The Calderón Problem — An Introduction to Inverse Problems

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2000 *Mathematics Subject Classification.* Primary

Research of J.F. supported in part by the Natural Sciences and Engineering Research Council of Canada.

Abstract.

Contents

Preface

Inverse problems are those where from "external" observations of a hidden, "black box" system (a patient's body, nontransparent industrial object, Earth interior, etc.) one needs to recover the unknown parameters of the system. Such problems lie at the heart of contemporary scientific inquiry and technological development. Applications include a vast variety of medical as well as other (geophysical, industrial, radar, sonar) imaging techniques, which are used for early detection of cancer and pulmonary edema, location of oil and mineral deposits in the Earth's interior, creation of astrophysical images from telescope data, finding cracks and interfaces within materials, shape optimization, model identification in growth processes and modeling in the life sciences among others. Most inverse problems arise from a physical situation modeled by a partial differential equation. The inverse problem is to determine coefficients of the equation given some information about the solutions. Analysis of such problems brings together diverse areas of mathematics such as complex analysis, differential geometry, harmonic analysis, integral geometry, microlocal analysis, numerical analysis, optimization, partial differential equations, probability etc. and is a fertile area for interaction between pure and applied mathematics.

A prototypical example of an inverse boundary problem for an elliptic equation is the by now classical Calderón problem, forming the basis of Electrical Impedance Tomography (EIT). Calderón proposed the problem in the mathematical literature in 1980. In EIT one attempts to determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. The information is encoded in the Dirichlet–to–Neumann (DN) map associated to the conductivity equation. EIT arises in several applications including geophysical prospection (the original motivation of Calderón) and in medical imaging. In the last 30 years or so there has been remarkable progress on this problem. This book includes a thorough account of many of these developments. It is intended for graduate students that have had a basic course in Real Analysis or its equivalent.

We briefly summarize the contents of this book. In Chapter 1 we give a motivation to Calderón's inverse problem as well as an introduction to other inverse problems like optical tomography and inverse scattering. Chapter 2 gives a precise formulation of Calderón's problem. We also analyze the linearized case at a constant conductivity, which is the case analyzed by Calderón. Also in this chapter one can find the reduction of Calderón's problem to a study of the DN map associated to the Schrödinger equation.

In Chapter 3 we show that one can determine, in a stable fashion, the conductivity at the boundary and the normal conductivity at the boundary from the DN map. In Chapter 4 we construct complex geometrical optics (CGO) solutions for the conductivity equation which have been the basis of many developments in the theory of inverse boundary problems for elliptic equations. We use these solutions to prove, in dimension greater than two, uniqueness for C^2 conductivities from the DN map, develop a reconstruction procedure of the conductivity from the DN map, and prove stability estimates for the inverse problem.

The two dimensional case is considered in detail in Chapter 5. For potentials in the class C^ε, $\varepsilon > 0$, a new class of CGO solutions are constructed that give uniqueness of the potential from the associated DN map for the Schrödinger equation. This gives a similar result for $C^{2+\varepsilon}$ conductivities. We also describe how to get a more general uniqueness result, just for bounded measurable conductivities, using quasiconformal maps.

The results described in Chapters 4 and 5 concern the case when the DN map is measured on the whole boundary. Chapter 6 describes several results for the case when the measurements are made on part of the boundary. A basic tool is to construct CGO solutions vanishing on an open subset of the boundary. This is done in dimension three or greater for special geometries using a reflection method and for some other cases using Carleman estimates.

The previous chapters have discussed isotropic conductivities, that is, conductivities that do not depend on direction. There are several important physical examples of anisotropic conductivities, including muscle tissue. This case is analyzed in detail in Chapter 7 in two dimensions.

A topic that has received a lot of attention in recent years is the subject of invisibility and cloaking. The method of transformation optics has been one of the main proposed techniques to achieve, at least theoretically, invisibility. This technique originated in the study of EIT and Calderón's problem, in constructing examples of non-uniqueness for Calderón's inverse problem. This construction leads to degenerate anisotropic conductivities. We give a detailed account of this in Chapter 8, together with other selected topics related to the Calderón problem.

Finally in Chapter 9 we consider an application of the methods developed in the previous chapters to inverse scattering at a fixed energy.

Chapter 1

Introduction

In this introduction we discuss a number of imaging methods for which the Calderón problem is relevant. In each situation we have a medium whose internal properties are unknown, and the objective is to determine internal properties of the medium from various measurements (electrical, optical, or acoustic) on its boundary or far away.

1.1. Electrical impedance tomography

The one-dimensional case. Consider a simple electric circuit consisting of two components: a resistor given by a metal wire occupying the interval $0 \leq x \leq \ell$ on the real line, and a voltage source attached to the resistor's terminals at $x = 0$ and $x = \ell$. We denote by $u(x)$ the voltage at x. By Ohm's law, the voltage difference between the points at x and $x + h$ is the current, I, flowing through the wire times the resistance between x and $x + h$. If the resistance density (or resistivity) $\rho(x)$ at each point x on the wire is continuous, then

$$u(x + h) - u(x) = -I\rho(x')h$$

for some x' between x and $x + h$. Dividing across by h and taking the limit $h \to 0$, we get

$$u'(x) = -I\rho(x).$$

There are no sources or sinks of charge inside the wire, so the current I is a constant. We may eliminate it from the equation just by dividing $\rho(x)$ across and differentiating. In terms of the conductivity $\gamma(x) = \frac{1}{\rho(x)}$, we have

$$\gamma(x)u'(x) = -I \implies \big(\gamma(x)u'(x)\big)' = 0. \tag{1.1}$$

Now suppose that the conductivity $\gamma(x)$ of the wire is unknown to us, and we may only measure the voltages and currents at the ends of the wire. That is, we may only measure $u(0), u(\ell)$, $\gamma(0)u'(0)$ and $\gamma(\ell)u'(\ell)$. By (1.1), $\gamma(x)u'(x)$ is a constant and so takes the value $\gamma(0)u'(0)$ everywhere. Thus

$$u'(x) = \gamma(0)u'(0)\frac{1}{\gamma(x)} \implies u(\ell) - u(0) = \gamma(0)u'(0)\int_0^\ell \frac{dx}{\gamma(x)}.$$

Consequently, the only property of the wire that one can determine by measurements at the ends of the wire is the total resistance $\int_0^\ell \frac{dx}{\gamma(x)}$.

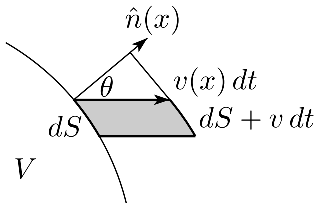
Derivation of the conductivity equation. Replacing the wire by a two or higher dimensional body changes the picture completely. In $\mathbb{R}^n$, $n \geq 2$, the current $i(x)$ is a vector and Ohm's Law is

$$i(x) = -\gamma(x)\nabla u(x). \tag{1.2}$$

Assuming that charge is still not allowed to accumulate anywhere in the body, the net rate of charge flow across the boundary ∂V of any region V must vanish, so that

$$\int_{\partial V} i(x) \cdot \hat{n}(x)\ dS = 0$$

where $\hat{n}(x)$ is the outward unit normal to ∂V at x. To derive this condition, concentrate on the charge that, at time t, is on an infinitesimal piece, dS, of the surface of V. If this charge has velocity $v(x)$, then at the end of an infinitesimal time interval dt it has moved to a surface element that is the translate by $v(x)\,dt$ of dS. In the figure, this surface element is denoted $dS + v\,dt$.



The charge that has left V through dS during this time interval now fills a tube whose ends are dS and $dS + v\,dt$. The tube has cross–sectional area dS and height $|v(x)|\,dt\cos\theta = \hat{n} \cdot v\,dt$. Hence the tube has volume $v(x) \cdot \hat{n}(x)\,dt\,dS$. If the charge density at x is $\kappa(x)$, the tube contains charge $\kappa(x)v(x) \cdot \hat{n}(x)\,dS\,dt = i(x) \cdot \hat{n}(x)\,dS\,dt$. The total charge that leaves V during the time interval dt is $dt\int_{\partial V} i(x) \cdot \hat{n}(x)\,dS$.

As we are not allowing charge to accumulate anywhere, $0 = \int_{\partial V} i(x) \cdot \hat{n}(x)\,dS = \int_V \nabla \cdot i(x)\,dx$, by the divergence theorem. This is true for all regions V. So, assuming that $\nabla \cdot i(x)$ is continuous,

$$\nabla \cdot i(x) = 0 \implies \nabla \cdot \big(\gamma(x)\nabla u(x)\big) = 0.$$

This equation will be called the *conductivity equation*.

The Calderón problem. Suppose now that we have a conductor filling a region Ω and that we apply a voltage f on the boundary $\partial\Omega$ and measure the current that then flows out of the region. By measuring the rate at which charge is leaving various parts of $\partial\Omega$, we are measuring the current flux through $\partial\Omega$, which determines the quantity $\gamma(x)\nabla u(x)\cdot\hat{n}(x) = \gamma(x)\partial_\nu u(x)$ on $\partial\Omega$ where $\partial_\nu u$ is the normal derivative of u.

For a given γ and f, we will see in §2 that the boundary value problem

$$\nabla\cdot\big(\gamma(x)\nabla u(x)\big) = 0 \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega$$

has a unique solution u in Ω. Let $\Lambda_\gamma(f)$ be the function $\gamma\partial_\nu u|_{\partial\Omega}$ on the boundary that results from a given γ and f. The map $f\mapsto\Lambda_\gamma(f)$, which clearly depends linearly on f, is called the *Dirichlet to Neumann map* (DN map). This map encodes the electrical boundary measurements for all possible functions f on the boundary. More precisely, we will see in §2 that Λ_γ is a bounded linear map between two Sobolev spaces on $\partial\Omega$,

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega).$$

The inverse problem of Calderón, also called the inverse conductivity problem or the inverse problem of electrical impedance tomography, is to determine the conductivity function γ from the knowledge of the map Λ_γ (that is, from the knowledge of $\Lambda_\gamma(f)$ for all $f\in H^{1/2}(\partial\Omega)$).

Formal variable count. When dealing with inverse problems it is sometimes informative to do a formal variable count. The Calderón problem for a domain in $\mathbb{R}^n$ asks to determine γ, which is a function depending on n variables in general, from the DN map Λ_γ. Pretend for a minute that the boundary $\partial\Omega$ contains only a finite number, m, of points and call the value of f at the j^{th} boundary point f_j and the value of $\Lambda_\gamma(f)$ at the i^{th} boundary point $\Lambda_\gamma(f)_i$. Then the map $f\mapsto\Lambda_\gamma(f)$ would correspond to a linear map taking $f\in\mathbb{R}^m$ to $\Lambda_\gamma(f)\in\mathbb{R}^m$, having of the form

$$\Lambda_\gamma(f)_i = \sum_{j=1}^m \lambda_{i,j} f_j$$

where $\lambda_{i,j}$ is the current that results at i^{th} boundary point when a unit voltage is applied at the j^{th} boundary point. The analogous formula for the true, continuous, boundary $\partial\Omega$ is

$$\Lambda_\gamma(f) = \int_{\partial\Omega} \lambda_\gamma(x,y) f(y)\, dS(y)$$

where dS is the surface measure on $\partial\Omega$ and $\lambda_\gamma(x,y)$ the current density that results at x when a unit voltage is applied at y. Knowing the DN map $\Lambda_\gamma(f)$ for all applied surface voltages f is equivalent to knowing the integral kernel $\lambda_\gamma(x,y)$ for all $x,y\in\partial\Omega$. This is a function of $2n-2$ variables.

Thus, in the inverse problem of Calderón we are hoping to determine a function of n variables (the conductivity γ) from a function of $2n-2$ variables (the integral kernel λ_γ of the DN map). For $n=1$, this problem is formally underdetermined since γ is a function of more variables than λ_γ, and we have already seen that for $n=1$ the DN map cannot possibly determine the conductivity. For $n=2$ the inverse problem is formally well-determined (γ and λ_γ are both functions depending on two variables), while for $n \geq 3$ the problem is formally overdetermined since the data has more degrees of freedom than the quantity that we wish to recover. The variable count suggests that in large dimensions there may be redundancy in the data, and this sometimes (but certainly not always) means that the inverse problem may be easier in high dimensions.

Different aspects of the Calderón problem. In connection with any inverse problem, there are a number of different questions that are of interest. In the following, we give a list of theorems proved in this book addressing these questions. The first result states that knowledge of the DN map determines the unknown coefficient γ uniquely on the boundary. This is often a first step in determining γ in the interior.

Theorem 1.1. *(Boundary uniqueness) Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain, and let $\gamma_1, \gamma_2 \in C(\overline{\Omega})$ be positive functions. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$.*

The next result states that, under certain conditions, the DN map uniquely determines the conductivity in the interior. The cases $n \geq 3$ and $n=2$ will require different proofs, as suggested by the variable count.

Theorem 1.2. *(Interior uniqueness) Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, and let γ_1, γ_2 be positive functions in $C^2(\overline{\Omega})$ if $n \geq 3$ and in $L^\infty(\Omega)$ if $n=2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.*

The first proof that we will obtain for the interior uniqueness result is not constructive, that is, it does not yield an algorithm for computing the values of γ in Ω from the knowledge of Λ_γ. However, with extra work we can also give a constructive procedure.

Theorem 1.3. *(Reconstruction) Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, $n \geq 3$, and let $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$ be positive functions. There is a convergent algorithm for determining the function γ from knowledge of Λ_γ.*

In practice one would like to have an efficient numerical implementation of the algorithm. This is out of the scope of this book, and is a challenging topic for several reasons. First, the imaging method is very diffuse and it is difficult to obtain high resolution images, and secondly the problem is ill-posed in the sense that small errors in the measurements may lead to a large error in the reconstructed image. The next stability result quantifies the degree of ill-posedness in the Calderón problem and states that, under some a priori assumptions, the inverse map taking Λ_γ to γ has a logarithmic modulus of continuity. It can also be proved that this modulus of continuity is optimal, and it cannot be improved to a Hölder or Lipschitz type continuity in general.

Theorem 1.4. *(Stability) Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain, $n \geq 3$, and let γ_j, $j = 1, 2$, be two positive functions in the Sobolev space $H^{s+2}(\Omega)$ with $s > n/2$, satisfying*

$$\frac{1}{M} \leq \gamma_j \leq M, \qquad\qquad \|\gamma_j\|_{H^{s+2}(\Omega)} \leq M.$$

There are constants $C = C(\Omega, n, M, s) > 0$ and $\sigma = \sigma(n, s) \in (0, 1)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)})$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C \left|\log t\right|^{-\sigma}, \quad 0 < t < 1/e.$$

The previous results considered the case of full data, where one can do measurements on the whole boundary $\partial\Omega$. In practice this is often not possible, and for instance in geophysical imaging one can only cover a tiny part of the Earth's surface with measurement devices. It is therefore of interest to consider partial data problems. For the Calderón problem this means that one has knowledge of $\Lambda_\gamma(f)$ on some subset of the boundary for functions f supported in some subset of the boundary. The next theorem gives a result of this type for the special case where the part of the boundary that is inaccessible to measurements is part of a sphere.

Theorem 1.5. *(Partial data) Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. Assume that $\Omega \subseteq B$ for some open ball B in $\mathbb{R}^n$, let $\Gamma_0 = \partial\Omega \cap \partial B$, and let $\Gamma = \partial\Omega \setminus \Gamma_0$. Assume also that $\partial B \setminus \partial\Omega \neq \emptyset$. If*

$$\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma \quad \textit{for all } f \in H^{1/2}(\partial\Omega) \textit{ with } \mathrm{supp}\,(f) \subset \Gamma,$$

then $\gamma_1 = \gamma_2$ in Ω.

There are certainly many other important aspects of inverse problems besides the ones listed above. A standard one is the problem of range characterization, which asks to find a set of necessary and sufficient conditions for a map acting on functions on $\partial\Omega$ to be the DN map of some conductivity. Not much is known about this problem at the time of writing this. Also, we have only considered the idealized mathematical problem where one can make infinitely many measurements (for all possible boundary voltages f) at infinitely many points on the boundary. On the practical side of things, of course only finitely many measurements at finitely many points are available and discrete versions of the problem need to be studied. One also needs a careful modeling of how the measurements are implemented with electrodes. There will always be some noise in the measured DN map, and the stability result above may not apply since the noisy version of the true DN map may not be a DN map corresponding to some conductivity. Therefore, an improved stability analysis possibly including a regularization strategy for the numerical algorithm would be of interest. These topics are out of the scope of this book.

The anisotropic Calderón problem. For certain materials, called *isotropic* materials, if you apply a voltage $u(x)$, then the current at x is in the direction opposite to the voltage gradient, $\nabla u(x)$, and has magnitude proportional to the magnitude of the voltage gradient, with the constant of proportionality called the conductivity and denoted $\gamma(x)$. So, for isotropic materials, $i(x) = -\gamma(x)\nabla u(x)$. The results mentioned previously in this section are concerned with isotropic materials. However, there are more complicated *anisotropic* materials where the current at x need not be parallel to $\nabla u(x)$ and the magnitude of the current depends on the direction as well as the magnitude of $\nabla u(x)$. For these materials, $i(x) = -\gamma(x)\nabla u(x)$, but with $\gamma(x)$ being an $n \times n$ matrix, rather than just a number. In general, $\gamma(x)$ is a positive definite, symmetric, $n \times n$ matrix.

Theorem 1.2 showed that, for $n \geq 2$, the map Λ_γ does indeed determine an isotropic conductivity satisfying suitable assumptions. However, it cannot possibly determine anisotropic conductivities for the following obvious reason. Let $\Psi : \bar{\Omega} \to \bar{\Omega}$ be any diffeomorphism that is the identity map in some neighbourhood of $\partial\Omega$ and set

$$\tilde{\gamma} = \Big[\frac{1}{|\det(D\Psi)|}(D\Psi)\gamma(D\Psi)^t\Big] \circ \Psi^{-1}, \qquad \tilde{u} = u \circ \Psi^{-1},$$

where $D\Psi$ is the Jacobian matrix (the matrix of first partial derivatives) of Ψ. In fact, a change of variables shows that

(1.3)

$$\left.\begin{aligned} \nabla\cdot\big[\gamma(x)\nabla u(x)\big] &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega \end{aligned}\right\} \iff \left\{\begin{aligned} \nabla\cdot\big[\tilde{\gamma}(x)\nabla \tilde{u}(x)\big] &= 0 \quad \text{in } \Omega \\ \tilde{u} &= f \quad \text{on } \partial\Omega \end{aligned}\right.$$

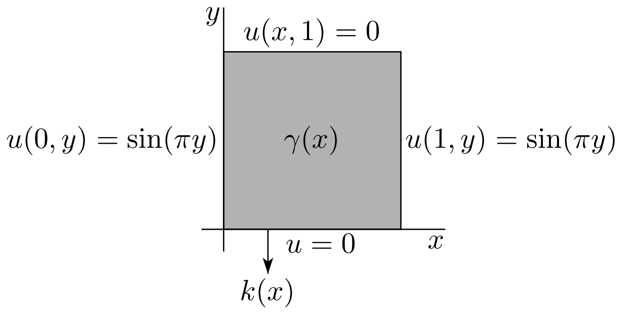
Let u_f and $\tilde{u}_f$ denote the solutions of the left and right hand boundary value problems of (1.3), respectively. By definition, $\Lambda_\gamma(f)(x) = \hat{n}(x) \cdot \gamma(x)\nabla u_f(x)_{\partial\Omega}$ and

$$\Lambda_{\tilde{\gamma}}(f)(x) = \hat{n}(x) \cdot \tilde{\gamma}(x)\nabla \tilde{u}_f(x)|_{\partial\Omega}$$

Since Ψ is the identity map in some neighbourhood of $\partial\Omega$, $D\Psi(x) = \mathbb{1}$, $\tilde\gamma(x) = \gamma(x)$ and $\tilde u_f(x) = u_f\big(\Psi^{-1}(x)\big) = u_f(x)$ for all x in that neighbourhood of $\partial\Omega$. Thus $\Lambda_\gamma = \Lambda_{\tilde\gamma}$. In §**??** we prove that, for $n = 2$, Λ_γ determines anisotropic conductivities up to diffeomorphisms like this. It is conjectured that this is also true for $n \geq 3$.

Problems and examples.

Example 1.6. Here is a much simplified example in which an isotropic conductivity is computed from a Dirichlet to Neumann map. The region is the square $\Omega = \big\{ \ (x, y) \in \mathbb{R}^2 \ \big| \ 0 < x, y < 1 \ \big\}$. To reduce the number of variables that we are dealing with, we assume that the conductivity is a function of x only. As $\gamma(x)$ is a function only of a single variable, we hope to be able to determine it by measuring just one function of a single variable. We choose to measure the current $k(x) = \gamma(x)\frac{\partial u}{\partial y}\big|_{y=0}$ at the base of the square that results from applying the boundary voltage function specified in the figure



So our boundary value problem is

$$(1.4) \qquad \begin{array}{lll} \text{(a)} & \nabla \cdot \big[\gamma(x)\nabla u(x, y)\big] = 0 & \text{in } \Omega \\ \text{(b)} & u(0, y) = u(1, y) = \sin(\pi y) & \text{for all } 0 \leq y \leq 1 \\ \text{(c)} & u(x, 0) = u(x, 1) = 0 & \text{for all } 0 \leq x \leq 1 \end{array}$$

The standard technique for solving the boundary value problem (1.4) is to Fourier expand $u(x, y) = \sum_{n=1}^{\infty} a_n(x)\sin(n\pi y)$. From the boundary condition (b), we would expect to only need the $n = 1$ term. So we look for a solution of the form $u(x, y) = a(x)\sin(\pi y)$. Boundary condition (c) is satisfied for all functions $a(x)$. Boundary condition (b) is satisfied if and only if $a(0) = a(1) = 1$. The differential equation (a) is satisfied if and only if

$$\begin{aligned} 0 &= \nabla \cdot \big(\gamma(x)a'(x)\sin(\pi y)\,,\ \pi\gamma(x)a(x)\cos(\pi y)\big) \\ &= \sin(\pi y)\big[\big(\gamma(x)a'(x)\big)' - \pi^2\gamma(x)a(x)\big] \end{aligned}$$

which is the case if and only if

$$(1.5) \qquad \big(\gamma(x)a'(x)\big)' - \pi^2\gamma(x)a(x) = 0 \qquad \text{for all } 0 < x < 1$$

We imagine that we have measured

$$k(x) = \gamma(x)\frac{\partial u}{\partial y}\Big|_{y=0} = \gamma(x)\pi a(x)\cos(\pi y)\ \Big|_{y=0} = \pi\gamma(x)a(x)$$

and that we wish to determine $\gamma(x)$. We can do so by subbing $\gamma(x) = \frac{k(x)}{\pi a(x)}$ into (1.5) and solving for a.

$$\begin{aligned}\big(k(x)\frac{a'(x)}{a(x)}\big)' = \pi^2 k(x) &\implies \frac{d}{dx}\big[k(x)\frac{d}{dx}\ln a(x)\big] = \pi^2 k(x)\\ &\implies k(x)\frac{d}{dx}\ln a(x) = \pi^2\int_0^x k(t)\ dt - \pi^2 C\\ &\implies \ln a(x) = \pi^2\int_0^x \frac{1}{k(s)}\Big[\int_0^s k(t)\ dt - C\Big]\ ds + D\end{aligned}$$

for some constants C and D. To satisfy the boundary condition $a(0) = 1$, we need $D = 0$ and to satisfy $a(1) = 1$, we need

$$C = \Big[\int_0^1 \frac{ds}{k(s)}\Big]^{-1}\Big[\int_0^1 \frac{ds}{k(s)}\int_0^s k(t)\ dt\Big]$$

This determines[1] $a(x)$ and hence $\gamma(x) = \frac{k}{\pi a(x)}$.

Exercise 1.7. Prove (1.3) by integrating the first equation against a test function $\varphi \in C_c^\infty(\Omega)$, the divergence theorem and a suitable change of variables.

Exercise 1.8. Find the Dirichlet to Neumann map when $\Omega = \big\{\ x \in \mathbb{R}^2\ \big|\ |x| < 1\ \big\}$ and the conductivity $\gamma(x) \equiv 1$.

Exercise 1.9. Let $\Omega = (-\infty, 0) \times S^1$. Functions on Ω can be identified with those functions $u(x,\theta)$ that are defined for $x < 0$ and all $\theta \in \mathbb{R}$ and that are periodic of period 2π in θ. The gradient operator for Ω is $\nabla = \big(\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta}\big)$. Find the Dirichlet to Neumann map when the conductivity $\gamma(x,\theta) \equiv 1$. Assume that potentials $u(x,\theta)$ must remain bounded in the limit $x \to -\infty$.

Exercise 1.10. Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the divergence theorem is applicable to Ω. Let $\gamma(x)$ be a real–valued C^∞ function on Ω all of whose derivatives are bounded. Suppose that the complex

[1] If you are worried about dividing by k in the integrals, you shouldn't be. We know that $0 \le u \le 1$ on $\partial\Omega$. By the maximum principle, this implies that $0 < u < 1$ in the interior of Ω. This in turn forces $\frac{\partial u}{\partial y} \ge 0$ when $y = 0$. In fact, by the strong maximum principle [Ev, §6.4.2], $\frac{\partial u}{\partial y} > 0$ for $y = 0$, which ensures that $k(x) > 0$ for all $0 \le x \le 1$.

numbers λ, μ and the, not identically zero, functions $\varphi,\ \psi \in C^2(\overline{\Omega})$ obey

$$\begin{aligned} &\nabla \cdot [\gamma(x)\nabla\varphi(x)] = \lambda\varphi(x) && \text{for all } x \in \Omega \\ &\varphi(x) = 0 && \text{for all } x \in \partial\Omega \\ &\nabla \cdot [\gamma(x)\nabla\psi(x)] = \mu\psi(x) && \text{for all } x \in \Omega \\ &\psi(x) = 0 && \text{for all } x \in \partial\Omega \end{aligned}$$

We say that φ and ψ are eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. The numbers λ and μ are the corresponding eigenvalues.

(a) Prove that $\lambda, \mu \in \mathbb{R}$.

(b) Prove that if $\lambda \neq \mu$ then φ and ψ are orthogonal in $L^2(\Omega)$. In other words, prove that $\int_\Omega \varphi(x)\overline{\psi(x)}\ d^n x = 0$.

(c) Suppose that $\gamma(x) > 0$ for all $x \in \Omega$. Prove that $\lambda, \mu < 0$.

(d) Let $\mathcal{H}$ be the closure of the subspace of $L^2(\Omega)$ spanned by the eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Prove that there is an orthonormal basis for $\mathcal{H}$ consisting of real–valued eigenfunctions.

Exercise 1.11. Let Ω and γ be as in Problem 1.10. Assume that we already know

- an orthonormal basis for $L^2(\Omega)$ consisting of C^2 eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma\nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Call the eigenfunctions and corresponding eigenvalues $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ and $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$.
- a linear map $E : C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$ such that $(Ef)(x) = f(x)$ for all $x \in \partial\Omega$.

Find the Dirichlet to Neumann map for conductivity γ.

Exercise 1.12. Apply the method of Problem 1.11 to find the Dirichlet to Neumann map for $\left\{\ x \in \mathbb{R}^2\ \middle|\ |x| < 1\ \right\}$ with conductivity $\gamma \equiv 1$. You may assume that a suitable orthonormal basis exists.

Exercise 1.13. Let Ω be an open subset of $\mathbb{R}^n$. Let $\gamma \in C^1(\Omega)$ be bounded away from zero. Find $q, \beta \in C^1(\Omega)$ such that

$$\nabla \cdot [\gamma\nabla u] = 0 \iff (-\Delta + q)v = 0 \text{ for } v = \beta u$$

Exercise 1.14. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\Lambda_\gamma(f)$ denote the Dirichlet to Neumann map for the conductivity $\gamma(x)$. Let $\beta(x)$ be a C^∞ function on Ω all of whose derivatives are bounded. Compute $\frac{d}{dt}\Lambda_{\gamma+t\beta}(f)\big|_{t=0}$. Assume that we already know

- a complete orthonormal basis for $L^2(\Omega)$ consisting of C^2 eigenfunctions for the differential operator $u \mapsto \nabla \cdot [\gamma \nabla u]$ with Dirichlet boundary conditions on $\partial\Omega$. Call the eigenfunctions and corresponding eigenvalues $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ and $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$.
- the solution to the boundary value problem $\nabla \cdot [\gamma \nabla u] = 0$ in Ω, $u = f$ on $\partial\Omega$. Call the solution $u_0(x)$.

1.2. Optical tomography

Optical tomography is concerned with the determination of spatially varying optical absorption and scattering properties of a medium by measuring the response of the medium to transmitted near–infrared light. This has been proposed as a diagnostic tool in medicine. The standard model for propagation of photons is the radiative transfer equation, also called the linear Boltzmann equation. The main point in this section is to indicate that in the diffusion approximation, which is often used to model highly scattering media, the optical tomography problem essentially reduces to the inverse problem of Calderón.

Consider photons propagating in a medium $\Omega \subset \mathbb{R}^n$ where absorption and scattering may occur, and assume that there are no sources of photons in Ω. The function $\phi(x, v, t)$, modeling the density of photons at point $x \in \Omega$ moving in direction $v \in S^{n-1}$ at time t, solves the radiative transfer equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + v \cdot \nabla_x + \mu_{tr}(x)\right)\phi(x, v, t) = \mu_s(x)\int_{S^{n-1}} p(v \cdot v')\phi(x, v', t)\,dv'.$$

Here c is the speed of light, $\mu_{tr} = \mu_a + \mu_s$ where $\mu_a, \mu_s \in C(\overline{\Omega})$ are the absorption and scattering coefficients, and $p(v, v') = p(v \cdot v')$ with $p \in C([-1, 1])$ is the scattering kernel representing the probability of photons traveling in direction v to scatter in direction v'. We will also make use of the photon density

$$\Phi(x, t) = \int_{S^{n-1}} \phi(x, v, t)\,dv$$

and the current $J = (J_1, \ldots, J_n)$ where

$$J_k(x, t) = \int_{S^{n-1}} v_k\phi(x, v, t)\,dv.$$

In the diffusion approximation, the following two assumptions that are valid in predominantly scattering media are made:

(1) The density $\phi(x, v, t)$ is only weakly dependent on the direction v.

(2) The current $J(x, t)$ is constant with respect to time.

Assume for simplicity that $\Omega \subset \mathbb{R}^2$, so unit vectors can be written as $v_\theta = (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi]$. We formally expand $\phi(x, v_\theta, t)$ in complex Fourier series with respect to θ,

$$\phi(x, v_\theta, t) = \sum_{k=-\infty}^{\infty} \phi_k(x, t) e^{ik\theta}.$$

A simple computation shows that

$$v_\theta \cdot \nabla_x \phi = e^{i\theta} \partial\phi + e^{-i\theta} \bar{\partial}\phi$$

where ∂ and $\bar{\partial}$ are the complex derivatives

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Exercise 1.15. Prove the above representation for $v_\theta \cdot \nabla_x \phi$.

Thus, the radiative transfer equation may formally be written as

(1.6)

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + e^{i\theta}\partial + e^{-i\theta}\bar{\partial} + \mu_{tr}\right) \sum_{k=-\infty}^{\infty} \phi_k(x, t) e^{ik\theta} = \mu_s \int_{S^1} p(v \cdot v')\phi(x, v', t)\, dv'.$$

To express also the right hand side as a series in $e^{i\theta}$, we will use two facts about Chebyshev polynomials T_k: the set $\{T_0/\sqrt{\pi}\} \cup \{\sqrt{2/\pi} T_k\}_{k=1}^{\infty}$ is an orthonormal basis of the L^2 space on $(-1, 1)$ with weight $1/\sqrt{1-t^2}$, and these polynomials have the important property that

$$T_k(\cos\theta) = \cos(k\theta). \tag{1.7}$$

Write the scattering kernel as the Chebyshev polynomial expansion

$$p(t) = \sum_{k=0}^{\infty} p_k T_k(t), \quad t \in (-1, 1).$$

We have $p_0 = 1/2\pi$ by the following problem:

Exercise 1.16. Assume that p is a probability density function in the sense that

$$\int_0^{2\pi} p(v_\theta \cdot v_{\theta'})\, d\theta' = 1 \text{ for all } \theta \in [0, 2\pi].$$

Show that $p_0 = 1/2\pi$.

Then formally, by using (1.7),

$$\int_0^{2\pi} p(v_\theta \cdot v_{\theta'})\phi(x, v_{\theta'}, t)\, d\theta' = \sum_{j=0}^{\infty}\sum_{l=-\infty}^{\infty}\int_0^{2\pi} p_j T_j(\cos(\theta-\theta'))\phi_l(x,t)e^{il\theta}\, d\theta'$$

$$= \sum_{j=0}^{\infty}\sum_{l=-\infty}^{\infty}\int_0^{2\pi} p_j \frac{e^{ij(\theta-\theta')}+e^{-ij(\theta-\theta')}}{2}\phi_l(x,t)e^{il\theta'}\, d\theta'$$

$$= 2\pi p_0\phi_0(x,t) + \pi\sum_{j=1}^{\infty} p_j\left[e^{ij\theta}\phi_j + e^{-ij\theta}\phi_{-j}\right].$$

We insert the last expression in (1.6) and consider the three equations obtained by collecting the $e^{ik\theta}$ terms for $k = 0, 1, -1$. In these equations, motivated by the assumption that ϕ is only weakly dependent on v, we also make the approximation that

$$\phi_k = 0 \quad \text{for } |k| \geq 2.$$

The resulting equations are

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu_{tr}(x)\right)\phi_0 + \bar{\partial}\phi_1 + \partial\phi_{-1} = 2\pi\mu_s p_0\phi_0,$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu_{tr}(x)\right)\phi_1 + \partial\phi_0 = \pi\mu_s p_1\phi_1,$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu_{tr}(x)\right)\phi_{-1} + \bar{\partial}\phi_0 = \pi\mu_s p_1\phi_{-1}.$$

Now, note that the photon density and current are given by

$$\Phi(x,t) = 2\pi\phi_0(x,t)$$

and

$$J(x,t) = \left(\int_0^{2\pi}\frac{e^{i\theta}+e^{-i\theta}}{2}\phi(x,v_\theta,t)\, d\theta, \int_0^{2\pi}\frac{e^{i\theta}-e^{-i\theta}}{2i}\phi(x,v_\theta,t)\, d\theta\right)$$

$$= 2\pi\left(\frac{\phi_1+\phi_{-1}}{2}, \frac{i(\phi_1-\phi_{-1})}{2}\right).$$

Consequently

$$2\pi\phi_{-1} = J_1 + iJ_2, \quad 2\pi\phi_1 = J_1 - iJ_2.$$

Using these expressions in the three equations obtained above and taking suitable combinations, and using that $p_0 = 1/2\pi$, we get the two equations

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu_a(x)\right)\Phi + \nabla_x \cdot J = 0,$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu_a(x) + (1-\pi p_1)\mu_s(x)\right)J + \frac{1}{2}\nabla_x\Phi = 0.$$

Exercise 1.17. Verify these two equations.

We now make the approximation that J is constant in time,

$$\frac{\partial J}{\partial t} = 0.$$

The second equation becomes

$$J = -D(x)\nabla_x \Phi$$

where $D = 1/(2(\mu_a + (1-\pi p_1)\mu_s))$ is the diffusion coefficient. Inserting this expression for J into the first equation, we obtain

$$-\nabla_x \cdot D(x)\nabla_x \Phi + \mu_a(x)\Phi + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0.$$

Finally, suppose that our transmitters emit light at a given frequency $\omega > 0$, so that

$$\Phi(x,t) = u(x)e^{i\omega t}.$$

Then u will solve the equation

$$-\nabla \cdot D\nabla u + \left(\mu_a + \frac{i\omega}{c}\right) u = 0 \text{ in } \Omega.$$

A similar equation may be derived in dimensions $n \geq 3$, if the Fourier series in the previous argument are replaced by expansions in spherical harmonics. The inverse problem in diffuse optical tomography (with infinitely many measurements) is then to determine D and μ_a from the Dirichlet to Neumann map associated to this equation. This problem is analogous to the inverse conductivity problem.

1.3. Inverse scattering

Suppose that we are interested in a system in which sound waves, for example, scatter off of some obstacle. Let $p(x,t)$ be the pressure at position x and time t. In (a somewhat idealized) free space, p obeys the wave equation $\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$, where c is the speed of sound. We shall assume that in most of the world, c takes a constant value c_0. But we introduce an obstacle by allowing c to depend on position in some compact region. We further allow for some absorption in that region. Then p obeys the damped wave equation

$$\frac{\partial^2 p}{\partial t^2} + \gamma(x)\frac{\partial p}{\partial t} = c(x)^2 \Delta p$$

where $\gamma(x)$ is the damping coefficient of the medium at x. If the solution p has fixed (temporal) frequency, then $p(x,t) = \operatorname{Re}\left[u(x)e^{-i\omega t}\right]$ where u satisfies

$$\Delta u + \frac{\omega^2}{c(x)^2}\left[1 + i\frac{\gamma(x)}{\omega}\right]u = 0.$$

Outside of some compact region, the coefficient in brackets is constant:

$$\frac{\omega^2}{c(x)^2}\big[1+i\frac{\gamma(x)}{\omega}\big]=\frac{\omega^2}{c_0^2}=k^2 \qquad \text{where} \qquad k=\frac{\omega}{c_0}>0.$$

If we define the index of refraction by

$$n(x)=\frac{c_0^2}{c(x)^2}\big[1+i\frac{\gamma(x)}{\omega}\big]$$

then

$$\Delta u+k^2 n(x)u=0 \tag{1.8}$$

with $n(x)=1$ outside of some compact region. For concreteness, we restrict to three dimensions for the rest of this section. We first consider two special cases.

Example 1.18 (Free space)**.** In the absence of any obstacle, we have $\Delta u+k^2u=0$ on all of $\mathbb{R}^3$. Any function $u=e^{ik\theta\cdot x}$, where θ is a unit vector, is a solution that represents a plane wave moving in direction θ. It is also true that any solution of this equation (say, in the class of tempered distributions) can be obtained as a superposition of these plane waves when interpreted in the right way.

