Next using
$$|\xi|^2 + 2(\Im\zeta) \cdot \xi = 0 \qquad \text{on} \qquad q_\zeta^{-1}(0)$$
and taking the inner product of the both hand sides of (5.21) with ξ, we have

(5.22) $$\alpha(\Im\zeta) \cdot \xi = 0 \qquad \text{on} \qquad q_\zeta^{-1}(0).$$

On the other hand, we have
$$\alpha|\Im\zeta|^2 + \alpha(\Im\zeta) \cdot \xi = 0 \qquad \text{on} \qquad q_\zeta^{-1}(0)$$
by taking the inner product of the both hand sides of (5.21) with $\Im\zeta$.

Hence we have $\alpha = 0$ from (5.20), (5.22). □

Remark 5.5. Note that Δ_ζ is by no means an elliptic pseudodifferential operator in terms of the definition of ellipticity given in Definition 5.3. By Lemma 5.1, it is like $\bar{\partial}$ near $q_\zeta^{-1}(0)$.

Next we formulate the intertwining property as a theorem.

Theorem 5.6 (intertwining property). For any $N \in \mathbf{N}$, there exist elliptic pseudodifferential operators $A_\zeta, B_\zeta \in L^0(\mathbf{R}^3, Z)$ such that the followings hold. That is, for any $\phi_1 \in C_0^\infty(\mathbf{R}^3)$, there exist $\phi_j \in C_0^\infty(\mathbf{R}^3)$ $(1 \leq j \leq 3)$ and $r > 0$ such that for any $\zeta \in Z$, $|\zeta| \geq r$, we have

(5.23) $$\phi_1 M_\zeta A_\zeta = \phi_1 B_\zeta \phi_2 (\Delta_\zeta I + \phi_3 R_\zeta^{(-N)} \phi_4),$$

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where for each $s \in \mathbf{R}$, $R_\zeta^{(-N)} : H^s(\mathbf{R}^3) \to H^{s+N}(\mathbf{R}^3)$ is an bounded linear operator whose operator norm has the estimate :

(5.24)
$$\|R_\zeta^{(-N)}\|_{s,s+N} \le C_s |\zeta|^{-N}$$

with some constant $C_s \ge 0$ depending on s.

Since the proof of Theorem 5.6 is very long, we postpone its details to the succeeding sections. We only explain here the outline of the proof of Theorem 5.6.

Neglecting the auxiliary functions ϕ_j $(1 \le j \le 4)$, (5.23) means

(5.25)
$$M_\zeta A_\zeta = B_\zeta \Delta_\zeta$$

modulo an element in $L^{(-N-1)}(\mathbf{R}^3, Z)$.

So we construct A_ζ, B_ζ in the following three steps.

Step 1   Construct A_ζ in a neighborhood of $q_\zeta^{-1}(0)$ such that $A_\zeta = B_\zeta$ is elliptic.

Step 2   Extend $\sigma(A_\zeta)$ constructed in Step 1 smoothly to $\{\xi \in \mathbf{R}^3\}$ from the neighborhood of $q_\zeta^{-1}(0)$ without destroying its ellipticity.

Step 3   Define $\widetilde{\sigma}(A_\zeta)$, $\sigma(B_\zeta)$ be equal to $\sigma(A_\zeta)$ in $\{\xi \in \mathbf{R}^3\} - q_\zeta^{-1}(0)$ constructed in Step 2 so that A_ζ, B_ζ are elliptic. Then, construct $\widetilde{\sigma}(B_\zeta) - \sigma(B_\zeta)$.

In each step, the neighborhood $q_\zeta^{-1}(0)$ and $\sigma(A_\zeta)(x, \xi)$, $\sigma(B_\zeta)(x, \xi)$ are chosen to be invariant under the scaling by $t > 0 : (\xi, \zeta) \mapsto (t\xi, t\zeta)$.

Remark 5.7. Since Δ_ζ is not elliptic on $q_\zeta^{-1}(0)$, the construction of the intertwining operator $A_\zeta = B_\zeta$ in Step 1 is the most important part throughout this lecture note.

5.2   Intertwining property (part 2)

In this subsection we give the proof of Theorem 5.6 formulated in the last section according to the the procedure given as Step 1–3 in the last subsection.

Before going into the construction of A_ζ, B_ζ, we note the following. By Lemma 5.1, taking $\varepsilon > 0$ small enough, $\nabla_\xi \Re q_\zeta$, $\nabla_\xi \Im q_\zeta$ are linearly independent on
(5.26)
$$N_{3\varepsilon|\zeta|}(q_\zeta^{-1}(0)) := \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3; \ |(\mathrm{Re}\,\zeta) \cdot \xi| < 3\varepsilon s^2, \ |\|\xi + \mathrm{Im}\,\zeta| - s| < 3\varepsilon s\},$$

where $|\mathrm{Re}\,\zeta| = |\mathrm{Im}\,\zeta| = s$. So define

(5.27)
$$U_{\zeta,2} := N_{2\varepsilon|\zeta|}(q_\zeta^{-1}(0)), \qquad U_{\zeta,1} := \mathbf{R}^3 - \overline{N_{\epsilon|\zeta|(q_\zeta^{-1}(0))}}$$

and we look for A_ζ, B_ζ in the form

(5.28)
$$A_\zeta = \sum_{j=1}^2 A_{\zeta,j} \chi_{\zeta,j}, \qquad B_\zeta = \sum_{j=1}^2 B_{\zeta,j} \chi_{\zeta,j},$$

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where $\chi_{\zeta,j} \in L^0(\mathbf{R}^3, Z)$ are the partition of unity subordinated to the open covering $U_{\zeta,j}$ $(j = 1, 2)$ of $\mathbf{R}^3$ and $A_{\zeta,j}, B_{\zeta,j} \in L^0(\mathbf{R}^3, Z)$.

Step 1 Let $N' \in \mathbf{N}$ and we look for $A_{\zeta,2}$, $B_{\zeta,2}$ in $\mathbf{R}^3 \times U_{\zeta,2}$ in the form :

$$(5.29) \qquad A_{\zeta,2} = B_{\zeta,2} = \sum_{k=0}^{N'-1} A_{\zeta,2}^{(-k)}.$$

Here we let the full symbol $(A_{\zeta,2}^{(-k)})$ $(0 \le k \le N'-1)$ of $A_{\zeta,2}^{(-k)} \in L^{-k}(\mathbf{R}^3, Z)$ satisfy

$$(5.30) \qquad (A_{\zeta,2}^{(-k)}) = \sigma(A_{\zeta,2}^{(-k)})$$

$$(5.31) \qquad H_{q_\zeta}(A_{\zeta,2}^{(0)}) + N_\zeta^{(1)}(A_{\zeta,2}^{(0)}) = 0 \quad \text{in} \quad \mathbf{R}^3 \times U_{\zeta,2}$$

$$(5.32) \qquad H_{q_\zeta}(A_{\zeta,2}^{(-k)}) + N_\zeta^{(1)}(A_{\zeta,2}^{(-k)}) + \sigma(J^{(-k)}) = 0 \quad \text{in} \quad \mathbf{R}^3 \times U_{\zeta,2}$$
$$(1 \le k \le N'-1),$$

where

$$(5.33) \qquad H_{q_\zeta} \quad := \quad -i\nabla_\xi q_\zeta \cdot \nabla_x$$

$$(5.34) \qquad J^{(-k)} \quad := \quad M_\zeta(x, D)(\sum_{l=0}^{k-1} A_{\zeta,2}^{(-l)}(x, D)) - (\sum_{l=0}^{k-1} A_{\zeta,2}^{(-l)}(x, D))\Delta_\zeta I.$$

The existence of $(A_{\zeta,2}^{(-k)}) \in S^{-k}(\mathbf{R}^3, Z)$ $(0 \le k \le N'-1)$ in $\mathbf{R}^3 \times U_{\zeta,2}$ is guaranteed by the following 2 lemmas.

Lemma 5.2. There exist a solution $(A_{\zeta,2}^{(0)}) \in S^0(\mathbf{R}^3, Z)$ to (5.31) in $\mathbf{R}^3 \times U_{\zeta,2}$ and it is elliptic there. That is

$$(5.35) \qquad \det(A_{\zeta,2}^{(0)})(x, \xi) \ne 0 \qquad ((x, \xi) \in \mathbf{R}^3 \times U_{\zeta,2}, \quad \zeta \in Z).$$

Furthermore, if $\sigma(J^{(-k)}) \in S^{-k+1}(\mathbf{R}^3, Z)$ $(1 \le k \le N'-1)$ in $\mathbf{R}^3 \times U_{\zeta,2}$, there exist solutions $(A_{\zeta,2}^{(-k)}) \in S^{-k}(\mathbf{R}^3, Z)$ $(1 \le k \le N'-1)$ to (5.32) in $\mathbf{R}^3 \times U_{\zeta,2}$.

The proof will be given later in the next two sections together with that of Lemma 5.4 which will be given later.

Lemma 5.3.

$$(5.36) \qquad J^{(-N')} \in L^{-N'+1}(\mathbf{R}^3, Z)$$

holds in $\mathbf{R}^3 \times U_{\zeta,2}$.

Proof. We use the induction argument on N'.

First we prove (5.36) for $N' = 1$. By (5.34),

$$J^{(-1)} \quad = \quad (\Delta_\zeta I + N_\zeta)A_{\zeta,2}^{(0)} - A_{\zeta,2}^{(0)}(\Delta_\zeta I)$$
$$= \quad [\Delta_\zeta I, A_{\zeta,2}^{(0)}] + N_\zeta A_{\zeta,2}^{(0)},$$

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where $[\Delta_\zeta I, A_{\zeta,2}^{(0)}] := (\Delta_\zeta I)A_{\zeta,2}^{(0)} - A_{\zeta,2}^{(0)}(\Delta_\zeta I)$. Hence, by (5.31), we have

$$\sigma(J^{(-1)}) = H_{q_\zeta}(A_{\zeta,2}^{(0)}) + N_\zeta^{(0)}(A_{\zeta,2}^{(0)}) = 0 \quad \text{in} \quad \mathbf{R}^3 \times U_{\zeta,2}.$$

This proves (5.36) for the case $N' = 1$.

Next we prove (5.36) for the case N' assuming that it holds up to $N' - 1$. By (5.34), we have

$$
\begin{aligned}
J^{(-N')} &= (\Delta_\zeta I + N_\zeta)\Big(\sum_{l=0}^{N'-2} A_{\zeta,2}^{(-l)} + A_{\zeta,2}^{(-N'+1)}\Big) - \Big(\sum_{l=0}^{N'-2} A_{\zeta,2}^{(-l)} + A_{\zeta,2}^{(-N'+1)}\Big)\Delta_\zeta I \\
&= J^{(-N'+1)} + [\{(\Delta_\zeta I)A_{\zeta,2}^{(-N'+1)} - A_{\zeta,2}^{(-N'+1)}(\Delta_\zeta I)\} + N_\zeta A_{\zeta,2}^{(-N'+1)}].
\end{aligned}
$$

Hence, by (5.32), we have

$$\sigma(J^{(-N')}) = H_{q_\zeta}(A_{\zeta,2}^{(-N'+1)}) + N_\zeta^{(1)}(A_{\zeta,2}^{(-N'+1)}) + \sigma(J^{(-N'+1)}) = 0 \quad \text{in} \quad \mathbf{R}^3 \times U_{\zeta,2}.$$

This proves (5.36) for the case N'. $\hfill\square$

Step 2 Let $\psi_1 \in C_0^\infty((-3\varepsilon, 3\varepsilon))$, $\psi_2 \in C_0^\infty(D(s^{-1}\mathrm{Im}\zeta, 3\varepsilon))$ satisfy

(5.37)
$$
\begin{cases}
\psi_1(\tau_1) = 1 & (|\tau_1| < 2\varepsilon) \\
\psi_2(\tau) = 1 & (\tau \in D(s^{-1}\mathrm{Im}\zeta, 2\varepsilon)),
\end{cases}
$$

where

(5.38) $\qquad D(s^{-1}\mathrm{Im}\zeta, r) := \{\tau \in \mathbf{R}^3; \ \tau \cdot \mathrm{Re}\zeta = 0, \ ||\tau + s^{-1}\mathrm{Im}\zeta| - 1| < r\}$

for $r > 0$.