Example 1.19 (Point source)**.** If we have free space everywhere except at the origin and we have a unit point source at the origin, then

$$\Delta u+k^2u=\delta(x)$$

where the Dirac delta function $\delta(x)$ is a distribution (generalized function) that is determined formally by the properties that $\delta(x)=0$ for all $x\neq 0$ and $\int_{\mathbb{R}^3}\delta(x)\,dx=1$. A rigorous version of "$\Delta u+k^2u=\delta(x)$" is provided in Problem 1.20 below. Except at the origin, where there is a singularity, we still have $\Delta u+k^2u=0$. The point source generates expanding spherical waves. So u should be a function of $r=|x|$ only and obey

$$u''(r)+\frac{2}{r}u'(r)+k^2u(r)=0.$$

This equation is easily solved by changing variables to $v(r)=ru(r)$, which obeys

$$v''(r)+k^2v(r)=0.$$

So $v(r)=\alpha\sin(kr)+\beta\cos(kr)$ and $u(r)=\alpha\frac{\sin(kr)}{r}+\beta\frac{\cos(kr)}{r}$. To be an outgoing (rather than incoming) wave, we require that $u(r)=\alpha'\frac{e^{ikr}}{r}$. (Note that $e^{ikr}e^{-i\omega t}$ is constant on $r=\frac{\omega}{k}t$, which is a sphere that is expanding with speed c_0.) To give the Dirac delta function on the right hand side of $\Delta u+k^2u=\delta(x)$, we need $u(x)=-\frac{e^{ik|x|}}{4\pi|x|}$.

Exercise 1.20. Set $\Phi(x)=-\frac{e^{ik|x|}}{4\pi|x|}$.

(a) Prove that $\Delta\Phi(x) + k^2\Phi(x) = 0$ for all $x \neq 0$.

(b) Let B_ε be the ball of radius ε centered on the origin and let dS be the surface measure on ∂B_ε. Prove that, for any continuous function $\psi(x)$,

$$\lim_{\varepsilon\to 0+} \int_{\partial B_\varepsilon} \frac{\psi(x)}{|x|^p}\, dS = \begin{cases} 4\pi\psi(0) & \text{if } p = 2 \\ 0 & \text{if } p < 2. \end{cases}$$

(c) (c) Let $\psi(x) \in C_c^\infty(\mathbb{R}^3)$. Prove that

$$\int_{\mathbb{R}^3} \Phi(x)\big[\Delta\psi(x) + k^2\psi(x)\big]\, dx = \psi(0).$$

Now let us return to the general case. We want to think of a physical situation in which we send a plane wave $u^i(x) = e^{ik\theta\cdot x}$ in from infinity. This plane wave shakes up the obstacle which then emits a bunch of expanding spherical waves $\frac{e^{ik|x-y|}}{|x-y|}$ emanating from various points y in the obstacle. So the full solution is of the form

$$u(x) = u^i(x) + u^s(x)$$

where the scattered wave, u^s, obeys the "radiation condition"

$$\frac{\partial}{\partial r}u^s(x) - iku^s(x) = O\Big(\frac{1}{|x|^2}\Big) \qquad \text{as} \qquad |x| \to \infty. \tag{1.9}$$

This condition is chosen to allow outgoing waves $\frac{e^{ik|x-y|}}{|x-y|}$ but not incoming waves $\frac{e^{-ik|x-y|}}{|x-y|}$.

Let, as in Problem 1.20,

$$\Phi(x) = -\frac{e^{ik|x|}}{4\pi|x|}.$$

Since $\delta(x-y)$ is the kernel of the identity operator, the equality

$$(\Delta_x + k^2)\Phi(x-y) = \delta(x-y)$$

says, roughly, that $u(x) \mapsto \int \Phi(x-y)u(y)\,dy$ is the inverse of the map $u(x) \mapsto (\Delta + k^2)u(x)$ for functions that obey the radiation condition. We can exploit this to convert (1.8), (1.9) into an equivalent integral equation

$$\begin{aligned} \Delta u + k^2 n(x) u = 0 &\iff \Delta u + k^2 u = k^2\big(1 - n(x)\big)u \\ &\iff \Delta u^s + k^2 u^s = k^2\big(1 - n(x)\big)u \end{aligned}$$

since $\Delta u^i + k^2 u^i = 0$. As u^s obeys the radiation condition, we have

$$u^s(x) = k^2 \int_{\mathbb{R}^3} \Phi(x-y)\big(1 - n(y)\big)u(y)\,dy$$

so that

$$u(x) = u^i(x) + k^2 \int_{\mathbb{R}^3} \big(1 - n(y)\big)\Phi(x-y)u(y)\,dy. \tag{1.10}$$

This is called the Lippmann–Schwinger equation. Observe that it is of the form $u = u^i + Fu$ or $(\mathbb{1} + F)u = u^i$ where F is the linear operator $u(x) \mapsto k^2 \int \Phi(x-y)\big(1-n(y)\big)u(y)\,dy$. This operator is compact (if you impose the appropriate norms) and so behaves much like a finite dimensional matrix. If F has operator norm smaller than one, which is the case if $k^2(1-n)$ is small enough, then $\mathbb{1}+F$ is trivially invertible and the equation $(\mathbb{1}+F)u = u^i$ has a unique solution. Even if F has operator norm larger than or equal to one, $(\mathbb{1}+F)u = u^i$ fails to have a unique solution only if F has eigenvalue minus one. One can show that this is impossible in the present setting. Thus, one can prove the following result.

Theorem 1.21. *If $n \in C^2(\mathbb{R}^3)$, $n(x)-1$ has compact support and $\operatorname{Re} n(x), \operatorname{Im} n(x) \geq 0$, then* (1.8), (1.9) *has a unique solution.*

For $|y|$ bounded and $|x|$ large, $\Phi(x-y)$ has the asymptotic behaviour

$$\Phi(x-y) = -\frac{e^{ik|x|}}{4\pi|x|}e^{-ik\hat{x}\cdot y} + O\big(\frac{1}{|x|^2}\big) \tag{1.11}$$

so that, when the incoming plane wave is moving in direction θ,

$$u(x;\theta) = u^i(x;\theta) + \frac{e^{ik|x|}}{4\pi|x|}u_\infty(\hat{x};\theta) + O\big(\frac{1}{|x|^2}\big) \tag{1.12}$$

where

$$u_\infty(\hat{x};\theta) = -k^2 \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y}\big(1-n(y)\big)u(y;\theta)\,dy.$$

If we are observing the scattered wave from vantage points far from the obstacle, we will only be able to measure $u_\infty(\hat{x};\theta)$ for $\hat{x} \in S^2$. This is called the *far field pattern*, or *scattering amplitude*, of n corresponding to the incoming wave $u^i(x) = e^{ik\theta\cdot x}$. Assuming that we can send incoming waves at a fixed frequency $k > 0$ from all possible directions θ, and that we can measure the corresponding far field patterns for all $\hat{x}$, the inverse problem may be formulated in the following way.

Fixed frequency inverse scattering problem: Given $u_\infty(\hat{x};\theta)$ for all $\hat{x}, \theta \in S^2$ and for fixed $k > 0$, can we determine n?

The answer is yes, as shown by the following theorem.

Theorem 1.22. *Fix $k > 0$. If $n_1, n_2 \in C^2(\mathbb{R}^3)$ with n_1-1, n_2-1 of compact support and $u_{1,\infty}(\hat{x};\theta) = u_{2,\infty}(\hat{x};\theta)$, for all $\hat{x}, \theta \in S^2$, then $n_1 = n_2$.*

We can get a rough idea why this theorem is true by looking at the Born approximation. In this approximation u^s is ignored in the computation of

u_∞ so that

$$u_\infty(\hat{x};\theta) \approx -k^2 \int e^{-ik\hat{x}\cdot y}\big(1-n(y)\big)u^i(y;\theta)\,dy$$
$$= -k^2 \int e^{-ik(\hat{x}-\theta)\cdot y}\big(1-n(y)\big)\,dy.$$

If we measure $u_\infty(\hat{x};\theta)$, then, in this approximation, we know the Fourier transform of $1-n(y)$ on the set $\{\ k(\hat{x}-\theta)\ \big|\ \hat{x},\theta \in S^2\ \}$ which is exactly the closed ball of radius $2k$ centered on the origin in $\mathbb{R}^3$. Since $1-n(y)$ is of compact support, its Fourier transform is analytic. So knowledge of the Fourier transform on any open ball uniquely determines it.

It turns out that the scattering amplitude for n at a fixed frequency is an analog of the Dirichlet to Neumann map, except that the measurements are made far away (on the sphere at infinity) instead of on the boundary of a domain. We shall discuss a quantum mechanical analog of the above classical inverse scattering problem in §7, and the methods for dealing with that problem will be very similar to those applied to the Calderón problem.

Exercise 1.23. Prove (1.11).

Exercise 1.24. Let $f \in C_c^\infty(\mathbb{R}^3)$. Prove that

$$F(x) = \int_{\mathbb{R}^3} \Phi(x-y)f(y)\,dy$$

obeys $\Delta F + k^2 F = f$ and the radiation condition.

1.4. Notes

Section 1.1. Example 1.6 is due to Kohn-Vogelius.

Section 1.2. See the survey of Arridge.

Section 1.3. See Colton-Kress for more information on acoustic inverse scattering.

Chapter 2

Formulation of the Calderón problem

In this chapter we formulate rigorously the inverse boundary value problem for the conductivity equation

$$\operatorname{div}(\gamma \nabla u) = 0$$

and for the related Schrödinger equation

$$(-\Delta + q)u = 0.$$

This will include a discussion of weak solutions of the corresponding Dirichlet problems, the definition of boundary measurements in terms of the Dirichlet-to-Neumann map (DN map for short), and basic properties of the DN map. We will also derive useful integral identities that allow to relate boundary measurements of solutions to the interior values of the coefficients. The section will end with a reduction of the inverse boundary value problem for conductivity equation to an inverse problem for Schrödinger equation.

2.1. Calculus facts

This preliminary section collects some basic notation and facts from multivariable calculus. In the course of the book we will frequently need to work locally in small sets and then patch up these local constructions into global ones. Therefore, we will also discuss partitions of unity that are the standard tool in such local arguments.

Convention. All functions will be complex valued unless stated otherwise.

Definition 2.1. (C^k spaces) Let Ω be an open set in $\mathbb{R}^n$ (not necessarily bounded). If $k \in \mathbb{N}_0$, we denote by $C^k(\Omega)$ the set of functions that are k times continuously differentiable in Ω. By $C^k_c(\Omega)$ we denote the set of compactly supported functions in $C^k(\Omega)$, and $C^k(\overline{\Omega})$ is the set of functions in $C^k(\Omega)$ whose partial derivatives up to order k extend continuously to $\overline{\Omega}$. By $C^\infty(\Omega)$, $C^\infty_c(\Omega)$, and $C^\infty(\overline{\Omega})$ we denote the corresponding sets of infinitely many times continuously differentiable functions.

For instance,

$$C^\infty_c(\Omega) = \{f \in C^\infty(\Omega)\,;\, \mathrm{supp}\,(f) \text{ is a compact subset of } \Omega\},$$

where the support is defined by $\mathrm{supp}\,(f) = \Omega \backslash \{x \in \Omega\,;\, f = 0 \text{ in some neighborhood of } x\}$. The next definition gives examples of functions in $C^\infty_c(\mathbb{R}^n)$, in particular showing that this set is not empty.

Definition 2.2. (Mollifiers) Define the function

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where the constant C is chosen so that $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$. If $\varepsilon > 0$ define the mollifier

$$\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon).$$

Exercise 2.3. Prove that η and η_ε are in $C^\infty_c(\mathbb{R}^n)$.

Note that $\eta_\varepsilon \in C^\infty_c(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = 1, \qquad \mathrm{supp}\,(\eta_\varepsilon) = \overline{B(0,\varepsilon)}.$$

If ε is small, the function η_ε looks like a sharp peak at the origin with area under the peak equal to one. It is thus a smooth approximation of the Dirac delta function at the origin, and it can be used to approximate a locally integrable function by smooth functions. Recall that the *convolution* of two functions $f, g : \mathbb{R}^n \to \mathbb{C}$ is the function $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

The convolution $f * g$ is well defined when $f, g \in L^1(\mathbb{R}^n)$, in which case $f * g \in L^1(\mathbb{R}^n)$ essentially by Fubini's theorem. If $f \in L^1_{loc}(\mathbb{R}^n)$ (meaning that $f|_K \in L^1(K)$ for any compact $K \subset \mathbb{R}^n$) but $g \in L^1(\mathbb{R}^n)$ has compact support, then similarly $f * g \in L^1_{loc}(\mathbb{R}^n)$. If additionally g is smooth, then also $f * g$ is smooth as in the next theorem.

Theorem 2.4. *(Mollifications) If $f \in L^1_{loc}(\mathbb{R}^n)$, define for $\varepsilon > 0$ the mollifications of f by*

$$f_\varepsilon = f * \eta_\varepsilon.$$

(a) $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ *for any* $\varepsilon > 0$, *and* $\partial^\alpha f_\varepsilon = f * \partial^\alpha \eta_\varepsilon$.

(b) $\mathrm{supp}\,(f_\varepsilon) \subset \{x \in \mathbb{R}^n\,;\,\mathrm{dist}(x, \mathrm{supp}\,(f)) \leq \varepsilon\}$.

(c) If $f \in C^0(\mathbb{R}^n)$, *then* $f_\varepsilon \to f$ *uniformly on compact sets in* $\mathbb{R}^n$ *as* $\varepsilon \to 0$.

(d) If $f \in L^p(\mathbb{R}^n)$ *where* $1 \leq p < \infty$, *then* $f_\varepsilon \to f$ *in* L^p *as* $\varepsilon \to 0$.

Proof. (a) Since $\eta_\varepsilon \in C_c^0(\mathbb{R}^n)$ and $|f|$ is integrable on compact sets, the function

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\eta_\varepsilon(x-y)\,dy$$

is well defined for all $x \in \mathbb{R}^n$. Fix $x \in \mathbb{R}^n$ and note that for any $h \in B(0,1)$,

$$f_\varepsilon(x+h) - f_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\left[\eta_\varepsilon(x+h-y) - \eta_\varepsilon(x-y)\right]\,dy.$$

The integral is over the set of all y for which $|x-y| \leq \varepsilon$ or $|x+h-y| \leq \varepsilon$. This is a compact set (depending on x and ε) that we denote by K. Also, since η_ε is C^∞, we have the Taylor expansion

$$\eta_\varepsilon(x-y+h) = \eta_\varepsilon(x-y) + \nabla\eta_\varepsilon(x-y)\cdot h + \frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2\eta_\varepsilon}{\partial x_j \partial x_k}(x-y+h')h_j h_k$$

where $h' \in \mathbb{R}^n$ is some point on the line segment between 0 and h. It follows that

$$f_\varepsilon(x+h) - f_\varepsilon(x) = \int_K f(y)\left[\nabla\eta_\varepsilon(x-y)\cdot h + \sum_{j,k=1}^n b_{jk}(y;x,h)h_j h_k\right]\,dy.$$

where $|b_{jk}(y;x,h)| \leq C$ uniformly over $y \in K$ and $h \in B(0,1)$. By dominated convergence we obtain, as $h \to 0$

$$f_\varepsilon(x+h) - f_\varepsilon(x) = \sum_{j=1}^n \left[\int_{\mathbb{R}^n} f(y)\partial_j\eta_\varepsilon(x-y)\,dy\right] h_j + O(h^2).$$

This shows that f_ε is differentiable. Repeating the argument for higher order derivatives gives that $f_\varepsilon \in C^\infty$ and $\partial^\alpha f_\varepsilon = f * \partial^\alpha \eta_\varepsilon$.

(b) Clear since $\mathrm{supp}\,(\eta_\varepsilon) \subset \overline{B(0,\varepsilon)}$.

(c) Let $x \in K$ where $K \subset \mathbb{R}^n$ is compact, and fix $\varepsilon_0 > 0$. Since $\int \eta_\varepsilon\,dy = 1$, we have

$$\begin{aligned}|f_\varepsilon(x) - f(x)| &= \left|\int_{\mathbb{R}^n} f(x-y)\eta_\varepsilon(y)\,dy - \int_{\mathbb{R}^n} f(x)\eta_\varepsilon(y)\,dy\right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)|\,\eta_\varepsilon(y)\,dy.\end{aligned}$$

Note that the integral is over $\overline{B(0,\varepsilon)}$. The uniform continuity of f on compact sets implies that there is $\delta_0 > 0$ such that

$$|f(x-y) - f(x)| < \varepsilon_0 \quad \text{whenever } x \in K \text{ and } |y| < \delta_0.$$

This shows that $\|f_\varepsilon - f\|_{L^\infty(K)} < \varepsilon_0$ whenever $\varepsilon < \delta_0$.

(d) To prove L^p convergence, we use the fact that $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ if $1 \leq p < \infty$. Fix $\varepsilon_0 > 0$ and choose some $g \in C_c(\mathbb{R}^n)$ with

$$\|f - g\|_{L^p(\mathbb{R}^n)} < \varepsilon_0/3.$$

By the triangle inequality

$$\|f_\varepsilon - f\|_{L^p} \leq \|f_\varepsilon - g_\varepsilon\|_{L^p} + \|g_\varepsilon - g\|_{L^p} + \|g - f\|_{L^p}\,.$$

The supports of g and g_ε are contained in some compact set K, and

$$\|g_\varepsilon - g\|_{L^p(\mathbb{R}^n)} = \|g_\varepsilon - g\|_{L^p(K)} \leq C_K \|g_\varepsilon - g\|_{L^\infty(K)}\,.$$

By part (c) in this theorem, there is $\delta_0 > 0$ such that $\|g_\varepsilon - g\|_{L^p} < \varepsilon_0/3$ whenever $\varepsilon < \delta_0$. Thus, for $\varepsilon < \delta_0$,

$$\|f_\varepsilon - f\|_{L^p} \leq \|(f-g)_\varepsilon\|_{L^p} + 2\varepsilon_0/3.$$

The result will now follow if we can show that for all $h \in L^p(\mathbb{R}^n)$,

$$\|h_\varepsilon\|_{L^p} \leq \|h\|_{L^p}\,.$$

This is a direct consequence of the Minkowski inequality in integral form (Problem 2.5). The usual Minkowski inequality reads

$$\|\sum_{j=1}^N f_j\|_{L^p} \leq \sum_{j=1}^N \|f_j\|_{L^p}\,,$$

and the integral form is the same inequality but where the sums are replaced by integrals. Thus we have

$$\begin{aligned}\|h_\varepsilon\|_{L^p} &= \left\| \int \eta_\varepsilon(y) h(\,\cdot\, - y)\,dy \right\|_{L^p} \leq \int \eta_\varepsilon(y)\,\|h(\,\cdot\, - y)\|_{L^p}\,dy \\ &= \|h\|_{L^p} \int \eta_\varepsilon(y)\,dy = \|h\|_{L^p}\,.\end{aligned}$$

□

Exercise 2.5. (Minkowski inequality in integral form) If (X,μ) and (Y,ν) are σ-finite measure spaces, $F : X \times Y \to \mathbb{C}$ is measurable, and $1 \leq p < \infty$, prove that

$$\left(\int_X \left(\int_Y |F(x,y)|\,d\nu(y) \right)^p d\mu(x) \right)^{1/p} \leq \int_Y \left(\int_X |F(x,y)|^p\,d\mu(x) \right)^{1/p} d\nu(y).$$

Mollifications yield an immediate proof of a variant of the du Bois-Reymond lemma, which is fundamental for the definition of weak derivatives.

Theorem 2.6. *(du Bois-Reymond lemma) Let $U \subset \mathbb{R}^n$ be an open set and $f \in L^1_{loc}(U)$. If*

$$\int_U f\varphi \, dx = 0 \qquad \text{for all } \varphi \in C_c^\infty(U),$$

then $f = 0$ almost everywhere in U.

Proof. Let $x \in U$ and choose $B = B(x,r)$ with $\overline{B} \subset U$. Define $\tilde{f} = \chi_B f \in L^1(\mathbb{R}^n)$ where χ_B is the characteristic function of the set B (that is, $\chi_B = 1$ on B and $\chi_B = 0$ elsewhere). If $y \in B$, then $\tilde{f} * \eta_\varepsilon(y) = 0$ for ε small by the assumption. Moreover, $\tilde{f} * \eta_\varepsilon \to \chi_B f$ in L^1 as $\varepsilon \to 0$ by Theorem 2.4. Thus $f = 0$ near x. □

The next result is an example of how mollification allows to create smooth bump functions with specified behavior.

Theorem 2.7. *(Smooth bump function) Let $K \subset U \subset \mathbb{R}^n$ where K is compact and U is open. There exists a function $\zeta \in C_c^\infty(U)$ with $0 \leq \zeta \leq 1$ in U and $\zeta = 1$ on K.*

Proof. Choose a compact set L with $K \subset \mathrm{int}(L) \subset L \subset U$. The characteristic function χ_L is in $L^1(\mathbb{R}^n)$. Since L is compact and $\mathbb{R}^n \setminus U$ is a closed set disjoint from L, there exists $\varepsilon > 0$ so that the set $\{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x,L) < \varepsilon\}$ is strictly contained in U. By Theorem 2.4 we have $\chi_L * \eta_\varepsilon \in C_c^\infty(U)$. By further decreasing ε we have $\chi_L * \eta_\varepsilon = 1$ on K, and $\zeta = \chi_L * \eta_\varepsilon$ satisfies the required properties. □

Theorem 2.8. *(Partition of unity) Let $K \subset \mathbb{R}^n$ be compact and let $K \subset \bigcup_{j=1}^N V_j$ where V_j are open sets. There exist functions $\zeta_j \in C_c^\infty(V_j)$ such that $0 \leq \zeta_j \leq 1$ and*

$$\sum_{j=1}^N \zeta_j \leq 1 \text{ in } \mathbb{R}^n, \qquad \sum_{j=1}^N \zeta_j = 1 \text{ on } K.$$

Definition 2.9. In the setting of Theorem 2.8, we say that $\{\zeta_j\}_{j=1}^N$ is a *partition of unity* on K subordinate to the cover $\{V_j\}_{j=1}^N$.

Exercise 2.10. Prove Theorem 2.8.

2.2. Integration by parts

In this section we give a brief discussion of one of the most fundamental and useful methods in mathematical analysis, namely *integration by parts*. In the most classical case, this amounts to the fundamental theorem of calculus: if f is a continuously differentiable real valued function on an interval $[a, b]$, then

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Observe that this formula relates information in the interior of the domain (the integral of f' over the interval) to information on the boundary (the "boundary integral", or the sum of values of f at the endpoints taken with opposite signs).

The integration by parts formulas in this section are multidimensional generalizations of the fundamental theorem of calculus. They underlie the theory of weak solutions for partial differential equations, and in fact the theory of weak solutions essentially amounts to taking the integration by parts formula as an *axiom* rather than a theorem. Integration by parts is also especially useful in inverse problems, since it allows to relate measurements at the boundary to information in the interior just as in the one-dimensional case above.

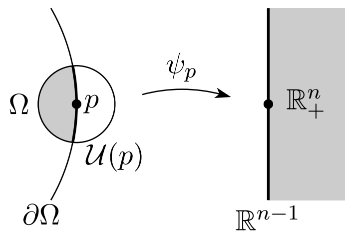
It is natural to first discuss the multidimensional domains over which we integrate. To have a reasonable boundary integral, we need to assume some regularity of the boundary of the domain. By definition, a *domain* in $\mathbb{R}^n$ is an open connected subset of $\mathbb{R}^n$. Much of the time connectedness will not be required.

Our definition of sets with C^k boundary is given in terms of mappings that flatten the boundary locally.

Definition 2.11. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $k \in \mathbb{Z}_+$.

(a) We say that Ω has C^k boundary (or that $\partial\Omega$ is C^k or that $\partial\Omega \in C^k$) if, for each $p \in \partial\Omega$, there is an open neighbourhood $U = U(p)$ of p and a C^k diffeomorphism $\Phi = \Phi_p : U \to \tilde{U}$ onto some open set $\tilde{U} \subset \mathbb{R}^n$ such that $\Phi(p) = 0$ and

$$\Phi\big(U(p) \cap \Omega\big) = \{\, x \in \tilde{U} \mid x_n > 0 \,\},\ \Phi\big(U(p) \cap \partial\Omega\big) = \{\, x \in \tilde{U} \mid x_n = 0 \,\}.$$



(b) We say that Ω has smooth boundary (or that $\partial\Omega$ is smooth or that $\partial\Omega \in C^\infty$) if each Φ_p, $p \in \partial\Omega$, of (a) is a C^∞ diffeomorphism.

(c) We call the system $\big(U(p), \Phi_p\big)_{p\in\partial\Omega}$ a coordinate system for $\partial\Omega$.

If $\partial\Omega$ has C^1 boundary, there is a well defined tangent space $T_p(\partial\Omega)$ at each point p of $\partial\Omega$; if $(U_q, \Phi_q)_{q\in\partial\Omega}$ is a coordinate system, a basis for this space is given by $\{\dot\gamma_1(0), \ldots, \dot\gamma_{n-1}(0)\}$ where $\gamma_\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ are the C^1 curves (here $\varepsilon > 0$ is sufficiently small, $\alpha = 1, \ldots, n-1$, and e_α is the αth coordinate vector)

$$\gamma_\alpha(t) = \Phi_p^{-1}(te_\alpha). \tag{2.1}$$

There are two other equivalent ways of looking at sets with C^k boundary, and we will mention these here since they will be useful below. The first way expresses $\partial\Omega$ locally as the graph of a C^k function.

Theorem 2.12. *(Local graph representation) Let Ω be a bounded open set in $\mathbb{R}^n$. Then Ω has C^k boundary if and only if for any point $p \in \partial\Omega$, there exist $r > 0$, an orthonormal coordinate system $x = (x', x_n)$ with origin at p, and a C^k function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$\Omega \cap B(p, r) = \{x \in B(p, r)\,;\, x_n > h(x')\}.$$

Proof. Suppose $(U(q), \Phi_q)$ is a coordinate system for $\partial\Omega$, fix $p \in \Omega$, and let $\Phi = \Phi_p$. We may translate and rotate the coordinate systems so that $p = 0$, $\Phi(0) = 0$, and the tangent space to $\partial\Omega$ at 0 is spanned by $\{e_1, \ldots, e_{n-1}\}$. Since the nth component of Φ vanishes on $\partial\Omega$, $\Phi^n(q) = 0$ for $q \in \partial\Omega$, we have

$$\Phi^n(\gamma_\alpha(t)) = 0$$

where γ_α are the curves (2.1). Differentiating in t and evaluating at $t = 0$ implies that

$$\partial_\alpha \Phi^n(0) = 0, \quad \alpha = 1, \ldots, n-1.$$

Since Φ is a C^k diffeomorphism ($k \geq 1$), the Jacobian matrix $D\Phi(0)$ is invertible. This implies that $\partial_n \Phi^n(0) \neq 0$. Changing x_n to $-x_n$ if necessary, we may assume that $\partial_n \Phi^n(0) > 0$.

By the implicit function theorem, there is a C^k function h defined near the origin in $\mathbb{R}^{n-1}$ such that near 0 we have

$$\Phi^n(x', x_n) = 0 \Longleftrightarrow x_n = h(x').$$

We also have the Taylor expansion of $\Phi^n(x', \cdot\,)$ at $x_n = h(x')$,

$$\Phi^n(x', x_n) = \Phi^n(x', h(x')) + \partial_n \Phi^n(x', h(x'))(x_n - h(x')) + o(x_n - h(x'))$$

as $x_n \to h(x')$. Here $\Phi^n(x', h(x')) = 0$ and $\partial_n \Phi^n(x', h(x')) > 0$ for x' sufficiently close to 0 by continuity. Thus, if x is close to 0, the conditions $x_n > h(x')$ and $\Phi^n(x', x_n) > 0$ are equivalent. This shows that locally Ω is given by $\{(x', x_n)\,;\, x_n > h(x')\}$.

The converse follows by choosing diffeomorphisms $\Phi(x', x_n) = (x', x_n - h(x'))$ in suitable neighborhoods of boundary points. $\square$

Theorem 2.13. *(Boundary defining function) Let Ω be a bounded open set in $\mathbb{R}^n$. Then Ω has C^k boundary if and only if there is a C^k function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\Omega = \{x \in \mathbb{R}^n\,;\, \rho(x) > 0\}, \qquad \partial\Omega = \{x \in \mathbb{R}^n\,;\, \rho(x) = 0\},$$

and $\nabla \rho \neq 0$ on $\partial\Omega$.

Proof. (Sketch) If $p \in \partial\Omega$ and if $\Omega \cap B(p, r) = \{x \in B(p, r)\,;\, x_n > h(x')\}$, the function ρ in $B(p, r)$ can be taken to be $\rho(x', x_n) = x_n - h(x')$. We can construct the global boundary defining function ρ from the local expressions by using a suitable partition of unity. $\square$

Exercise 2.14. Prove Theorem 2.13 in detail.

We can use any of the above definitions to define $C^l(\partial\Omega)$ functions if Ω has C^k boundary and $l \leq k$.

Definition 2.15. If Ω is a bounded open set with C^k boundary and if $l \leq k$, we say that $f : \partial\Omega \to \mathbb{C}$ is C^l and write $f \in C^l(\partial\Omega)$ if there is a coordinate system $\big(U(p), \Phi_p\big)$ such that $f \circ \Phi_p^{-1} : \mathbb{R}^{n-1} \to \mathbb{C}$ is C^l.

There are two quantities associated with C^k domains that will be used frequently. The first is the outer unit normal vector of $\partial\Omega$.

Definition 2.16. Let Ω be a C^k domain and let ρ be a boundary defining function for $\partial\Omega$. The *outer unit normal vector* of $\partial\Omega$ at a point $p \in \partial\Omega$ is

$$\nu(p) = -\frac{\nabla \rho(p)}{|\nabla \rho(p)|}.$$

The definition of ν is independent of the choice of boundary defining function. We write the vector field $\nu : \partial\Omega \to \mathbb{R}^n$ in terms of its components as $\nu = (\nu_1, \ldots, \nu_n)$, where each ν_j is a function in $C^{k-1}(\partial\Omega)$ if the boundary is C^k. If $\partial\Omega$ is locally given as the graph $x' \mapsto (x', h(x'))$, the unit outer normal has the expression

$$\nu(x', h(x')) = \frac{(\nabla_{x'} h(x'), -1)}{(1 + |\nabla_{x'} h(x')|^2)^{1/2}}.$$

Exercise 2.17. Verify the claims about ν in the preceding paragraph.

The second useful quantity on a C^k boundary $\partial\Omega$ is its (Euclidean) surface measure dS, induced by the usual Lebesgue measure dx in $\mathbb{R}^n$. The surface measure is a constant multiple of the $(n-1)$-dimensional Hausdorff measure restricted to $\partial\Omega$. Another way to obtain this measure is as follows: for any $f \in C^0(\partial\Omega)$, define a function $\tilde{f}$ near $\partial\Omega$ by

$$\tilde{f}(p + t\nu(p)) = f(p), \qquad p \in \partial\Omega, |t| < \varepsilon.$$

If $\varepsilon > 0$ is small enough, $\tilde{f}$ is a well-defined continuous function in the set

$$V_\varepsilon = \{x \in \mathbb{R}^n \,;\, \mathrm{dist}(x, \partial\Omega) < \varepsilon\}.$$

Theorem 2.18. *(Surface measure) There is a unique positive Borel measure on $\partial\Omega$, acting on functions $f \in C^0(\partial\Omega)$ by $f \mapsto \int_{\partial\Omega} f\, dS$, that satisfies*

$$\int_{\partial\Omega} f\, dS = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{V_\varepsilon} \tilde{f}(x)\, dx.$$

If $\partial\Omega$ is expressed as the graph $x' \mapsto (x', h(x'))$ near some $p \in \partial\Omega$ and if $f \in C^0(\partial\Omega)$ is supported near p , then

$$\int_{\partial\Omega} f\, dS = \int_{\mathbb{R}^{n-1}} f(x', h(x'))(1 + \left|\nabla h(x')\right|^2)^{1/2}\, dx'.$$

Exercise 2.19. Verify that the extension $\tilde{f}$ is well defined near $\partial\Omega$, and prove Theorem 2.18.

We are now ready to state the integration by parts formulas that will be used in this book. Most of them are equivalent, and all are consequences of the next result:

Theorem 2.20. *(Gauss-Green formula) If $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary and if $u \in C^1(\overline{\Omega})$, then for $j = 1, \ldots, n$*

$$\int_\Omega \partial_j u\, dx = \int_{\partial\Omega} u\nu_j\, dS.$$

Proof. For each $p \in \partial\Omega$, we choose an orthonormal coordinate system (x', x_n), a ball $B(p, r)$ and a C^1 function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Omega \cap B(p, r) = \{x \in B(p, r)\,;\, x_n > h(x')\}$. These balls cover the compact set $\partial\Omega$, and there is a finite subcover $\{B_k\}_{k=1}^N$. Choose some open set $B_0 \subset \Omega$ such that the sets $\{B_k\}_{k=0}^N$ cover $\overline{\Omega}$. By Theorem 2.8, we may find a partition of unity $\{\zeta_j\}_{k=0}^N$ on $\overline{\Omega}$ subordinate to the cover $\{B_k\}$. Then

$$\int_\Omega \partial_j u \, dx = \sum_{k=0}^N \int_\Omega \partial_j(\zeta_k u) \, dx.$$

Since $\zeta_0 u \in C_c^1(\Omega)$, we have

$$\int_\Omega \partial_j(\zeta_0 u) \, dx = \int_{\mathbb{R}^n} \partial_j(\zeta_0 u) \, dx = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \partial_j(\zeta_0 u) \, dx_j \, dy$$

where y contains all the variables in x except for x_j. The fundamental theorem of calculus implies that

$$\int_{-\infty}^{\infty} \partial_j(\zeta_0 u)(x_1, \ldots, x_n) \, dx_j = 0.$$

Thus $\int_\Omega \partial_j(\zeta_0 u) \, dx = 0$.

Let now $v = \zeta_k u$ where $1 \leq k \leq N$, and write $B_k = B(p, r)$ and $\Omega \cap B_k = \{x \in B_k \,;\, x_n > h(x')\}$. Since $\zeta_k \in C_c^\infty(B_k)$, we have

$$\int_\Omega \partial_j v \, dx = \int_{\mathbb{R}^{n-1}} \int_{h(x')}^{\infty} \partial_j v(x) \, dx_n \, dx'.$$

Choose a function $\psi \in C^\infty(\mathbb{R})$ with $\psi(t) = 0$ for $t < 0$ and $\psi(t) = 1$ for $t > 1$. (Such a function can be obtained by mollifying the function which is zero for $t < 1/2$ and equals one for $t > 1/2$.) Define also $\rho(x) = x_n - h(x')$ in B_k. It follows that

$$\begin{aligned}
\int_\Omega \partial_j v \, dx &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \partial_j v(x) \psi\left(\frac{\rho(x)}{\varepsilon}\right) dx \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \left[\partial_j\left(v\psi\left(\frac{\rho}{\varepsilon}\right)\right) - v\psi'\left(\frac{\rho}{\varepsilon}\right)\frac{\partial_j \rho}{\varepsilon}\right] dx.
\end{aligned}$$

Since $v\psi(\rho/\varepsilon) \in C_c^1(\mathbb{R}^n)$, the integral $\int_{\mathbb{R}^n} \partial_j(v\psi(\rho/\varepsilon)) \, dx$ vanishes by the fundamental theorem of calculus. Consequently

$$\int_\Omega \partial_j v \, dx = -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} v\psi'\left(\frac{\rho}{\varepsilon}\right)\frac{\partial_j \rho}{\varepsilon} \, dx.$$

Using that $\psi'(t) = 0$ for $t < 0$ and $t > 1$ and writing $x_n = h(x') + \varepsilon t$, we may write

$$\begin{aligned}
\int_\Omega \partial_j v \, dx &= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-1}} \int_{h(x')}^{h(x')+\varepsilon} v(x)\psi'\left(\frac{x_n - h(x')}{\varepsilon}\right)\frac{\partial_j \rho(x)}{\varepsilon} \, dx_n \, dx' \\
&= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-1}} \int_0^1 v(x', h(x') + \varepsilon t)\psi'(t)\partial_j \rho(x', h(x') + \varepsilon t) \, dt \, dx'.
\end{aligned}$$

We may take the limit by dominated convergence, and the fact that $\int_0^1 \psi'(t)\,dt = 1$ implies

$$\int_\Omega \partial_j v\,dx = -\int_{\mathbb{R}^{n-1}} v(x',h(x'))\partial_j \rho(x',h(x'))\,dt\,dx'.$$

From the definitions of the unit outer normal and surface measure, we see that $\nu_j(x',h(x')) = -\partial_j\rho(x',h(x'))/\left|\nabla\rho(x',h(x'))\right|$ and $dS(x') = \left|\nabla\rho(x',h(x'))\right|\,dx'$. This shows that

$$\int_\Omega \partial_j(\zeta_k u)\,dx = \int_{\partial\Omega} \zeta_k u \nu_j\,dS, \quad k = 1,\ldots,N.$$

The result follows by summing over k from 0 to N and using the fact that $\{\zeta_k\}$ is a partition of unity. $\square$

Theorem 2.21. *Let Ω have C^1 boundary.*

(1) *(Integration by parts) If $u, v \in C^1(\overline{\Omega})$, then*
$$\int_\Omega u\partial_j v\,dx = -\int_\Omega (\partial_j u)v\,dx + \int_{\partial\Omega} uv\nu_j\,dS.$$

(2) *(Divergence theorem) If $F : \overline{\Omega} \to \mathbb{R}^n$ is a C^1 vector field, then*
$$\int_\Omega \mathrm{div}(F)\,dx = \int_{\partial\Omega} F\cdot\nu\,dS.$$

(3) *(Green formula) If $u \in C^1(\overline{\Omega}), v \in C^2(\overline{\Omega})$, then*
$$\int_{\partial\Omega} u\partial_\nu v\,dS = \int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Omega u\Delta v\,dx.$$

(4) *(Green formula) If $u, v \in C^2(\overline{\Omega})$, then*
$$\int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u)\,dS = \int_\Omega (u\Delta v - v\Delta u)\,dx.$$

Exercise 2.22. Prove Theorem 2.21.

2.3. Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^n$, and let $\gamma \in L^\infty(\Omega)$ be a positive function representing the electrical conductivity of the medium Ω. We wish to consider a suitable function space for solutions of the conductivity equation

$$\mathrm{div}(\gamma\nabla u) = 0 \qquad \text{in } \Omega.$$

The function spaces that will be used for this purpose are called *Sobolev spaces*, and they turn out to be appropriate for describing weak solutions of a large class of partial differential equations.