Now, consider an equation which interpolates (5.31) and $H_{q_\zeta}(A_{\zeta,2}^{(0)}) = 0$:

(5.39)
$$H_{q_\zeta}(A_{\zeta,2}^{(0)}) + \psi_1(s^{-1}(\mathrm{Re}\zeta) \cdot \xi)\psi_2(s^{-1}\xi)N_\zeta^{(1)}(A_{\zeta,2}^{(0)}) = 0 \quad \text{in} \quad \{(x,\xi) \in \mathbf{R}^3 \times \mathbf{R}^3\}.$$

Then, we have the following lemma.

Lemma 5.4. There exists a solution $(A_{\zeta,2}^{(0)}) \in S^0(\mathbf{R}^3, Z)$ of (5.39) which is elliptic in $\mathbf{R}^3 \times \mathbf{R}^3$. That is

(5.40) $\qquad \det(A_{\zeta,2}^{(0)})(x,\xi) \neq 0 \qquad ((x,\xi) \in \mathbf{R}^3 \times \mathbf{R}^3, \quad \zeta \in Z).$

the proof is given in the next two sections.

Step 3 We first define $A_{\zeta,1} \in L^0(\mathbf{R}^3, Z)$ by

(5.41) $\qquad\qquad \widetilde{\sigma}(A_{\zeta,1}) = (A_{\zeta,2}^{(0)}) \quad \text{in} \quad \mathbf{R}^3 \times U_{\zeta,1}$

in $\mathbf{R}^3 \times U_{\zeta,1}$. Then, for $N' \in \mathbf{N}$, we define $B_{\zeta,1} \in L^0(\mathbf{R}^3, Z)$ by

(5.42)
$$B_{\zeta,1} = \sum_{k=0}^{N'} B_{\zeta,1}^{(-k)}$$

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in $\mathbf{R}^3 \times U_{\zeta,1}$. Here, we let each $B_{\zeta,1}^{(-k)} \in L^{-k}(\mathbf{R}^3, Z)$ satisfy

$$(5.43) \qquad \widetilde{\sigma}(B_{\zeta,1}^{(0)}) \;\; = \;\; (A_{\zeta,2}^{(0)}) \quad \text{in} \quad \mathbf{R}^3 \times U_{\zeta,1}$$

$$(5.44) \qquad \widetilde{\sigma}(B_{\zeta,1}^{(-k)}) \;\; = \;\; -q_\zeta^{-1}\sigma(I^{(-k+1)}) \quad \text{in} \quad \mathbf{R}^3 \times U_{\zeta,1} \quad (1 \le k \le N'),$$

where

$$(5.45) \qquad I^{(-k)} := (\sum_{l=0}^{k} B_{\zeta,1}^{(-l)}(x, D))(\Delta_\zeta I) - M_\zeta(x, D)A_{\zeta,1}(x, D)$$

for each k ($0 \le k \le N' - 1$).

The following lemma shows that $B_{\zeta,1}^{(-k)}$ defined in this way satisfies $B_{\zeta,1}^{(-k)} \in L^{-k}(\mathbf{R}^3, Z)$ ($0 \le k \le N'$).

Lemma 5.5.

$$(5.46) \qquad\qquad\qquad I^{(-N')} \in L^{-N'+1}(\mathbf{R}^3, Z)$$

in $\mathbf{R}^3 \times U_{\zeta,1}$

Proof. We prove by using the induction argument on N'.

First we prove (5.5) for the case $N' = 0$. By (5.44),

$$I^{(0)} = B_{\zeta,1}^{(0)}(\Delta_\zeta I) - (\Delta_\zeta I + N_\zeta)A_{\zeta,1}.$$

Hence, by (5.43),

$$\sigma(I^{(0)}) = (A_{\zeta,2}^{(0)})q_\zeta - q_\zeta(A_{\zeta,2}^{(0)}) = 0 \qquad \text{in} \qquad \mathbf{R}^3 \times U_{\zeta,1}.$$

This proves (5.46) for the case $N' = 0$.

Next assuming that (5.46) holds up to $N' - 1$, we prove (5.46) for the case N'. By (5.44),

$$\begin{aligned} I^{(-N')} \;\; &= \;\; (\sum_{k=0}^{N'-1} B_{\zeta,1}^{(-k)} + B_{\zeta,1}^{(-N')})(\Delta_\zeta I) - (\Delta_\zeta I + N_\zeta)A_{\zeta,1} \\ &= \;\; I^{(-N'+1)} + B_{\zeta,1}^{(-N')}(\Delta_\zeta I). \end{aligned}$$

Hence, by (5.44),

$$\sigma(I^{(-N')}) = \sigma(I^{(-N'+1)}) + q_\zeta(B_{\zeta,1}^{(-N')}) = 0 \qquad \text{in} \qquad \mathbf{R}^3 \times U_{\zeta,1}.$$

This proves (5.46) for the case N'. □

Lemma 5.6.

$$(5.47) \qquad\qquad R_\zeta^{(-N')} := M_\zeta A_\zeta - B_\zeta(\Delta_\zeta I) \in L^{-N'+1}(\mathbf{R}^3, Z)$$

holds.

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Proof. By (5.36), (5.46) and $\chi_{\zeta,j}(\Delta_\zeta I) = (\Delta_\zeta I)\chi_{\zeta,j}$,

$$
\begin{aligned}
R_\zeta^{(-N')} &= M_\zeta(\sum_{j=1}^{2} A_{\zeta,j}\chi_{\zeta,j}) - (\sum_{j=1}^{2} B_{\zeta,j}\chi_{\zeta,j})(\Delta_\zeta I) \\
&= \sum_{j=1}^{2}\{M_\zeta A_{\zeta,j} - B_{\zeta,j}(\Delta_\zeta I)\}\chi_{\zeta,j} \in L^{-N'+1}(\mathbf{R}^3, Z).
\end{aligned}
$$

$\square$

To rewrite (5.47) in the form of (5.23), we need the following two lemmas.

Lemma 5.7. Let $Q_\zeta \in L^l(\mathbf{R}^3, Z)$ be elliptic in $\mathbf{R}^3 \times \mathbf{R}^3$. Then, for any $\varphi \in C_0^\infty(\mathbf{R}^3)$ there exist $r > 0$ and a linear operator $\widetilde{Q}_\zeta^{-1}$ with ζ satisfying $|\zeta| \geq r$ such that

$$(5.48) \qquad \varphi Q_\zeta \widetilde{Q}_\zeta^{-1} = \varphi \qquad \text{or} \qquad \varphi Q_\zeta^{-1}\widetilde{Q}_\zeta = \varphi.$$

Moreover, for any $s \in \mathbf{R}$, $k \geq -l$, $\widetilde{Q}_\zeta^{-1} : H^s(\mathbf{R}^3) \to H^{s-k}(\mathbf{R}^3)$ satisfies the estimate :

$$(5.49) \qquad \|\widetilde{Q}_\zeta^{-1}\|_{s,s-k} \leq C_{s,k} \begin{cases} |\zeta|^{-l} & (k \leq 0) \\ |\zeta|^{-l-k} & (k < 0) \end{cases} \qquad (\zeta \in Z,\ |\zeta| \geq r),$$

where $C_{s,k} > 0$ is a constant depending on s, k. For later reference, we call $\widetilde{Q}_\zeta^{-1}$ the semi-inverse of Q_ζ.

Proof. We only prove (5.48) for the former one and the corresponding (5.49), because the latter one can be proven similarly.

If we can prove for the case $l = 0$, we can prove for the case $l \neq 0$ by first applying the result for the case $l = 0$ to $Q'_\zeta := Q_\zeta \Lambda_\zeta^{-l} \in L^0(\mathbf{R}^3, Z)$ and then using $\Lambda_\zeta^{-l}\Lambda_\zeta - I \in L^{-\infty}(\mathbf{R}^3, Z)$ and Theorem 5.2, (iv).

So we assume $l = 0$ in the rest of the proof.

Since Q_ζ is elliptic in $\mathbf{R}^3 \times \mathbf{R}^3$, by Theorem 5.4,(i), there exists a right parametrix $Q_\zeta^{-1} \in L^0(\mathbf{R}^3, Z)$ of Q_ζ. So we take $\varphi_0 \in C_0^\infty(\mathbf{R}^3)$ which is one in a neighborhood of supp φ and define

$$(5.50) \qquad S_\zeta := \varphi_0(I - Q_\zeta Q_\zeta^{-1}) \in L^{-\infty}(\mathbf{R}^3, Z).$$

Since Q_ζ, Q_ζ^{-1} are properly supported, there exist a compact set $K \subset \mathbf{R}^6$ such that

$$(5.51) \qquad \text{supp } K_{S_\zeta} \subset K.$$

Hence, by Theorem 5.2, (iv), there exist $r > 0$ and $C > 0$ such that for any ζ satisfying $|\zeta| \geq r$, the bounded linear operator

$$I - S_\zeta : L^2(\mathbf{R}^3) \to L^2(\mathbf{R}^3)$$

has a bounded inverse $(I - S_\zeta)^{-1}$ and it satisfies the estimate :

(5.52) $$\|(I - S_\zeta)^{-1}\|_{0,0} \leq C \qquad (|\zeta| \geq r).$$

Moreover, by (5.50), (5.51) and (5.52), we can easily see that for any $s \in \mathbf{R}$, there exists a constant $C_s > 0$ depending on s such that

(5.53) $$\|(I - S_\zeta)^{-1}\|_{s,s} \leq C_s \qquad (|\zeta| \geq r).$$

Reminding the choice of φ_0, we have

(5.54) $$\varphi Q_\zeta Q_\zeta^{-1} = \varphi(I - S_\zeta).$$

Hence, defining

(5.55) $$\widetilde{Q_\zeta^{-1}} := Q_\zeta^{-1}(I - S_\zeta)^{-1}$$

when $|\zeta| \geq r$, it is clear that the first formula of (5.48) holds. Also, by (5.53) and Theorem 5.2, (iv), we have (5.49). $\qquad\square$

Now we have prepared everything which is necessary to prove (5.23). First note the following two things.

(i) B_ζ is properly supported.

(ii) By (5.29), Lemma 5.4, (5.43), B_ζ is elliptic in $\mathbf{R}^3 \times \mathbf{R}^3$.

Hence, by Lemma 5.7, there exist the semi-inverse $\widetilde{B_\zeta^{-1}}$ of B_ζ and $\phi_j \in C_0^\infty(\mathbf{R}^3)$ $(1 \leq j \leq 4)$ such that

(5.56) $$\phi_1 B_\zeta \phi_2 = \phi_1 B_\zeta, \ \phi_1 \phi_3 = \phi_1, \ \phi_1 B_\zeta \widetilde{B_\zeta^{-1}} = \phi_1 I.$$

Therefore

(5.57) $$\phi_1 B_\zeta \widetilde{B_\zeta^{-1}} = \phi_1 B_\zeta \phi_2 \widetilde{B_\zeta^{-1}}.$$

By (5.47), (5.56), (5.57), we have

(5.58) $$\phi_1 M_\zeta A_\zeta = \phi_1(B_\zeta(\Delta_\zeta I) + \phi_3 R_\zeta^{(-N')}) = \phi_1 B_\zeta(\Delta_\zeta + \phi_2 \widetilde{B_\zeta^{-1}} \phi_3 R_\zeta^{(-N')}).$$

To finish the proof, we only have to argue as follows. First, reminding $R^{(-N')}$ is properly supported, we can take $\phi_4 \in C_0^\infty(\mathbf{R}^3)$ satisfying

$$\phi_3 R^{(-N')} \phi_4 = \phi_3 R^{(-N')}.$$

Next, rewrite Then, we only have to define

$$R_\zeta^{(-N)} := \widetilde{B_\zeta^{-1}} \phi_3 R^{(-N')}.$$

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5.3 Intertwining property (part 3)–Some reductions

Now, we are going to start proving Lemmas 5.2, 5.4. Since the proof of Lemma 5.2 can be done in a similar way as that of Lemma 5.4, we omit its proof.