If $\gamma \in C^1(\overline{\Omega})$ and $u \in C^2(\overline{\Omega})$, then one can interpret the equation $\mathrm{div}(\gamma\nabla u) = 0$ in the classical pointwise sense since the derivatives exist pointwise. Solutions in $C^2(\overline{\Omega})$ are often called *classical solutions*. However, when the conductivity γ is only in $L^\infty(\Omega)$ classical solutions do not make sense. Also, our aim is to use energy methods and Hilbert space theory to produce solutions, and for this purpose it is more natural to use spaces based on $L^2(\Omega)$ rather than the C^k spaces.

To define weak solutions of the equation $\mathrm{div}(\gamma\nabla u) = 0$, one first needs the concept of weak derivatives of functions that may not be differentiable in the classical sense. Weak derivatives will be defined via a suitable test function space.

Definition 2.23. (Test functions) The elements of $C_c^\infty(\Omega)$, that is, infinitely differentiable functions $\varphi : \Omega \to \mathbb{C}$ with compact support in Ω, are called *test functions.*

Motivation 2.24. To motivate the definition of weak derivatives, let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary and let $u \in C^k(\overline{\Omega})$. Consider the classical αth derivative of u, $\partial^\alpha u$, where $\alpha \in \mathbb{N}_0^n$ is a multi-index with $|\alpha| \leq k$. If $\varphi \in C_c^\infty(\Omega)$ is a test function, integrating by parts repeatedly using Theorem 2.21(a) implies that

$$\int_\Omega u\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_\Omega (\partial^\alpha u)\varphi\,dx.$$

We used the fact that φ and its derivatives vanish near $\partial\Omega$, so there are no boundary terms. Now the left hand side is well defined if $u \in L^1_{loc}(\Omega)$, and in this case we say that u has αth weak partial derivative in $L^1_{loc}(\Omega)$ if the above identity remains true for all test functions φ when $\partial^\alpha u$ on the right hand side is replaced by some locally integrable function v.

Definition 2.25. (Weak derivatives) Let $\Omega \subset \mathbb{R}^n$ be open, let $u, v \in L^1_{loc}(\Omega)$, and let $\alpha \in \mathbb{N}^n$ be a multi-index. We say that v is the αth *weak partial derivative* of u, written

$$v = \partial^\alpha u,$$

if

$$\int_\Omega u\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_\Omega v\varphi\,dx \qquad \text{for all } \ \varphi \in C_c^\infty(\Omega).$$

By Theorem 2.6, the αth weak partial derivative (whenever it exists) is uniquely defined as an L^1_{loc} function. Having given the definition of weak derivatives, we proceed to discuss the spaces of functions relevant for weak solutions of the equation $\mathrm{div}(\gamma\nabla u) = 0$ in Ω.

Definition 2.26. (Sobolev spaces) If $\Omega \subset \mathbb{R}^n$ is open and $k \in \mathbb{N}_0$, the *Sobolev space* $H^k(\Omega)$ consists of all functions $u \in L^2(\Omega)$ for which the weak partial derivative $\partial^\alpha u$ is in $L^2(\Omega)$ whenever $\alpha \in \mathbb{N}_0^N$ and $|\alpha| \leq k$. We equip this space with the inner product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \le k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$$

and with the norm

$$\|u\|_{H^k(\Omega)} = (u, u)^{1/2}_{H^k(\Omega)}.$$

Examples of Sobolev functions are given next; the problem below shows that some rather pathological functions can still belong to $H^1(\Omega)$.

Example 2.27. (a) If $\Omega \subset \mathbb{R}^n$ is a bounded open set, the space $C^k(\overline{\Omega})$ is contained in $H^k(\Omega)$.

(b) If $\Omega \subset \mathbb{R}^n$ is open, $C_c^\infty(\Omega)$ is contained in $H^k(\Omega)$ for all $k \ge 0$.

(c) If $\Omega = B(0, 1) \subset \mathbb{R}^n$, then the function

$$u(x) = |x|^{-\alpha}, \qquad |x| < 1$$

is in $H^1(\Omega)$ if and only if $\alpha < n/2 - 1$. Indeed, this function has gradient

$$\nabla u(x) = -\alpha\, |x|^{-\alpha-2}\, x, \qquad x \in \Omega.$$

A computation in polar coordinates shows that $x \mapsto |x|^{-\beta}$ is integrable near 0 if and only if $\beta < n$. Using these facts, it is not hard to see that the weak gradient is equal to the pointwise gradient and that $u \in H^1(\Omega)$ for any $\alpha < n/2 - 1$.

Exercise 2.28. If $\Omega = B(0, 1) \subset \mathbb{R}^n$, $n \ge 3$, give an example of a function $u \in H^1(\Omega)$ that is not essentially bounded in any open subset of Ω. Give a similar example for $n = 2$. Can you find an example of this type for $n = 1$?

The next result shows that $H^k(\Omega)$ is a Hilbert space, as the notation already suggests.

Theorem 2.29. *$H^k(\Omega)$ is a Hilbert space for each $k \in \mathbb{N}_0$.*

Exercise 2.30. Prove Theorem 2.29.

Exercise 2.31. (Pointwise multipliers) If $\Omega \subset \mathbb{R}^n$ is a bounded open set and $a \in C^k(\overline{\Omega})$, $u \in H^k(\Omega)$, show that $au \in H^k(\Omega)$. Show also that

$$\partial^\alpha (au) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta a \partial^{\alpha-\beta} u, \quad \alpha \in \mathbb{N}_0^n, |\alpha| \le k,$$

where $\beta \le \alpha$ means that $\beta_j \le \alpha_j$ for $j = 1, \ldots, n$, $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$, and $\alpha! = \alpha_1! \cdots \alpha_n!$. (Hint: use induction on $|\alpha|$. It is enough to do the case $k = 1$ if the general case is difficult.)

To deal with boundary value problems for the equation $\mathrm{div}(\gamma \nabla u) = 0$ in Ω, we need to consider the space $H_0^1(\Omega)$ consisting of those functions in $H^1(\Omega)$ which vanish on the boundary $\partial\Omega$ in some sense. Functions in $C_c^\infty(\Omega)$ always vanish near the boundary, which motivates the following definition.

Definition 2.32. We denote by $H_0^1(\Omega)$ the closure of the set $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Its dual space is denoted by

$$H^{-1}(\Omega) = (H_0^1(\Omega))^* = \{F : H_0^1(\Omega) \to \mathbb{C} \text{ bounded linear functional}\}.$$

There is one potentially confusing point in the previous definition: Hilbert space theory tells that any Hilbert space is isomorphic to its dual, so one might wonder why the dual space $H^{-1}(\Omega)$ is needed. In fact, the Riesz representation theorem shows that any $F \in H^{-1}(\Omega)$ can be uniquely represented in the form

$$F(v) = (v, w)_{H_0^1(\Omega)}, \quad v \in H_0^1(\Omega),$$

for some $w \in H_0^1(\Omega)$. The point is that this representation involves the $H_0^1(\Omega)$ inner product, whereas the definition of weak derivatives is given in terms of the L^2 inner product. The above representation can be written in the weak sense as

$$F(v) = (v, w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{L^2(\Omega)} = (v, w - \Delta w)_{L^2(\Omega)}.$$

Thus $H^{-1}(\Omega)$ can be identified with the set $\{w - \Delta w \,;\, w \in H_0^1(\Omega)\}$, if these functions are understood to act on $H_0^1(\Omega)$ functions with respect to the L^2 inner product. In this interpretation $H^{-1}(\Omega)$ contains all functions in $L^2(\Omega)$, since any function $g \in L^2(\Omega)$ gives rise to a bounded linear functional on $H_0^1(\Omega)$ by

$$g(v) = \int_\Omega gv \, dx, \quad v \in H_0^1(\Omega).$$

Theorem 2.33. *$H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are Hilbert spaces, and $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$.*

Proof. These claims follow immediately from Hilbert space theory. □

We will need only one nontrivial fact about Sobolev spaces to obtain weak solutions to the partial differential equations that we are interested in. This is the fact that the inclusion map $i : H_0^1(\Omega) \to L^2(\Omega)$ is compact, or in other words, every bounded sequence in $H_0^1(\Omega)$ has a subsequence that converges in $L^2(\Omega)$. After showing this we will have access to powerful tools in the theory of compact operators, such as the Fredholm alternative and the spectral theorem, in the analysis of weak solutions.

The fundamental principle that allows us to extract a convergent subsequence is the Arzelà-Ascoli theorem. This result is an extension of the fact that any bounded sequence of complex numbers has a convergent subsequence.

Theorem 2.34. *(Arzelà-Ascoli) Let (X, d) be a compact metric space, and let (f_j) be a sequence of functions $X \to \mathbb{C}$. Assume that (f_j) is pointwise bounded and equicontinuous, that is, $\sup_{j\in\mathbb{Z}_+} |f_j(x)| < \infty$ for each $x \in X$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|f_j(x) - f_j(y)| < \varepsilon \quad \text{whenever } d(x, y) < \delta \text{ and } j = 1, 2, \ldots.$$

Then (f_j) has a subsequence that converges uniformly on X.

Exercise 2.35. Prove Theorem 2.34 in the case where additionally (X, d) is separable. (Hint: choose a countable dense subset $\{x_l\}_{l=1}^{\infty}$ of X, and first find a subsequence that converges at each x_l.)

Exercise 2.36. Prove Theorem 2.34 in general.

Theorem 2.37. *(Compact Sobolev embedding) If $\Omega \subset \mathbb{R}^n$ is a bounded open set, then the inclusion map $i : H_0^1(\Omega) \to L^2(\Omega)$ is compact.*

Proof. Suppose that $(u_j) \subset H_0^1(\Omega)$ is a bounded sequence, so

$$\|u_j\|_{H^1(\Omega)} \leq C, \qquad j = 1, 2, \ldots.$$

Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, for any j there is $\varphi_j \in C_c^\infty(\Omega)$ with $\|u_j - \varphi_j\|_{H^1(\Omega)} < 1/j$. Then also (φ_j) is a bounded sequence in $H_0^1(\Omega)$, and if we can find a subsequence (φ_{j_k}) that converges in $L^2(\Omega)$ then also (u_{j_k}) will converge in $L^2(\Omega)$ to the same limit. Consequently, we may assume that $(u_j) \subset C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}(u_j) \subset \Omega$ for each j, and

$$\|u_j\|_{H^1(\mathbb{R}^n)} \leq C, \qquad j = 1, 2, \ldots. \tag{2.2}$$

Assume for the moment that instead of (2.2) we have uniform bounds for $p = \infty$,

$$\|u_j\|_{L^\infty(\mathbb{R}^n)} + \|\nabla u_j\|_{L^\infty(\mathbb{R}^n)} \leq C, \qquad j = 1, 2, \ldots.$$

Then the sequence $(u_j|_{\overline{\Omega}})$ would be pointwise bounded and also equicontinuous since

$$|u_j(x) - u_j(y)| \leq \left(\sup_{t\in[0,1]} |\nabla u_j(x + t(y - x)| \right) |x - y| \leq C\,|x - y|\,, \qquad x, y \in K.$$

By the Arzelà-Ascoli theorem we could find a uniformly convergent subsequence on $\overline{\Omega}$.

The device that will be used to pass from the uniform bounds with $p = 2$ to the case $p = \infty$ described above is mollification. Define

$$u_j^\varepsilon = u_j * \eta_\varepsilon.$$

We compute

$$\begin{aligned}
u_j^\varepsilon(x) - u_j(x) &= \int_{\mathbb{R}^n} \eta_\varepsilon(y)\,[u_j(x-y) - u_j(x)]\,dy = \int_{\mathbb{R}^n} \eta_\varepsilon(y)\left[\int_0^1 \frac{d}{dt} u_j(x-ty)\,dt\right] dy \\
&= \int_{\mathbb{R}^n}\int_0^1 \eta_\varepsilon(y)\nabla u_j(x-ty)\cdot(-y)\,dt\,dy \\
&= -\varepsilon \int_{B(0,1)}\int_0^1 \eta(y)\nabla u_j(x-t\varepsilon y)\cdot y\,dt\,dy.
\end{aligned}$$

The Minkowski inequality in integral form (Problem 2.5) implies that

$$\left\|u_j^\varepsilon - u_j\right\|_{L^2(\mathbb{R}^n)} \leq \varepsilon \int_{B(0,1)}\int_0^1 \eta(y)\left\|\nabla u_j(\,\cdot\, - t\varepsilon y)\right\|_{L^2(\mathbb{R}^n)} |y|\,dt\,dy.$$

Since $\|\nabla u_j(\,\cdot\, - t\varepsilon y)\|_{L^2} = \|\nabla u_j\|_{L^2}$, the uniform bound (2.2) shows that

$$\left\|u_j^\varepsilon - u_j\right\|_{L^2(\mathbb{R}^n)} \leq C'\varepsilon, \qquad j = 1, 2, \ldots$$

for some $C' > 0$. The point is that these bounds are uniform with respect to j.

We will now prove the theorem by showing that (u_j) has a subsequence that is Cauchy in $L^2(\Omega)$. Fix $\varepsilon_0 > 0$, and choose ε so small that

$$\left\|u_j - u_j^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \leq \varepsilon_0/3, \qquad j = 1, 2, \ldots.$$

For this ε, the sequence (u_j^ε) is uniformly bounded and equicontinuous. In fact, by (2.2) and Cauchy-Schwarz we have

$$\left|u_j^\varepsilon(x)\right| = \left|\int \eta_\varepsilon(x-y)u_j(y)\,dy\right| \leq \|\eta_\varepsilon(x-\,\cdot\,)\|_{L^2}\,\|u_j\|_{L^2} \leq C_\varepsilon$$

and similarly

$$\left|\nabla u_j^\varepsilon(x)\right| = \left|\int \nabla\eta_\varepsilon(x-y)u_j(y)\,dy\right| \leq \|\nabla\eta_\varepsilon(x-\,\cdot\,)\|_{L^2}\,\|u_j\|_{L^2} \leq C_\varepsilon,$$

where the constants are uniform over $x \in \mathbb{R}^n$ and $j = 1, 2, \ldots$. The Arzelà-Ascoli theorem shows that there is a subsequence $(u_{j_k}^\varepsilon)_{k=1}^\infty$ that converges uniformly on $\overline{\Omega}$. It follows that

$$\begin{aligned}
\|u_{j_k} - u_{j_l}\|_{L^2(\Omega)} &\leq \left\|u_{j_k} - u_{j_k}^\varepsilon\right\|_{L^2(\Omega)} + \left\|u_{j_k}^\varepsilon - u_{j_l}^\varepsilon\right\|_{L^2(\Omega)} + \left\|u_{j_l}^\varepsilon - u_{j_l}\right\|_{L^2(\Omega)} \\
&\leq 2\varepsilon_0/3 + C_\Omega \left\|u_{j_k}^\varepsilon - u_{j_l}^\varepsilon\right\|_{L^\infty(\Omega)}.
\end{aligned}$$

Here we used that Ω is bounded.

It follows that for any $\varepsilon_0 > 0$ there is a subsequence (u_{j_k}) such that

$$\limsup_{k,l\to\infty} \|u_{j_k} - u_{j_l}\|_{L^2(\Omega)} \leq \varepsilon_0.$$

We apply this argument with $\varepsilon_0 = 1$ to obtain a subsequence $(u_j^{(1)})$ of (u_j) with

$$\limsup_{k,l\to\infty} \left\| u_k^{(1)} - u_l^{(1)} \right\|_{L^2(\Omega)} \leq 1.$$

Now, repeat the argument for $\varepsilon_0 = 1/2$, but with (u_j) replaced by the sequence $(u_j^{(1)})$, to obtain a further subsequence $(u_j^{(2)})$ with

$$\limsup_{k,l\to\infty} \left\| u_k^{(2)} - u_l^{(2)} \right\|_{L^2(\Omega)} \leq \frac{1}{2}.$$

We continue this for $\varepsilon_0 = \frac{1}{3}, \frac{1}{4}, \ldots$ and use the diagonal procedure to obtain a sequence (v_m), where $v_m = u_m^{(m)}$, that is a subsequence of the original (u_j) and satisfies

$$\limsup_{k,l\to\infty} \|v_k - v_l\|_{L^2(\Omega)} = 0.$$

By the Cauchy criterion, we have found a subsequence that converges in $L^2(\Omega)$. □

As the first application of compact Sobolev embedding, we prove a Poincaré inequality that will be crucial in showing existence of weak solutions. The proof is quite general and adapts to other situations, but it does not give any bounds on the constant C. A more direct proof is given in Proposition **??**.

Theorem 2.38 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. There is a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \qquad u \in H_0^1(\Omega).$$

Proof. By density it is enough to prove this for all $u \in C_c^\infty(\Omega)$. We argue by contradiction and assume that for any $k \in \mathbb{Z}_+$ there is $u_k \in C_c^\infty(\Omega)$ with

$$\|u_k\|_{L^2(\Omega)} > k \|\nabla u_k\|_{L^2(\Omega)}.$$

By dividing each u_k by $\|u_k\|_{L^2(\Omega)}$, we may assume that

$$\|u_k\|_{L^2(\Omega)} = 1, \quad \|\nabla u_k\|_{L^2(\Omega)} < \frac{1}{k}. \tag{2.3}$$

Then (u_k) is a bounded sequence in $H_0^1(\Omega)$, and by compact Sobolev embedding there is a subsequence, also denoted by (u_k), converging to some u in $L^2(\Omega)$. By (2.3), we also have $\nabla u_k \to 0$ in $L^2(\Omega)$.

We claim that $u \in H_0^1(\Omega)$ and $\nabla u = 0$ in the weak sense. In fact, if $\varphi \in C_c^\infty(\Omega)$ then

$$\int_\Omega u \partial_j \varphi \, dx = \lim_{k\to\infty} \int_\Omega u_k \partial_j \varphi \, dx = -\lim_{k\to\infty} \int_\Omega (\partial_j u_k) \varphi \, dx = 0$$

and so $\nabla u = 0$. (The integration by parts for a general open set Ω is easily justified since $u_k, \varphi \in C_c^\infty(\Omega)$.) Thus $u_k \to u$ in $H^1(\Omega)$, and since $H_0^1(\Omega)$ is a closed subspace we have $u \in H_0^1(\Omega)$.

It is proved in Problem 2.39 that any $u \in H_0^1(\Omega)$ with vanishing gradient must be identically zero. But then

$$0 = \|u\|_{L^2(\Omega)} = \lim_{k\to\infty} \|u_k\|_{L^2(\Omega)} = 1,$$

and we have arrived at a contradiction. $\square$

Exercise 2.39. Show that any $u \in H^1(\Omega)$ whose weak gradient vanishes is constant on each connected component of Ω. If additionally $u \in H_0^1(\Omega)$, show that $u = 0$.

Remark 2.40. The Poincaré inequality implies that $\|\nabla \cdot\|_{L^2(\Omega)}$ is an equivalent norm on $H_0^1(\Omega)$:

$$C^{-1}\|u\|_{H^1(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)}, \quad u \in H_0^1(\Omega).$$

(The first inequality follows from Poincaré, and the second one is trivial.) This will be useful for the existence of weak solutions.

We proceed to describe Sobolev spaces on the boundary $\partial\Omega$ that will serve as appropriate function spaces for boundary values of weak solutions. As mentioned above, we think of $H_0^1(\Omega)$ as the set of those functions in $H^1(\Omega)$ whose boundary value, also called *trace*, on $\partial\Omega$ vanishes. In the same spirit, we consider two functions $u, v \in H^1(\Omega)$ to have the same boundary value on $\partial\Omega$ if $u - v \in H_0^1(\Omega)$. This motivates the following definition of an abstract trace space of $H^1(\Omega)$.

Definition 2.41. Define $H^{1/2}(\partial\Omega)$ as the quotient space

$$H^{1/2}(\partial\Omega) = H^1(\Omega)/H_0^1(\Omega).$$

The elements of $H^{1/2}(\Omega)$ are the equivalence classes $[u] = \{u + \varphi\,;\, \varphi \in H_0^1(\Omega)\}$ where u runs through all elements of $H^1(\Omega)$. Also define the *trace operator*

$$R : H^1(\Omega) \to H^{1/2}(\Omega), \quad Ru = [u].$$

We also write $u|_{\partial\Omega} = Ru$.

We will see later that if Ω is a bounded open set with C^1 boundary, the abstract space $H^{1/2}(\partial\Omega)$ can be identified with a subspace of $L^2(\partial\Omega)$ (the space of square integrable functions on $\partial\Omega$ with respect to surface measure). This identification and a more precise description of $H^{1/2}(\partial\Omega)$ is most conveniently done via the Fourier transform. At this point, we only motivate the notation $H^{1/2}(\partial\Omega)$ with an example of a function $u \in H^1(\Omega)$ whose boundary value $u|_{\partial\Omega}$ is in $L^2(\partial\Omega)$ but not in $H^1(\partial\Omega)$. In this example, one can heuristically think that $u|_{\partial\Omega}$ has half a derivative in $L^2(\partial\Omega)$.

Example 2.42. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^∞ boundary such that $\Omega \subset \{x_n > 0\}$ and $B(0,1) \cap \{x_n > 0\} \subset \Omega$, and consider the function in Example 2.46(c),

$$u(x) = |x|^{-\alpha}, \qquad x \in \Omega.$$

We fix $\alpha < n/2 - 1$, so that $u \in H^1(\Omega)$.

Writing $x = (x', x_n)$, the pointwise restriction of u to $\partial\Omega \cap \{x_n = 0\}$ is

$$u(x', 0) = |x'|^{-\alpha}.$$

The pointwise gradient of this function is

$$\nabla_{x'} u(x', 0) = -\alpha \, |x'|^{-\alpha-2} \, x'.$$

Thus, the function $x' \mapsto u(x', 0)$ is in $L^2(\partial\Omega \cap \{x_n = 0\})$ (since $\alpha < \frac{n-1}{2}$), but its pointwise gradient is L^2 integrable only if $\alpha < \frac{n-1}{2} - 1$. Heuristically, interpolating the expressions for $u(x', 0)$ and $\nabla_{x'} u(x', 0)$ suggests that the absolute value of the "fractional gradient" $\left|\nabla^\theta_{x'} u(x', 0)\right|$ would behave like $|x'|^{-\alpha-\theta}$. This is always L^2 integrable if $0 \leq \theta \leq 1/2$, suggesting that $u|_{\partial\Omega}$ has half a derivative in $L^2(\partial\Omega)$ but not a full derivative in general.

The benefit of the abstract definition of $H^{1/2}(\partial\Omega)$ is that this definition is valid without any regularity assumptions on the boundary $\partial\Omega$. For many results considered in this book, this abstract setup is actually sufficient to formulate the corresponding inverse problems and their solutions. We now describe some further properties of the abstract space $H^{1/2}(\partial\Omega)$. Note first the orthogonal decomposition

$$H^1(\Omega) = H^1_0(\Omega) \oplus H^1_0(\Omega)^\perp.$$

This is valid since $H^1_0(\Omega)$ is a closed subspace of $H^1(\Omega)$. The next result gives the standard Hilbert space structure on $H^{1/2}(\partial\Omega)$.

Theorem 2.43. *The orthogonal projection $P : H^1(\Omega) \to H^1_0(\Omega)^\perp$ induces a bijective linear map*

$$T : H^{1/2}(\partial\Omega) \to H^1_0(\Omega)^\perp, \quad T([u]) = P(u).$$

The space $H^{1/2}(\partial\Omega)$ becomes a Hilbert space when equipped with the inner product

$$([u], [v])_{H^{1/2}(\partial\Omega)} = (T([u]), T([v]))_{H^1(\Omega)}, \qquad u, v \in H^1(\Omega)$$

and with the norm

$$\|[u]\|_{H^{1/2}(\partial\Omega)} = \|T([u])\|_{H^1(\Omega)} = \inf_{v \in H^1_0(\Omega)} \|u + v\|_{H^1(\Omega)}, \qquad u \in H^1(\Omega).$$

Proof. The map T is well defined since $P(u+v) = P(u)$ for $v \in H_0^1(\Omega)$. If $T([u]) = 0$, then $P(u) = 0$ so $u \in H_0^1(\Omega)$ and $[u] = 0$, and given $w \in H_0^1(\Omega)^\perp$ we have $w = P(w) = T([w])$. Thus T is linear and bijective.

The inner product on $H^{1/2}(\partial\Omega)$ is clearly sesquilinear, conjugate symmetric and positive definite. If $([u_j])$ is a Cauchy sequence, then $(T[u_j]) = (P(u_j))$ is Cauchy in $H^1(\Omega)$ and converges in $H^1(\Omega)$. Since $H_0^1(\Omega)^\perp$ is closed it follows that $P(u_j) \to P(u)$ in $H^1(\Omega)$ for some $u \in H_0^1(\Omega)$, so that $[u_j] \to [u]$ in $H^{1/2}(\partial\Omega)$. $\square$

Theorem 2.44. *(Right inverse of trace operator) There is a bounded linear map*

$$E_{\partial\Omega} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$$

that satisfies

$$RE_{\partial\Omega} f = f, \qquad f \in H^{1/2}(\partial\Omega).$$

In particular, for any $f \in H^{1/2}(\partial\Omega)$ there is $v_f \in H^1(\Omega)$ with

$$\|v_f\|_{H^1(\Omega)} \leq C \|f\|_{H^{1/2}(\partial\Omega)}, \quad v_f|_{\partial\Omega} = f.$$

Proof. It is enough to take $E_{\partial\Omega}([u]) = P(u)$ for $u \in H^1(\Omega)$. Then $RE_{\partial\Omega}([u]) = [P(u)] = [u]$ and $\|E_{\partial\Omega}([u])\|_{H^1(\Omega)} = \|P(u)\|_{H^1(\Omega)} = \|[u]\|_{H^{1/2}(\partial\Omega)}$. $\square$

Let us finally define the negative order Sobolev space $H^{-1/2}(\partial\Omega)$ as a dual space:

Definition 2.45. Define $H^{-1/2}(\partial\Omega)$ as the Hilbert dual

$$H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^* = \{T : H^{1/2}(\Omega) \to \mathbb{C} \text{ bounded linear functional}\}.$$

Example 2.46. We will see later that if Ω has C^1 boundary, any function $f \in L^2(\partial\Omega)$ can be identified with the element $T_f \in H^{-1/2}(\partial\Omega)$ defined by

$$T_f : H^{1/2}(\partial\Omega) \to \mathbb{R}, \quad T_f(g) = \int_{\partial\Omega} f g \, dS.$$

Thus in this case $L^2(\partial\Omega)$ will be a subspace of $H^{-1/2}(\partial\Omega)$.

2.4. Weak solutions

In this section, Ω will be a bounded open subset of $\mathbb{R}^n$ (no regularity of the boundary $\partial\Omega$ is required). Consider a second order differential operator L, acting on functions u on Ω, given by

$$Lu = -\sum_{j,k=1}^{\infty} \frac{\partial}{\partial x_j} \left(a^{jk} \frac{\partial u}{\partial x_k} \right) + qu. \tag{2.4}$$

Here the coefficients are assumed to satisfy the following conditions:

$$
(2.5)\qquad
\begin{aligned}
&a^{jk}, q \in L^\infty(\Omega) \text{ are real valued,}\\
&a^{jk} = a^{kj} \text{ for all } j,k = 1,\ldots,n,\\
&\sum_{j,k=1}^n a^{jk}(x)\xi_j\xi_k \geq c\,|\xi|^2 \text{ for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n, \text{ where } c > 0.
\end{aligned}
$$

The last condition is an *ellipticity condition* for the operator L, and it ensures that the operator L will have similar properties as the Laplace operator Δ.

Later in this book, we will mostly consider L to be one of the following special cases:

(1) The conductivity operator
$$Lu = -\mathrm{div}(\gamma\nabla u)$$
where $\gamma \in L^\infty(\Omega)$ is positive.

(2) The anisotropic conductivity operator
$$Lu = -\mathrm{div}(\gamma\nabla u)$$
where $\gamma = (\gamma^{jk})_{j,k=1}^n$ satisfies $\gamma^{jk} = \gamma^{kj} \in L^\infty(\Omega)$ and $\sum_{j,k=1}^n \gamma^{jk}(x)\xi_j\xi_k \geq c\,|\xi|^2$.

(3) The Schrödinger operator
$$Lu = (-\Delta + q)u$$
where $q \in L^\infty(\Omega)$.

Remark 2.47. The ellipticity condition implies a similar condition for complex vectors: if $A = (a^{jk}(x))_{j,k=1}^n$ and $\zeta \in \mathbb{C}^n$, then writing $\zeta = \xi + i\eta$ where $\xi, \eta \in \mathbb{R}^n$ and using the symmetry of A gives
$$A\zeta \cdot \zeta = A(\xi + i\eta)\cdot(\xi - i\eta) = A\xi\cdot\xi + A\eta\cdot\eta.$$
The ellipticity condition thus implies
$$\sum_{j,k=1}^n a^{jk}(x)\zeta_j\bar{\zeta}_k \geq c\,|\zeta|^2 \text{ for a.e. } x \in \Omega \text{ and all } \zeta \in \mathbb{C}^n. \tag{2.6}$$
We will use this stronger condition below.

Remark 2.48. Notice that if $a^{jk} \in C^1(\overline{\Omega})$, one can write Lu in the form
$$Lu = -\sum_{j,k=1}^n a^{jk}\partial_{jk}u - \sum_{j,k=1}^n (\partial_j a^{jk})\partial_k u + qu.$$
This operator is said to be in *nondivergence form*, while the operator (2.4) is in *divergence form*. We will consider divergence form operators in this

section since they are better suited to the energy method related to weak solutions.

Motivation 2.49. (Weak solutions) Suppose Ω has C^1 boundary, L has C^1 coefficients, and $u \in C^2(\overline{\Omega})$ is a classical solution of $Lu = F$ in Ω which satisfies the boundary condition $u|_{\partial\Omega} = f$, where $F \in L^2(\Omega)$ and $f \in C^0(\partial\Omega)$. Multiplying the equation $Lu = F$ by $\bar{v}$ where $v \in C^1(\overline{\Omega})$ satisfies $v|_{\partial\Omega} = 0$, an integration by parts implies that

$$\int_\Omega \left(\sum_{j,k=1}^n a^{jk} \partial_j u \overline{\partial_k v} + qu\bar{v} \right) dx = \int_\Omega F\bar{v}\, dx.$$

Now, the left hand side makes sense for any $u, v \in H^1(\Omega)$ (note however that the integration by parts above made use of the vanishing of v on $\partial\Omega$). We can use this identity to define weak solutions of the equation $Lu = F$ in Ω with $u|_{\partial\Omega} = f$. More generally, we can consider any right hand side F that is a continuous linear functional on $H_0^1(\Omega)$.

Definition 2.50. Let L be the differential operator (2.4). The sesquilinear form related to L is given by

$$B[u,v] = \int_\Omega \left(\sum_{j,k=1}^n a^{jk} \partial_j u \overline{\partial_k v} + qu\bar{v} \right) dx, \quad u, v \in H^1(\Omega). \tag{2.7}$$

If $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$, we say that a function $u \in H^1(\Omega)$ is a weak solution of the Dirichlet problem

$$\begin{cases} Lu = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

if one has

$$B[u,v] = F(\bar{v}) \quad \text{for all } v \in H_0^1(\Omega)$$

and if $Ru = f$, where R is the trace operator in Definition 2.41.

Remark 2.51. The condition $u = f$ on $\partial\Omega$ is called the *Dirichlet boundary condition*. It is understood in an abstract sense, having the following equivalent interpretations:

(a) $Ru = f$ where R is the trace operator,

(b) $u - v \in H_0^1(\Omega)$ for some $v \in H^1(\Omega)$ with $Rv = f$,

(c) $u - v \in H_0^1(\Omega)$ for any $v \in H^1(\Omega)$ with $Rv = f$.

We are ready to give the first solvability result for boundary value problems.

Theorem 2.52. *(Weak solutions) Let Ω be a bounded open set in $\mathbb{R}^n$, and let L be as in (2.4), (2.5). Assume in addition that*

$$q \geq 0 \quad \text{a.e. in } \Omega.$$

For any $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem

$$\begin{cases} Lu = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

There is a constant C independent of F and f such that

$$\|u\|_{H^1(\Omega)} \leq C\big(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)} \big).$$

This result follows readily from the next theorem.

Theorem 2.53. *If $s \in \mathbb{R}$ is a constant such that $q+s \geq 0$ almost everywhere, then the sesquilinear form $B_s[u,v] = B[u,v]+s(u,v)_{L^2(\Omega)}$ is an inner product on the space $H_0^1(\Omega)$ that induces a norm equivalent to the original one:*

$$C^{-1} \|u\|^2_{H^1(\Omega)} \leq B_s[u,u] \leq C \|u\|^2_{H^1(\Omega)}, \quad u \in H_0^1(\Omega).$$

Proof. It is clear that the map $(u,v) \mapsto B_s[u,v]$ is sesquilinear, and $B_s[u,v] = \overline{B_s[v,u]}$ since $a^{jk} = a^{kj}$ and a^{jk}, q, s are real. The ellipticity condition (2.6) and the assumption that $q + s \geq 0$ imply that

$$B_s[u,u] = \int_\Omega \left(\sum_{j,k=1}^n a^{jk} \partial_j u \overline{\partial_k u} + (q+s)\,|u|^2 \right) dx \geq c \int_\Omega |\nabla u|^2 \, dx.$$

Thus $B_s[u,u] \geq 0$, and if $B_s[u,u] = 0$ then $\nabla u = 0$ a.e. and thus $u = 0$ for instance by the Poincaré inequality (Theorem 2.38). We have proved that $B_s[\,\cdot\,,\,\cdot\,]$ is an inner product.

The triangle inequality and the fact that $a^{jk}, q \in L^\infty(\Omega)$ imply that

$$B_s[u,u] \leq C_{a^{jk},q,s} \int_\Omega \left(|\nabla u|^2 + |u|^2 \right) \leq C \|u\|^2_{H^1(\Omega)}, \quad u \in H_0^1(\Omega).$$

Moreover, the previous argument and the Poincaré inequality show that

$$B_s[u,u] \geq c \|\nabla u\|^2_{L^2(\Omega)} \geq c \|u\|^2_{H^1(\Omega)}, \quad u \in H_0^1(\Omega).$$

Thus $B_s[\,\cdot\,,\,\cdot\,]$ gives an equivalent norm on $H_0^1(\Omega)$. □

Proof of Theorem 2.52. Consider first the case of zero boundary values, where we want to solve

$$\begin{cases} Lu = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The assumption that $q \geq 0$ together with Theorem 2.53 show that $B[\,\cdot\,,\,\cdot\,] = B_0[\,\cdot\,,\,\cdot\,]$ is an inner product on $H_0^1(\Omega)$ giving a norm equivalent to the original one. It follows that the inner product space $(H_0^1(\Omega), B[\,\cdot\,,\,\cdot\,])$ has the same Cauchy sequences as the original $H_0^1(\Omega)$ and therefore is a Hilbert space. Also, F is a bounded linear functional on this space, satisfying

$$|F(v)| \leq \|F\|_{H^{-1}(\Omega)} \|v\|_{H^1(\Omega)} \leq C \|F\|_{H^{-1}(\Omega)} B[v,v]^{1/2}, \quad v \in H_0^1(\Omega).$$

The Riesz representation theorem implies that there exists a unique $u \in H_0^1(\Omega)$, whose norm is equal to the norm of F as a bounded linear functional on $(H_0^1(\Omega), B[\,\cdot\,,\,\cdot\,])$, such that

$$B[u,v] = F(\bar{v}), \quad v \in H_0^1(\Omega).$$

Since this function satisfies $Ru = 0$, we have found the unique solution of our boundary value problem. The solution also satisfies $\|u\|_{H^1(\Omega)} \leq C \|F\|_{H^{-1}(\Omega)}$.

We move to the case of nonzero boundary values, and want to find $u \in H^1(\Omega)$ with

$$B[u,v] = F(\bar{v}) \;\text{ for } v \in H_0^1(\Omega), \quad Ru = f.$$

Choose $e_f \in H^1(\Omega)$ with $\|e_f\|_{H^1(\Omega)} \leq C \|f\|_{H^{1/2}(\partial\Omega)}$ (this is possible by Theorem 2.44). Writing $u = e_f + \tilde{u}$, the boundary value problem is equivalent with

$$B[\tilde{u},v] = F(\bar{v}) - B[e_f,v] \;\text{ for } v \in H_0^1(\Omega), \quad R\tilde{u} = 0.$$

The map $\tilde{F} : w \mapsto F(w) - B[e_f,w]$ is a bounded linear functional on $H_0^1(\Omega)$ since by the triangle inequality and Cauchy-Schwarz

$$|B[e_f,w]| \leq C \int_\Omega (|\nabla e_f|\,|\nabla w| + |e_f|\,|w|)\,dx \leq C \|e_f\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}.$$

It follows that

$$\|\tilde{F}\|_{H^{-1}(\Omega)} \leq C(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}).$$

The result for zero boundary values proved above shows that there is a unique solution $\tilde{u} \in H_0^1(\Omega)$ satisfying

$$\|\tilde{u}\|_{H^1(\Omega)} \leq C(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}).$$

Thus, the original boundary value problem for u also has a unique solution with

$$\|u\|_{H^1(\Omega)} \leq C(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}).$$

□

Since the conductivity operator $-\mathrm{div}(\gamma\nabla u)$ is of the form (2.4) with $q = 0$, the previous result implies the basic solvability result for the conductivity equation. Next we wish to deal with the case where q may be negative somewhere. This is different from the previous case since the Dirichlet problem is not always uniquely solvable, and there may be nontrivial solutions of the equation $(-\Delta + q)u = 0$ with $u|_{\partial\Omega} = 0$. This is due to the existence of eigenfunctions, as illustrated by the following example.

Example 2.54. Let $\Omega = (0, \pi) \subset \mathbb{R}$, and consider the Schrödinger operator $-\Delta+q$ in Ω in the special case where the function q happens to be a constant, $q = -\lambda$ where $\lambda > 0$. A function u solves $(-\Delta + q)u = 0$ with vanishing boundary values if

$$u''(x) + \lambda u(x) = 0 \text{ for } 0 < x < \pi, \qquad u(0) = u(\pi) = 0.$$

The general solution to the ordinary differential equation $u''(x) + \lambda u(x) = 0$ is

$$u(x) = A \sin\left(\sqrt{\lambda}\, x\right) + B \cos\left(\sqrt{\lambda}\, x\right).$$

The boundary condition $u(0) = 0$ is satisfied if and only if $B = 0$. With $B = 0$, the boundary condition $u(\pi) = 0$ is satisfied if and only if either $A = 0$ or $\sin\left(\sqrt{\lambda}\,\pi\right) = 0$. The latter condition is equivalent to

$$\sqrt{\lambda} \in \mathbb{Z} \iff \lambda = k^2, \quad k \in \mathbb{Z}_+.$$

Thus there is a nontrivial solution $u \in H^1(\Omega)$ with vanishing boundary value whenever $\lambda = k^2$ for some positive integer k.