We first find a global transformation which transforms $H_{q\zeta} = \ell_1 \cdot \nabla_x + \sqrt{-1}\ell_2 \cdot \nabla_x$ to $\bar{\partial}$ in two variables, where

$$(5.59) \quad \begin{cases} \ell_1(x, \xi) & := {}^t(\ell_{11}(\xi, \zeta), \ell_{12}(\xi, \zeta), \ell_{13}(\xi, \zeta)) \\ & = 2\mathrm{Re}\zeta \\ \ell_2(\xi, \zeta): & = {}^t(\ell_{21}(\xi, \zeta), \ell_{22}(\xi, \zeta), \ell_{23}(\xi, \zeta)) \\ & = 2(\xi + \mathrm{Im}\zeta). \end{cases}$$

Lemma 5.8. *There exist an invertible matrix $T(\xi, \zeta) \in C^\infty(N_{3\varepsilon|\zeta|}(q_\zeta^{-1}(0)) \times Z)$ such that*

(i) $T(\rho\xi, \rho\zeta) = \rho^{-1}T(\xi, \zeta) \quad (\rho \geq 1)$,

(ii) In terms of $z = {}^t(z_1, z_2, z_3) = T(\xi, \zeta)x$ where $x = {}^t(x_1, x_2, x_3)$,

$$(5.60) \quad H_{q\zeta} = \ell_1 \cdot \nabla_x + i\ell_2 \cdot \nabla_x = \frac{1}{2}(\partial_{z_1} + i\partial_{z_2})(= \bar{\partial}).$$

$T(\xi, \zeta)$ is given more precisely by the following relations.

$$(5.61) \quad \begin{cases} z_1 & = \frac{1}{2}(w_1 - \lambda_{12}w_2), \quad z' = {}^t(z_2, z_3) = \frac{1}{2}w' \\ w_1 & = |\ell_1|^{-2}(\ell_1 \cdot x), \quad w' = |\lambda_2'|^{-1}|\ell_1|^{-1t}P'({}^tMx)', \end{cases}$$

where

$$(5.62) \quad ({}^tMx)' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}({}^tMx)$$

and λ_2', P', M are defined as follows.

$$(5.63) \quad \begin{cases} M: & = [m_1, m_2, m_3], \quad K: = |\ell_1|M \\ m_1 & = |\ell_1|^{-1}\ell_1, \quad m_2 = |\tilde{m}_2|^{-1}\tilde{m}_2, \quad m_3 = m_1 \times m_2 \\ \tilde{m}_2 & = \ell_2 - (\ell_2 \cdot m_1)m_1, \end{cases}$$

$$(5.64) \quad \begin{cases} P' & = [|\lambda_2'|^{-1}\lambda_2', |\tilde{\lambda}_3'|^{-1}\tilde{\lambda}_3'], \quad \tilde{\lambda}_3' = \lambda_3' - |\lambda_2'|^{-2}(\lambda_3' \cdot \lambda_2')\lambda_2' \\ \lambda_2': & = {}^t(\lambda_{22}, \lambda_{32}), \quad \lambda_3': = {}^t(\lambda_{23}, \lambda_{33}) \\ K^{-1}\ell_2 & = {}^t(\lambda_{12}, \lambda_{22}, \lambda_{32}), \quad |\ell_1|K^{-1}m_3 = {}^t(\lambda_{13}, \lambda_{23}, \lambda_{33}). \end{cases}$$

Proof. We will make the change of variables 3 times i.e. $x = (x_1, x_2, x_3)$ to $y = (y_1, y_2, y_3)$, y to $w = (w_1, w_2, w_3)$ and w to $z = (z_1, z_2, z_3)$. Define matrix K by $K = [k_1, k_2, k_3]$. Clearly, $K = |\ell_1|M$ and M is an orthogonal matrix. Consider the change of variables $x = Ky$. We easily see $\partial_{y_1} = \ell_1 \cdot \nabla_x = L_1$. Also,

$$\begin{aligned} y_1 & = e_1 \cdot y = e_1 \cdot K^{-1}x = |\ell_1|^{-1}(e_1 \cdot {}^tMx) \\ & = |\ell_1|^{-1}((Me_1) \cdot x) = |\ell_1|^{-2}(\ell_1 \cdot x), \end{aligned}$$

where $e_1 = {}^t(1,0,0)$. Let

$$L_2 = \ell_2 \cdot \nabla_x = \lambda_{12}\partial_{y_1} + \lambda_{22}\partial_{y_2} + \lambda_{32}\partial_{y_3},$$

$$L_3 = k_3 \cdot \nabla_x = \lambda_{13}\partial_{y_1} + \lambda_{23}\partial_{y_2} + \lambda_{33}\partial_{y_3}.$$

Then, since L_1, L_2, L_3 are linearly independent due to the linearly independency of ℓ_1, ℓ_2, k_3, λ_2', λ_3' are linearly independent nonzero real vectors. Let $Q' = |\lambda_2'|P'$. Consider the change of variables

$$\begin{cases} y_1 & = w_1 \\ y' & = {}^t(y_2, y_3) = Q'w' \\ w' & = {}^t(w_2, w_3). \end{cases}$$

Since P' is an orthogonal matrix, we easily see

$$L_1 = \partial_{w_1}, \ \ L_2 = \lambda_{12}\partial_{w_1} + \partial_{w_2}$$

and

$$\begin{aligned} w_2 &= e_2' \cdot w' = e_2' \cdot Q'^{-1}y' = |\lambda_2'|^{-1}(e_2' \cdot {}^tP'y') \\ &= |\lambda_2'|^{-1}((P'e_2') \cdot y') = |\lambda_2'|^{-2}(p_2' \cdot y'), \end{aligned}$$

aligned where $e_2' = (1,0)$.

Finally, consider the change of variables

$$\begin{cases} z_1 & = \frac{1}{2}(w_1 - \lambda_{12}w_2) \\ z' & = {}^t(z_2, z_3) = \frac{1}{2}w'. \end{cases}$$

Then, we easily see that $L_1 = \frac{1}{2}\partial_{z_1}$, $L_2 = \frac{1}{2}\partial_{z_2}$.