The next theorem shows that there is only a countable set of eigenvalues where unique solvability of the Dirichlet problem for L may fail. Outside of these eigenvalues, we recover the same solvability result as before. The proof uses compact Sobolev embedding and the spectral theorem for compact operators.

Theorem 2.55 (Weak solutions)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$, and let L be as in* (2.4)*,* (2.5)*.*

(1) There is a set of real numbers

$$Spec(L) = \{\lambda_j\}_{j=1}^{\infty}$$

with $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$, such that whenever $\lambda \notin Spec(L)$ the boundary value problem

$$\begin{cases} Lu = \lambda u + F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in H^1(\Omega)$ for any $F \in H^{-1}(\Omega)$ and any $f \in H^{1/2}(\partial\Omega)$.

(2) If $\lambda \notin Spec(L)$, then the map

$$\begin{aligned} H^1(\Omega) &\to H^{-1}(\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \\ u &\mapsto \quad \big(Lu - \lambda u \ , \ Ru\big) \end{aligned}$$

is an isomorphism (1–1, onto, bounded with bounded inverse). There is a constant C independent of F and f (but depending on λ) such that

$$\|u\|_{H^1(\Omega)} \leq C\big(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)} \big).$$

(3) If $\lambda \in Spec(L)$, there is a nontrivial solution $u \in H_0^1(\Omega)$ to the Dirichlet problem

$$\begin{cases} Lu = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The space of such solutions is finite dimensional.

(4) If $a \in \mathbb{R}$ is a constant such that

$$q(x) \geq a \quad \text{a.e. in } \Omega,$$

then $Spec(L) \subset (a, \infty)$.

Definition 2.56. The set $Spec(L)$ is called the *spectrum* of the operator L with Dirichlet boundary condition. The elements of $Spec(L)$ are called *Dirichlet eigenvalues.*

Proof. 1. Consider first the case of zero boundary values. Since $q \in L^\infty(\Omega)$, we may choose $s \in \mathbb{R}$ such that $q + s \geq 0$ a.e. in Ω. Let L_s be the operator obtained from L by replacing q by $q + s$, so that $L_s = L + s$, and let B_s be the corresponding sesquilinear form. Theorem 2.53 shows that $B_s[\,\cdot\,,\,\cdot\,]$ is an inner product on $H_0^1(\Omega)$ giving a norm equivalent to the original one.

Theorem 2.52 implies that the Dirichlet problem for L_s with zero boundary values has unique solutions, and the map $L_s : H_0^1(\Omega) \to H^{-1}(\Omega)$ is invertible with bounded inverse L_s^{-1} taking $F \in H^{-1}(\Omega)$ to the unique solution $u \in H_0^1(\Omega)$ of $Lu = F$ in Ω with $Ru = 0$. We now have, for $u \in H_0^1(\Omega)$,

$$Lu = \lambda u + F \Longleftrightarrow L_s u = (\lambda + s)u + F \Longleftrightarrow u - (\lambda + s)L_s^{-1}u = L_s^{-1}F.$$

If $\lambda \neq -s$, the last part may be equivalently written as

$$(\mu I - K)(i(u)) = \tilde{F}$$

with $\mu = \frac{1}{\lambda + s}$, $K = i \circ L_s^{-1} \circ j$, and $\tilde{F} = \mu i(L_s^{-1}F)$, where $i : H_0^1(\Omega) \to L^2(\Omega)$ and $j : L^2(\Omega) \to H^{-1}(\Omega)$ are the inclusion maps.

We claim that

$K : L^2(\Omega) \to L^2(\Omega)$ is a compact, self-adjoint, positive definite operator.

By Theorem 2.37 the map i is compact, and consequently K is compact. It is also self-adjoint, since for $F, G \in L^2(\Omega)$, writing $v = L_s^{-1}F$ and $w = L_s^{-1}G$ gives that

$$(KF, G)_{L^2} = (L_s^{-1}(F), G)_{L^2} = (v, L_s w)_{L^2} = \overline{B_s[w, v]}.$$

Similarly

$$(KG, F)_{L^2} = \overline{B_s[v, w]}.$$

Since B_s is conjugate symmetric, $\overline{B_s[v, w]} = B_s[w, v]$, we have $(KF, G)_{L^2} = (F, KG)_{L^2}$. Finally, K is positive definite since by the above computation

$$(KF, F)_{L^2} = B_s[v, v]$$

where B_s is a positive definite inner product.

By the spectral theorem for self-adjoint compact operators (Proposition A.73), $\mathrm{Spec}(K)$ is an at most countable subset of $\mathbb{R}$ that may only accumulate at 0, and each element of $\mathrm{Spec}(K)$ (except possibly 0) is an eigenvalue with finite dimensional eigenspace. Each eigenvalue is positive since K is positive definite. Since $L^2(\Omega)$ is not finite dimensional, $\mathrm{Spec}(K)$ is in fact countably infinite and contains 0, and we may write $\mathrm{Spec}(K) = \{\mu_j\}_{j=1}^{\infty} \cup \{0\}$ where $\mu_1 \geq \mu_2 \geq \ldots$ and $\mu_j \to 0$ as $j \to \infty$. Now, $\mu I - K$ is invertible on $L^2(\Omega)$ for all $\mu \in \mathbb{R} \setminus \mathrm{Spec}(K)$.

We now return to solvability of the Dirichlet problem. We wrote earlier that $\mu = \frac{1}{\lambda+s}$, so we define

$$\lambda_j = \frac{1}{\mu_j} - s, \qquad j = 1, 2, \ldots.$$

Then $\lambda_1 \leq \lambda_2 \leq \ldots$ and $\lambda_j \to \infty$. If $F \in H^{-1}(\Omega)$ and $\lambda + s \neq 0$, we saw above that for $u \in H_0^1(\Omega)$ one has

$$Lu = \lambda u + F \iff (\mu I - K)(i(u)) = \mu i(L_s^{-1}(F))$$

where $\mu = \frac{1}{\lambda+s} \notin \mathrm{Spec}(K)$. If we assume that $\lambda \in \mathbb{R} \setminus \{\lambda_j\}_{j=1}^{\infty}$, then $\mu \notin \mathrm{Spec}(K)$ and for any $F \in H^{-1}(\Omega)$ there is a unique solution $\tilde{u} \in L^2(\Omega)$ of

$$(\mu I - K)(\tilde{u}) = \mu i(L_s^{-1}(F)).$$

This function satisfies $\tilde{u} = \mu^{-1} L_s^{-1} \tilde{u} + L_s^{-1} F$, so $\tilde{u} = i(u)$ for some $u \in H_0^1(\Omega)$ with $\|u\|_{H^1(\Omega)} \leq C(\|\tilde{u}\|_{L^2(\Omega)} + \|F\|_{H^{-1}(\Omega)}) \leq C \|F\|_{H^{-1}(\Omega)}$, using that $\mu I - K$ and L_s are invertible. This shows the existence of a unique solution if $\lambda + s \neq 0$. In the remaining case where $\lambda + s = 0$, the equation is $L_s u = F$ which has unique solutions in $H_0^1(\Omega)$ by Theorem 2.52.

The case of nonzero boundary values is handled in the same way as in Theorem 2.52. This proves part 1 in the theorem.

2. The fact that $u \mapsto (Lu-\lambda u, Ru)$ is an isomorphism follows easily (only boundedness remains to be proved, but the inequality $\|Lu - \lambda u\|_{H^{-1}(\Omega)} + \|Ru\|_{H^{1/2}(\partial\Omega)} \leq C\,\|u\|_{H^1(\Omega)}$ follows from the arguments above).

3. If $\lambda = \lambda_j \in \mathrm{Spec}(L)$, then the proof in part 1 shows that the equation $Lu = \lambda u$ for $u \in H_0^1(\Omega)$ is equivalent with

$$(\mu I - K)(i(u)) = 0$$

for $\mu = \mu_j = \frac{1}{\lambda+s}$. Since $\mu_j \in \mathrm{Spec}(K)$ and K is compact, there is a nontrivial finite dimensional space consisting of those $\tilde{u} \in L^2(\Omega)$ with $(\mu I - K)\tilde{u} = 0$. This space is contained in $H_0^1(\Omega)$ because any such $\tilde{u}$ satisfies $\tilde{u} = \mu^{-1}L_s^{-1}\tilde{u}$, and gives rise to a finite dimensional space of solutions to $Lu = \lambda u$ in Ω.

4. If $q \geq a$ a.e., then we may choose $s = -a$ above and each eigenvalue satisfies $\lambda_j > a$ by definition. □

Exercise 2.57. Prove Theorem 2.52 in the simpler case of the Dirichlet problem for the Laplacian,

$$\begin{cases} -\Delta u = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

by using Remark 2.40 instead of Theorem 2.53.

Exercise 2.58. Show that Theorem 2.55, except for part 4, remains true if L is the following operator containing first order terms,

$$Lu = -\sum_{j,k=1}^{\infty} \frac{\partial}{\partial x_j}\left(a^{jk}\frac{\partial u}{\partial x_k}\right) + i\sum_{j=1}^{n}\left(b^j\frac{\partial u}{\partial x_j} + \frac{\partial}{\partial x_j}(b^j u)\right) + qu,$$

where the coefficients satisfy (2.5) and additionally $b^j \in L^\infty(\Omega)$ are real valued. The sesqulinear form corresponding to L is given by

$$B[u,v] = \int_\Omega \left(\sum_{j,k=1}^{n} a^{jk}\partial_j u\overline{\partial_k v} + i\sum_{j=1}^{n}(b^j(\partial_j u)\bar{v} - b^j u\partial_j\bar{v}) + qu\bar{v}\right)\,dx.$$

2.5. Higher regularity

We will end this chapter with a discussion of higher order regularity of solutions. The philosophy is that a solution of the second order elliptic equation $Lu = F$ should always be two derivatives smoother than the right hand side F, unless this gain of regularity is prevented by lack of smoothness in the coefficients of L, the boundary values of u, or in the boundary $\partial\Omega$.

Theorem 2.59. *(Elliptic regularity) Let Ω be a bounded open set in $\mathbb{R}^n$, and let L be the second order operator* (2.4)*. Assume that $u \in H^1(\Omega)$ solves the Dirichlet problem*

$$\begin{cases} Lu = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

(1) Assume Ω has C^2 boundary, $\gamma \in C^1(\overline{\Omega})$, and $q \in L^\infty(\Omega)$. If $F \in L^2(\Omega)$ and $f \in H^{3/2}(\Omega)$, then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}).$$

(2) Let $l \geq 1$ and assume that Ω has C^{l+2} boundary, $\gamma \in C^{l+1}(\overline{\Omega})$, and $q \in C^l(\Omega)$. If $F \in H^l(\Omega)$ and $f \in H^{l+3/2}(\partial\Omega)$, then $u \in H^{l+2}(\Omega)$ and

$$\|u\|_{H^{l+2}(\Omega)} \leq C(\|F\|_{H^l(\Omega)} + \|f\|_{H^{l+3/2}(\partial\Omega)}).$$

Theorem 2.59 has two useful consequences. The first concerns interior regularity of solutions.

Theorem 2.60. *(Interior regularity) Let Ω and Ω' be bounded open subsets of $\mathbb{R}^n$ with $\overline{\Omega} \subset \Omega'$. Let L be the second order operator* (2.4) *in Ω', and let $\ell \in \mathbb{N}$. There is a constant C, depending only on ℓ, Ω, Ω' and L such that, for all $u \in H^1(\Omega')$ and $F \in H^{\ell-2}(\Omega')$ obeying*

$$Lu = F \qquad \text{in } \Omega'$$

we have $u|_\Omega \in H^\ell(\Omega)$ and

$$\|u\|\,|_{H^\ell(\Omega)} \leq C\Big(\ \|F\|_{H^{\ell-2}(\Omega')} + \|u\|_{L^2(\Omega')}\ \Big).$$

The second consequence of Theorem 2.59 shows that if all quantities are C^∞, then also the weak solution u is C^∞ up to the boundary.

Theorem 2.61. *(Smoothness up to the boundary) Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary, and let L be the second order operator* (2.4)*. Assume that $\gamma \in C^\infty(\overline{\Omega})$ and $q \in C^\infty(\overline{\Omega})$. If $u \in H^1(\Omega)$ solves the Dirichlet problem*

$$\begin{cases} Lu = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $F \in C^\infty(\overline{\Omega})$ and $f \in C^\infty(\partial\Omega)$, then $u \in C^\infty(\overline{\Omega})$.

2.6. The DN map and inverse problems

Let Ω be a bounded open set in $\mathbb{R}^n$, and let $\gamma \in L^\infty(\Omega)$ be a positive function. We would like to define the Dirichlet-to-Neumann map mapping a boundary voltage f to the current flux at the boundary,

$$\Lambda_\gamma : f \mapsto \gamma \partial_\nu u_f|_{\partial\Omega}$$

where u_f is the solution the conductivity equation $\mathrm{div}(\gamma \nabla u_f) = 0$ in Ω with $u_f|_{\partial\Omega} = f$.

In fact we can consider the more general operators (2.4),

$$Lu = -\sum_{j,k=1}^{\infty} \frac{\partial}{\partial x_j}\left(a^{jk}\frac{\partial u}{\partial x_k}\right) + qu,$$

where $a^{jk}, q \in L^\infty(\Omega)$ satisfy the conditions (2.5). The corresponding DN map for L would be formally given by

$$\Lambda_L : f \mapsto \sum_{j,k=1}^{n} a^{jk}(\partial_j u)\nu_k\Big|_{\partial\Omega}. \tag{2.8}$$

Motivation 2.62. (Definition of DN map) Assume that Ω has smooth boundary and $a^{jk}, q \in C^\infty(\overline{\Omega})$, and suppose that u_f solves the Dirichlet problem $Lu = 0$ in Ω, $u_f|_{\partial\Omega} = f$, for some $f \in C^\infty(\partial\Omega)$. Then $u_f \in C^\infty(\overline{\Omega})$ by Theorem 2.61, and we may define $\Lambda_L f$ by the right hand side of (2.8) as a function in $C^\infty(\partial\Omega)$. Let $g \in C^\infty(\partial\Omega)$ and let $e_g \in C^\infty(\overline{\Omega})$ be any function such that $e_g|_{\partial\Omega} = g$. An integration by parts, using that all quantities are smooth, shows that

$$\begin{aligned}
\int_{\partial\Omega} (\Lambda_L f) g \, dS &= \sum_{j,k=1}^{n} \int_{\partial\Omega} a^{jk}(\partial_j u_f) e_g \nu_k \, dS \\
&= \sum_{j,k=1}^{n} \int_{\Omega} \partial_k \left(a^{jk}(\partial_j u_f) e_g\right) dx \\
&= \int_{\Omega} \Big[\sum_{j,k=1}^{n} a^{jk} \partial_j u_f \partial_k e_g + q u_f e_g \Big] \, dx.
\end{aligned}$$

In the last step we used that $Lu = 0$.

Notice that the expression on the last line is just $B[u_f, \bar{e}_g]$ where B is the sesquilinear form corresponding to L. This expression is well defined even when $a^{jk}, q \in L^\infty(\Omega)$ and $u_f, e_g \in H^1(\Omega)$. We use this observation to define the DN map in a weak sense even when the quantity $a^{jk}(\partial_j u)\nu_k$ may not be defined pointwise.

Recall that $H^{-1/2}(\partial\Omega)$ is the dual space of $H^{1/2}(\partial\Omega) = H^1(\Omega)/H_0^1(\Omega)$. If $f \in H^{-1/2}(\partial\Omega)$, we will express the duality by the notation

$$\langle f, g\rangle_{\partial\Omega} = f(g), \qquad g \in H^{1/2}(\partial\Omega).$$

If Ω has C^1 boundary and $f \in L^2(\partial\Omega)$, this reduces to the usual integral

$$\langle f, g\rangle_{\partial\Omega} = \int_{\partial\Omega} f g \, dS.$$

Theorem 2.63. *(DN map for L) Let Ω be a bounded open set in $\mathbb{R}^n$ and let L satisfy* (2.4), (2.5)*. Assume that* 0 *is not a Dirichlet eigenvalue of L in Ω. There is a unique bounded linear map*

$$\Lambda_L : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

that satisfies

$$\langle \Lambda_L f, g\rangle_{\partial\Omega} = B[u_f, \bar{e}_g] = \int_\Omega \Big[\sum_{j,k=1}^n a^{jk} \partial_j u_f \partial_k e_g + q u_f e_g \Big] \, dx \tag{2.9}$$

where $u_f \in H^1(\Omega)$ is the unique solution of $Lu = 0$ in Ω with $u|_{\partial\Omega} = f$, and e_g is any function in $H^1(\Omega)$ with $e_g|_{\partial\Omega} = g$.

If Ω has C^∞ boundary and $a^{jk}, q \in C^\infty(\overline{\Omega})$, then Λ_L restricts to a linear map

$$\Lambda_L : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$

which satisfies (2.8) *for all $f \in C^\infty(\partial\Omega)$.*

Proof. 1. The first step is to show that the right hand side of (2.9) does not depend on the particular choice of the extension e_g of g. That is, if $e_g, \tilde{e}_g$ are two functions in $H^1(\Omega)$ with $e_g|_{\partial\Omega} = \tilde{e}_g|_{\partial\Omega} = g$, then

$$B[u_f, \bar{e}_g] = B[u_f, \overline{\tilde{e}_g}].$$

But this follows from the fact that $\tilde{e}_g = e_g + \varphi$ for some $\varphi \in H_0^1(\Omega)$, since the definition of weak solutions implies $B[u_f, \bar{\varphi}] = 0$.

2. Fix $f \in H^{1/2}(\partial\Omega)$, and define a linear functional $T_f : H^{1/2}(\partial\Omega) \to \mathbb{C}$ by

$$T_f(g) = B[u_f, \bar{e}_g], \qquad g \in H^{1/2}(\partial\Omega),$$

where $e_g \in H^1(\Omega)$ is the extension of g provided by Theorem 2.44 satisfying $e_g|_{\partial\Omega} = g$ and $\|e_g\|_{H^1(\Omega)} \leq C \|g\|_{H^{1/2}(\partial\Omega)}$. Then by Cauchy-Schwarz

$$|T_f(g)| \leq \int_\Omega \left(\sum_{j,k=1}^n \left| a^{jk} \right| |\nabla u_f| \, |\nabla e_g| + |q| \, |u_f| \, |e_g| \right) dx \leq C \|u_f\|_{H^1(\Omega)} \|e_g\|_{H^1(\Omega)} .$$

By Theorem 2.55 we have

$$\|u_f\|_{H^1(\Omega)} \leq C \|f\|_{H^{1/2}(\partial\Omega)}$$

Consequently

$$|T_f(g)| \leq C \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)} .$$

Thus T_f is a bounded linear functional on $H^{1/2}(\partial\Omega)$, or in other words $T_f \in H^{-1/2}(\partial\Omega)$, and T_f has norm less than or equal to $C \|f\|_{H^{1/2}(\partial\Omega)}$. We define

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad f \mapsto T_f.$$

This map satisfies (2.9) and is the unique map with this property.

3. If Ω has C^∞ boundary and $\gamma \in C^\infty(\overline{\Omega})$, and if $f, g \in C^\infty(\partial\Omega)$, then $u_f \in C^\infty(\overline{\Omega})$ by Theorem 2.61. The computation done in Motivation 2.62 implies that

$$\langle \Lambda_L f, g\rangle_{\partial\Omega} = \sum_{j,k=1}^{n} \int_{\partial\Omega} a^{jk}(\partial_j u_f) e_g \nu_k \, dS.$$

Thus $\langle \Lambda_L f - \sum_{j,k=1}^n a^{jk}(\partial_j u_f)\nu_k|_{\partial\Omega}, g\rangle_{\partial\Omega} = 0$ for all $g \in C^\infty(\partial\Omega)$. Since $C^\infty(\partial\Omega)$ is dense in $H^{1/2}(\partial\Omega)$, this shows that $\Lambda_L f = \sum_{j,k=1}^n a^{jk}(\partial_j u_f)\nu_k|_{\partial\Omega}$ as elements of $H^{-1/2}(\partial\Omega)$. Consequently $\Lambda_\gamma f$ can be identified with the C^∞ function $\sum_{j,k=1}^n a^{jk}(\partial_j u_f)\nu_k|_{\partial\Omega}$. □

We now obtain the DN maps for the conductivity equation, anisotropic conductivity equation, and Schrödinger equation as special cases of the previous result.

Theorem 2.64. *(DN map for conductivity and Schrödinger equations) Let Ω be a bounded open set in $\mathbb{R}^n$, and let $c > 0$. Assume that*

(1) $\gamma \in L^\infty(\Omega)$ and $\gamma(x) \geq c$ for a.e. $x \in \Omega$,

(2) $G = (\gamma^{jk})_{j,k=1}^n$ is a symmetric matrix of $L^\infty(\Omega)$ functions and

$$\sum_{j,k=1}^{n} \gamma^{jk}(x)\xi_j\xi_k \geq c\,|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n,$$

(3) $q \in L^\infty(\Omega)$ is real valued and 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω.

There are bounded linear maps

$$\Lambda_\gamma, \Lambda_G, \Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

that satisfy for all $f, g \in H^{1/2}(\partial\Omega)$

$$\langle \Lambda_\gamma f, g\rangle_{\partial\Omega} = \int_\Omega \gamma \nabla u_f \cdot \nabla e_g \, dx, \langle \Lambda_G f, g\rangle_{\partial\Omega} = \int_\Omega G\nabla v_f \cdot \nabla e_g \, dx, \langle \Lambda_q f, g\rangle_{\partial\Omega} = \int_\Omega (\nabla w_f \cdot \nabla e_g + q w_f e_g$$

where $u_f, v_f, w_f \in H^1(\Omega)$ are the unique solutions of $\mathrm{div}(\gamma\nabla u) = 0$, $\mathrm{div}(G\nabla v) = 0$ *and $(-\Delta + q)w = 0$ in Ω with boundary value f, and e_g is any function in $H^1(\Omega)$ with $e_g|_{\partial\Omega} = g$.*

If Ω has C^∞ boundary and $\gamma, \gamma^{jk}, q \in C^\infty(\overline{\Omega})$, then these maps restrict to linear maps

$$\Lambda_\gamma, \Lambda_G, \Lambda_q : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$

which satisfy for $f \in C^\infty(\partial\Omega)$

$$\Lambda_\gamma f = \gamma \partial_\nu u_f|_{\partial\Omega}, \quad \Lambda_G f = G\nabla u_f \cdot \nu|_{\partial\Omega}, \quad \Lambda_q f = \partial_\nu u_f|_{\partial\Omega}.$$

Proof. By our assumptions, the operators $-\mathrm{div}(\gamma\nabla\,\cdot\,)$, $-\mathrm{div}(G\nabla\,\cdot\,)$, and $-\Delta + q$ are of the form (2.4), (2.5). In the first two cases there is no zero order term and we see from Theorem 2.55 that 0 is not a Dirichlet eigenvalue, and for the third case this is explicitly assumed. The result follows from Theorem 2.63. □

Exercise 2.65. Assume the conditions of Theorem 2.63, and show that knowledge of the DN map Λ_L is equivalent to knowing the quadratic form

$$Q_L : H^{1/2}(\partial\Omega) \to \mathbb{R}, \ \ Q_L(f) = B[u_f, u_f] = \int_\Omega \Big[\sum_{j,k=1}^n a^{jk}\partial_j u_f \overline{\partial_k u_f} + q\,|u_f|^2 \Big]\,dx.$$

(Physically, for the conductivity equation, the quadratic form

$$Q_\gamma(f) = \int_\Omega \gamma\,|\nabla u_f|^2\,dx$$

expresses the power needed to maintain the voltage f at the boundary.)

The next result shows that the DN map is a symmetric operator.

Theorem 2.66. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let L satisfy* (2.4), (2.5)*. Assume that* 0 *is not a Dirichlet eigenvalue of L in Ω. Then*

$$\langle \Lambda_L f, g\rangle_{\partial\Omega} = \langle f, \Lambda_L g\rangle_{\partial\Omega}, \quad f, g \in H^{1/2}(\partial\Omega).$$

In particular, the maps Λ_γ, Λ_G, Λ_q in Theorem 2.64 also have this property.

Proof. By definition

$$\langle \Lambda_L f, g\rangle_{\partial\Omega} = B[u_f, \bar{e}_g]$$

where $u_f \in H^1(\Omega)$ is the unique solution of $Lu = 0$ in Ω with $u_f|_{\partial\Omega} = f$, and e_g is any function $H^1(\Omega)$ with $e_g|_{\partial\Omega} = g$. We choose $e_g = u_g$, the solution of $Lu = 0$ with boundary value g. Then, since $B[\,\cdot\,,\,\cdot\,]$ is conjugate symmetric and a^{jk}, q are real valued,

$$\langle \Lambda_L f, g\rangle_{\partial\Omega} = B[u_f, \bar{u}_g] = \overline{B[\bar{u}_g, u_f]} = B[u_g, \bar{u}_f] = \langle \Lambda_L g, f\rangle_{\partial\Omega}.$$

□

We are now in a position to give mathematically precise formulations for the inverse problems considered in this book.

2.6.1. Calderón problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We think of Ω as an electrical conductor, and assume that the conductivity at each point of Ω is given by a function $\gamma \in L^\infty(\Omega)$ satisfying $\gamma(x) \geq c > 0$ a.e. in Ω. By Theorem 2.64 there is a bounded linear map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

which formally associates to a function $f \in H^{1/2}(\partial\Omega)$ an element $\Lambda_\gamma f \in H^{-1/2}(\partial\Omega)$ that may be thought of as the electrical current $\gamma\partial_\nu u_f|_{\partial\Omega}$ corresponding to boundary voltage f. (If Ω has C^∞ boundary and $\gamma \in C^\infty(\overline{\Omega})$, we saw in by Theorem 2.64 that $\Lambda_\gamma f = \gamma\partial_\nu u_f|_{\partial\Omega}$ for any $f \in C^\infty(\partial\Omega)$ in the classical sense.)

We think that for each boundary voltage $f \in H^{1/2}(\partial\Omega)$, we can measure the corresponding current $\Lambda_\gamma f \in H^{-1/2}(\partial\Omega)$. This leads to the following inverse problem.

Calderón problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\gamma \in L^\infty(\Omega)$ satisfy $\gamma \geq c > 0$ a.e. in Ω. From the knowledge of the map Λ_γ, determine the function γ in Ω.

2.6.2. Inverse BVP for Schrödinger equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $q \in L^\infty(\Omega)$. We consider an inverse problem for the equation $(-\Delta + q)u = 0$ analogous to the Calderón problem. However, in order to have a well defined DN map we need to assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. If this is the case, then by Theorem 2.64 there is a bounded linear map

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

that associates to any function $f \in H^{1/2}(\partial\Omega)$ an element $\Lambda_q f \in H^{-1/2}(\partial\Omega)$ that corresponds (in the case where everything is smooth) to the normal derivative $\partial_\nu u_f|_{\partial\Omega}$, where $u_f \in H^1(\Omega)$ is the unique solution of $(-\Delta+q)u = 0$ in Ω with boundary value f. The inverse problem is as follows.

Inverse BVP for Schrödinger equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $q \in L^\infty(\Omega)$, and assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. From the knowledge of the map Λ_q, determine the function q in Ω.

It is also possible to formulate this problem without the assumption that 0 is not a Dirichlet eigenvalue, by using the notion of Cauchy data sets. To do this, we observe that even though the normal derivative on $\partial\Omega$ does not make sense for a general function $u \in H^1(\Omega)$, we can still define the normal derivative weakly if we assume that u is additionally a solution. The proof is similar to that of Theorem 2.63.

Exercise 2.67. If $u \in H^1(\Omega)$ is a solution of $(-\Delta + q)u = 0$ in Ω, show that the following identity defines $\partial_\nu u|_{\partial\Omega}$ as an element of $H^{-1/2}(\partial\Omega)$:

$$\langle \partial_\nu u|_{\partial\Omega}, g\rangle_{\partial\Omega} = \int_\Omega (\nabla u \cdot \nabla e_g + q u e_g)\, dx, \quad g \in H^{1/2}(\partial\Omega),$$

where e_g is any function in $H^1(\Omega)$ with $e_g|_{\partial\Omega} = g$. Show that if $u \in C^\infty(\overline{\Omega})$, then this definition of $\partial_\nu u|_{\partial\Omega}$ coincides with the usual one.

Definition 2.68. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $q \in L^\infty(\Omega)$. The *Cauchy data set* associated with the operator $-\Delta + q$ in Ω is the set

$$C_q = \{ \ (u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) \mid u \in H^1(\Omega), (-\Delta + q)u = 0 \text{ in } \Omega \ \}.$$

By Problem 2.67, C_q is a subset of $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. If 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω, the next problem shows that knowing the Cauchy data set C_q is equivalent to knowing the DN map Λ_q.

Exercise 2.69. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $q \in L^\infty(\Omega)$, and assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. Show that the Cauchy data set is the graph of the DN map acting on $H^{1/2}(\partial\Omega)$:

$$C_q = \{ \ (f, \Lambda_q f) \mid f \in H^{1/2}(\partial\Omega) \ \}.$$

The next question generalizes the inverse BVP for Schrödinger equation given above.

Inverse BVP for Schrödinger equation, Cauchy data set version. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $q \in L^\infty(\Omega)$. From the knowledge of the set C_q, determine the function q in Ω.

2.6.3. Anisotropic Calderón problem. Again, $\Omega \subset \mathbb{R}^n$ is a bounded open set that is thought of as an electrical conductor, but this time the conductivity at each point of Ω is given by a symmetric matrix function $G = (\gamma^{jk})_{j,k=1}^n$. For this problem we assume that Ω has C^∞ boundary and that each γ^{jk} is in $C^\infty(\overline{\Omega})$. We also assume the ellipticity condition for some $c > 0$,

$$\sum_{j,k=1}^n \gamma^{jk}(x)\xi_j\xi_k \geq c\,|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n.$$

By Theorem 2.64 there is a linear map

$$\Lambda_G : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$

which associates to a function $f \in C^\infty(\partial\Omega)$ the boundary current $\Lambda_\gamma f = G\nabla u_f \cdot \nu|_{\partial\Omega} \in C^\infty(\partial\Omega)$.

Anisotropic Calderón problem. From the knowledge of the map Λ_G, determine a symmetric matrix function $\hat{G}$ with elements in $C^\infty(\overline{\Omega})$ such that $\hat{G} = F_* G$ for some diffeomorphism $F : \overline{\Omega} \to \overline{\Omega}$ with $F|_{\partial\Omega} = Id$.

2.7. Integral identities and reductions

The purpose of this section is to introduce certain integral identities for differences of two DN maps. These identities can be used to relate boundary measurements to interior information about the coefficients, and they allow to reduce uniqueness questions in inverse problems to questions about the density of products of (gradients of) solutions. We also give two simple reductions that will be useful later.

Theorem 2.70. *(Integral identity for $\Lambda_{L_1} - \Lambda_{L_2}$) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let L_1, L_2 be two operators of the form*

$$L_m u = -\sum_{j,k=1}^{\infty} \frac{\partial}{\partial x_j}\left(a_m^{jk} \frac{\partial u}{\partial x_k}\right) + q_m u,$$

where a_m^{jk}, q_m are as in (2.5) for $m = 1, 2$. Assume that 0 is not a Dirichlet eigenvalue for L_1 or L_2 in Ω. Then for any $f_1, f_2 \in H^{1/2}(\partial\Omega)$,

$$\langle(\Lambda_{L_1} - \Lambda_{L_2})f_1, f_2\rangle_{\partial\Omega} = \int_\Omega \left(\sum_{j,k=1}^{n} (a_1^{jk} - a_2^{jk})\partial_j u_1 \partial_k u_2 + (q_1 - q_2)u_1 u_2\right) dx,$$

where $u_m \in H^1(\Omega)$ is the unique solution of $L_m u_m = 0$ in Ω with $u_m|_{\partial\Omega} = f_m$.

Proof. Let u_1 and u_2 be as described. By Theorem 2.64 we have

$$\langle\Lambda_{L_1} f_1, f_2\rangle_{\partial\Omega} = \int_\Omega \left(\sum_{j,k=1}^{n} a_1^{jk}\partial_j u_1 \partial_k v_2 + q_1 u_1 v_2\right) dx$$

where v_2 is any function in $H^1(\Omega)$ with $v_2|_{\partial\Omega} = f_2$. Similarly, also using Theorem 2.66, we have

$$\langle\Lambda_{L_2} f_1, f_2\rangle_{\partial\Omega} = \langle\Lambda_{L_2} f_2, f_1\rangle_{\partial\Omega} = \int_\Omega \left(\sum_{j,k=1}^{n} a_2^{jk}\partial_j u_2 \partial_k v_1 + q_2 u_2 v_1\right) dx$$

where v_1 is any function in $H^1(\Omega)$ with $v_1|_{\partial\Omega} = f_1$. Now, we may choose $v_1 = u_1$ and $v_2 = u_2$. Subtracting the two identities above, we obtain the theorem. □

As an immediate consequence of the above result, we obtain integral identities for the differences of DN maps in the case of the conductivity and Schrödinger equation.

Theorem 2.71. *(Integral identity for $\Lambda_{\gamma_1} - \Lambda_{\gamma_2}$) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ satisfy $\gamma_1, \gamma_2 \geq c > 0$ a.e. in Ω.*

(a) One has the integral identity

$$\langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1, f_2 \rangle_{\partial\Omega} = \int_\Omega (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 \, dx, \quad f_1, f_2 \in H^{1/2}(\partial\Omega),$$

where $u_j \in H^1(\Omega)$ is the unique solution of $\mathrm{div}(\gamma_j \nabla u_j) = 0$ in Ω with $u_j|_{\partial\Omega} = f_j$.

(b) If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then

$$\int_\Omega (\gamma_1 - \gamma_2) \nabla w_1 \cdot \nabla w_2 \, dx = 0$$

for all $w_j \in H^1(\Omega)$ with $\mathrm{div}(\gamma_j \nabla w_j) = 0$ in Ω.

Theorem 2.72. *(Integral identity for $\Lambda_{q_1} - \Lambda_{q_2}$) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $q_1, q_2 \in L^\infty(\Omega)$, and assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q_j$ in Ω.*

(a) One has the integral identity

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle_{\partial\Omega} = \int_\Omega (q_1 - q_2) u_1 u_2 \, dx, \quad f_1, f_2 \in H^{1/2}(\partial\Omega),$$

where $u_j \in H^1(\Omega)$ is the unique solution of $(-\Delta + q_j) u_j = 0$ in Ω with $u_j|_{\partial\Omega} = f_j$.

(b) If $\Lambda_{q_1} = \Lambda_{q_2}$, then

$$\int_\Omega (q_1 - q_2) w_1 w_2 \, dx = 0$$

for all $w_j \in H^1(\Omega)$ with $(-\Delta + q_j) u_j = 0$ in Ω.

Exercise 2.73. Show that Theorem 2.72(b) remains true in the case of Cauchy data sets: if $\Omega \subset \mathbb{R}^n$ is a bounded open set and $q_1, q_2 \in L^\infty(\Omega)$, then

$$\int_\Omega (q_1 - q_2) w_1 w_2 \, dx = \langle \partial_\nu w_1, w_2 \rangle_{\partial\Omega} - \langle w_1, \partial_\nu w_2 \rangle_{\partial\Omega}$$

for all $w_j \in H^1(\Omega)$ with $(-\Delta + q_j) w_j = 0$ in Ω, where the normal derivatives are interpreted as in Problem 2.67. If additionally $C_{q_1} = C_{q_2}$, then

$$\int_\Omega (q_1 - q_2) w_1 w_2 \, dx = 0$$

for all such w_j.

Suppose now that γ_1, γ_2 are two conductivities such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. We would like to conclude that $\gamma_1 = \gamma_2$. By Theorem 2.71(b), this would follow if one can show that the set

$$\{ \nabla w_1 \cdot \nabla w_2 \mid w_j \in H^1(\Omega), \operatorname{div}(\gamma_j \nabla w_j) = 0 \text{ in } \Omega \}$$

is dense in $L^1(\Omega)$. In the same way, if $q_1, q_2 \in L^\infty(\Omega)$ and $\Lambda_{q_1} = \Lambda_{q_2}$ then it would follow from Theorem 2.72(b) that $q_1 = q_2$ if one can show the set

$$\{ w_1 w_2 \mid w_j \in H^1(\Omega), (-\Delta + q_j) w_j = 0 \text{ in } \Omega \}$$

is dense in $L^1(\Omega)$. This implies that uniqueness in the Calderón problem would follow from the density of products of gradients of solutions to conductivity equations. Similarly, uniqueness in the inverse BVP for Schrödinger equation would follow from the density of products of solutions to Schrödinger equations. Most of the interior uniqueness results in this book will be proved by following this route.

Next we give a result showing that the Calderón problem can be reduced to the inverse BVP for the Schrödinger equation, provided that the conductivity has two derivatives. This is based on a simple argument, sometimes called a *Liouville transformation*, where the substitution $w = \gamma^{1/2} u$ reduces the conductivity equation $\operatorname{div}(\gamma \nabla u) = 0$ to the Schrödinger equation $(-\Delta + q_\gamma) w = 0$, where the potential q_γ is given by

$$q_\gamma = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}.$$

Theorem 2.74. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\gamma \in C^2(\overline{\Omega})$ be strictly positive, and set $q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$.*

(a) If $f \in H^{1/2}(\partial\Omega)$, then $u \in H^1(\Omega)$ is a solution of

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega$$

if and only if $w = \gamma^{1/2} u$ is a solution of

$$-\Delta w + q w = 0 \text{ in } \Omega, \qquad w = \gamma^{1/2} f \text{ on } \partial\Omega.$$

In particular, 0 is not a Dirichlet eigenvalue for $-\Delta + q$ in Ω if q arises from a C^2 conductivity.

(b) If Ω has C^∞ boundary and $\gamma \in C^\infty(\overline{\Omega})$, then the DN maps Λ_γ and Λ_q for q as defined above are related by

$$\Lambda_q f = \gamma^{-1/2} \Lambda_\gamma (\gamma^{-1/2} f) + \tfrac{1}{2} \gamma^{-1} (\partial_\nu \gamma) f \big|_{\partial\Omega}, \quad f \in C^\infty(\partial\Omega).$$

Proof. (a) Assume first that $w \in C^2(\overline{\Omega})$. To reduce the Schrödinger equation to a conductivity equation, we attempt to find $a \in C^2(\overline{\Omega})$ such that

$\nabla \cdot (\gamma \nabla (aw)) = \gamma a(\Delta w + rw)$ for some function r. By the product rule

$$\begin{aligned}\nabla \cdot (\gamma \nabla (aw)) &= \nabla \cdot (\gamma (\nabla a) w + \gamma a \nabla w) \\ &= \gamma a \Delta w + (\nabla (\gamma a) + \gamma \nabla a) \cdot \nabla w + (\nabla \cdot (\gamma \nabla a)) w.\end{aligned}$$

The first order term will disappear if we can choose a so that

$$\nabla(\gamma a) + \gamma \nabla a = 0.$$

This is equivalent with $2\gamma \nabla a + (\nabla \gamma) a = 0$, and dividing by γa we obtain the equation

$$\nabla(2 \log a + \log \gamma) = 0.$$

Thus, by the properties of logarithms $\log(a^2 \gamma)$ should be constant, so that $a = C\gamma^{-1/2}$ for some constant C.

We choose $a = \gamma^{-1/2}$, and the previous computation implies

$$\nabla \cdot (\gamma \nabla (aw)) = \gamma a \Delta w + (\nabla \cdot (\gamma \nabla a)) w.$$

Here $\gamma \nabla a = -\nabla(\gamma a)$, so we obtain

$$\nabla \cdot (\gamma \nabla (\gamma^{-1/2} w)) = \gamma^{1/2} (\Delta w - \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} w).$$

This shows part (a) in the case where $u, w \in C^2(\overline{\Omega})$. The case where $u, w \in H^1(\Omega)$ follows either from the definition of weak solutions or by approximation, and is left as a problem.

(b) Set $g = \gamma^{-1/2} f$ and $u_g = \gamma^{-1/2} w_f$, where w_f is the unique solution of $(-\Delta + q) w = 0$ in Ω with boundary value f. Then, by part (a), u_g is the solution of $\nabla \cdot \gamma \nabla u_g = 0$ in Ω with $u_g|_{\partial\Omega} = g$, and

$$\begin{aligned}\Lambda_q(f) &= \sum_{j=1}^n (\partial_j w_f) \hat{n}_j \Big|_{\partial\Omega} = \sum_{j=1}^n \partial_j (\gamma^{1/2} u_g) \hat{n}_j \Big|_{\partial\Omega} \\ &= \sum_{j=1}^n \gamma^{1/2} (\partial_j u_g) \hat{n}_j + \sum_{j=1}^n \tfrac{1}{2} u_g \gamma^{-1/2} (\partial_j \gamma) \hat{n}_j \Big|_{\partial\Omega} \\ &= \gamma^{-1/2} \Lambda_\gamma(g) + \sum_{j=1}^n \tfrac{1}{2} g \gamma^{-1/2} (\partial_j \gamma) \hat{n}_j \Big|_{\partial\Omega} \\ &= \gamma^{-1/2} \Lambda_\gamma(\gamma^{-1/2} f) + \tfrac{1}{2} \gamma^{-1} (\partial_\nu \gamma) f \Big|_{\partial\Omega}.\end{aligned}$$

□

Exercise 2.75. Verify Theorem 2.74(a) in detail for $u, w \in H^1(\Omega)$.

The final reduction states that if two DN maps for operators in some domain Ω are equal, then the DN maps corresponding to extensions of the operators to a larger domain $\tilde{\Omega}$ are also equal provided that the coefficients agree in $\tilde{\Omega} \setminus \Omega$.

Theorem 2.76. *Let Ω and $\tilde{\Omega}$ be bounded open sets in $\mathbb{R}^n$ with $\Omega \subset \tilde{\Omega}$. Suppose that L_1 and L_2 are two operators in Ω of the form*

$$L_m u = -\sum_{j,k=1}^{\infty} \frac{\partial}{\partial x_j}\left(a_m^{jk}\frac{\partial u}{\partial x_k}\right) + q_m u.$$

Let $\tilde{a}_m^{jk}, \tilde{q}_m$ be extensions of a_m^{jk}, q_m to $\tilde{\Omega}$, and denote by $\tilde{L}_1$ and $\tilde{L}_2$ the corresponding operators in $\tilde{\Omega}$,

$$\tilde{L}_m u = -\sum_{j,k=1}^{\infty} \frac{\partial}{\partial x_j}\left(\tilde{a}_m^{jk}\frac{\partial u}{\partial x_k}\right) + \tilde{q}_m u.$$

Assume that $a_m^{jk}, q_m, \tilde{a}_m^{jk}, \tilde{q}_m$ satisfy (2.5) *for $m = 1, 2$, and assume that* 0 *is not a Dirichlet eigenvalue for L_1 or L_2 in Ω or for $\tilde{L}_1$ or $\tilde{L}_2$ in Ω.*

If

$$\tilde{a}_1^{jk} = \tilde{a}_2^{jk} \text{ in } \tilde{\Omega}\setminus\Omega, \qquad \tilde{q}_1 = \tilde{q}_2 \text{ in } \tilde{\Omega}\setminus\Omega,$$

then

$$\langle(\Lambda_{\tilde{L}_1} - \Lambda_{\tilde{L}_2})\tilde{f}_1, \tilde{f}_2\rangle_{\partial\tilde{\Omega}} = \langle(\Lambda_{L_1} - \Lambda_{L_2})(\tilde{u}_1|_{\partial\Omega}), \tilde{u}_2|_{\partial\Omega}\rangle_{\partial\Omega}, \quad \tilde{f}_1, \tilde{f}_2 \in H^{1/2}(\partial\tilde{\Omega}),$$

where $\tilde{u}_m$ is the unique solution in $H^1(\tilde{\Omega})$ of $\tilde{L}_m\tilde{u}_m = 0$ in $\tilde{\Omega}$ with $\tilde{u}_m|_{\partial\tilde{\Omega}} = \tilde{f}_m$.

In particular,

$$\Lambda_{L_1} = \Lambda_{L_2} \implies \Lambda_{\tilde{L}_1} = \Lambda_{\tilde{L}_2}.$$

Proof. By Theorem 2.70, and using the fact that $\tilde{a}_1^{jk} = \tilde{a}_2^{jk}$ and $\tilde{q}_1 = \tilde{q}_2$ outside of Ω, we have

$$\begin{aligned}\langle(\Lambda_{\tilde{L}_1} - \Lambda_{\tilde{L}_2})\tilde{f}_1, \tilde{f}_2\rangle_{\partial\tilde{\Omega}} &= \int_{\tilde{\Omega}}\left(\sum_{j,k=1}^{n}(\tilde{a}_1^{jk} - \tilde{a}_2^{jk})\partial_j\tilde{u}_1\partial_k\tilde{u}_2 + (\tilde{q}_1 - \tilde{q}_2)\tilde{u}_1\tilde{u}_2\right)dx \\ &= \int_{\Omega}\left(\sum_{j,k=1}^{n}(a_1^{jk} - a_2^{jk})\partial_j\tilde{u}_1\partial_k\tilde{u}_2 + (q_1 - q_2)\tilde{u}_1\tilde{u}_2\right)dx \\ &= \langle(\Lambda_{L_1} - \Lambda_{L_2})(\tilde{u}_1|_{\partial\Omega}), \tilde{u}_2|_{\partial\Omega}\rangle_{\partial\Omega}\end{aligned}$$

since $\tilde{u}_m|_{\Omega}$ solves $L_m\tilde{u}_m = 0$ in Ω. □

2.8. Notes

Section 2.1. For more on convolutions we refer to Hörmander, The analysis of linear partial differential operators, Vol. I.

Sections 2.3–2.5. The treatment here partly follows Evans, Partial differential equations, which contains further material on Sobolev spaces and weak solutions.

Chapter 3

Boundary determination

The goal of this chapter is to show that if two conductivities γ_1 and γ_2 are in $C^\infty(\overline{\Omega})$ and give rise to the same boundary measurements (i.e., $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$) on the entire boundary, then the conductivities and their normal derivatives of all orders agree on $\partial\Omega$. This was the first identifiability theorem proved for the conductivity equation and it seems to remain a necessary ingredient in many proofs of identifiability in the interior.

The critical observation is that by choosing the Dirichlet boundary data f to be highly oscillatory and supported near a point $p \in \partial\Omega$, we can arrange that the solution to $\mathrm{div}(\gamma\nabla u) = 0$, $u|_{\partial\Omega} = f$, is concentrated near p. Solutions of this type can be used to extract the Taylor series of the conductivity at p from the knowledge of Λ_γ.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let Γ be an open subset of $\partial\Omega$. Suppose that $\gamma \in C^\infty(\overline{\Omega})$ is a positive function, and that one has knowledge of the measurements*

$$\Lambda_\gamma f\big|_\Gamma \quad \textit{for all } f \in C^\infty(\partial\Omega) \textit{ supported in } \Gamma.$$

From this information it is possible to determine $(\partial/\partial\nu)^l\gamma$ on Γ for any integer $l \geq 0$.

The precise definition of the l^{th} order normal derivative $(\partial/\partial\nu)^l\gamma$ is given below in Section 3.3. Note that Theorem 3.1 is a constructive and local result: from the knowledge of the Dirichlet-to-Neumann map on a small subset Γ of the boundary, one can constructively determine the conductivity and its normal derivatives on Γ. In particular, the following uniqueness result is an immediate corollary.

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and suppose that γ_1 and γ_2 are two positive functions in $C^\infty(\overline{\Omega})$. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then for any integer $l \geq 0$

$$\left(\frac{\partial}{\partial \nu}\right)^l \gamma_1 = \left(\frac{\partial}{\partial \nu}\right)^l \gamma_2 \quad \text{on } \partial\Omega.$$

We wish to give one heuristic explanation as to why oscillating boundary data are useful in boundary determination. This explanation is based on symbol calculus for (pseudo)differential operators, which will not be used anywhere in the book.

Recall that we are trying to determine the boundary values of the conductivity from the map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega).$$

As the notation suggests, this map acts like a first order differential operator on $\partial\Omega$ in the sense that it takes away one derivative from any function that it is applied to. Pretend for the moment that $\partial\Omega = \mathbb{R}^{n-1}$. First order differential operators on $\mathbb{R}^{n-1}$ have the form

$$A(x', D') = \sum_{j=1}^{n-1} a_j(x') D_j, \qquad x' \in \mathbb{R}^{n-1},$$

where we write $D' = (D_1, \ldots, D_{n-1})$ and $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. This operator is characterized by its symbol, which is the function

$$a(x', \xi') = \sum_{j=1}^{n-1} a_j(x') \xi_j, \qquad x', \xi' \in \mathbb{R}^{n-1}.$$

The symbol can be obtained from the operator by testing against oscillatory functions:

$$a(x', \xi') = N^{-1} e^{-iNx'\cdot\xi'} A(x', D') e^{iNx'\cdot\xi'}, \quad N \text{ large}.$$

Now, the operator Λ_γ is not a differential operator of order one, but belongs to the more general class of *classical pseudodifferential operators* of order one. This means that the main behavior of Λ_γ is also governed by a symbol, but this time the symbol is given by an infinite asymptotic sum

$$a(x', \xi') \sim a_1(x', \xi') + a_0(x', \xi') + a_{-1}(x', \xi') + \ldots$$

where each a_j is a smooth function that is positively homogeneous of degree j in ξ' for ξ' away from the origin. As for differential operators, the functions

a_j can be obtained from the operator Λ_γ by testing against (localized) highly oscillatory functions. It turns out that one can recover $\gamma|_{\partial\Omega}$ from a_1, $\partial_\nu\gamma|_{\partial\Omega}$ from a_0, and so on. The (distributional) integral kernel $\lambda(x,y)$ of Λ_γ has an expansion corresponding to the symbol expansion,

$$\lambda(x,y) \sim \lambda_1(x,y) + \lambda_0(x,y) + \lambda_{-1}(x,y) + \ldots,$$

where $\lambda_1(x,y)$ corresponds to the strongest singularities of the kernel $\lambda(x,y)$, $\lambda_0(x,y)$ corresponds to the next strongest singularities, and so on. It turns out that from the singularities of the kernel of Λ_γ one can only recover the Taylor series of the conductivity at boundary points, but not at interior points. The values of γ at interior points are hidden in the C^∞ part of the kernel, which makes the interior uniqueness problem rather subtle.

In this chapter, instead of using the theory of pseudodifferential operators to recover the boundary values, we will employ elementary direct methods. In the next two sections we will show that the boundary value $\gamma|_{\partial\Omega}$ and the normal derivative $\partial_\nu\gamma|_{\partial\Omega}$ are determined by the DN map in a local and stable way. These arguments are valid also when the conductivity and the boundary have limited regularity. The proof of Theorem 3.1 follows similar ideas, but is longer and requires a higher order asymptotic construction. The proof is divided in three parts. The first step is to flatten the boundary near a fixed boundary point p by a suitable change of coordinates. Next, one constructs the solutions which concentrate near the boundary point and oscillate rapidly on the flat boundary piece, the speed of oscillations depending on a large parameter $s > 0$. The third step is to use the boundary values ϕ_s of these solutions in the expression $\langle \Lambda_\gamma \phi_s, \bar{\phi}_s \rangle$, where Λ_γ is our given data and ϕ_s will be explicit functions. The Taylor series of γ can now be read off from the large s asymptotics of this expression.

3.1. Recovering boundary values

The main result in this section states that the boundary values $\gamma|_{\partial\Omega}$ can be determined from the knowledge of the DN map Λ_γ.

Theorem 3.3. *(Recovering γ on $\partial\Omega$) Let Ω be a bounded open set with C^1 boundary, and let $\gamma \in C^0(\overline{\Omega})$ be positive. Given a point $x_0 \in \partial\Omega$, there exists a sequence of functions $(f_M) \subset C^1(\partial\Omega)$ for which*

$$\lim_{M\to\infty} \langle \Lambda_\gamma f_M, \bar{f}_M \rangle_{\partial\Omega} = \gamma(x_0).$$

The functions f_M do not depend on γ and they are supported in $B(x_0, 1/M) \cap \partial\Omega$.

In fact, since f_M are independent of γ and are supported in small balls, the result gives a constructive method for finding $\gamma(x_0)$ from the local DN map evaluated in a small neighborhood of x_0 on $\partial\Omega$. The method also allows to show the following stability result at the boundary.

Theorem 3.4. *(Stability of γ on $\partial\Omega$) Let Ω be a bounded open set with C^1 boundary, and let $\gamma_1, \gamma_2 \in C^0(\overline{\Omega})$ be positive. Then*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}.$$

Both of these results follow immediately from the existence of special solutions to the conductivity equation that concentrate at the boundary point x_0:

Theorem 3.5. *(Solutions concentrating at a boundary point) Let Ω be a bounded open set with C^1 boundary, and let $\gamma \in C^0(\overline{\Omega})$ be positive. Given a point $x_0 \in \partial\Omega$, there is a sequence of solutions $(u_M) \subset H^1(\Omega)$ of the conductivity equation*

$$\mathrm{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega$$

such that

$$\lim_{M\to\infty} \int_\Omega \gamma \left|\nabla u_M\right|^2 \, dx = \gamma(x_0).$$

Further, $f_M = u_M|_{\partial\Omega}$ are functions in $C^1(\partial\Omega)$ supported in $B(x_0, 1/M) \cap \partial\Omega$ that do not depend on γ, and they satisfy

$$\|f_M\|_{H^{1/2}(\partial\Omega)} = O(1) \quad \text{as } M \to \infty \text{ uniformly over } x_0 \in \partial\Omega.$$

Proof of Theorem 3.3. Let u_M be as in Theorem 3.5. By the definition of the DN map (Theorem 2.64), we have

$$\langle \Lambda_\gamma f_M, \bar{f}_M \rangle_{\partial\Omega} = \int_\Omega \gamma \nabla u_M \cdot \nabla \bar{u}_M \, dx$$

and consequently

$$\lim_{M\to\infty} \langle \Lambda_\gamma f_M, \bar{f}_M \rangle_{\partial\Omega} = \lim_{M\to\infty} \int_\Omega \gamma \left|\nabla u_M\right|^2 \, dx = \gamma(x_0).$$

□

Proof of Theorem 3.4. Let u_M and v_M be solutions provided by Theorem 3.5 of the equations

$$\mathrm{div}(\gamma_1 \nabla u_M) = 0, \qquad \mathrm{div}(\gamma_2 \nabla v_M) = 0.$$

Since the boundary values of u_M and v_M only depend on M and $\partial\Omega$, we have $u_M|_{\partial\Omega} = v_M|_{\partial\Omega} = f_M$. Then by the definition of the DN maps,

$$\langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_M, \bar{f}_M \rangle_{\partial\Omega} = \int_\Omega \gamma_1 \left|\nabla u_M\right|^2 \, dx - \int_\Omega \gamma_2 \left|\nabla v_M\right|^2 \, dx.$$

Taking the limit as $M \to \infty$ and taking absolute values, we have

$$\begin{aligned}|\gamma_1(x_0) - \gamma_2(x_0)| &= \lim_{M\to\infty} \left|\langle(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})f_M, \bar{f}_M\rangle_{\partial\Omega}\right| \\ &\leq \lim_{M\to\infty} \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)} \|f_M\|^2_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

Since $\|f_M\|_{H^{1/2}(\partial\Omega)}$ is bounded uniformly with respect to M and x_0, it follows that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)}.$$

□

To prove Theorem 3.5, we choose coordinates so that $x_0 = 0$ and $h : \mathbb{R}^{n-1} \to \mathbb{R}$ is a C^1 function such that for some $r > 0$,

$$\Omega = \left\{\, x \in B(0,r) \;\middle|\; x_n > h(x') \,\right\}, \qquad \partial\Omega = \left\{\, x \in B(0,r) \;\middle|\; x_n = h(x') \,\right\},$$

and moreover $h(0) = 0$ and $\nabla_{x'} h(0) = 0$. Consider the local boundary defining function

$$\rho : B(0,r) \to \mathbb{R}, \quad \rho(x) = x_n - h(x').$$

Then $\rho(0) = 0$ and $\nabla\rho(0) = e_n$. Also choose some unit tangent vector α to $\partial\Omega$ at 0, that is, $\alpha \in \mathbb{R}^n$ is a unit vector with $\alpha \cdot e_n = 0$. We wish to use oscillating boundary data $e^{iN\alpha\cdot x}$ for $x \in \partial\Omega$, where $N > 0$ is a large number, to determine the conductivity on the boundary. However, in order to focus on the value of γ at the origin, we need to multiply by a cutoff function.

We will eventually choose the boundary data to be

$$f_M = c_{M,N}\eta(Mx)e^{iN\alpha\cdot x}, \quad x \in \partial\Omega,$$

where η is a cutoff function supported in the unit ball, M and N are large numbers, and $c_{M,N}$ is a scaling constant. The boundary value f_M oscillates with period $2\pi/N$, and we should choose N so that there are many oscillations in the ball of radius $1/M$. For this reason, we will choose $N = N(M)$ such that

$$M/N = o(1) \quad \text{as } M \to \infty. \tag{3.1}$$

For example, $N(M) = M^\beta$ for $\beta > 1$ satisfies this, and we will actually fix the choice $N(M) = M^3$ in the end of the proof.

The next lemma will be useful in estimating the size of the corresponding solutions.

Lemma 3.6. *Let η be continuous and supported in $B(0,1)$. Then*

$$\lim_{M\to\infty} M^{n-1}N \int_\Omega \eta(Mx)e^{-2N\rho(x)}\,dx = \frac{1}{2}\int_{\mathbb{R}^{n-1}} \eta(x',0)\,dx',$$

and for M large, for some constant C only depending on η,

$$\left| \int_\Omega \eta(Mx)e^{2N\rho(x)}\,dx \right| \le CM^{1-n}N^{-1}.$$

Proof. If M is large, changing variables $x_n = t + h(x')$ and scaling gives

$$\begin{aligned}
\int_\Omega \eta(Mx)e^{-2N\rho(x)}\,dx &= \int_{\mathbb{R}^{n-1}} \int_{h(x')}^\infty \eta(Mx', Mx_n)e^{-2N(x_n - h(x'))}\,dx_n\,dx' \\
&= \int_{\mathbb{R}^{n-1}} \int_0^\infty \eta(Mx', M(t+h(x'))e^{-2Nt}\,dt\,dx' \\
&= M^{1-n}N^{-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \eta(x', \frac{M}{N}t + Mh(\frac{x'}{M}))e^{-2t}\,dt\,dx'.
\end{aligned}$$

Note that

$$\lim_{M\to\infty} Mh(\frac{x'}{M}) = \lim_{s\to 0} \frac{h(sx') - h(0)}{s} = \nabla_{x'}h(0)\cdot x' = 0.$$

Since the integral over $\mathbb{R}^{n-1}$ is actually over the unit ball, dominated convergence and (3.1) imply, as $M \to \infty$,

$$M^{n-1}N \int_\Omega \eta(Mx)e^{-2N\rho(x)}\,dx \to \int_{\mathbb{R}^{n-1}} \int_0^\infty \eta(x',0)e^{-2t}\,dt\,dx' = \tfrac{1}{2}\int_{\mathbb{R}^{n-1}} \eta(x',0)\,dx'.$$

This shows the first claim, and the second one is an immediate consequence. □

Proof of Theorem 3.5. 1. With M and N as in (3.1), we define

$$v_0(x) = \eta_M(x)h_N(x), \quad x \in \mathbb{R}^n,$$

where h_N is the complex exponential

$$h_N(x) = e^{N(i\alpha\cdot x - \rho(x))}$$

and η_M is a cutoff function

$$\eta_M(x) = \eta(Mx)$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$, $0 \le \eta \le 1$, $\eta = 1$ for $|x| \le 1/2$, and $\eta = 0$ for $|x| \ge 1$. Note that $v_0 \in C_c^1(\mathbb{R}^n)$ is supported in a small ball $B(0, 1/M)$.

(The function v_0 calls for some explanation. First of all, we will think of $\partial\Omega$ as being almost flat in small neighborhoods near 0 (this is justified since the boundary is C^1). In the case where $\partial\Omega$ is exactly flat near 0, we have $v_0(x) = \eta_M(x)e^{N(i\alpha - e_n)\cdot x}$ for M large. The exponential $e^{N(i\alpha-e_n)\cdot x}$ is harmonic in Ω:

$$\Delta(e^{N(i\alpha-e_n)\cdot x}) = N^2(i\alpha - e_n)\cdot(i\alpha - e_n)e^{N(i\alpha-e_n)\cdot x} = 0$$

since $|\alpha|^2 = 1$ and $\alpha \cdot e_n = 0$. Consequently, this exponential solves the conductivity equation $\mathrm{div}(\gamma(0)\nabla v) = 0$ with coefficient frozen at 0. Multiplying by the cutoff η_M concentrates the exponential in a small neighborhood of

0, and in this neighborhood the conductivity equation $\operatorname{div}(\gamma\nabla v) = 0$ can be approximated by the same equation with γ replaced by $\gamma(0)$. Thus, v_0 is an *approximate solution* of the conductivity equation that concentrates near the boundary point 0 and eventually allows to determine $\gamma(0)$.)

2. We establish two basic properties of v_0: as $M \to \infty$,

$$\int_\Omega |\nabla v_0|^2 \, dx = O(M^{1-n}N), \tag{3.2}$$

$$\lim_{M\to\infty} M^{n-1}N^{-1} \int_\Omega \gamma \, |\nabla v_0|^2 \, dx = c_\eta \gamma(0) \tag{3.3}$$

where $c_\eta = \int_{\mathbb{R}^{n-1}} \eta(x',0)^2 \, dx'$. For the proof we first write

$$\nabla v_0 = \underbrace{N(i\alpha - \nabla\rho)\eta_M h_N}_{F_1} + \underbrace{M(\nabla\eta)(M\,\cdot)h_N}_{F_2}. \tag{3.4}$$

Lemma 3.6 implies that, as $M \to \infty$,

(3.5)

$$\|F_1\|^2_{L^2(\Omega)} = O(M^{1-n}N), \qquad \|F_2\|^2_{L^2(\Omega)} = O(M^{1-n}N(M/N)^2) = o(M^{1-n}N)$$

using that $M/N = o(1)$. This shows (3.2). The second claim follows by writing

$$\int_\Omega \gamma \, |\nabla v_0|^2 \, dx = \gamma(0) \int_\Omega |\nabla v_0|^2 \, dx + \int_\Omega (\gamma - \gamma(0)) \, |\nabla v_0|^2 \, dx$$

$$= \gamma(0) \int_\Omega |F_1|^2 \, dx + \gamma(0) \int_\Omega (F_1 \cdot \bar{F}_2 + \bar{F}_1 \cdot F_2 + |F_2|^2) \, dx + \int_\Omega (\gamma - \gamma(0)) \, |\nabla v_0|^2 \, dx.$$

By continuity of γ we have

$$\sup_{x\in B(0,1/M)} |\gamma(x) - \gamma(0)| = o(1) \text{ as } M \to \infty. \tag{3.6}$$

Since $\operatorname{supp}(v_0) \subset B(0,1/M)$, all the terms above except the first one are $o(M^{1-n}N)$ by (3.5) and Cauchy-Schwarz. For the first term we have

$$M^{n-1}N^{-1} \int_\Omega |F_1|^2 \, dx = M^{1-n}N \int_\Omega \eta(Mx)^2 e^{-2N\rho(x)} (1 + |\nabla\rho(x)|^2) \, dx$$

$$= 2M^{1-n}N \int_\Omega \eta(Mx)^2 e^{-2N\rho} \, dx + M^{1-n}N \int_\Omega \eta(Mx)^2 e^{-2N\rho} (|\nabla\rho(x)|^2 - |\nabla\rho(0)|^2) \, dx.$$

In the last expression, the first term has limit c_η as $M \to \infty$ by Lemma 3.6, and the second term is $o(1)$ since $x \mapsto |\nabla\rho(x)|^2$ is continuous near 0. We have proved (3.3).

3. In addition to the approximate solution v_0, we will make use of the exact solution v of the conductivity equation obtained by solving the Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma\nabla v) = 0 & \text{in } \Omega, \\ v = f_0 & \text{on } \partial\Omega \end{cases}$$

where

$$f_0 = v_0|_{\partial\Omega}.$$

Since $v_0 \in H^1(\Omega)$, we have $f_0 \in H^{1/2}(\partial\Omega)$ and the Dirichlet problem above has a unique solution $v \in H^1(\Omega)$ by Theorem **??**. We also write

$$v = v_0 + v_1$$

where $v_1 = v - v_0$ is (by Theorem **??**) the unique $H^1_0(\Omega)$ solution of

$$\begin{cases} \mathrm{div}(\gamma\nabla v_1) = -\mathrm{div}(\gamma\nabla v_0) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The right hand side $-\mathrm{div}(\gamma\nabla v_0)$ is in $H^{-1}(\Omega)$, and it acts on functions $\varphi \in H^1_0(\Omega)$ by

$$\langle -\mathrm{div}(\gamma\nabla v_0), \varphi\rangle = \int_\Omega \gamma\nabla v_0 \cdot \nabla\varphi\,dx.$$

4. Now we consider the function v_1, and claim that as $M \to \infty$

$$\int_\Omega |\nabla v_1|^2\,dx = o(M^{1-n}N). \tag{3.7}$$

This estimate justifies calling v_0 an approximate solution, since it says that the difference v_1 between the exact solution v and v_0 is asymptotically smaller than v_0. By Theorem **??**, to prove (3.7) it is enough to show that

$$\|-\mathrm{div}(\gamma\nabla v_0)\|^2_{H^{-1}(\Omega)} = o(M^{1-n}N),$$

or equivalently,

$$\left|\int_\Omega \gamma\nabla v_0 \cdot \nabla\varphi\,dx\right| \leq o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1(\Omega)}\,, \quad \varphi \in C^\infty_c(\Omega). \tag{3.8}$$

Let $\varphi \in C^\infty_c(\Omega)$. We begin by writing

$$\int_\Omega \gamma\nabla v_0 \cdot \nabla\varphi\,dx = \gamma(0)\int_\Omega \nabla v_0 \cdot \nabla\varphi\,dx + \int_\Omega (\gamma - \gamma(0))\nabla v_0 \cdot \nabla\varphi\,dx.$$

Using the continuity of γ and (3.2), the second term is $o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1(\Omega)}$. For the first term, write ∇v_0 as in (3.4) and use (3.5) to obtain

$$\int_\Omega \gamma\nabla v_0 \cdot \nabla\varphi\,dx = \gamma(0)\int_\Omega N(i\alpha - \nabla\rho)\eta_M h_N \cdot \nabla\varphi\,dx + o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1(\Omega)}\,.$$

In the first term on the right write $\nabla\rho = \nabla\rho(0) + (\nabla\rho - \nabla\rho(0))$. Since $\nabla\rho$ is continuous and the integral is over $B(0, 1/M)$, we get (recall that $\nabla\rho(0) = e_n$)

$$\int_\Omega \gamma\nabla v_0 \cdot \nabla\varphi\,dx = \gamma(0)\int_\Omega N(i\alpha - e_n)\eta_M h_N \cdot \nabla\varphi\,dx + o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1(\Omega)}\,.$$

Integrating by parts in the first term on the right, which is possible since φ has compact support, yields

$$\int_\Omega \gamma\nabla v_0\cdot\nabla\varphi\,dx = -\gamma(0)\int_\Omega N^2(i\alpha-e_n)\cdot(i\alpha-\nabla\rho)\eta_M h_N\varphi\,dx$$
$$-\gamma(0)\int_\Omega MN(i\alpha-e_n)\cdot(\nabla\eta)(M\,\cdot\,)h_N\varphi\,dx + o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1(\Omega)}\,.$$

Here comes a key point in the proof: we can now use the fact that $e^{N(i\alpha-e_n)\cdot x}$ is a harmonic function, or equivalently that $(i\alpha-e_n)\cdot(i\alpha-e_n)=0$, to write

(3.9)
$$\int_\Omega \gamma\nabla v_0\cdot\nabla\varphi\,dx = \gamma(0)\int_\Omega N^2(i\alpha-e_n)\cdot(\nabla\rho-\nabla\rho(0))\eta_M h_N\varphi\,dx$$
$$-\gamma(0)\int_\Omega MN(i\alpha-e_n)\cdot(\nabla\eta)(M\,\cdot\,)h_N\varphi\,dx + o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1(\Omega)}\,.$$

Recall that we want $\|\varphi\|_{H^1(\Omega)}$ on the right. For this purpose, we write

$$h_N = -\frac{1}{N}\partial_n h_N. \tag{3.10}$$

This is valid since $\partial_n\rho = 1$. Using (3.10) and integrating by parts, the second integral on the right hand side of (3.9) becomes

$$\int_\Omega M(i\alpha-e_n)\cdot(\nabla\eta)(M\,\cdot\,)(\partial_n h_N)\varphi\,dx = -\int_\Omega M(i\alpha-e_n)\cdot(\nabla\eta)(M\,\cdot\,)h_N\partial_n\varphi\,dx$$
$$-\int_\Omega M^2(i\alpha-e_n)\cdot(\nabla\partial_n\eta)(M\,\cdot\,)h_N\varphi\,dx.$$

The first integral on the right is of the form $\int_\Omega F_2\cdot(i\alpha-e_n)\partial_n\varphi\,dx$ and is bounded by $o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1}$ by Cauchy-Schwarz and (3.5). Similarly, the second integral is bounded by $O(M^{(1-n)/2}N^{1/2}(M^2/N))\,\|\varphi\|_{L^2}$. If we make the choice

$$N(M) = M^3,$$

then this is $o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1}$.

We have proved that if $N(M)=M^3$, then

$$\int_\Omega \gamma\nabla v_0\cdot\nabla\varphi\,dx = \int_\Omega N^2(i\alpha-e_n)\cdot(\nabla\rho-\nabla\rho(0))\eta_M h_N\varphi\,dx + o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1(\Omega)}\,.$$

Inserting (3.10) in the integral on the right and integrating by parts, this integral becomes

$$\int_\Omega N(i\alpha-e_n)\cdot(\nabla\rho-\nabla\rho(0))\eta_M h_N\partial_n\varphi\,dx$$
$$+\int_\Omega N(i\alpha-e_n)\cdot(\nabla\rho-\nabla\rho(0))M\partial_n\eta(M\,\cdot\,)h_N\varphi\,dx.$$

Here we used that $\nabla\rho(x) = (-\nabla h(x'), 1)$ is independent of x_n, which justifies that one can integrate by parts with respect to x_n even though $\nabla\rho$ is only continuous. The second integral is essentially of the same form as the second integral on the right hand side of (3.9), and the argument above shows that it is $o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1}$ (use again that $\nabla\rho$ is independent of x_n). Also the first integral is $o(M^{(1-n)/2}N^{1/2})\,\|\varphi\|_{H^1}$ by Cauchy-Schwarz, Lemma 3.6 and the continuity of $\nabla\rho$. This shows (3.7).

5. We can now finish the proof of the theorem. Define

$$u_M = c_{M,N} v, \quad f_M = c_{M,N} f_0, \quad c_{M,N} = \sqrt{\frac{M^{n-1}N^{-1}}{c_\eta}}.$$

Since $f_0 = v_0|_{\partial\Omega}$, clearly f_M is in $C^1(\partial\Omega)$ and supported in $B(0,1/M)\cap\partial\Omega$ and

$$\|f_M\|_{H^{1/2}(\partial\Omega)} \leq C c_{M,N} \|v_0\|_{H^1(\Omega)}.$$

We saw in (3.2) that $c_{M,N}\|\nabla v_0\|_{L^2(\Omega)} \leq C$ uniformly over M, and since the constant only depends on the choice of η and the C^1 norm of ρ it can be chosen uniform over $x_0 \in \partial\Omega$. Similarly $c_{M,N}\|v_0\|_{L^2(\Omega)} \leq C$ uniformly over M and x_0. Furthermore,

$$\int_\Omega \gamma\nabla u_M\cdot\nabla\bar{u}_M\,dx = \frac{M^{n-1}N^{-1}}{c_\eta}\int_\Omega \gamma(|\nabla v_0|^2+\nabla\bar{v}_0\cdot\nabla\bar{v}_1+\nabla v_0\cdot\nabla v_1+|\nabla v_1|^2)\,dx.$$

The first term satisfies by (3.3)

$$\lim_{M\to\infty}\frac{M^{n-1}N^{-1}}{c_\eta}\int_\Omega \gamma\,|\nabla v_0|^2\,dx = \gamma(0).$$

The other terms may be estimated by Cauchy-Schwarz and (3.2), (3.7), so that

$$\begin{aligned}\left|\int_\Omega \gamma(\nabla v_0\cdot\nabla\bar{v}_1 + \nabla v_0\cdot\nabla v_1 + |\nabla v_1|^2)\,dx\right| &\leq C(\|\nabla v_0\|_{L^2(\Omega)} + \|\nabla v_1\|_{L^2(\Omega)})\,\|\nabla v_1\|_{L^2(\Omega)} \\ &= o(M^{1-n}N).\end{aligned}$$

This proves the result. □

Remark 3.7. For later purposes, we make the following remarks about the proof. If Ω has C^k boundary, it is clear that the approximate solution is in $C^k(\overline{\Omega})$ and consequently $f_M \in C^k(\partial\Omega)$ and $u_M \in H^k(\Omega)$. Inspecting the proof, we have actually shown that the sequence (u_M) satisfies

$$\lim_{M\to\infty}\int_\Omega g\,|\nabla u_M|^2\,dx = g(x_0)$$

for any function $g \in C^0(\overline{\Omega})$ (not just $g = \gamma$).

Exercise 3.8. Verify the details in Remark 3.7.

The end part of the proof of Theorem 3.5 may be simplified by using the Hardy inequality. Writing

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \inf_{z \in \partial\Omega} |x - z|\,,$$

this inequality is as follows:

Theorem 3.9. *(Hardy inequality) Let Ω be a bounded open set with C^1 boundary. There is a constant $C > 0$ such that*

$$\|\varphi/\delta\|_{L^2(\Omega)} \leq C\,\|\nabla\varphi\|_{L^2(\Omega)}\,, \quad \varphi \in H_0^1(\Omega).$$

The following problems contain a proof of the Hardy inequality and discuss how it is used in boundary determination.

Exercise 3.10. (Hardy inequality on the half line) If $f \in C_c^\infty((0,\infty))$, define

$$Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0.$$

For $1 < p < \infty$ prove that

$$\|Tf\|_{L^p((0,\infty))} \leq C_p\,\|f\|_{L^p((0,\infty))}\,, \quad f \in C_c^\infty((0,\infty)).$$

Exercise 3.11. (Hardy inequality in half space) Let $\mathbb{R}^n_+ = \left\{\, x \in \mathbb{R}^n \;\middle|\; x_n > 0 \,\right\}$. If $1 < p < \infty$, prove that

$$\|u/x_n\|_{L^p(\mathbb{R}^n_+)} \leq C_p\,\|\partial_n u\|_{L^p(\mathbb{R}^n_+)}\,, \quad u \in C_c^\infty(\mathbb{R}^n_+).$$

Exercise 3.12. Let Ω be a bounded open set with C^1 boundary. Show that for any $x_0 \in \partial\Omega$, there is $r > 0$, a C^1 function $h : \mathbb{R}^{n-1} \to \mathbb{R}$, and a constant $c > 0$ such that $\Omega \cap B(x_0, r) = \left\{\, x \in B(x_0, r) \;\middle|\; x_n > h(x') \,\right\}$ and

$$c(x_n - h(x')) \leq \delta(x) \leq x_n - h(x'), \quad x \in B(x_0, r) \cap \Omega.$$

Exercise 3.13. Prove Theorem 3.9.

Exercise 3.14. Consider the situation before the proof of Theorem 3.5, and show that if η is continuous and supported in $B(0,1)$, then

$$\left|\int_\Omega \delta(x)^k \eta(Mx) e^{-2N\rho(x)}\,dx\right| \leq CM^{1-n}N^{-k-1}.$$

Exercise 3.15. Give an alternative proof of Theorem 3.7, by using the Hardy inequality and Problem 3.14 in the part following (3.9).