$\square$

For simplicity we write

(5.65)
$$\begin{cases} N & = N(x, \xi, \zeta) & = \psi_1(s^{-1}(\mathrm{Re}\zeta) \cdot \xi)\psi_2(s^{-1}\xi)N_\zeta^{(1)}(x, \xi) \\ A & = A(x, \xi, \zeta) & = (A_{\zeta,2}^{(0)})(x, \xi). \end{cases}$$

By Lemma 5.8, we can write (5.39)

(5.66)
$$\bar{\partial}A + NA = 0$$

in terms of the variables z_j ($1 \le j \le 3$). Here, we have used the same N for simplicity. To avoid too many notations, we write

(5.67)
$$x_j = z_j \quad (1 \le j \le 3)$$

Also, we will sometimes write only the variable $x' := (x_1, x_2)$ and suppress all the other variables.

5.4 Construction of pseudoanalytic matrices

Let $N(x) \in C_0^\infty(\mathbf{R}^2)$ be a $\ell \times \ell$ matrix. Consider the equation :

$$(5.68) \qquad \overline{\partial}A + NA = 0 \quad \text{in} \quad \mathbf{R}^2,$$

where $\overline{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. We look for an invertible $C^\infty(\mathbf{R}^3)$ matrix solution $A(x)$. We call this equation a pseudoanalytic matrix in analogy with pseudoanalytic functions in complex analysis. For a different approach see [6].

Lemma 5.9. Let $N \in C_0^\infty(\mathbf{R}^2)$. Then, there exist a negatively large constant $\tau \in \mathbf{R}$ and a constant $C > 0$ such that for any $f \in L_{\tau+2}^2(\mathbf{R}^2)$,

$$(5.69) \qquad \overline{\partial}u + Nu = f \quad \text{in} \quad \mathbf{R}^2$$

admits a solution $u \in L_\tau^2(\mathbf{R}^2)$ with the estimate

$$(5.70) \qquad \|u\|_\tau \leq C\|f\|_{\tau+2}.$$

Here $u \in L_\tau^2(\mathbf{R}^2)$ if and only if

$$(5.71) \qquad \|u\|_\tau^2 := \int_{\mathbf{R}^2} (1 + |x|^2)^\tau |u(x)|^2 dx < \infty.$$

Proof. First of all note that we can easily prove

$$(5.72) \qquad (L_\sigma^2(\mathbf{R}^2))^* = L_{-\sigma}(\mathbf{R}^2)$$

for any $\sigma \in \mathbf{R}$. More precisely, for any $T \in (L_\sigma^2(\mathbf{R}^2))^*$, there exists a unique $u \in L_{-\sigma}^2(\mathbf{R}^2)$ such that

$$(5.73) \qquad T(\phi) = (u, \phi)_{L^2(\mathbf{R}^2)} \quad (\sigma \in L_\sigma^2(\mathbf{R}^2))$$

and

$$(5.74) \qquad \|T\| = \|u\|_{-\sigma},$$

where $(u, \phi)_{L^2(\mathbf{R}^2)}$ is the $L^2(\mathbf{R}^2)$ inner product.

A. Bukhgeim [2] prove that, for any real valued function $\rho \in C^\infty(\mathbf{R}^2)$,

$$(5.75) \qquad \int_{\mathbf{R}^2} (\Delta\rho)|\phi|^2 e^\rho dx \leq 4 \int_{\mathbf{R}^2} |\partial\phi|^2 e^\rho dx \quad (\phi \in C_0^\infty(\mathbf{R}^2)),$$

where $\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$.

By taking $\rho = \alpha \log(1 + |x|^2)$ with a constant $\alpha > 0$, a direct computation gives

$$(5.76) \qquad \sqrt{\alpha}\|\phi\|_\alpha \leq \|\partial\phi\|_\alpha \quad (\phi \in C_0^\infty(\mathbf{R}^2)).$$

Combine this with the estimate:

$$(5.77) \qquad \|(\partial - N^*)\phi\|_{\alpha-2} \geq \|\partial\phi\|_\alpha - \| < x >^2 N^*\|_{L^\infty(\mathbf{R}^2)}\|\phi\|_{\alpha-2}.$$

Then, reminding that N is compactly supported, we have for large enough α,

(5.78) $$\|\phi\|_{\alpha-2} \le C\|(\partial - N^*)\phi\|_\alpha \quad (\phi \in C_0^\infty(\mathbf{R}^2))$$

for some constant $C > 0$ independent of ϕ.

Now for given $f \in L^2_{-\alpha+2}(\mathbf{R}^2)$, consider the map $T : E \longrightarrow \mathbf{C}$ defined by

(5.79) $$T(-(\partial - N^*)\phi) = (f, \phi)_{L^2(\mathbf{R}^2)} \quad (\phi \in C_0^\infty(\mathbf{R}^2)),$$

where

(5.80) $$E = \{-(\partial - N^*)\phi \; ; \; \phi \in C_0^\infty(\mathbf{R}^2)\} \subset L^2_\alpha(\mathbf{R}^2).$$

From (5.78), (5.79), we have
(5.81)
$$|T(-(\partial - N^*)\phi)| \le \|f\|_{-\alpha+2}\|\phi\|_{\alpha-2} \le C\|f\|_{-\alpha+2}\|(\partial - N^*)\phi\|_\alpha \quad (\phi \in C_0^\infty(\mathbf{R}^2)).$$

By Hahn-Banach theorem, T can be extended to $L^2_\alpha(\mathbf{R}^2)$ and its extension $\tilde{T}$ satisfies

(5.82) $$\|\tilde{T}\| \le C\|f\|_{-\alpha+2}.$$

By (5.72), there exists a unique $u \in L^2_{-\alpha}(\mathbf{R}^2)$ such that

(5.83) $$\tilde{T}(\psi) = (u, \psi)_{L^2(\mathbf{R}^2)} \; (\psi \in L^2_\alpha(\mathbf{R}^2)),$$

(5.84) $$\|T\| = \|u\|_{-\alpha}.$$

Hence,

(5.85) $$(u, -(\partial - N^*)\phi) = T(-(\partial - N^*)\phi) = (f, \phi)_{L^2(\mathbf{R}^2)} \quad (\phi \in C_0^\infty(\mathbf{R}^2),$$

(5.86) $$\|u\|_{-\alpha} \le C\|f\|_{-\alpha+2}.$$

(5.85) means that u satisfies $(\overline{\partial} + N)u = f$. So the proof is done if we take $\tau = -\alpha$. $\qquad\square$

Now let $k \in \mathbf{N}$ be $k > -\tau$. By 5.9, there exists a matrix solution $U \in L^2_\tau(\mathbf{R}^2)$ to

(5.87) $$\overline{\partial}U + NU = -z^k N,$$

where $z = x_1 + ix_2$.