Exercise 3.16. In this problem we construct harmonic functions on $\mathbb{R}^n_+$ that are concentrated near the origin. We denote $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. Let

$$P_{x_n}(x') = \frac{\Gamma(\frac{n}{2})}{\pi^{n/2}}\frac{x_n}{(x_n^2 + |x'|^2)^{n/2}}$$

be the Poisson kernel.

(a) Prove that $\Delta P_{x_n}(x') = 0$ for all $x_n > 0$. Here Δ is the Laplacian on $\mathbb{R}^n$.

(b) Prove that $\int_{\mathbb{R}^{n-1}} P_{x_n}(x')\, dx' = 1$ for all $x_n > 0$.

(c) Let $\varphi(x')$ be a bounded continuous function on $\mathbb{R}^{n-1}$. Prove that

$$\Phi(x) = \int_{\mathbb{R}^{n-1}} P_{x_n}(x' - y')\varphi(y')\, dy'$$

obeys $\Delta\Phi(x) = 0$ in $\mathbb{R}^n_+$ and

$$\lim_{\substack{x\to(z',0)\\ x_n>0}} \Phi(x) = \varphi(z').$$

(d) Now suppose that $\varphi(x') = \partial^\alpha\psi(x')$ with $\psi \in C^{|\alpha|}(\mathbb{R}^{n-1})$ supported in the ball of radius 1 centred on the origin. Prove that there is a constant, which depends only on $|\alpha|$ and n, such that

$$|\Phi(x)| \leq \frac{C}{1 + |x|^{|\alpha|+n-1}}.$$

3.2. Recovering normal derivatives

The next results imply uniqueness, reconstruction, and Hölder type stability for determining the normal derivative of γ on the boundary from the DN map.

Theorem 3.17. *(Recovering $\partial_\nu\gamma$ on $\partial\Omega$) Let Ω be a bounded open set with C^2 boundary, and let $\gamma \in C^1(\overline{\Omega})$ be positive. Given a point $x_0 \in \partial\Omega$ and an open set $\Gamma \subset \partial\Omega$ containing x_0, the quantity $\partial_\nu\gamma(x_0)$ can be determined from the knowledge of $\Lambda_\gamma f|_\Gamma$ for all $f \in C^2(\partial\Omega)$ with* $\operatorname{supp}(f) \subset \Gamma$.

Theorem 3.18. *(Stability of $\partial_\nu\gamma$ on $\partial\Omega$) Let Ω be a bounded open set with C^2 boundary, and let $\gamma_j \in C^2(\overline{\Omega})$ for $j = 1, 2$. Let $E > 0$ be a constant so that*

$$1/E \leq \gamma_j \leq E \text{ in } \Omega,$$
$$\|\gamma_j\|_{C^2(\overline{\Omega})} \leq E.$$

Then

$$\|\partial_\nu\gamma_1 - \partial_\nu\gamma_2\|_{L^\infty(\partial\Omega)} \leq C(E)\, \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{1/4}_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}.$$

In the previous section, we recovered the value of γ at a boundary point x_0 by using that

$$\gamma(x_0) = \lim_{M\to\infty} \int_\Omega \gamma\, |\nabla u_M|^2\, dx = \lim_{M\to\infty} \langle\Lambda_\gamma f_M, \bar{f}_M\rangle_{\partial\Omega}.$$

Here $u_M \in H^1(\Omega)$ are special solutions to the conductivity equation which concentrate near x_0, and f_M is the boundary value of u_M. These solutions can also be used to recover the normal derivative of γ at x_0. By Remark 3.7, if $\gamma \in C^1(\overline{\Omega})$ and if $\alpha \in \mathbb{R}^n$ is a constant vector then one has

$$\alpha \cdot \nabla \gamma(x_0) = \lim_{M \to \infty} \int_\Omega (\alpha \cdot \nabla \gamma) \, |\nabla u_M|^2 \, dx. \tag{3.11}$$

By choosing $\alpha = \nu(x_0)$, Theorem 3.17 will follow if we can somehow determine the right hand side of the above identity from boundary measurements. This will be done by the following Rellich type identity.

Lemma 3.19. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary, and let $\gamma \in C^1(\overline{\Omega})$. If $u \in H^2(\Omega)$ satisfies $\mathrm{div}(\gamma \nabla u) = 0$ in Ω, then for any $\alpha \in \mathbb{R}^n$ one has*

$$\int_\Omega (\alpha \cdot \nabla \gamma) \, |\nabla u|^2 \, dx = \int_{\partial\Omega} (\alpha \cdot \nu) \gamma \, |\nabla u|^2 \, dS - 2 \int_{\partial\Omega} \mathrm{Re}((\alpha \cdot \nabla u) \gamma \overline{\partial_\nu u}) \, dS.$$

Proof. Let first $w \in C^\infty(\overline{\Omega})$. Integrating by parts, rearranging terms and integrating by parts again gives that

$$\int_\Omega (\alpha \cdot \nabla \gamma) \, |\nabla w|^2 \, dx = \int_{\partial\Omega} (\alpha \cdot \nu) \gamma \, |\nabla w|^2 \, dS - \sum_{j=1}^n \int_\Omega \gamma \alpha_j \partial_j (\nabla w \cdot \overline{\nabla w}) \, dx$$

$$= \int_{\partial\Omega} (\alpha \cdot \nu) \gamma \, |\nabla w|^2 \, dS - 2 \sum_{j=1}^n \int_\Omega \alpha_j \, \mathrm{Re}(\nabla \partial_j w \cdot \gamma \overline{\nabla w}) \, dx$$

$$= \int_{\partial\Omega} (\alpha \cdot \nu) \gamma \, |\nabla w|^2 \, dS - 2 \sum_{j=1}^n \int_{\partial\Omega} \alpha_j \, \mathrm{Re}(\nu \partial_j w \cdot \gamma \overline{\nabla w}) \, dS$$

$$+ 2 \sum_{j=1}^n \int_\Omega \alpha_j \, \mathrm{Re}((\partial_j w) \mathrm{div}(\gamma \overline{\nabla w})) \, dx$$

$$= \int_{\partial\Omega} (\alpha \cdot \nu) \gamma \, |\nabla w|^2 \, dS - 2 \int_{\partial\Omega} \mathrm{Re}((\alpha \cdot \nabla w) \gamma \partial_\nu \bar{w}) \, dS + 2 \int_\Omega \mathrm{Re}((\alpha \cdot \nabla w) \mathrm{div}(\gamma \nabla \bar{w})) \, dx.$$

Let now $u \in H^2(\Omega)$ solve $\mathrm{div}(\gamma \nabla u) = 0$ in Ω. Since Ω has C^2 boundary, we may choose $(w_j) \subset C^\infty(\overline{\Omega})$ such that $w_j \to u$ in $H^2(\Omega)$. Multiplication by $\gamma \in C^1(\overline{\Omega})$ is a bounded operator on $H^1(\Omega)$, which shows that $\gamma \nabla w_j \to \gamma \nabla u$ in $H^1(\Omega)$ and

$$\mathrm{div}(\gamma \nabla w_j) \to \mathrm{div}(\gamma \nabla u) = 0 \text{ in } L^2(\Omega).$$

By the trace theorem $\partial_l w_j|_{\partial\Omega} \to \partial_l u|_{\partial\Omega}$ in $L^2(\partial\Omega)$ for each l. Thus, the theorem follows by applying the integral identity derived above to w_j and taking the limit as $j \to \infty$. $\square$

Proof of Theorem 3.17. By Remark 3.7, the solutions u_M in Theorem 3.5 are in $H^2(\Omega)$ and their boundary values f_M are in $C^2(\partial\Omega)$. Applying

Lemma 3.19 to u_M, we have

$$\int_\Omega (\alpha\cdot\nabla\gamma)\,|\nabla u_M|^2\,dx = \int_{\partial\Omega}(\alpha\cdot\nu)\gamma\,|\nabla u_M|^2\,dS - 2\int_{\partial\Omega}\mathrm{Re}((\alpha\cdot\nabla u_M)\gamma\overline{\partial_\nu u_M})\,dS.$$

Denote the tangential part of ∇u_M on $\partial\Omega$ by

$$\nabla_T u_M = \nabla u_M - (\nabla u_M\cdot\nu)\nu\Big|_{\partial\Omega}.$$

Since $u_M|_{\partial\Omega} = f_M$, the tangential gradient of u_M on $\partial\Omega$ is just the tangential gradient of f_M and we have

$$\nabla u_M|_{\partial\Omega} = \nabla_T u_M + (\partial_\nu u_M)\nu\Big|_{\partial\Omega} = \nabla_T f_M + \gamma^{-1}\Lambda_\gamma f_M\Big|_{\partial\Omega}.$$

Similarly, writing $\alpha = \alpha_T + (\alpha\cdot\nu)\nu$ we obtain

$$\int_\Omega (\alpha\cdot\nabla\gamma)\,|\nabla u_M|^2\,dx = \int_{\partial\Omega}(\alpha\cdot\nu)\gamma\,|\nabla_T f_M|^2\,dS + \int_{\partial\Omega}(\alpha\cdot\nu)\gamma^{-1}\,|\Lambda_\gamma f_M|^2\,dS$$
$$-2\int_{\partial\Omega}\mathrm{Re}((\alpha\cdot\nabla_T f_M)\overline{\Lambda_\gamma f_M})\,dS - 2\int_{\partial\Omega}\mathrm{Re}((\alpha\cdot\nu)\gamma\,|\partial_\nu u_M|^2)\,dS.$$

Thus

(3.12)

$$\int_\Omega (\alpha\cdot\nabla\gamma)\,|\nabla u_M|^2\,dx = \int_{\partial\Omega}(\alpha\cdot\nu)\gamma\,|\nabla_T f_M|^2\,dS - \int_{\partial\Omega}(\alpha\cdot\nu)\gamma^{-1}\,|\Lambda_\gamma f_M|^2\,dS$$
$$-2\int_{\partial\Omega}\mathrm{Re}((\alpha\cdot\nabla_T f_M)\overline{\Lambda_\gamma f_M})\,dS.$$

The right hand side of (3.11) is therefore determined by the knowledge of $\Lambda_\gamma f_M$ and the restriction of γ to $\mathrm{supp}\,(f_M)$. By Theorem 3.3 the boundary values of γ near x_0 are determined by the DN map near x_0, and it follows from (3.11) that $\alpha\cdot\nabla\gamma(x_0)$ can be determined from the DN map near x_0 for any constant vector α. The normal derivative is obtained by choosing $\alpha = \nu(x_0)$. □

The proof of the stability result, Theorem 3.18, follows by comparing the expressions in the previous theorem for two different conductivities. To do this properly we will need certain further facts about the approximate solutions constructed in 3.1, and these facts will be proved in the problems in the end of this section.

Proof of Theorem 3.18. In this proof, the constants C will only depend on Ω and E. Fix a point $x_0 \in \partial\Omega$ (the constants C will also be independent of the choice of x_0), and for ease of notation assume that $x_0 = 0$. Let u_M and v_M be the solutions provided by Theorem 3.5 to the equations $\mathrm{div}(\gamma_1\nabla u_M) = 0$ and $\mathrm{div}(\gamma_2\nabla v_M) = 0$ in Ω. By Remark 3.7, these solutions are in $H^2(\Omega)$ and their boundary values satisfy $u|_{\partial\Omega} = v|_{\partial\Omega} = f_M \in C^2(\partial\Omega)$.

Applying (3.12) to u_M and v_M and subtracting the resulting expressions shows that

$$\begin{aligned}
&\int_\Omega (\alpha \cdot \nabla \gamma_1) \left|\nabla u_M\right|^2 \, dx - \int_\Omega (\alpha \cdot \nabla \gamma_2) \left|\nabla v_M\right|^2 \, dx \\
&= \int_{\partial\Omega} (\alpha \cdot \nu)(\gamma_1 - \gamma_2) \left|\nabla_T f_M\right|^2 \, dS - \int_{\partial\Omega} (\alpha \cdot \nu)(\gamma_1^{-1} - \gamma_2^{-1}) \left|\Lambda_{\gamma_1} f_M\right|^2 \, dS \\
&\quad + \int_{\partial\Omega} (\alpha \cdot \nu)\gamma_2^{-1} \left[(\Lambda_{\gamma_1} f_M - \Lambda_{\gamma_2} f_M)\overline{\Lambda_{\gamma_1} f_M} + \Lambda_{\gamma_2} f_M \overline{(\Lambda_{\gamma_1} f_M - \Lambda_{\gamma_2} f_M)}\right] dS \\
&\quad - 2 \int_{\partial\Omega} \mathrm{Re} \left[(\alpha \cdot \nabla_T f_M)\overline{(\Lambda_{\gamma_1} f_M - \Lambda_{\gamma_2} f_M)}\right] dS.
\end{aligned}$$

Notice that

$$\left|\gamma_1^{-1} - \gamma_2^{-1}\right| = \left|\frac{\gamma_1 - \gamma_2}{\gamma_1 \gamma_2}\right| \leq E^2 \left|\gamma_1 - \gamma_2\right|.$$

Assuming that α has unit length, we obtain

$$\begin{aligned}
&\left|\int_\Omega (\alpha \cdot \nabla \gamma_1) \left|\nabla u_M\right|^2 \, dx - \int_\Omega (\alpha \cdot \nabla \gamma_2) \left|\nabla v_M\right|^2 \, dx\right| \\
&\leq C \left\|\gamma_1 - \gamma_2\right\|_{L^\infty(\partial\Omega)} \left[\left\|f_M\right\|_{H^1}^2 + \left\|\Lambda_{\gamma_1} f_M\right\|_{L^2}^2\right] \\
&\quad + C \left\|(\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_M\right\|_{H^{-1/2}} \left[\left\|\Lambda_{\gamma_1} f_M\right\|_{H^{1/2}} + \left\|\Lambda_{\gamma_2} f_M\right\|_{H^{1/2}} + \left\|f_M\right\|_{H^{3/2}}\right].
\end{aligned}$$

Using the bounds for γ_j, we have (see Problem 3.20)

$$\left\|\Lambda_{\gamma_j} f_M\right\|_{H^{1/2}} \leq C \left\|f_M\right\|_{H^{3/2}}, \quad \left\|\Lambda_{\gamma_j} f_M\right\|_{L^2} \leq C \left\|f_M\right\|_{H^1}.$$

Using also Theorem 3.4, it follows that

$$\begin{aligned}
&\left|\int_\Omega (\alpha \cdot \nabla \gamma_1) \left|\nabla u_M\right|^2 \, dx - \int_\Omega (\alpha \cdot \nabla \gamma_2) \left|\nabla v_M\right|^2 \, dx\right| \\
&\qquad \leq C \left\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\right\|_{H^{1/2} \to H^{-1/2}} \left[\left\|f_M\right\|_{H^1}^2 + \left\|f_M\right\|_{H^{1/2}} \left\|f_M\right\|_{H^{3/2}}\right].
\end{aligned}$$

Since f_M is an explicit function, we have the bounds (see Problem 3.21)

$$\left\|f_M\right\|_{H^{1/2}} \leq C, \quad \left\|f_M\right\|_{H^1} \leq C N^{1/2}, \quad \left\|f_M\right\|_{H^{3/2}} \leq C N.$$

It follows that

$$\begin{aligned}
&\left|\int_\Omega (\alpha \cdot \nabla \gamma_1) \left|\nabla u_M\right|^2 \, dx - \int_\Omega (\alpha \cdot \nabla \gamma_2) \left|\nabla v_M\right|^2 \, dx\right| \\
&\qquad \leq C N \left\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\right\|_{H^{1/2} \to H^{-1/2}}.
\end{aligned}$$

Inspecting the construction in Section 3.1 carefully, for any $g \in C^1(\overline{\Omega})$ we have the quantitative estimate (see Problem 3.22)

$$\left|\int_\Omega g \left|\nabla u_M\right|^2 \, dx - g(0)\right| \leq C(\left|g(0)\right| M/N + \left\|\nabla g\right\|_{L^\infty(\Omega)} M^{-1}).$$

Applying this with $g = \alpha \cdot \nabla\gamma_j$, it follows that

$$|\alpha \cdot \nabla\gamma_1(0) - \alpha \cdot \nabla\gamma_2(0)| \leq C(N \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} + M/N + M^{-1}).$$

Let $\alpha = \nu(0)$. Recalling that we eventually made the choice $N(M) = M^3$ in 3.1, this implies

$$|\partial_\nu\gamma_1(0) - \partial_\nu\gamma_2(0)| \leq C(M^3 \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} + M^{-1}).$$

We now fix a suitable large number M, of the form

$$M = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}^s,$$

for some $s > 0$ optimized so that both terms on the right hand side of the last estimate are comparable. This condition results in the equation

$$3s + 1 = -s,$$

or $s = -1/4$. With these choices, we have

$$|\partial_\nu\gamma_1(0) - \partial_\nu\gamma_2(0)| \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}^{1/4}.$$

Since this applies at any boundary point with a uniform constant C, we have proved the result. □

Exercise 3.20. Let $\Omega \subset \mathbb{R}^n$ is a bounded open set, and let $\gamma \in L^\infty(\Omega)$ be such that $1/E \leq \gamma \leq E$ a.e. in Ω for some $E > 0$. Show that for any $f \in H^{1/2}(\partial\Omega)$ the unique solution of $\mathrm{div}(\gamma\nabla u) = 0$ in Ω with $u|_{\partial\Omega} = f$ satisfies

$$\|u\|_{H^1(\Omega)} \leq C(E,\Omega) \|f\|_{H^{1/2}(\partial\Omega)}.$$

Show also that

$$\|\Lambda_\gamma f\|_{H^{-1/2}(\partial\Omega)} \leq C(E,\Omega) \|f\|_{H^{1/2}(\partial\Omega)}.$$

Moreover, if Ω has C^2 boundary and $\|\gamma\|_{C^1(\overline{\Omega})} \leq E$ and $f \in H^{3/2}(\partial\Omega)$, show that

$$\|u\|_{H^2(\Omega)} \leq C(E,\Omega) \|f\|_{H^{3/2}(\partial\Omega)}$$

and

$$\|\Lambda_\gamma f\|_{H^{1/2}(\partial\Omega)} \leq C(E,\Omega) \|f\|_{H^{3/2}(\partial\Omega)}.$$

Use also interpolation to show that

$$\|\Lambda_\gamma f\|_{H^{s-1}(\partial\Omega)} \leq C(E,\Omega) \|f\|_{H^s(\partial\Omega)}, \quad 1/2 \leq s \leq 3/2.$$

Exercise 3.21. Let $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^2 boundary. Show that the function f_M satisfies

$$\|f_M\|_{H^s(\partial\Omega)} \leq C(\Omega) N^{s-1/2}, \quad 0 \leq s \leq 2.$$

Exercise 3.22. Let $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary. Show that there is $C > 0$ such that for any $g \in C(\overline{\Omega})$ with modulus of continuity ω, meaning that

$$|g(x) - g(y)| \leq \omega(|x - y|), \quad x, y \in \overline{\Omega},$$

one has

$$\left| \int_\Omega g \, |\nabla u_M|^2 \, dx - g(0) \right| \leq C(|g(0)| \, M/N + \omega(M^{-1})).$$

3.3. Boundary normal coordinates

To recover higher order derivatives of the conductivity, it will be useful to reduce to the case where the boundary is flat. Flattening the boundary can be carried out in different ways, and typically the scalar conductivity is transformed into a matrix conductivity in the process. Here it is convenient to choose the change of coordinates so that the new conductivity matrix has special form. These coordinates are called *boundary normal coordinates*, since (as may be seen from the proof) the point $F^{-1}(y', y_n)$ is obtained by choosing a boundary point according to y' and then moving y_n units in the direction of the inner unit normal.

Proposition 3.23. *Let Ω be a bounded domain with C^∞ boundary in $\mathbb{R}^n$, and let $p \in \partial\Omega$. There is a C^∞ diffeomorphism $F : U \to V$ between open sets of $\mathbb{R}^n$ where U is a neighborhood of p and V is a neighborhood of 0, such that*

$$F(p) = 0, \quad F(\Omega \cap U) = V \cap \{y_n > 0\}, \quad F(\partial\Omega \cap U) = V \cap \{y_n = 0\},$$

and further for any $y \in V \cap \{y_n \geq 0\}$,

$$(DF)(DF)^t|_{F^{-1}(y)} = \begin{bmatrix} h(y) & 0 \\ 0 & 1 \end{bmatrix}$$

for some C^∞ symmetric positive definite matrix $h(y) = \left[h^{\alpha\beta}(y)\right]_{\alpha,\beta=1}^{n-1}$.

Proof. Since Ω has smooth boundary, there is a system of coordinates where $p = 0$ and where

$$\Omega \cap B(0, 2r) = \left\{ \, y \in B(0, 2r) \;\middle|\; y_n > h(y') \, \right\}$$

for some C^∞ map $h : \mathbb{R}^{n-1} \to \mathbb{R}$ with $h(0) = 0$. Write $q(y') = (y', h(y'))$, and define a map $\Phi : B(0, r) \to \mathbb{R}^n$ by

$$\Phi(y', y_n) = q(y') - y_n \nu(q(y'))$$

This is a C^∞ map in some neighborhood of 0, and its Jacobian matrix is given in terms of columns by

(3.13)

$$D\Phi(y', y_n) = \Big[\partial_1 q - y_n\partial_1\big(\nu(q(y'))\big),\ \ldots,\ \partial_{n-1}q - y_n\partial_{n-1}\big(\nu(q(y'))\big), -\nu(q(y'))\Big]$$

Thus $D\Phi(0,0) = [\partial_1 q(0), \ldots, \partial_{n-1}q(0), -\nu(0)]$. Since q parametrizes $\partial\Omega$, the vectors $\{\partial_1 q(0), \ldots, \partial_{n-1}q(0)\}$ form a basis for all the tangent vectors to $\partial\Omega$ at 0. Consequently $D\Phi(0,0)$ is invertible. By the inverse function theorem, there is a neighborhood V of 0 and a neighborhood U of 0 such $\Phi : V \mapsto U$ is a diffeomorphism.

We set $F = \Phi^{-1} : U \to V$. This is a diffeomorphism with $F(0) = 0$, and $F(\Omega \cap U)$ coincides with $V \cap \{y_n > 0\}$ since ν was the unit outer normal. By (3.13) we have

$$D\Phi(y)^t D\Phi(y) = \begin{bmatrix} g(y) & v(y) \\ v(y)^t & 1 \end{bmatrix}$$

where $g(y)$ is some smooth $(n-1)\times(n-1)$ matrix. But $v = 0$ since

$$\begin{aligned} v_j(y) &= \big[\partial_j q(y') - y_n\partial_j\big(\nu(q(y'))\big)\big]\cdot\big[-\nu(q(y'))\big] \\ &= \frac{y_n}{2}\partial_j\big[\nu(q(y'))\cdot\nu(q(y'))\big] = 0 \end{aligned}$$

using that $\partial_j q(y')$ is tangent to $\partial\Omega$ and $|\nu| = 1$. This concludes the proof since $D\Phi(y) = DF(F^{-1}(y))^{-1}$, and thus $(DF)(DF)^t|_{F^{-1}(y)}$ has the required form where $h(y) = g(y)^{-1}$ is positive definite because $(DF)(DF)^t$ is. □

Remark 3.24. A brief discussion about the precise meaning of "$\big(\frac{\partial}{\partial\nu}\big)^\ell f$" is in order. Let f be any C^∞ function that is defined in a neighbourhood of $\partial\Omega$. For each point x, which is sufficiently close to $\partial\Omega$, there is a unique point $\pi(x) \in \partial\Omega$ that is nearest to x. The vector from x to $\pi(x)$ is normal to $\partial\Omega$ at $\pi(x)$. See Problem 3.25. Let $\hat{n}(x)$ be a unit vector that is parallel to the vector from x to $\pi(x)$ and points from inside Ω to outside Ω. In the event that $x \in \partial\Omega$, so that $\pi(x) = x$, choose $\hat{n}(x)$ to be the unit outward normal to $\partial\Omega$ at x. The vector $\hat{n}(x) = \hat{n}\big(\pi(x)\big)$ is a C^∞ function of x in a neighbourhood of $\partial\Omega$. Again, see Problem 3.25. We define $\frac{\partial}{\partial\nu}f(x) = \hat{n}(x)\cdot\nabla f(x)$, for all x in a neighbourhood of $\partial\Omega$. Then we may define $\big(\frac{\partial}{\partial\nu}\big)^\ell f$ by ℓ applications of $\frac{\partial}{\partial\nu}$.

Exercise 3.25. Let $p \in \partial\Omega$. Let $x'(\xi')$ be a C^∞ parametrization of a neighbourhood of p in $\partial\Omega$ with $x'(0) = p$. Denote by $\hat{n}(x')$ the unit outward normal to $\partial\Omega$ at $x' \in \partial\Omega$. Define $x(\xi', \xi_n) = x'(\xi') - \xi_n\hat{n}\big(x'(\xi')\big)$.

(a) Prove that $x(\xi', \xi_n)$ is a C^∞ diffeomorphism from a neighbourhood of $0 \in \mathbb{R}^n$ to a neighbourhood of $p \in \mathbb{R}^n$.

(b) Prove that, for all sufficiently small (ξ', ξ_n), $x'(\xi')$ is the point of $\partial\Omega$ that is nearest $x(\xi', \xi_n)$, so that the distance from $x(\xi', \xi_n)$ to $\partial\Omega$ is $|\xi_n|$.

We now give a precise definition of higher order normal derivatives at the boundary.

Definition 3.26. ??? MS:Possible exercises: relation to Remark 3.24, proof that the definition does not depend on choice of F. ??? Let Ω be a bounded domain with C^∞ boundary, let $\gamma \in C^\infty(\overline{\Omega})$, and let $p \in \partial\Omega$. For $l \geq 0$ we define

$$\left(\frac{\partial}{\partial \nu}\right)^l \gamma(p) = \partial_{y_n}^l (\gamma(F^{-1}(y)))|_{y=0}$$

where F is as in Proposition 3.23.

Exercise 3.27. Show that the definition above is independent of the choice of F.

??? MS:Could remark that coordinate invariance is explained in more detail elsewhere in the book. ??? We will next determine how the conductivity equation transforms under F. The idea is that solutions to $L_\gamma u = 0$ transform into solutions to $L_{\tilde{\gamma}} \tilde{u} = 0$ by the rule $\tilde{u} = u \circ F^{-1}$, where $\tilde{\gamma}$ is a certain matrix conductivity given below. However, since the change of coordinates is only defined near p we need to restrict our attention to functions defined in a neighborhood of p.

Lemma 3.28. *Let Ω, p, and $F : U \to V$ be as in Proposition 3.23, and let $\gamma \in C^\infty(\overline{\Omega})$ be a positive function. Assume that $v \in C^\infty(U \cap \overline{\Omega})$ and let $\tilde{v}(y) = v(F^{-1}(y))$ for $y \in V \cap \{y_n \geq 0\}$. If $\varphi \in C_c^\infty(U \cap \Omega)$ and if $\tilde{\varphi}(y) = \varphi(F^{-1}(y))$, then* **??? MS:Is this notation consistent? ???**

$$\langle L_\gamma v, \varphi \rangle = \langle L_{\tilde{\gamma}} \tilde{v}, \tilde{\varphi} \rangle$$

where

$$\tilde{\gamma}(y) = \begin{bmatrix} ch & 0 \\ 0 & c \end{bmatrix}$$

Here $h = \left[h^{\alpha,\beta}\right]_{\alpha,\beta=1}^{n-1}$ is the matrix in Proposition 3.23 and

$$c(y) = \frac{\gamma(F^{-1}(y))}{|\det DF(F^{-1}(y))|}.$$

Proof. ??? MS:Are references needed to these basic things such as changing coordinates in integrals, chain rule, ...? ??? Since φ is compactly supported in $U \cap \Omega$, we can make the change of coordinates

$x = F^{-1}(y)$ to obtain

$$\begin{aligned}\langle L_\gamma v, \varphi\rangle &= \int_{U\cap\Omega} \gamma(x)\nabla v(x)\cdot\nabla\varphi(x)\,dx \\ &= \int_{V\cap\{y_n>0\}} \gamma(F^{-1}(y))\nabla v(F^{-1}(y))\cdot\nabla\varphi(F^{-1}(y))|\det D(F^{-1})(y)|\,dy\end{aligned}$$

The chain rule implies $Dv(x) = D(\tilde v\circ F)(x) = D\tilde v(F(x))DF(x)$, thus

$$\nabla v(x) = DF(x)^t\nabla\tilde v(F(x))$$

Using the analogous result for φ, we have

$$\begin{aligned}\langle L_\gamma v, \varphi\rangle &= \int_{V\cap\{y_n>0\}} \frac{\gamma(F^{-1}(y))DF(F^{-1}(y))^t\nabla\tilde v(y)\cdot DF(F^{-1}(y))^t\nabla\tilde\varphi(y)}{|\det DF(F^{-1}(y))|}\,dy \\ &= \int_{V\cap\{y_n>0\}} \frac{\gamma}{|\det DF|}(DF)(DF)^t\Big|_{F^{-1}(y)}\nabla\tilde v(y)\cdot\nabla\tilde\varphi(y)\,dy\end{aligned}$$

$\langle L_{\tilde\gamma}\tilde v, \tilde\varphi\rangle$ where $\tilde\gamma$ is as required. □

3.4. Oscillating solutions

Using the change of coordinates in Proposition 3.23, we may assume that we are working in a domain $\tilde\Omega$ which is flat near 0 and for some $r > 0$ one has

$$B(0,2r)\cap\tilde\Omega = B(0,2r)\cap\{y_n>0\}.$$

We will determine the conductivity in the set

$$\tilde\Gamma = B(0,r)\cap\{y_n=0\}$$

Below, we will also think of $\tilde\Gamma$ as a subset of $\mathbb{R}^{n-1}$. Motivated by Lemma 3.28, we consider a matrix conductivity $\tilde\gamma$ having the form

$$\tilde\gamma(y) = \begin{bmatrix} ch & 0 \\ 0 & c\end{bmatrix} \tag{3.14}$$

where $h(y) = (h^{\alpha,\beta}(y))_{\alpha,\beta=1}^{n-1}$ is a symmetric positive definite matrix and $c(y)$ is a positive scalar function, both depending smoothly on y in $B(0,2r)\cap\{y_n\geq 0\}$. We assume that the matrix h is known, and that c is an unknown function which needs to be determined.

The next step is to construct approximate solutions to the conductivity equation in $\tilde\Omega$ which are supported in a small neighborhood of $\tilde\Gamma$ and oscillate rapidly on $\tilde\Gamma$, the speed of oscillation depending on a large parameter $s > 0$. To recover high order derivatives of the conductivity, we need to do an asymptotic construction in terms of powers of s. The proof is rather long but quite elementary.

We will use the following result several times below:

Exercise 3.29 (Leibniz rule)**.** If $u(t)$ and $v(t)$ are C^m functions on some real interval I, then

(3.15)
$$\left(\frac{d}{dt}\right)^m \big(u(t)v(t)\big) = u^{(m)}(t)v(t) + u(t)v^{(m)}(t) + \sum_{j=1}^{m-1} \begin{bmatrix} m \\ j \end{bmatrix} u^{(j)}(t)v^{(m-j)}(t)$$

Note that the approximate solutions in the next result are exponentially decaying in $\tilde{\Omega}$. They are related to the exponentially growing solutions which are used later for proving uniqueness results for the Calderón problem in the interior.

Proposition 3.30. *Let $N > 0$, let t' be a unit vector in $\mathbb{R}^{n-1}$, and assume that $\eta \in C^\infty(\mathbb{R}^{n-1})$ is a compactly supported function in $\tilde{\Gamma}$. For some small $\delta > 0$ and for any $s \geq 1$, there is a function $\tilde{v}_s \in C^\infty(\overline{\tilde{\Omega}})$ satisfying*

$$\begin{aligned} &\tilde{v}_s(y', 0) = e^{isy'\cdot t'}\eta(y') \quad \textit{on} \;\; \tilde{\Gamma} \\ &\operatorname{supp}(\tilde{v}_s) \subset \tilde{\Gamma} \times [0, \delta] \end{aligned} \tag{3.16}$$

and

$$\|\tilde{v}_s\|_{H^1(\tilde{\Omega})} \leq Cs^{1/2} \tag{3.17}$$

$$\|L_{\tilde{\gamma}}\tilde{v}_s\|_{L^2(\tilde{\Omega})} \leq Cs^{-N+3/2} \tag{3.18}$$

where the constant C is independent of s. Further, this function has the form

$$\tilde{v}_s = e^{s\Phi}(a_0 + s^{-1}a_{-1} + \ldots + s^{-N}a_{-N}) \tag{3.19}$$

where Φ is a smooth complex function satisfying for some $\sigma > 0$

$$\begin{aligned} \Phi(y', 0) &= iy' \cdot t' && \textit{for} \;\; y' \in \tilde{\Gamma} \\ \partial_n \Phi(y', 0) &= f_1(y') && \textit{for} \;\; y' \in \tilde{\Gamma} \\ \operatorname{Re}(\Phi(y', y_n)) &\leq -\sigma y_n && \textit{for} \;\; y \in \tilde{\Gamma} \times [0, \delta] \end{aligned}$$

with

$$f_1(y') = -\left(\sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta}(y', 0)t_\alpha t_\beta\right)^{1/2}$$

and $a_0, a_{-1}, \ldots, a_{-N}$ are smooth complex functions independent of s and supported in the set $\tilde{\Gamma} \times [0, \delta]$ and they satisfy

$$\begin{aligned} a_0(y', 0) &= \eta(y') && \textit{for} \;\; y' \in \tilde{\Gamma} \\ a_{-l}(y', 0) &= 0 && \textit{for} \;\; y' \in \tilde{\Gamma} \; \textit{and} \;\; l \geq 1 \end{aligned}$$

Proof. ??? MS:Should this long proof be broken into separate lemmas? ??? We try to find an approximate solution of the equation $L_{\tilde\gamma}\tilde v = 0$ having the form $\tilde v_s = e^{s\Phi}a$. One has the identity

$$\partial_j(e^{s\Phi}w) = e^{s\Phi}(\partial_j + s\partial_j\Phi)w.$$

Using this identity, the function $\tilde v_s$ satisfies

$$\begin{aligned} L_{\tilde\gamma}\tilde v_s &= \sum_{j,k=1}^{n} \partial_j(\tilde\gamma^{j,k}\partial_k\tilde v_s) \\ &= \sum_{j,k=1}^{n} e^{s\Phi}(\partial_j + s\partial_j\Phi)(\tilde\gamma^{jk}(\partial_k + s\partial_k\Phi)a) \\ &= e^{s\Phi}\Big\{s^2\Big[\Big(\sum_{j,k=1}^{n}\tilde\gamma^{j,k}\partial_j\Phi\partial_k\Phi\Big)a\Big] \\ &\qquad + s\Big[\sum_{j,k=1}^{n}\Big(2\tilde\gamma^{j,k}\partial_j\Phi\partial_k a + \partial_j\big(\tilde\gamma^{j,k}\partial_k\Phi\big)\Big)a\Big] + [L_{\tilde\gamma}a]\Big\} \end{aligned} \tag{3.20}$$

Note that we have grouped the terms corresponding to different powers of s. The idea is to choose Φ and a so that the terms involving the largest powers of s are small, finally resulting in the estimate (3.18). Since we are only interested in finding approximate solutions in a small neighborhood of $\tilde\Gamma$, *it is sufficient to arrange that the terms vanish to high order on* $\tilde\Gamma$ *instead of vanishing in a full neighborhood of* $\tilde\Gamma$.

Finding Φ. 1. Looking at the s^2 term in (3.20), the first task is to find a complex function Φ which satisfies

$$\partial_n^j\Big(\sum_{j,k=1}^{n}\tilde\gamma^{j,k}\partial_j\Phi\partial_k\Phi\Big)\Big|_{\tilde\Gamma} = 0 \qquad \textit{for } j = 0,1,\ldots,N-1 \tag{3.21}$$

We will look for Φ in the form

$$\Phi(y', y_n) = f_0(y') + y_n f_1(y') + \frac{y_n^2}{2}f_2(y') + \ldots + \frac{y_n^N}{N!}f_N(y') \tag{3.22}$$

where $f_j(y') = \partial_n^j\Phi(y',0)$ are functions to be determined. To ensure the boundary condition (3.16), we choose

$$f_0(y') = iy'\cdot t'$$

2. Next we find $f_1(y') = \partial_n\Phi(y',0)$ so that (3.21) is satisfied for $j = 0$. Using the special form of $\tilde\gamma$ given in 3.14, we need that

$$c(\partial_n\Phi)^2 + c\sum_{\alpha,\beta=1}^{n-1}h^{\alpha,\beta}\partial_\alpha\Phi\partial_\beta\Phi\Big|_{\tilde\Gamma} = 0$$

Since $\partial_\alpha \Phi|_{\tilde{\Gamma}} = it_\alpha$, this reduces to

$$f_1^2 - \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta} t_\alpha t_\beta = 0.$$

Recalling that we want $\mathrm{Re}(\Phi) \leq 0$ in $\tilde{\Gamma} \times [0,\delta]$, it makes sense to require that $f_1 = \partial_n \Phi|_{\tilde{\Gamma}}$ is negative. Thus, we make the choice

$$f_1(y') = -\left(\sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta}(y',0) t_\alpha t_\beta \right)^{1/2}$$

$(h^{\alpha,\beta}(y',0))$ is positive definite.