Lemma 5.10. For large $|x|$, U admits an expansion:

(5.88) $$U(x) = \sum_{n=0}^\infty U_n z^{k-n-1}.$$

Proof. Let $\chi \in C^\infty(\mathbf{R}^2)$ be a function satisfying

(5.89) $$1 - \chi \in C_0^\infty(\mathbf{R}^2), \quad \chi = 0 \quad \text{near} \quad x = 0.$$

From (5.87),

(5.90) $$\bar{\partial}(\chi z^{-k} U) = (\bar{\partial}\chi) z^{-k} U + \chi z^{-k} \bar{\partial} U = G$$

with

(5.91) $$G = (\bar{\partial}\chi) z^{-k} U - \chi z^{-k} N U - \chi N \in C_0^\infty(\mathbf{R}^2).$$

For further argument, we need the following well known fact (see [40]).

Let $-1 < \delta < 0$. Then, for given $f \in L^2_{\delta+1}(\mathbf{R}^2)$ such that

(5.92) $$\bar{\partial} u = f \quad \text{in} \quad \mathbf{R}^2,$$

(5.93) $$u(x) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{1}{(x_1 + i x_2) - (y_1 + i y_2)} f(y) dy$$

with $x = (x_1, x_2)$, $y = (y_1, y_2)$ and

(5.94) $$\|u\|_\delta \leq C \|f\|_{\delta+1}$$

for some constant $C > 0$ independent of f.

Note that $\chi z^{-k} U = \chi z^{-k} < x >^{-\tau} (< x >^\tau U) \in L^2(\mathbf{R}^2) \subset L^2_\delta(\mathbf{R}^2)$ for any $-1 < \delta < 0$, because $-(\tau + k) < 0$. Hence, by 5.4,

(5.95) $$\chi z^{-k} U(x) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{G(y)}{(x_1 + i x_2) - (y_1 + i y_2)} dy.$$

Put $z = x_1 + i x_2$, $\zeta = y_1 + i y_2$. Then, for large $|z|$,

(5.96) $$\chi z^{-k} U(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} z^{-n-1} \int_{\mathbf{R}^2} \zeta^n G(y) dy.$$

Therefore, for large $|z|$,

(5.97) $$\begin{aligned} U(x) &= \frac{1}{\pi} \sum_{n=0}^{\infty} z^{k-n-1} \int_{\mathbf{R}^2} \zeta^n G(y) dy \\ &= \sum_{n=1}^{\infty} z^{k-n} U_n \end{aligned}$$

with

(5.98) $$U_n = \frac{1}{\pi} \int_{\mathbf{R}^2} \zeta^n G(y) dy.$$

This proves 5.10. $\qquad \square$

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Since $(\bar{\partial} + N)(z^k I) = z^k N$, A' defined by

(5.99)
$$A' = z^k I + U$$

satisfies

(5.100)
$$\bar{\partial} A' + N A' = 0 \quad \text{in} \quad \mathbf{R}^2$$

and A' admits an expansion

(5.101)
$$A' = \sum_{n=0}^{\infty} A'_n z^{k-n}, \quad A'_0 = I, \quad A'_n = U_n \quad (n = 1, 2, \cdots)$$

for large $|z|$.

Next we investigate the zeros of $\det A'$.

Lemma 5.11. $\det A'$ satisfies

(5.102)
$$\bar{\partial} \det A' = -(\text{trace} N) \det A' \quad \text{in} \quad \mathbf{R}^2,$$

where $\text{trace} N$ is the trace of N.

Proof. Let $A' = (a'_{ij})$, $N = (n_{ij})$. By (5.100), we have

(5.103)
$$\bar{\partial} a'_{ij} + \sum_{k=1}^{\ell} n_{ik} a'_{kj} = 0 \quad (1 \le i, j \le \ell).$$

Hence

(5.104)
$$\bar{\partial} \det A' = \sum_{i=1}^{\ell} \det \begin{bmatrix} a'_1 \\ . \\ . \\ \bar{\partial} a'_i \\ . \\ . \\ a'_\ell \end{bmatrix} = -\sum_{i=1}^{\ell} \sum_{k=1}^{\ell} n_{ik} \det \begin{bmatrix} a'_1 \\ . \\ . \\ a'_k \\ . \\ . \\ a'_\ell \end{bmatrix}$$

where a'_k in the matrix of the right-hand side is the i-th row vector and a'_i $(1 \le i \le \ell)$ are the row vectors of A'. This immediately implies (5.102). $\quad\square$

Lemma 5.12. The number of zeros of $\det A'$ is $k\ell$.

Proof. Let $-1 < \delta < 0$ and $\beta \in L^2_\delta(\mathbf{R}^2)$ be the solution of

(5.105)
$$\bar{\partial} \beta = -\text{trace } N \quad \text{in} \quad \mathbf{R}^2.$$

Also, write $\det A'$ in the form:

(5.106)
$$\det A' = \alpha \, e^\beta.$$

Then, we easily see that α satisfies

(5.107) $$\overline{\partial}\,\alpha = 0 \quad \text{in} \quad \mathbf{R}^2$$

and β satisfies

(5.108) $$\beta(x) = O(|x|^{-1}) \ (|x| \to \infty).$$

By (5.101), we have

(5.109) $$\det A' - z^{k\ell} = O(|x|^{k\ell-1}) \ (|x| \to \infty).$$

Hence, by (5.106),(5.107), we have

(5.110) $$\alpha - z^{k\ell}e^{-\beta} = O(|x|^{k\ell-1}) \ (|x| \to \infty).$$

So, by taking $R > 0$ large, we have

(5.111) $$|\alpha - z^{k\ell}| \le |\alpha - z^{k\ell}e^{-\beta}| + |z^{k\ell}||e^{-\beta} - 1| < |z^{k\ell}| \ \ (|z| = R).$$

Hence, reminding (5.107), we have from Rouche's theorem, the number of zeros of α is $k\ell$. Therefore, by (5.106), the number of zeros of $\det A'$ is $k\ell$. This proves 5.12. $\qquad\square$

Next we want to modify A' to get an invertible $C^\infty(\mathbf{R}^2)$ matrix solution A of (5.68) by dividing out the zeros of $\det A'$. For this we prove the following.

Lemma 5.13. Fix $y \in \mathbf{R}^2$. Let $m \in \mathbf{R}$ and $a(x) \in C^\infty(\mathbf{R}^2)$ be a function satisfying

(5.112) $$a(x) = O(|x - y|^m) \ (x \to y).$$

Then, we have

(5.113) $$\overline{\partial}(z^{-m}a) = z^{-m}\overline{\partial}a \quad \text{in} \quad \mathbf{R}^2,$$

where $z = (x_1 - y_1) + i(x_2 - y_2)$ for $x = (x_1, x_2), \ y = (y_1, y_2)$.

Proof. In this proof, $<\,,\,>$ denotes the pairing between a distribution and a test function. Let $\phi \in C_0^\infty(\mathbf{R}^2)$. Then,
(5.114)
$$\begin{aligned}
< \overline{\partial}(z^{-m}a), \phi > \ &= \ - < z^{-m}a, \overline{\partial}\phi > \\
&= -\lim_{\varepsilon \to +0} \int_{|x-y|>\varepsilon} z^{-m}a(x)\overline{\partial}\phi(x)dx \\
&= -\lim_{\varepsilon \to +0} \int_{|x-y|>\varepsilon} \{\overline{\partial}(z^{-m}a(x)\phi(x))\} - z^{-m}\overline{\partial}a(x)\phi(x)\}dx \\
&= \int_{\mathbf{R}^2} z^{-m}\overline{\partial}a(x)\phi(x)dx - \lim_{\varepsilon \to +0} I_\varepsilon(y),
\end{aligned}$$

where

(5.115) $$I_\varepsilon(y) = \int_{|x-y|>\varepsilon} \overline{\partial}(z^{-m}a(x)\phi(x))dx.$$

By Stokes' theorem, we have

$$(5.116) \qquad \begin{aligned} I_\varepsilon(y) &= \tfrac{1}{2} \int_{|x-y|=\varepsilon} z^{-m} a(x) \phi(x)(dx_2 - i dx_1) \\ &= -\tfrac{1}{2} \int_0^{2\pi} (\varepsilon e^{i\theta})^{-m+1} (a\phi)(y_1 + \varepsilon\cos\theta, y_2 + \varepsilon\sin\theta) d\theta, \end{aligned}$$

where the first integral in the right-hand side of (5.116) is a contour integral along the circle $|x - y| = \varepsilon$ in the clockwise direction. Hence, by (5.112), we have

$$(5.117) \qquad \lim_{\varepsilon \to +0} I_\varepsilon(y) = 0.$$

Therefore, by (5.114), (5.117),

$$(5.118) \qquad < \overline{\partial}(z^{-m}a), \phi > = < z^{-m}\overline{\partial}a, \phi > \quad (\phi \in C_0^\infty(\mathbf{R}^2)).$$

This proves 5.13. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Combining the local regularity property of the elliptic equations with 5.13, we have the following.

Lemma 5.14. If a vector valued function $a \in C^\infty(\mathbf{R}^2)$ satisfies

$$\overline{\partial}a + Na = 0 \quad \text{in} \quad \mathbf{R}^2$$

and $a(x) = O(|x - y|^m)$ $(x = (x_1, x_2) \to y)$ at a point $y = (y_1, y_2) \in \mathbf{R}^2$, then $a(x)$ can be written in the form

$$(5.119) \qquad a(x) = z^m a'(x), \ z = (x_1 - y_1) + i(x_2 - y_2)$$

with some $a'(x) \in C^\infty(\mathbf{R}^2)$.

Now we will use 5.14 to divide out the zeros of $\det A'$ to make A' invertible. First of all divide out the zero column vectors of A' at any point by using 5.14. Let a'_j $(1 \le j \le \ell)$ be the resultant nonzero column vectors such that

$$(5.120) \qquad a'_j = z^{d_j} e_j, \ as \ |z| \to \infty, \ 1 \le j \le \ell, \ \text{with} \ z = x_1 + ix_2,$$

where $e_i = {}^t(0, \cdots, 1, \cdots, 1)$ with 1 in the i-th component and 0 in the rest of the components.

Suppose there exists a point $x^0 \in \mathbf{R}^2$ such that

$$(5.121) \qquad \det A' = 0 \quad \text{at} \quad x = x^0.$$

Let $a'_{j_r}(x^0)$ $(1 \le r \le k)$ be the linearly independent vectors such that

$$(5.122) \qquad \sum_{r=1}^k c_r a'_{j_r}(x^0) = 0 \quad \text{with} \quad c_r \ne 0 \ (1 \le r \le k).$$

Multiply, a'_{j_r} $(1 \le r \le k,\ r \ne r_1)$ by $c_{r_1}^{-1} c_r$ and add to $a'_{j_{r_1}}$. Then, denoting the new j_{r_0} th column vector by the same symbols $a'_{j_{r_1}}$, we of course have $a'_{j_{r_1}}(x^0) = 0$. Dividing out the zero of $a'_{j_{r_1}}$ at $x = x^0$ and denote the resultant nonzero j_{r_1} th column vector again by the same symbol $a'_{j_{r_1}}$. We refer this procedure as dividing out the zero due to the linear dependency. With this new $a'_{j_{r_1}}$, we again denote $[a'_1, a'_2, \cdots, a'_\ell]$ by A'. Then, the column vectors a'_j $(1 \le j \le \ell)$ of this new A' are nonzero and the number of zeros of $\det A'$ is reduced. So we have returned to the starting point when we divided out the zero due to the linear dependency. We repeat this argument until we get a new $A' = [a'_1, a'_2, \cdots, a'_\ell]$ such that $\det A' \ne 0$.

Applications

The CGO solutions constructed in this section have been applied to solve inverse boundary problems for the magnetic Schrödinger equation with different regularity in the coefficients [30], [35], [36], [42], [19], [26]. Also applications to an inverse boundary problem for linear elasticity are given in [31] (see also [7]). See also applications to inverse problems for higher order operators [23], [24], [25].

6   Bibliography

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