3. We continue with (3.21) for $j = 1$ and try to find f_2 such that

$$\partial_n \Big(\sum_{j,k=1}^{n} \tilde{\gamma}^{j,k} \partial_j \Phi \partial_k \Phi \Big)\Big|_{\tilde{\Gamma}} = 0$$

Since $(\partial_n \Phi)^2 + \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta} \partial_\alpha \Phi \partial_\beta \Phi|_{\tilde{\Gamma}} = 0$ by (3.21) for $j = 0$, we have

$$\begin{aligned} \partial_n \Big(\sum_{j,k=1}^{n} \tilde{\gamma}^{j,k} \partial_j \Phi \partial_k \Phi \Big)\Big|_{\tilde{\Gamma}} &= c\partial_n \Big((\partial_n \Phi)^2 + \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta} \partial_\alpha \Phi \partial_\beta \Phi \Big)\Big|_{\tilde{\Gamma}} \\ &= c\Big(2f_1 \partial_n^2 \Phi - \sum_{\alpha,\beta=1}^{n-1} (\partial_n h^{\alpha,\beta}) t_\alpha t_\beta + T_0(h) \Big)\Big|_{\tilde{\Gamma}} \end{aligned}$$

where $T_0(h)$ is a quantity only depending on t' and on y'-derivatives of $h^{\alpha,\beta}(y',0)$. The last expression vanishes if we choose $f_2 = \partial_n^2 \Phi|_{\tilde{\Gamma}}$ as

$$f_2 = \frac{1}{2f_1} \sum_{\alpha,\beta=1}^{n-1} (\partial_n h^{\alpha,\beta}) t_\alpha t_\beta - \frac{1}{2f_1} T_0(h)$$

4. Let f_0 and f_1 be as above, and suppose that we have found $f_2, \ldots, f_m$ such that **??? MS: The precise form of f_j is actually not needed, the only thing that one needs to know is that f_j exists and is independent of c. ???**

$$\partial_n^j \Big(\sum_{j,k=1}^{n} \tilde{\gamma}^{j,k} \partial_j \Phi \partial_k \Phi \Big)\Big|_{\tilde{\Gamma}} = 0 \qquad \textit{for } 0 \leq j \leq m-1 \tag{3.23}$$

$$f_j = \frac{1}{2f_1} \sum_{\alpha,\beta=1}^{n-1} (\partial_n^{j-1} h^{\alpha,\beta}) t_\alpha t_\beta + T_{j-2}(h) \qquad \textit{for } 2 \leq j \leq m \tag{3.24}$$

where $T_l(h)$ is an expression only depending on t' and on y'-derivatives of $\partial_n^k h^{\alpha,\beta}(y',0)$ for $0 \leq k \leq l$. We wish to find f_{m+1} of the form (3.24) such that (3.23) is valid also for $j = m$. Since

$$\sum_{j,k=1}^{n} \tilde{\gamma}^{j,k} \partial_j \Phi \partial_k \Phi = c\Big((\partial_n \Phi)^2 + \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta} \partial_\alpha \Phi \partial_\beta \Phi \Big)$$

the identity (3.15) and the hypothesis (3.23) imply that

$$\partial_n^j \Big((\partial_n \Phi)^2 + \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta} \partial_\alpha \Phi \partial_\beta \Phi \Big)\Big|_{\tilde{\Gamma}} = 0 \qquad \textit{for } 0 \leq j \leq m-1.$$

Consequently, by (3.15) again and by (3.24),

$$\begin{aligned} \partial_n^m \Big(\sum_{j,k=1}^{n} \tilde{\gamma}^{j,k} \partial_j \Phi \partial_k \Phi \Big)\Big|_{\tilde{\Gamma}} &= c \partial_n^m \Big((\partial_n \Phi)^2 + \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta} \partial_\alpha \Phi \partial_\beta \Phi \Big)\Big|_{\tilde{\Gamma}} \\ &= c\Big(2 f_1 f_{m+1} - \sum_{\alpha,\beta=1}^{n-1} (\partial_n^m h^{\alpha,\beta}) t_\alpha t_\beta + T_{m-1}(h) \Big) \end{aligned}$$

We can make the last expression vanish by choosing f_{m+1}, as required, as

$$f_{m+1} = \frac{1}{2 f_1} \sum_{\alpha,\beta=1}^{n-1} (\partial_n^m h^{\alpha,\beta}) t_\alpha t_\beta + T_{m-1}(h)$$

5. We use the previous steps to inductively find $f_0, f_1, \ldots, f_N$ so that Φ given by (3.22) satisfies the condition (3.21). Further, we have $\Phi(y', 0) = iy' \cdot t'$, and

$$\mathrm{Re}(\Phi(y', 0)) = y_n \Big[f_1(y') + \mathrm{Re} \Big(\frac{y_n}{2} f_2(y') + \ldots + \frac{y_n^{N-1}}{N!} f_N(y') \Big) \Big]$$

If $\sigma > 0$ is such that $f_1|_{\tilde{\Gamma}} \leq -2\sigma$, we can find $\delta > 0$ such that

$$\Big| \frac{y_n}{2} f_2(y') + \ldots + \frac{y_n^{N-1}}{N!} f_N(y') \Big| \leq \sigma \quad \textit{for } y \in \tilde{\Gamma} \times [0, \delta]$$

Then, for $y \in \tilde{\Gamma} \times [0, \delta]$ we have $\mathrm{Re}(\Phi(y)) \leq -\sigma y_n$. This completes the construction of Φ.

Finding a. 1. We have constructed a function Φ so that the s^2 term in (3.20) vanishes to high order on $\tilde{\Gamma}$. The next task is to find a function a so that the s^1 and s^0 terms in (3.20) have the same property. We will in fact look for a in the form

$$a = a_0 + s^{-1} a_{-1} + \ldots + s^{-N} a_{-N}.$$

Here $a_0, a_{-1}, \ldots, a_{-N}$ will be complex functions functions supported in $\tilde{\Gamma} \times [0, \delta]$ which are independent of s. For such a, the s^1 and s^0 terms in (3.20) become

$$s\Big[\sum_{j,k=1}^{n}(2\tilde{\gamma}^{j,k}\partial_j\Phi\partial_k a+\partial_j(\tilde{\gamma}^{j,k}\partial_k\Phi)a\Big]+[L_{\tilde{\gamma}}a]$$
$$=s[Ma_0]+[Ma_{-1}+L_{\tilde{\gamma}}a_0]+s^{-1}[Ma_{-2}+L_{\tilde{\gamma}}a_{-1}]+\dots$$
$$+s^{-(N-1)}[Ma_{-N}+L_{\tilde{\gamma}}a_{-(N-1)}]+s^{-N}[L_{\tilde{\gamma}}a_{-N}]$$

where M is the first order differential operator given by

$$Mb=\sum_{j,k=1}^{n}\Big(2\tilde{\gamma}^{j,k}\partial_j\Phi\partial_k b+\partial_j(\tilde{\gamma}^{j,k}\partial_k\Phi)b\Big)$$

Note that the quantities in brackets are independent of s. We shall find $a_0,a_{-1},\dots,a_{-N}$ successively so that one has

$$\partial_n^j\big(Ma_0\big)\big|_{\tilde{\Gamma}}=0\qquad \textit{for } j=0,1,\dots,N-1 \tag{3.25}$$

$$\partial_n^j\big(Ma_{-1}+L_{\tilde{\gamma}}a_0\big)\big|_{\tilde{\Gamma}}=0\qquad \textit{for } j=0,1,\dots,N-1 \tag{3.26}$$

$$\vdots \tag{3.27}$$

$$\partial_n^j\big(Ma_{-N}+L_{\tilde{\gamma}}a_{-(N-1)}\big)\big|_{\tilde{\Gamma}}=0\qquad \textit{for } j=0,1,\dots,N-1 \tag{3.28}$$

2. The function a_0 is constructed in a similar way as Φ. We look for a_0 in the form

$$a_0(y',y_n)=\left(g_0(y')+y_ng_1(y')+\dots+\frac{y_n^N}{N!}g_N(y')\right)\zeta(y_n/\delta)$$

where $g_j(y')=\partial_n^j a_0(y',0)$ are functions in $C_c^\infty(\tilde{\Gamma})$ to be determined, and $\zeta\in C^\infty(\mathbb{R})$ is a fixed cutoff function with $\zeta(t)=1$ for $|t|\le 1/2$ and $\zeta(t)=0$ for $|t|\ge 1$. It follows that a_0 is compactly supported in $\tilde{\Gamma}\times[0,\delta]$. Motivated by the boundary condition (3.16), we choose g_0 as

$$g_0(y')=\eta(y')$$

3. Using the special form 3.14 for $\tilde{\gamma}$, it follows that

(3.29)
$$Mb=\sum_{j,k=1}^{n}\Big(2\tilde{\gamma}^{j,k}\partial_j\Phi\partial_k b+\tilde{\gamma}^{j,k}(\partial_j\partial_k\Phi)b+(\partial_j\tilde{\gamma}^{j,k})(\partial_k\Phi)b\Big)$$
$$=2c\partial_n\Phi\partial_n b+2c\sum_{\alpha,\beta=1}^{n-1}h^{\alpha,\beta}\partial_\alpha\Phi\partial_\beta b+c(\partial_n^2\Phi)b+c\sum_{\alpha,\beta=1}^{n-1}h^{\alpha,\beta}(\partial_\alpha\partial_\beta\Phi)b$$
$$+(\partial_n c)(\partial_n\Phi)b+\sum_{\alpha,\beta=1}^{n-1}(\partial_\alpha c)h^{\alpha,\beta}(\partial_\beta\Phi)b+c\sum_{\alpha,\beta=1}^{n-1}(\partial_\alpha h^{\alpha,\beta})(\partial_\beta\Phi)b$$

For the equation (3.25) with $j = 0$, we have

$$Ma_0\big|_{\tilde\Gamma} = 2cf_1\partial_n a_0 + (\partial_n c)f_1\eta + c(\partial_n^2\Phi)\eta + T_0(c,h)$$

where $T_k(c,h)$ denotes an expression depending only on t', η, and on y'-derivatives of $\partial_n^j c|_{\tilde\Gamma}$ and $\partial_n^j h^{\alpha,\beta}|_{\tilde\Gamma}$ for $0 \le j \le k$. Consequently, we have $Ma_0|_{\tilde\Gamma} = 0$ if $g_1 = \partial_n a_0|_{\tilde\Gamma}$ is chosen as **??? MS: The precise form of g_j for $j \ge 1$ (similarly $\partial_n^j a_{-l}|_{\tilde\Gamma}$ for $j \ge 1$) is actually not needed anywhere. ???**

$$\begin{aligned} g_1(y') &= -\frac{\partial_n c}{2c}\eta - \frac{1}{2f_1}f_2\eta + T_0(c,h) \\ &= -\frac{\partial_n c}{2c}\eta - \Big(\frac{1}{2f_1}\Big)^2\Big(\sum_{\alpha,\beta=1}^{n-1}(\partial_n h^{\alpha,\beta})t_\alpha t_\beta\Big)\eta + T_0(c,h) \end{aligned}$$

by (3.24).

4. Let g_0 be as above, and suppose that we have found $g_1, \dots, g_m$ such that (3.25) holds for $j = 0, 1, \dots, m-1$, and

$$g_j = -\frac{\partial_n^j c}{2c}\eta - \Big(\frac{1}{2f_1}\Big)^2\Big(\sum_{\alpha,\beta=1}^{n-1}(\partial_n^j h^{\alpha,\beta})t_\alpha t_\beta\Big)\eta + T_{j-1}(c,h),\ 1 \le j \le m \tag{3.30}$$

We will find g_{m+1} having the form (3.30) so that (3.25) is satisfied also for $j = m$. By (3.29) and (3.15),

$$\partial_n^m(Ma_0)|_{\tilde\Gamma} = 2cf_1\partial_n^{m+1}a_0 + (\partial_n^{m+1}c)f_1\eta + cf_{m+2}\eta + T_m(c,h).$$

We can make this vanish by choosing $g_{m+1} = \partial_n^{m+1}a_0|_{\tilde\Gamma}$ having the required form (3.30), using the expression for f_{m+2} in (3.24).

5. By the previous steps, we can find $g_0, g_1, \dots, g_N$ inductively so that (3.25) is satisfied and each g_j for $j = 1, \dots, N$ has the form (3.30). This completes the construction of a_0.

6. The construction of a_{-1} is similar to that of a_0. The function a_{-1} will have the form

$$a_{-1}(y', y_n) = \left(g_{-1,0}(y') + y_n g_{-1,1}(y') + \ldots + \frac{y_n^N}{N!}g_{-1,N}(y')\right)\zeta(y_n/\delta)$$

for suitable $g_{-1,j} \in C_0^\infty(\tilde\Gamma)$ and with ζ as above. Considering the boundary condition (3.16), we choose

$$g_{-1,0}(y') = 0.$$

Considering (3.26) for $j = 0$, since $a_{-1}|_{\tilde\Gamma} = 0$ we have by (3.29)

$$\begin{aligned} Ma_{-1} + L_{\tilde\gamma}a_0\big|_{\tilde\Gamma} &= 2cf_1\partial_n a_{-1} + L_{\tilde\gamma}a_0\big|_{\tilde\Gamma} \\ &= 2cf_1\partial_n a_{-1}\big|_{\tilde\Gamma} + cg_2 + T_1(c,h) \end{aligned}$$

This vanishes if $g_{-1,1}$ is chosen as

$$
\begin{aligned}
g_{-1,1} &= -\frac{1}{2f_1} g_2 + T_1(c,h) \\
&= -\Big(-\frac{1}{2f_1}\Big)\frac{\partial_n^2 c}{2c}\eta - \Big(-\frac{1}{2f_1}\Big)^3\Big(\sum_{\alpha,\beta=1}^{n-1}(\partial_n^2 h^{\alpha,\beta})t_\alpha t_\beta\Big)\eta + T_1(c,h)
\end{aligned}
$$

Here we used (3.30). The next case to consider is

$$
\partial_n\big(Ma_{-1} + L_{\tilde{\gamma}} a_0\big)\big|_{\tilde{\Gamma}} = 2cf_1 g_{-1,2} + cg_3 + T_2(c,h)
$$

This vanishes if one defines

$$
\begin{aligned}
g_{-1,2} &= -\frac{1}{2f_1} g_3 + T_2(c,h) \\
&= -\Big(-\frac{1}{2f_1}\Big)\frac{\partial_n^3 c}{2c}\eta - \Big(-\frac{1}{2f_1}\Big)^3\Big(\sum_{\alpha,\beta=1}^{n-1}(\partial_n^3 h^{\alpha,\beta})t_\alpha t_\beta\Big)\eta + T_2(c,h)
\end{aligned}
$$

Continuing, we obtain $g_{-1,0}, g_{-1,1}, \dots, g_{-1,N}$ such that a_{-1} satisfies (3.26) and

$$
\begin{aligned}
g_{-1,j} = -\Big(-\frac{1}{2f_1}\Big)\frac{\partial_n^{1+j} c}{2c}\eta - \Big(-\frac{1}{2f_1}\Big)^3\Big(\sum_{\alpha,\beta=1}^{n-1}(\partial_n^{1+j} h^{\alpha,\beta})t_\alpha t_\beta\Big)\eta \\
+ T_{1+j-1}(c,h), \qquad 1 \le j \le N
\end{aligned}
$$

We have now constructed a_{-1}.

7. The construction of $a_{-2}, \dots, a_{-N}$ is completely analogous to that of a_{-1}. We leave it as an exercise to check that one can find $a_{-2}, \dots, a_{-N}$ such that (3.25)–(3.28) are satisfied, and

$$
\begin{aligned}
a_{-l}|_{\tilde{\Gamma}} &= 0, \quad 1 \le l \le N \\
\partial_n^j a_{-l}|_{\tilde{\Gamma}} &= -\frac{1}{2f_1}\partial_n^{j+1} a_{-(l-1)}|_{\tilde{\Gamma}} + T_{l+j-1}(c,h) \\
&= -\Big(-\frac{1}{2f_1}\Big)^l \frac{\partial_n^{l+j} c}{2c}\eta - \Big(-\frac{1}{2f_1}\Big)^{l+2}\Big(\sum_{\alpha,\beta=1}^{n-1}(\partial_n^{l+j} h^{\alpha,\beta})t_\alpha t_\beta\Big)\eta \\
&\qquad + T_{l+j-1}(c,h), \qquad 1 \le l \le N,\ 1 \le j \le N
\end{aligned}
$$

where, as before, $T_k(c,h)$ denotes an expression only depending on t', η, and on y'-derivatives of $\partial_n^j c|_{\tilde{\Gamma}}$ and $\partial_n^j h^{\alpha,\beta}|_{\tilde{\Gamma}}$ for $0 \le j \le k$.

End of proof. We have proved all the statements in the proposition except for the estimates (3.17) and (3.18). To verify (3.17), we first use that

a is uniformly bounded in $\tilde{\Gamma} \times [0,\delta]$ and that $\mathrm{Re}(\Phi) \leq -\sigma y_n$ to obtain

$$\begin{aligned}\|\tilde{v}_s\|_{L^2(\tilde{\Gamma}\times[0,\delta])} &\leq C \left\| e^{s\,\mathrm{Re}(\Phi)} \right\|_{L^2(\tilde{\Gamma}\times[0,\delta])} \\ &\leq C\Big(\int_{\tilde{\Gamma}} \int_0^\delta e^{-2s\sigma y_n}\,dy'\,dy_n \Big)^{1/2} \\ &\leq C s^{-1/2}\end{aligned}$$

by changing variables $y_n = t/s$. The derivatives of $\tilde{v}_s$ are given by $\partial_j \tilde{v}_s = e^{s\Phi}\big(s(\partial_j\Phi)a + \partial_j a\big)$, and a similar argument as above shows that $\|\partial_j \tilde{v}_s\|_{L^2(\tilde{\Gamma}\times[0,\delta])} \leq Cs^{1/2}$ which implies (3.17).

To prove (3.18), we note that (3.20) and the construction of Φ and a imply that

$$L_{\tilde{\gamma}}\tilde{v}_s = e^{s\Phi}\big(s^2 b_2 + s b_1 + b_0 + \ldots + s_{-(N-1)} b_{-(N-1)} + s^{-N} r_{-N}\big)$$

where each b_k satisfies

$$\partial_n^j b_k|_{\tilde{\Gamma}} = 0, \qquad 0 \leq j \leq N-1, \quad -(N-1) \leq k \leq 2,$$

and the remainder term r_{-N} has the form

$$r_{-N} = L_{\tilde{\gamma}} a_{-N}.$$

??? MS: This property of Taylor series could be formulated as a problem. ??? It follows that for $y \in \tilde{\Gamma} \times [0,\delta]$, one has

$$\big|b_k(y', y_n)\big| \leq C y_n^N, \qquad -(N-1) \leq k \leq 2,$$

where the constant C depends on the N^{th} derivatives of b_k and is independent of s. Also $|r_N(y', y_n)| \leq C$ with C independent of s. Since $s \geq 1$, we can estimate

$$\begin{aligned}\big|L_{\tilde{\gamma}}\tilde{v}_s(y', y_n)\big| &\leq C e^{s\,\mathrm{Re}(\Phi)}(s^2 y_n^N + s^{-N}) \\ &\leq C e^{-\sigma s y_n}(s^2 y_n^N + s^{-N})\end{aligned}$$

The function $L_{\tilde{\gamma}}\tilde{v}_s$ is supported in $\tilde{\Gamma} \times [0,\delta]$ and satisfies

$$\|L_{\tilde{\gamma}}\tilde{v}_s\|^2_{L^2(\tilde{\Omega})} \leq 2C^2 \int_{\tilde{\Gamma}} \int_0^\delta e^{-2\sigma s y_n}\big(s^4 y_n^{2N} + s^{-2N}\big)\,dy'\,dy_n$$

We used the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b \geq 0$. The integrand is independent of y' and consequently

$$\begin{aligned}\|L_{\tilde{\gamma}}\tilde{v}_s\|^2_{L^2(\tilde{\Omega})} &\leq C' \int_0^\infty e^{-2\sigma s y_n}\big(s^4 y_n^{2N} + s^{-2N}\big)\,dy_n \\ &= C' \int_0^\infty e^{-2\sigma t}\big(s^{4-2N} t^{2N} + s^{-2N}\big) s^{-1}\,dt\end{aligned}$$

by the change of variable $y_n = t/s$. Finally,

$$\begin{aligned}\|L_{\tilde\gamma}\tilde v_s\|^2_{L^2(\tilde\Omega)} &\le C' s^{3-2N}\int_0^\infty e^{-2\sigma t}t^{2N}\,dt + C' s^{-1-2N}\int_0^\infty e^{-2\sigma t}\,dt\\ &\le C'' s^{3-2N}\end{aligned}$$

We have proved (3.18). □

3.5. Recovering higher order derivatives

Let now Ω be a bounded domain with smooth boundary, and let p be a fixed point on $\partial\Omega$. Assume that $\gamma \in C^\infty(\overline{\Omega})$ is a positive function. We will prove that γ and its normal derivatives at p can be recovered from the Dirichlet–to–Neumann map measured on a small neighborhood of p.

First, let $F : U \to V$ be the boundary flattening change of coordinates given in Proposition 3.23. We choose $r > 0$ so small that $B(0, 2r) \cap \{y_n > 0\} \subset V$, and define as in Lemma 3.28

$$\tilde\gamma(y) = \begin{bmatrix} ch & 0 \\ 0 & c\end{bmatrix}$$

where $h(y) = \left[h^{\alpha,\beta}(y)\right]_{\alpha,\beta=1}^{n-1}$ is the matrix in Proposition 3.23 and where

$$c(y) = \frac{\gamma(F^{-1}(y))}{|\det DF(F^{-1}(y))|}$$

Note that c and h are only defined in $V \cap \{y_n \ge 0\}$. We also consider the flat boundary piece $\tilde\Gamma = B(0, r) \cap \{y_n = 0\}$, and $\Gamma = F^{-1}(\tilde\Gamma)$ will be the corresponding neighborhood of p in $\partial\Omega$.

Let $t' \in \mathbb{R}^{n-1}$ be a unit vector, let $\eta \in C^\infty(\mathbb{R}^{n-1})$ be supported in $\tilde\Gamma$, and let N be large. We take $\tilde v_s$ to be the approximate solution given in Proposition 3.30, and we transport $\tilde v_s$ to the original domain Ω by

$$v_s(x) = \tilde v_s(F(x)), \quad x \in U \cap \overline{\Omega}.$$

Since $\tilde v_s$ is compactly supported in $\tilde\Gamma \times [0, \delta]$, it follows that we may extend v_s by zero to $\overline{\Omega}$ and obtain a new function, also denoted by v_s, in $C^\infty(\overline{\Omega})$ with $\operatorname{supp}(v_s)$ contained in a small neighborhood of p in $\overline{\Omega}$. We define the function ϕ_s on $\partial\Omega$ by

$$\phi_s = v_s|_{\partial\Omega}$$

The next result shows that one can construct an exact solution of $L_\gamma u = 0$ in Ω which looks like v_s when s is large.

Lemma 3.31. *There is a unique solution $u_s \in H^1(\Omega)$ of*

$$\begin{aligned} L_\gamma u_s &= 0 && \text{in } \Omega\\ u_s &= \phi_s && \text{on } \partial\Omega\end{aligned}$$

This solution has the form $u_s = v_s + r_s$ with

$$\|r_s\|_{H^1(\Omega)} \leq Cs^{-N+3/2} \tag{3.31}$$

Proof. The existence of a unique solution follows from the well–posedness of the Dirichlet problem. **??? MS: Insert reference for Dirichlet problem. ???** To show the estimate (3.31) for the correction term r_s, we note that r_s solves

$$\begin{aligned} L_\gamma r_s &= -L_\gamma v_s && \textit{in } \Omega \\ r_s &= 0 && \textit{on } \partial\Omega \end{aligned}$$

Again by well–posedness of the Dirichlet problem, we have

$$\|r_s\|_{H^1(\Omega)} \leq C\,\|L_\gamma v_s\|_{H^{-1}(\Omega)} \leq C\,\|L_\gamma v_s\|_{L^2(\Omega)}$$

Now Lemma 3.28 together with a change of coordinates shows that for any $\varphi \in C_c^\infty(U \cap \Omega)$, with $\tilde\varphi = \varphi \circ F^{-1}$, one has

$$\begin{aligned} |\langle L_\gamma v_s, \varphi\rangle| = |\langle L_{\tilde\gamma}\tilde v_s, \tilde\varphi\rangle| &\leq \|L_{\tilde\gamma}\tilde v_s\|_{L^2(V\cap\{y_n>0\})}\,\|\tilde\varphi\|_{L^2(V\cap\{y_n>0\})} \\ &\leq C\,\|L_{\tilde\gamma}\tilde v_s\|_{L^2(V\cap\{y_n>0\})}\,\|\varphi\|_{L^2(\Omega)} \end{aligned}$$

??? MS: Should we give a reference to the density statement? ??? Since $C_c^\infty(U \cap \Omega)$ is dense in $L^2(U \cap \Omega)$, the same result remains true for $\varphi \in L^2(U \cap \Omega)$. This implies that

$$\|L_\gamma v_s\|_{L^2(\Omega)} = \|L_\gamma v_s\|_{L^2(U\cap\Omega)} \leq C\,\|L_{\tilde\gamma}\tilde v_s\|_{L^2(V\cap\{y_n>0\})} \leq Cs^{-N+3/2}$$

by (3.18). Thus

$$\|r_s\|_{H^1(\Omega)} \leq C\,\|L_\gamma v_s\|_{L^2(\Omega)} \leq Cs^{-N+3/2}$$

□

Note that the boundary value ϕ_s is explicit and is given by

$$\phi_s(x) = \begin{cases} e^{isy'(x)\cdot t'}\eta(y'(x)) & \text{if x} \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

where $y'(x) = \big(F_1(x), \ldots, F_{n-1}(x)\big)$ is the representation of Γ as a flat boundary piece. Thus, the next proposition proves that the boundary values of γ on Γ can be determined from the knowledge of Λ_γ.

We first give a simple result needed for the proof:

Lemma 3.32. *If Φ is as in Proposition 3.30 and if b is a smooth function with* $\operatorname{supp}(b) \subset \tilde\Gamma \times [0,\delta]$*, then for any $k \geq 0$*

$$\lim_{s\to\infty} s^{k+1} \int_{\tilde\Gamma\times[0,\delta]} e^{2s\,\mathrm{Re}(\Phi(y))} y_n^k b(y)\,dy = k! \int_{\tilde\Gamma} \Big(-\frac{1}{2f_1(y')}\Big)^{k+1} b(y',0)\,dy'$$

Proof. ??? MS:This proof could also be left as a problem. ??? From Proposition 3.30 we have that $\mathrm{Re}(\Phi(y)) = y_n\psi(y', y_n)$ where $\psi \leq -\sigma$ in $\tilde{\Gamma} \times [0, \delta]$ and $\psi(y', 0) = f_1(y')$. Consequently, the change of variable $y_n = t/s$ shows that

$$s^{k+1} \int_{\tilde{\Gamma}\times[0,\delta]} e^{2s\,\mathrm{Re}(\Phi(y))} y_n^k b(y)\,dy = s^{k+1} \int_{\tilde{\Gamma}} \int_0^{\delta} e^{2sy_n\psi(y',y_n)} y_n^k b(y', y_n)\,dy'\,dy_n$$
$$= \int_{\tilde{\Gamma}} \int_0^{s\delta} e^{2t\psi(y',t/s)} t^k b(y', t/s)\,dy'\,dt$$

Since $\psi \leq -\sigma$ the integral is absolutely convergent, and we may apply dominated convergence to obtain

$$\lim_{s\to\infty} s^{k+1} \int_{\tilde{\Gamma}\times[0,\delta]} e^{2s\,\mathrm{Re}(\Phi(y))} y_n^k b(y)\,dy = \int_{\tilde{\Gamma}} \int_0^{\infty} e^{2t\psi(y',0)} t^k b(y', 0)\,dy'\,dt$$

The result follows by noting that for any $\lambda > 0$,

$$\int_0^{\infty} e^{-\lambda t} t^k\,dt = \lambda^{-k-1} \int_0^{\infty} e^{-t} t^k\,dt = k!\,\lambda^{-k-1}.$$

$\square$

Proposition 3.33. *We have*

$$\lim_{s\to\infty} s^{-1}\langle \Lambda_\gamma \phi_s, \bar{\phi}_s\rangle = -\int_{\tilde{\Gamma}} c(y', 0) f_1(y')\eta(y')^2\,dy'$$

Proof. Assume that $N > 1$ in the construction of $\tilde{v}_s$. Using that $u_s = v_s + r_s$, we have **??? MS:Reference for this identity? ???**

$$\langle \Lambda_\gamma \phi_s, \bar{\phi}_s\rangle = \int_\Omega \gamma\nabla u_s \cdot \nabla \bar{v}_s\,dx = \int_\Omega \gamma\nabla v_s \cdot \nabla \bar{v}_s\,dx + R_s$$

where $R_s = \int_\Omega \gamma\nabla r_s \cdot \nabla \bar{v}_s\,dx$ satisfies

$$|R_s| \leq C\,\|r_s\|_{H^1(\Omega)}\,\|v_s\|_{H^1(\Omega)} \leq Cs^{-N+2} \tag{3.32}$$

by (3.31) and the fact that $\|v_s\|_{H^1(\Omega)} \leq Cs^{1/2}$ which follows from (3.17). Lemma 3.28 implies that

$$\int_\Omega \gamma\nabla v_s \cdot \nabla \bar{v}_s\,dx = \int_{\tilde{\Gamma}\times[0,\delta]} \tilde{\gamma}\nabla\tilde{v}_s \cdot \overline{\nabla\tilde{v}_s}\,dy.$$

From the explicit form of $\tilde{v}_s$ in (3.19), we see that

$$\partial_j \tilde{v}_s = e^{s\Phi}\big(s(\partial_j\Phi)a_0 + r_{0,j}\big)$$

where $|r_{0,j}| \leq C$ uniformly in s. Combining all these facts, one has the identity

$$s^{-1}\langle \Lambda_\gamma \phi_s, \bar{\phi}_s\rangle = s\int_{\tilde{\Gamma}\times[0,\delta]} e^{2s\,\mathrm{Re}(\Phi)}(\tilde{\gamma}\nabla\Phi \cdot \nabla\bar{\Phi})|a_0|^2\,dy + \int_{\tilde{\Gamma}\times[0,\delta]} e^{2s\,\mathrm{Re}(\Phi)} q\,dy + s^{-1}R_s$$

where $|q| \leq C$ uniformly in s. By Lemma 3.32 for $k = 0$ the second term converges to zero as $s \to \infty$, and this is also true for the last term by (3.32) since $N > 1$. Thus by Lemma 3.32 again

$$\lim_{s\to\infty} s^{-1}\langle \Lambda_\gamma \phi_s, \bar{\phi}_s\rangle = -\frac{1}{2}\int_{\tilde{\Gamma}} \frac{(\tilde{\gamma}\nabla\Phi\cdot\nabla\bar{\Phi})|a_0|^2}{f_1}\Big|_{\tilde{\Gamma}}\, dy'$$

It remains to observe that

$$(\tilde{\gamma}\nabla\Phi\cdot\nabla\bar{\Phi})|a_0|^2|_{\tilde{\Gamma}} = c(y',0)\Big[|\partial_n\Phi(y',0)|^2 + \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta}(y',0)t_\alpha t_\beta\Big]\eta(y')^2$$

where the expression in brackets is equal to $2f_1(y')^2$. □

Since $\eta \in C_c^\infty(\tilde{\Gamma})$ can be chosen arbitrarily and f_1 is known, it follows from the previous result that $c(y',0)$ is determined by Λ_γ on Γ. Finally, because $c(y) = |\det DF|^{-1}\gamma|_{F^{-1}(y)}$ where F only depends on p and Ω and can therefore be considered as known, we recover $\gamma(F^{-1}(y',0))$ for $y' \in \tilde{\Gamma}$. Thus we have determined $\gamma|_\Gamma$.

The last result recovered the boundary value of γ. The simplest way to recover the normal derivatives of γ is by comparing Λ_γ to Λ_{γ^k} where the normal derivatives of γ and γ^k agree up to order $k-1$.

Proposition 3.34. *Let $k \geq 1$ and let $\gamma^k \in C^\infty(\overline{\Omega})$ be any positive function such that for $0 \leq j \leq k-1$,*

$$\left(\frac{\partial}{\partial\nu}\right)^j \gamma^k = \left(\frac{\partial}{\partial\nu}\right)^j \gamma \quad \text{on } \Gamma \tag{3.33}$$

and

$$\left(\frac{\partial}{\partial\nu}\right)^k \gamma^k = 0 \quad \text{on } \Gamma \tag{3.34}$$

Then

$$\lim_{s\to\infty} s^{k-1}\langle(\Lambda_\gamma - \Lambda_{\gamma^k})\phi_s, \bar{\phi}_s\rangle = \frac{1}{2}\int_{\tilde{\Gamma}} \left(-\frac{1}{2f_1(y')}\right)^{k-1} \frac{\partial_n^k(\gamma\circ F^{-1})(y',0)}{|\det DF(F^{-1}(y',0))|}\eta(y')^2\, dy'$$

Proof. Let $N > k+1$, let $u_s = v_s + r_s$ be the solution given in Lemma 3.31, and let $u_s^k = v_s^k + r_s^k$ be the corresponding solution in the case where γ is replaced by γ^k. Then we have $v_s^k = \tilde{v}_s^k \circ F$, where

$$\tilde{v}_s^k = e^{s\Phi}(a_0^k + s^{-1}a_{-1}^k + \ldots + s^{-N}a_{-N}^k)$$

is the function in Proposition 3.30 with c replaced by c^k, and

$$c^k(y) = \frac{\gamma^k(F^{-1}(y))}{|\det DF(F^{-1}(y))|}.$$

Note that Φ in Proposition 3.30 does not depend on c (see (3.22) and (3.24)), therefore Φ is the same both for $\tilde{v}_s$ and $\tilde{v}_s^k$. We also have $a_0|_{\tilde{\Gamma}} = a_0^k|_{\tilde{\Gamma}} = \eta$.

As in the proof of Proposition 3.33, we have **??? MS: Reference for this identity? ???**

$$\begin{aligned}\langle(\Lambda_\gamma - \Lambda_{\gamma^k})\phi_s, \bar{\phi}_s\rangle &= \int_\Omega (\gamma - \gamma^k)\nabla u_s \cdot \nabla \bar{u}_s^k \, dx \\ &= \int_{\tilde{\Gamma}\times[0,\delta]} (\tilde{\gamma} - \tilde{\gamma}^k)\nabla \tilde{v}_s \cdot \overline{\nabla \tilde{v}_s^k} \, dy + R_s\end{aligned}$$

where R_s involves r_s and r_s^k and $|R_s| \le Cs^{-N+2}$. Here

$$\tilde{\gamma} - \tilde{\gamma}^k = (c - c^k)\begin{bmatrix} h & 0 \\ 0 & 1\end{bmatrix}$$

By the condition (3.33) we have $\partial_n^j(c-c^k)|_{\tilde{\Gamma}} = 0$ for $0 \le j \le k-1$. Therefore **??? MS: Reference for this fact on Taylor series? ???**

$$(c - c^k)(y', y_n) = y_n^k \psi_k(y', y_n)$$

where ψ_k is a smooth function satisfying

$$\psi_k(y', 0) = \frac{\partial_n^k(c - c^k)(y', 0)}{k!} = \frac{1}{k!}\frac{\partial_n^k(\gamma \circ F^{-1})(y', 0)}{|\det DF(F^{-1}(y', 0))|} \tag{3.35}$$

by the Leibniz rule, (3.33), and (3.34). Using that

$$\begin{aligned}\partial_j \tilde{v}_s &= e^{s\Phi}(s(\partial_j\Phi)a_0 + r_{0,j}) \\ \partial_j \tilde{v}_s^k &= e^{s\Phi}(s(\partial_j\Phi)a_0^k + r_{0,j}^k)\end{aligned}$$

where $|r_{0,j}|$ and $|r_{0,j}^k|$ are bounded uniformly in s, it follows that

$$\begin{aligned}\langle(\Lambda_\gamma - \Lambda_{\gamma^k})\phi_s, \bar{\phi}_s\rangle = s^2 \int_{\tilde{\Gamma}\times[0,\delta]} & e^{2s\,\mathrm{Re}(\Phi)} y_n^k \psi_k \big[|\partial_n\Phi|^2 \\ &+ \sum_{\alpha,\beta=1}^{n-1} h^{\alpha,\beta}\partial_\alpha\Phi\partial_\beta\bar{\Phi}\big] a_0\bar{a}_0^k \, dy + s\int_{\tilde{\Gamma}\times[0,\delta]} e^{2s\,\mathrm{Re}(\Phi)} y_n^k q \, dy + R_s\end{aligned}$$

Here $|q| \le C$ uniformly in s, and Lemma 3.32 and the condition $N > k+1$ imply

$$\lim_{s\to\infty} s^{k-1}\left[s\int_{\tilde{\Gamma}\times[0,\delta]} e^{2s\,\mathrm{Re}(\Phi)} y_n^k q \, dy + R_s\right] = 0.$$

From Lemma 3.32 we conclude that

$$\lim_{s\to\infty} s^{k-1}\langle(\Lambda_\gamma - \Lambda_{\gamma^k})\phi_s, \bar{\phi}_s\rangle = k! \int_{\tilde{\Gamma}} \left(-\frac{1}{2f_1(y')}\right)^{k+1} \psi_k(y', 0)[2f_1(y')^2]\eta(y')^2 \, dy'$$

This finishes the proof by using (3.35). □

The main result of this section follows immediately.

Proof of Theorem 3.1. If Γ is as stated and p is a fixed point on Γ, then it is enough to determine $(\partial/\partial\nu)^l\gamma$ near p. Thus, by reducing Γ if necessary we may assume that we are working in the setting described in the beginning of this section.

If $l \geq 0$ is given, we choose $N > l+1$ and consider the solution u_s given in Lemma 3.31 such that the function $\phi_s = u_s|_{\partial\Omega}$ is supported in Γ. Since we have knowledge of $\Lambda_\gamma f|_\Gamma$ for f supported in Γ, we also know the quantities $\langle \Lambda_\gamma \phi_s, \bar{\phi}_s \rangle$ for $s \geq 1$. By Proposition 3.33 this determines γ on Γ by varying the function η. From this knowledge we can construct a conductivity γ^1 with $\gamma = \gamma^1$ near p, and then Proposition 3.34 allows to recover $(\partial/\partial\nu)\gamma$ near p. Continuing in this way, one finds the normal derivatives of γ up to order l near p and thus on all of Γ by varying p.

??? MS:The construction of γ^k given in this paragraph could be left as a problem. ??? The previous argument used that if $(\partial/\partial\nu)^j\gamma$ is known on Γ for $0 \leq j \leq k-1$, one can construct a smooth conductivity γ^k satisfying (3.33) and (3.34) near p. To see this, define the function

$$\tilde{\gamma}^k(y', y_n) = \sum_{j=0}^{k-1} \frac{\partial_{y_n}^j (\gamma \circ F^{-1})(y', 0)}{j!} y_n^j.$$

This function is known since it involves $(\partial/\partial\nu)^j\gamma|_\Gamma$ for $0 \leq j \leq k-1$. Also, since γ is positive in $\overline{\Omega}$, we have $\tilde{\gamma}^k \geq \varepsilon_0 > 0$ in $\tilde{\Gamma} \times [0, \delta_0]$ for some sufficiently small $\delta_0 > 0$. Let $\chi_1(y')$ and $\chi_2(y_n)$ be smooth cutoff functions such that $\chi_1(y') = 1$ for $|y'|$ small and $\operatorname{supp}(\chi_1) \subset \tilde{\Gamma}$, and $\chi_2(y_n)$ is 1 for $|y_n| \leq \delta_0/2$ and 0 for $|y_n| \geq \delta_0$. We then define $\chi(y', y_n) = \chi_1(y')\chi_2(y_n)$ and

$$\gamma^k(x) = \chi(F(x))\tilde{\gamma}^k(F(x)) + (1 - \chi(F(x))).$$

This gives a smooth positive function in $\overline{\Omega}$ for which (3.33) holds near p, as desired. $\square$

3.6. Stability

??? MS: Insert discussion about the norm of DN map (maybe also local DN map, since the proofs immediately give also local stability results). ??? A slightly more careful argument can be used to prove a stability estimate for the inverse problem at the boundary. For that purpose we define **??? ... ???**

The main stability result is as follows.

Theorem 3.35. *Let γ_1 and γ_2 be two positive functions in $C^\infty(\overline{\Omega})$. Then*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C_0 \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \tag{3.36}$$

Also
(3.37)
$$\left\|\frac{\partial}{\partial\nu}(\gamma_1-\gamma_2)\right\|_{L^\infty(\partial\Omega)} \le C_0 \left\|\Lambda_{\gamma_1}-\Lambda_{\gamma_1^1}-(\Lambda_{\gamma_2}-\Lambda_{\gamma_2^1})\right\|_{H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)}$$
The constant C_0 only depends on Ω, and γ_j^1 are as in Proposition 3.34.

The proof depends on two lemmas. The first one is of independent interest and states that even if the operators Λ_γ and Λ_1 (the Dirichlet–to–Neumann map for the constant conductivity) only map $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, the difference $\Lambda_\gamma-\gamma\Lambda_1$ has better regularity properties and maps $H^{1/2}(\partial\Omega)$ to itself. The same holds for $\Lambda_\gamma-\Lambda_{\gamma^1}$, which shows that the norm in (3.37) is well defined.

Lemma 3.36. *Let γ be a positive function in $C^\infty(\overline{\Omega})$, and let $\hat\gamma$ be another such function which satisfies $\hat\gamma|_{\partial\Omega}=\gamma|_{\partial\Omega}$. Then $\Lambda_\gamma-\gamma\Lambda_1$ and $\Lambda_\gamma-\Lambda_{\hat\gamma}$ are bounded operators from $H^{1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$.*

Proof. Let first $f\in C^\infty(\partial\Omega)$. We have **??? MS: Reference to the fact that $\Lambda_\gamma f=\gamma\partial u/\partial\nu$ if $f\in C^\infty$... ???**
$$(\Lambda_\gamma-\gamma\Lambda_1)f=\gamma\frac{\partial}{\partial\nu}(u-u_0)\Big|_{\partial\Omega}$$
where $u,u_0\in H^1(\Omega)$ solve the equations
$$\nabla\cdot\gamma\nabla u=0 \quad \textit{in } \Omega$$
$$\Delta u_0=0 \quad \textit{in } \Omega$$
and $u|_{\partial\Omega}=u_0|_{\partial\Omega}=f$. Then the function $w=u-u_0$ satisfies
$$\nabla\cdot\gamma\nabla w=-\nabla\cdot\gamma\nabla u_0=-\gamma\Delta u_0-\nabla\gamma\cdot\nabla u_0=-\nabla\gamma\cdot\nabla u_0$$
Thus $w\in H^1(\Omega)$ solves the Dirichlet problem
$$\begin{aligned}\nabla\cdot\gamma\nabla w&=-\nabla\gamma\cdot\nabla u_0 && \textit{in } \Omega\\ w&=0 && \textit{on } \partial\Omega\end{aligned}$$
Since $\nabla\gamma\cdot\nabla u_0\in L^2(\Omega)$, elliptic regularity **??? MS: Reference for elliptic regularity. ???** implies that $w\in H^2(\Omega)$ and
$$\|w\|_{H^2(\Omega)}\le C\,\|\nabla\gamma\cdot\nabla u_0\|_{L^2(\Omega)}\le C\,\|\gamma\|_{C^1(\overline{\Omega})}\,\|u_0\|_{H^1(\Omega)}\le C\,\|\gamma\|_{C^1(\overline{\Omega})}\,\|f\|_{H^{1/2}(\partial\Omega)}$$
Therefore
$$\begin{aligned}\|(\Lambda_\gamma-\gamma\Lambda_1)f\|_{H^{1/2}(\partial\Omega)}&=\left\|\gamma\frac{\partial w}{\partial\nu}\right\|_{H^{1/2}(\partial\Omega)}\le C\,\|\gamma\|_{C^1(\partial\Omega)}\left\|\frac{\partial w}{\partial\nu}\right\|_{H^{1/2}(\partial\Omega)}\le C\,\|w\|_{H^2(\Omega)}\\&\le C\,\|f\|_{H^{1/2}(\partial\Omega)}\end{aligned}$$
where C depends on γ. This is true for all $f\in H^{1/2}(\partial\Omega)$ by density.

Finally, we have

$$\Lambda_\gamma - \Lambda_{\hat\gamma} = (\Lambda_\gamma - \gamma\Lambda_1) - (\Lambda_{\hat\gamma} - \hat\gamma\Lambda_1)$$

and this operator also maps $H^{1/2}(\partial\Omega)$ to itself boundedly. $\square$

The next lemma concerns norm estimates for ϕ_s where the dependence on the choice of η is made explicit. We make the same assumptions as in the beginning of Section 3.5, and suppose that $p \in \partial\Omega$ is a boundary point with a neighborhood Γ in $\partial\Omega$ corresponding to a flat boundary piece $\tilde\Gamma$.

Lemma 3.37.

$$\|\phi_s\|_{H^{1/2}(\partial\Omega)} \le C_0\big(s^{1/2}\,\|\eta\|_{L^2(\mathbb{R}^{n-1})} + \|\eta\|_{H^{1/2}(\mathbb{R}^{n-1})}\big) \tag{3.38}$$

$$\|\phi_s\|_{H^{-1/2}(\partial\Omega)} \le C_0\big(s^{-1/2}\,\|\eta\|_{L^2(\mathbb{R}^{n-1})} + s^{-1}\,\|\eta\|_{H^{1}(\mathbb{R}^{n-1})}\big) \tag{3.39}$$

where C_0 only depends on Ω.

Proof. ??? MS:Are references for these Sobolev facts needed? ??? The function ϕ_s is supported in Γ which corresponds to $\tilde\Gamma$ in the change of coordinates F. The invariance of Sobolev norms under changes of coordinates implies $\|\phi_s\|_{H^\alpha(\partial\Omega)} \le C_0\,\|f_s\|_{H^\alpha(\mathbb{R}^{n-1})}$ where C_0 only depends of $\partial\Omega$ and α, and

$$f_s(y') = e^{isy'\cdot t'}\eta(y').$$

By definition, $\|f_s\|_{H^\alpha(\mathbb{R}^{n-1})} = (2\pi)^{-\frac{n-1}{2}}\left\|(1+|\xi'|^2)^{\alpha/2}\hat f_s\right\|_{L^2(\mathbb{R}^{n-1})}$. But we have $\hat f_s(\xi') = \hat\eta(\xi' - st')$. This satisfies

$$\left\|\hat f_s\right\|_{L^2(\mathbb{R}^{n-1})} = (2\pi)^{\frac{n-1}{2}}\,\|f_s\|_{L^2(\mathbb{R}^{n-1})} = (2\pi)^{\frac{n-1}{2}}\,\|\eta\|_{L^2(\mathbb{R}^{n-1})}$$

and, using the inequality $(a+b)^{1/2} \le a^{1/2} + b^{1/2}$ for $a, b \ge 0$,

$$\begin{aligned}
\left\||\xi'|^{1/2}\hat f_s\right\|_{L^2(\mathbb{R}^{n-1})} &\le \left\|(|\xi' - st'|^{1/2} + |st'|^{1/2})\hat\eta(\xi' - st')\right\|_{L^2(\mathbb{R}^{n-1})} \\
&\le \left\||z'|^{1/2}\hat\eta(z')\right\|_{L^2(\mathbb{R}^{n-1})} + s^{1/2}\,\|\hat\eta(z')\|_{L^2(\mathbb{R}^{n-1})} \\
&\le (2\pi)^{\frac{n-1}{2}}[\|\eta\|_{H^{1/2}(\mathbb{R}^{n-1})} + s^{1/2}\,\|\eta\|_{L^2(\mathbb{R}^{n-1})}]
\end{aligned}$$

These results imply (3.38). To show (3.39) we note that

$$
\begin{aligned}
(2\pi)^{\frac{n-1}{2}} \|f_s\|_{H^{-1/2}(\mathbb{R}^{n-1})} &= \left\| (1+|\xi'|^2)^{-1/4} \hat{\eta}(\xi' - st') \right\|_{L^2(\mathbb{R}^{n-1})} \\
&\le \Big(\int_{|\xi'|\le s/2} (1+|\xi'|^2)^{-1/2} \left|\hat{\eta}(\xi'-st')\right|^2 d\xi' \Big)^{1/2} \\
&\qquad + \Big(\int_{|\xi'|\ge s/2} (1+|\xi'|^2)^{-1/2} \left|\hat{\eta}(\xi'-st')\right|^2 d\xi' \Big)^{1/2} \\
&\le \Big(\int_{|\xi'|\le s/2} \left|\hat{\eta}(\xi'-st')\right|^2 d\xi' \Big)^{1/2} + (s/2)^{-1/2} \Big(\int_{|\xi'|\ge s/2} \left|\hat{\eta}(\xi'-st')\right|^2 d\xi' \Big)^{1/2}
\end{aligned}
$$

If $|\xi'| \le s/2$ then $|\xi' - st'| \ge s - |\xi'| \ge s/2$. Thus

$$
\Big(\int_{|\xi'|\le s/2} \left|\hat{\eta}(\xi'-st')\right|^2 d\xi' \Big)^{1/2} \le \frac{2}{s} \Big(\int_{\mathbb{R}^{n-1}} |z'|^2 \left|\hat{\eta}(z')\right|^2 dz' \Big)^{1/2}
$$

This proves (3.39). □

Proof of Theorem 3.35. Fix $N > 2$, $t' \in \mathbb{R}^{n-1}$ with $|t'| = 1$, and $\eta \in C_c^\infty(\tilde{\Gamma})$. Proposition 3.33 shows that for $j = 1, 2$ and for $s \ge 1$,

$$
s^{-1} \langle \Lambda_{\gamma_j} \phi_s, \bar{\phi}_s \rangle = - \int_{\tilde{\Gamma}} c_j(y', 0) f_1(y') \eta(y')^2 \, dy' + \varepsilon_j(s)
$$

where c_j corresponds to γ_j, and $\varepsilon_j(s) \to 0$ as $s \to \infty$. The estimate (3.38) implies

$$
\begin{aligned}
&\left| \int_{\tilde{\Gamma}} (c_1 - c_2)(y', 0) f_1(y') \eta(y')^2 \, dy' \right| \\
&\qquad \le \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}} s^{-1} \|\phi_s\|^2_{H^{1/2}(\partial\Omega)} + |\varepsilon_1(s)| + |\varepsilon_2(s)| \\
&\qquad \le C_0 \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}} \left(\|\eta\|^2_{L^2(\tilde{\Gamma})} + s^{-1} \|\eta\|^2_{H^{1/2}(\tilde{\Gamma})} \right) + |\varepsilon_1(s)| + |\varepsilon_2(s)|
\end{aligned}
$$

Taking the limit as $s \to \infty$, we obtain

$$
\left| \int_{\tilde{\Gamma}} (c_1 - c_2)(y', 0) f_1(y') \eta(y')^2 \, dy' \right| \le C_0 \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}} \left\| \eta^2 \right\|_{L^1(\tilde{\Gamma})}
$$

Since η is an arbitrary test function this implies

$$
\|(c_1 - c_2) f_1\|_{L^\infty(\tilde{\Gamma})} \le C_0 \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \to H^{-1/2}}
$$

The stability estimate (3.36) now follows since

$$
\left| (c_1 - c_2)(y', 0) f_1(y') \right| \ge \alpha_0 \left| (\gamma_1 - \gamma_2)(F^{-1}(y', 0)) \right|
$$

for some $\alpha_0 > 0$ which only depends on Ω.

The proof of (3.37) is analogous. Proposition 3.34 shows that

$$
\langle (\Lambda_{\gamma_j} - \Lambda_{\gamma_j^1}) \phi_s, \bar{\phi}_s \rangle = \frac{1}{2} \int_{\tilde{\Gamma}} \frac{\partial_n (\gamma_j \circ F^{-1})(y', 0)}{|\det DF(F^{-1}(y', 0))|} \eta(y')^2 \, dy' + \varepsilon_j(s)
$$

where $\varepsilon_j(s) \to 0$ as $s \to \infty$. Substracting these identities for $j = 1, 2$ gives

$$\begin{aligned}&\left|\frac{1}{2}\int_{\tilde{\Gamma}} \frac{\partial_n((\gamma_1-\gamma_2)\circ F^{-1})(y',0)}{|\det DF(F^{-1}(y',0))|}\eta(y')^2\,dy'\right| \\ &\qquad \leq \left\|\Lambda_{\gamma_1}-\Lambda_{\gamma_1^1}-(\Lambda_{\gamma_2}-\Lambda_{\gamma_2^1})\right\|_{H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)} \|\phi_s\|_{H^{1/2}(\partial\Omega)} \|\phi_s\|_{H^{-1/2}(\partial\Omega)} \\ &\qquad\qquad + |\varepsilon_1(s)| + |\varepsilon_2(s)|\end{aligned}$$

The estimates (3.38) and (3.39) show, upon taking $s \to \infty$, that

$$\begin{aligned}&\left|\frac{1}{2}\int_{\tilde{\Gamma}} \frac{\partial_n((\gamma_1-\gamma_2)\circ F^{-1})(y',0)}{|\det DF(F^{-1}(y',0))|}\eta(y')^2\,dy'\right| \\ &\qquad \leq C_0 \left\|\Lambda_{\gamma_1}-\Lambda_{\gamma_1^1}-(\Lambda_{\gamma_2}-\Lambda_{\gamma_2^1})\right\|_{H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)} \left\|\eta^2\right\|_{L^1(\tilde{\Gamma})}\end{aligned}$$

This implies (3.37). □

Later, when proving an interior stability result, we will need an alternative version of (3.37) which involves the $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ norm of the difference of Dirichlet-to-Neumann maps. We outline the proof as a problem.

Exercise 3.38. Let γ_1 and γ_2 be two positive functions in $C^\infty(\overline{\Omega})$, and assume that **??? MS: More details should be given here. This might be tricky... ???**

(i) $1/E \leq \gamma_j \leq E$,

(ii) $\|\gamma_j\|_{C^8(\overline{\Omega})} \leq E$.

There is $0 < \sigma < 1$ such that

$$\|(\partial/\partial\nu)(\gamma_1-\gamma_2)\|_{L^\infty(\partial\Omega)} \leq C\,\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|^{\sigma}_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$$

where C and σ only depend on Ω, E, and n.

(The proof of this result is more involved than that of Theorem 3.35 partly because one needs to consider behavior for large fixed s instead of just taking the limit as $s \to \infty$. For this reason one needs precise information on certain constants appearing in Proposition 3.30 and Lemma 3.32 ...)

3.7. Anisotropic conductivities

??? Note: This section should be in the chapter on anisotropic Calderón problem, after the discussion of boundary normal coordinates and Laplace-Beltrami operator. ???

The point is that the Laplace-Beltrami operator Δ_g in boundary normal coordinates looks like the operator $|g|^{-1/2} L_{\tilde{\gamma}}$ with anisotropic conductivity

$$\tilde{\gamma}(y) = c(y) \begin{bmatrix} h(y) & 0 \\ 0 & 1 \end{bmatrix}$$

where $c = |g|^{1/2}$ and $h^{\alpha,\beta} = g^{\alpha,\beta}$. We can then use the results in Sections 3.4 and 3.5, with the exception that here both the scalar function c and the matrix h depend on the unknown metric g. The proof of the boundary determination result needs to take this into account.

Theorem 3.39. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary, and let g be a Riemannian metric in $\overline{\Omega}$. Assume that p is a point on $\partial\Omega$, and let (y', y_n) be boundary normal coordinates for $(\overline{\Omega}, g)$ near p. If $n \geq 3$, then from the knowledge of Λ_g it is possible to determine $\partial_n^l g_{j,k}(y', 0)$ near p for any integer $l \geq 0$ and for all $1 \leq j, k \leq n$.*

We begin with some preparations. Let p be identified with 0, and suppose that the boundary normal coordinates (y', y_n) are defined in $B(0, 2r) \cap \{y_n \geq 0\}$. Let $\tilde{\Gamma} = B(0, r) \cap \{y_n = 0\}$, which is identified with the corresponding set in $\mathbb{R}^{n-1}$. In boundary normal coordinates, the inverse of the metric has the form

$$g^{-1}(y) = \begin{bmatrix} h(y) & 0 \\ 0 & 1 \end{bmatrix}$$

where $h(y) = \left[g^{\alpha,\beta}(y)\right]_{\alpha,\beta=1}^{n-1}$ is a symmetric positive definite matrix depending smoothly on y in $\tilde{\Gamma} \times [0, r]$. Then, if v is supported in $\tilde{\Gamma} \times [0, r]$, one has

$$\Delta_g v = |g|^{-1/2} L_{\tilde{\gamma}} v$$

where $\tilde{\gamma}$ is the anisotropic conductivity in $\tilde{\Gamma} \times [0, r]$ defined by

$$\tilde{\gamma}(y) = c(y) \begin{bmatrix} h(y) & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$c(y) = |g(y)|^{1/2}$$

We have now reduced matters to the situation in Sections 3.4 and 3.5. Choose a unit vector $t' \in \mathbb{R}^{n-1}$ and a function $\eta \in C_c^\infty(\tilde{\Gamma})$, and let ϕ_s be the explicit boundary value supported in $\tilde{\Gamma}$ and defined by

$$\phi_s(y', 0) = e^{isy' \cdot t'} \eta(y')$$

The next result states that certain quantities can be recovered from the boundary measurements Λ_g by using the method in Section 3.5.

Proposition 3.40. *In any dimension $n \geq 2$, one has*

$$\lim_{s\to\infty} s^{-1}\langle \Lambda_g \phi_s, \bar{\phi}_s\rangle = \int_{\tilde{\Gamma}} \Big[\sum_{\alpha,\beta=1}^{n-1} (|g|\, g^{\alpha,\beta})(y',0)t_\alpha t_\beta\Big]^{1/2} \eta(y')^2\, dy' \tag{3.40}$$

Further, if $l \geq 1$ and if g^l is any C^∞ Riemannian metric in $\overline{\Omega}$ so that in the (y', y_n) coordinates

$$g^l(y) = \begin{bmatrix} h^l(y) & 0 \\ 0 & 1 \end{bmatrix}$$

for some matrix h^l, and if for all $1 \leq j,k \leq n$

$$\partial_n^p(\left|g^l\right|^{1/2} (g^l)^{j,k}) = \partial_n^p(|g|^{1/2}\, g^{j,k}) \quad \text{on } \tilde{\Gamma} \text{ for } 0 \leq p \leq l-1 \tag{3.41}$$

$$\partial_n^l(\left|g^l\right|^{1/2} (g^l)^{j,k}) = 0 \qquad \text{on } \tilde{\Gamma} \tag{3.42}$$

then

(3.43)

$$\lim_{s\to\infty} s^{l-1}\langle(\Lambda_g - \Lambda_{g^l})\phi_s, \bar{\phi}_s\rangle$$
$$= \int_{\tilde{\Gamma}} \Big(-\frac{1}{2f_1(y')}\Big)^{l+1}\Big[\sum_{\alpha,\beta=1}^{n} (\partial_n^l(|g|^{1/2})g^{\alpha,\beta} + \partial_n^l(|g|^{1/2}\, g^{\alpha,\beta}))(y',0)t_\alpha t_\beta\Big]\eta(y')^2\, dy'$$

Proof. We first take $N > 1$ and let $\tilde{v}_s$ be the approximate solution in Proposition 3.30 with $\tilde{\gamma}^{j,k} = |g|^{1/2}\, g^{j,k}$, satisfying

$$\|\Delta_g \tilde{v}_s\|_{L^2(\tilde{\Gamma}\times[0,\delta])} = \left\||g|^{-1/2}\, L_{\tilde{\gamma}}\tilde{v}_s\right\|_{L^2(\tilde{\Gamma}\times[0,\delta])} \leq Cs^{-N+3/2}$$

Following the proof of Lemma 3.31, there is a solution $u_s = v_s + r_s$ of $\Delta_g u_s = 0$ in Ω with $u_s|_{\partial\Omega} = \phi_s$ and $\|r_s\|_{H^1(\Omega)} \leq Cs^{-N+3/2}$. Then, as in the proof of Proposition 3.33,

$$\begin{aligned}\langle \Lambda_g \phi_s, \bar{\phi}_s\rangle &= \int_\Omega \sum_{j,k=1}^n |g|^{1/2}\, g^{j,k}\partial_j u_s \partial_k \bar{v}_s\, dx \\ &= \int_{\tilde{\Gamma}\times[0,\delta]} \sum_{j,k=1}^n \tilde{\gamma}^{j,k}\partial_j \tilde{v}_s \partial_k \overline{\tilde{v}_s}\, dx + R_s\end{aligned}$$

where $|R_s| \leq Cs^{-N+2}$. The computation in the proof of Proposition 3.33 implies that

$$\begin{aligned}\lim_{s\to\infty} s^{-1}\langle \Lambda_g \phi_s, \bar{\phi}_s\rangle &= -\frac{1}{2}\int_{\tilde{\Gamma}} \frac{(\tilde{\gamma}\nabla\Phi\cdot\nabla\bar{\Phi})\,|a_0|^2}{f_1}\bigg|_{\tilde{\Gamma}} dy' \\ &= -\int_{\tilde{\Gamma}} c(y',0)f_1(y')\eta(y')^2\, dy'\end{aligned}$$

Since $c = |g|^{1/2}$, the expression for f_1 in Proposition 3.30 implies (3.40).

We move to the proof of (3.43), which proceeds in the same way as the proof of Proposition 3.34. Let g^l be the Riemannian metric in $\overline{\Omega}$ given above, and note that

$$\langle(\Lambda_g - \Lambda_{g^l})\phi_s, \bar{\phi}_s\rangle = \int_\Omega \sum_{j,k=1}^n |g|^{1/2}\, g^{j,k}\partial_j u_s \partial_k \bar{u}_s^l\, dx - \int_\Omega \sum_{j,k=1}^n \left|g^l\right|^{1/2} (g^l)^{j,k}\partial_j u_s \partial_k \bar{u}_s^l\, dx$$

where $\Delta_g u_s = 0$ and $\Delta_{g^l} u_s^l = 0$ in Ω and $u_s = u_s^l = \phi_s$ on $\partial\Omega$. Let $\tilde{v}_s$ be the approximate solution of Proposition 3.30 where $N > l+1$ and $\tilde{\gamma}^{j,k} = |g|^{1/2}\, g^{j,k}$, and let $\tilde{v}_s^l$ be the corresponding function where g is replaced by g^l. Then

$$\langle(\Lambda_g - \Lambda_{g^l})\phi_s, \bar{\phi}_s\rangle = \int_{\tilde{\Gamma}\times[0,\delta]} \sum_{j,k=1}^n (|g|^{1/2}\, g^{j,k} - \left|g^l\right|^{1/2} (g^l)^{j,k})\partial_j \tilde{v}_s \overline{\partial_k \tilde{v}_s^l}\, dy + R_s$$

with $|R_s| \le Cs^{-N+2}$. As a consequence of (3.41), we obtain that

$$\partial_n^p(|g|^{1/2}\, g^{j,k} - \left|g^l\right|^{1/2} (g^l)^{j,k})|_{\tilde{\Gamma}} = 0 \qquad \text{for } 0 \le p \le l-1$$

This implies

$$\langle(\Lambda_g - \Lambda_{g^l})\phi_s, \bar{\phi}_s\rangle = \int_{\tilde{\Gamma}\times[0,\delta]} \sum_{j,k=1}^n y_n^l \psi_l^{j,k}(y', y_n)\partial_j \tilde{v}_s \overline{\partial_k \tilde{v}_s^l}\, dy + R_s$$

where $\psi_l^{j,k}$ is a smooth function satisfying

$$\psi_l^{j,k}(y', 0) = \partial_n^l(|g|^{1/2}\, g^{j,k})(y', 0)/l!$$

Here we used (3.42). Then, as in Proposition 3.34,

$$\begin{aligned}&\lim_{s\to\infty} s^{l-1}\langle(\Lambda_g - \Lambda_{g^l})\phi_s, \bar{\phi}_s\rangle \\ &\qquad = l! \int_{\tilde{\Gamma}} \big(-\frac{1}{2f_1(y')}\big)^{l+1}\Big[\sum_{j,k=1}^n \psi_l^{j,k}(y', 0)\partial_j\Phi(y', 0)\partial_k\bar{\Phi}(y', 0)\Big]\eta(y')^2\, dy'\end{aligned}$$

The quantity in brackets is equal to

$$\frac{1}{l!}\Big[\partial_n^l(|g|^{1/2})(y', 0) f_1(y')^2 + \sum_{\alpha,\beta=1}^{n-1} \partial_n^l(|g|^{1/2}\, g^{\alpha,\beta})(y', 0)t_\alpha t_\beta\Big]$$

The identity (3.43) follows upon substituting the expression for f_1. □

The next step is to show that the information contained in (3.40) and (3.43) is sufficient to determine all derivatives of $g^{\alpha,\beta}$ on $\tilde{\Gamma}$. To do this, we use a basic identity concerning the derivative of the logarithm of a determinant.

Exercise 3.41. If $A(t)$ is a symmetric $n \times n$ invertible matrix depending smoothly on t in some interval, then

$$\frac{d}{dt}\big(\log\,\det(A(t))\big) = \sum_{j,k=1}^{n} \big(A(t)^{-1}\big)_{j,k} \frac{d}{dt} A(t)_{j,k}$$

Proposition 3.42. *Let $n \geq 3$ and $l \geq 1$. From the knowledge of*

$$g^{\alpha,\beta}\big|_{\tilde{\Gamma}},\ \partial_n g^{\alpha,\beta}\big|_{\tilde{\Gamma}},\ \ldots,\ \partial_n^{l-1} g^{\alpha,\beta}\big|_{\tilde{\Gamma}},\ \text{and } [\partial_n^l(|g|^{1/2}) g^{\alpha,\beta} + \partial_n^l(|g|^{1/2}\, g^{\alpha,\beta})]|_{\tilde{\Gamma}}$$

it is possible to determine $\partial_n^l g^{\alpha,\beta}|_{\tilde{\Gamma}}$.

Proof. Fix l and assume that one has knowledge of the stated quantities. The Leibniz rule implies that

$$\partial_n^l\big(\,|g|^{1/2}\,\big) g^{\alpha,\beta} + \partial_n^l\big(\,|g|^{1/2}\, g^{\alpha,\beta}\big) = |g|^{1/2}\, \partial_n^l g^{\alpha,\beta} + 2\partial_n^l\big(\,|g|^{1/2}\,\big) g^{\alpha,\beta} + T_{l-1}$$

where T_{l-1} denotes an expression depending on $\partial_n^j g^{\alpha,\beta}$ where $0 \leq j \leq l-1$ and $1 \leq \alpha, \beta \leq n-1$. Since

$$\partial_n(|g|^{1/2}) = \frac{1}{2}\,|g|^{-1/2}\,\partial_n(|g|)$$

we have again by the Leibniz rule

$$\begin{aligned} \partial_n^l(|g|^{1/2}) g^{\alpha,\beta} + \partial_n^l(|g|^{1/2}\, g^{\alpha,\beta}) &= |g|^{1/2}\, \partial_n^l g^{\alpha,\beta} + \partial_n^{l-1}\big(\,|g|^{1/2}\,\partial_n(\log\,|g|)\big) g^{\alpha,\beta} + T_{l-1} \\ &= |g|^{1/2}\Big[\partial_n^l g^{\alpha,\beta} - \partial_n^l\big(\log\,\big|g^{-1}\big|\big) g^{\alpha,\beta}\Big] + T_{l-1} \end{aligned}$$

The last expression is then known on $\tilde{\Gamma}$. Denoting the expression in brackets by $k^{\alpha,\beta}$, the fact that $|g|^{1/2}|_{\tilde{\Gamma}}$ and $T_{l-1}|_{\tilde{\Gamma}}$ are known implies that $k^{\alpha,\beta}|_{\tilde{\Gamma}}$ is also known. By Problem 3.41 we have in fact

$$\begin{aligned} k^{\alpha,\beta} &= \partial_n^l g^{\alpha,\beta} - \partial_n^{l-1}\Big(\sum_{\gamma,\delta=1}^{n-1} g_{\gamma,\delta}\partial_n g^{\gamma,\delta}\Big) g^{\alpha,\beta} \\ &= \partial_n^l g^{\alpha,\beta} - \Big(\sum_{\gamma,\delta=1}^{n-1} g_{\gamma,\delta}\partial_n^l g^{\gamma,\delta}\Big) g^{\alpha,\beta} + T_{l-1} \end{aligned}$$

Since $\sum_{\alpha,\beta=1}^{n-1} g_{\alpha,\beta} g^{\alpha,\beta} = n-1$, it follows that

$$\sum_{\alpha,\beta=1}^{n-1} g_{\alpha,\beta} k^{\alpha,\beta} = (2-n) \sum_{\gamma,\delta=1}^{n-1} g_{\gamma,\delta}\partial_n^l g^{\gamma,\delta} + T_{l-1}.$$

Therefore (using that $n \geq 3$)

$$\partial_n^l g^{\alpha,\beta} = k^{\alpha,\beta} + \frac{1}{2-n}\Big(\sum_{\gamma,\delta=1}^{n-1} g_{\gamma,\delta} k^{\gamma,\delta}\Big) g^{\alpha,\beta} + T_{l-1}$$

Since $k^{\alpha,\beta}|_{\tilde{\Gamma}}$ was known, this determines $\partial_n^l g^{\alpha,\beta}|_{\tilde{\Gamma}}$ as required. $\square$

The main result now follows from Propositions 3.40 and 3.42.

Proof of Theorem 3.39. We use boundary normal coordinates (y', y_n) to identify p with 0 as before, and denote by $\tilde{\Gamma}$ a flat neighborhood of 0 in $\partial\Omega$. Since $g_{\alpha,n} = 0$ for $1 \le \alpha \le n-1$ and $g_{n,n} = 1$, it is enough to determine $\partial_n^l g_{\alpha,\beta}|_{\tilde{\Gamma}}$ for all $l \ge 0$ and $1 \le \alpha, \beta \le n-1$. Further, to determine the matrix $(g_{\alpha,\beta})|\tilde{\Gamma}$ it is enough to determine the inverse matrix $(g^{\alpha,\beta})|_{\tilde{\Gamma}}$. For higher order derivatives the identity

$$\sum_{\beta=1}^{n} g_{\alpha,\beta} g^{\beta,\gamma} = \delta_\alpha^\gamma$$

and the Leibniz rule show that

$$\sum_{\beta=1}^{n} \left(\partial_n^l g_{\alpha,\beta}\right) g^{\beta,\gamma} = T_\alpha^\gamma$$

where T_α^γ contains terms depending on $\partial_n^j g_{\alpha,\beta}$ for $0 \le j \le l-1$ and $\partial_n^j g^{\beta,\gamma}$ for $0 \le j \le l$. Consequently

$$\partial_n^l g_{\alpha,\delta} = \sum_{\beta,\gamma=1}^{n} \left(\partial_n^l g_{\alpha,\beta}\right) g^{\beta,\gamma} g_{\gamma,\delta} = \sum_{\gamma=1}^{n} T_\alpha^\gamma g_{\gamma,\delta}$$

This shows that it is sufficient to prove that Λ_g determines $\partial_n^l g^{\alpha,\beta}|_{\tilde{\Gamma}}$ for all l and α, β.

By (3.40), since the test function η can be chosen arbitrarily, it follows that Λ_g determines

$$\sum_{\alpha,\beta=1}^{n-1} \left(|g|\, g^{\alpha,\beta}\right)(y', 0) t_\alpha t_\beta$$

Also the unit vector $t' \in \mathbb{R}^{n-1}$ was arbitrary, and therefore we determine for all $1 \le \alpha, \beta \le n-1$

$$\left|g(y', 0)\right| g^{\alpha,\beta}(y', 0)$$

The $(n-1)\times(n-1)$ matrix $|g|\,(g^{\alpha,\beta})$ has determinant $|g|^{n-1}\,|g|^{-1} = |g|^{n-2}$, which is also known on $\tilde{\Gamma}$. Since $n \ge 3$ we know $|g|$ on $\tilde{\Gamma}$ and thus also

$$g^{\alpha,\beta}|_{\tilde{\Gamma}} = |g|^{-1}\,(|g|\, g^{\alpha,\beta})|_{\tilde{\Gamma}}.$$

This determines the boundary values of $g^{\alpha,\beta}$ on $\tilde{\Gamma}$. For higher order derivatives, we note that (3.43) shows upon varying η and t' that Λ_g determines on $\tilde{\Gamma}$ the quantity

$$\partial_n^l(|g|^{1/2}) g^{\alpha,\beta} + \partial_n^l(|g|^{1/2}\, g^{\alpha,\beta})$$

Taking $l = 1$ and using that $g^{\alpha,\beta}|_{\tilde{\Gamma}}$ was known, Proposition 3.42 shows that $\partial_n g^{\alpha,\beta}|_{\tilde{\Gamma}}$ is determined by Λ_g. Proceeding inductively, we recover $\partial_n^l g^{\alpha,\beta}|_{\tilde{\Gamma}}$ for all $l \ge 0$ and $1 \le \alpha, \beta \le n-1$. $\square$

3.8. Notes

Section 3.2. The treatment is based on an unpublished argument due to Russell Brown, whom we would like to thank for making this argument available to us.

Chapter 4

The Calderón problem in three and higher dimensions

In this chapter, we prove interior uniqueness, stability, and reconstruction results for the Calderón problem in dimensions three and higher. To describe the contents of this chapter, we assume that Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, with C^2 boundary, and all conductivities are positive functions in $C^2(\overline{\Omega})$.

The linearized version of the Calderón problem is considered in Section 4.1, and we present an argument showing uniqueness in the linearized problem. The proof is based on special harmonic functions given by the complex exponentials $e^{i\zeta\cdot x}$, where $\zeta \in \mathbb{C}^n$ is a vector satisfying $\zeta \cdot \zeta = 0$, In Section 4.2, we construct complex geometrical optics solutions that resemble the harmonic exponentials, and use these in Section 4.3 to prove that if two isotropic conductivities γ_1 and γ_2 in Ω give rise to the same boundary measurements, then $\gamma_1 = \gamma_2$ throughout Ω. This follows from the corresponding identifiability result for Schrödinger operators.

4.1. The linearized Calderón problem

The "plane wave" exponential function

$$u = e^{x\cdot\zeta}, \qquad \zeta \in \mathbb{C}^n,$$

is a solution to Laplace's equation,

$$\Delta u = 0,$$

if and only if

$$\zeta \cdot \zeta = 0.$$

If $\zeta = \eta + ik$ with $\eta, k \in \mathbb{R}^n$, then

$$0 = \zeta \cdot \zeta = |\eta|^2 - |k|^2 + 2i\eta \cdot k \iff |\eta| = |k| \text{ and } \eta \perp k. \tag{4.1}$$

So any non–zero ζ obeying $\zeta \cdot \zeta = 0$ will have non–zero real and imaginary parts and the corresponding solution, $e^{x \cdot \zeta}$, will grow or decay exponentially in some directions in $\mathbb{R}^n$ and will oscillate in some directions. The utility of exponentially growing solutions in solving the inverse conductivity problem was first observed by Calderón, and we begin by exhibiting his proof of injectivity of the linearized inverse boundary value problem.

Theorem 4.1 (Uniqueness of the linearized Calderón problem)**.** *The Fréchet derivative of Λ at $\gamma = 1$, $\delta\gamma \mapsto D\Lambda_1[\delta\gamma]$, is injective. That is, if*

$$D\Lambda_1[\delta\gamma] = 0$$

for some $\delta\gamma \in L^\infty(\Omega)$, then

$$\delta\gamma = 0.$$

Let us first show that the Fréchet derivative exists.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\gamma \in L^\infty(\Omega)$ satisfy $\gamma \geq c > 0$ a.e. in Ω. For $\delta\gamma \in L^\infty(\Omega)$, the identity*

$$\langle D\Lambda_\gamma[\delta\gamma] f, g\rangle_{\partial\Omega} = \int_\Omega \delta\gamma \nabla u \cdot \nabla v \, dx, \qquad f, g \in H^{1/2}(\partial\Omega),$$

where $u, v \in H^1(\Omega)$ satisfy $\operatorname{div}(\gamma\nabla u) = \operatorname{div}(\gamma \nabla v) = 0$ in Ω with $u|_{\partial\Omega} = f$, $v|_{\partial\Omega} = g$, defines a bounded linear map

$$D\Lambda_\gamma[\delta\gamma] : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega).$$

This map gives the Fréchet derivative of the map $\gamma \mapsto \Lambda_\gamma$ in the sense that

$$\lim_{t \to 0} \frac{1}{t}(\Lambda_{\gamma + t\delta\gamma} - \Lambda_\gamma) = D\Lambda_\gamma[\delta\gamma]$$

in the space of bounded operators from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

Proof. Let $\gamma(t) = \gamma_t$ be a C^1 curve of functions in $L^\infty(\Omega)$ that are uniformly bounded from below. This means that $1/E \leq \gamma(t) \leq E$ in Ω for t near 0, and

$$\gamma(t) = \gamma(0) + t\dot\gamma(0) + t\varepsilon(t)$$

where $\dot\gamma(0) \in L^\infty(\Omega)$ and

$$\lim_{t\to 0} \|\varepsilon(t)\|_{L^\infty(\Omega)} = 0.$$

Let $f, g \in H^{1/2}(\partial\Omega)$, and let u_t and v_t satisfy

$$\operatorname{div}(\gamma_t \nabla u_t) = \operatorname{div}(\gamma_t \nabla v_t) = 0 \text{ in } \Omega, \quad u_t|_{\partial\Omega} = f, v_t|_{\partial\Omega} = g.$$

By the definition of DN maps and symmetry of Λ_0,

$$\begin{aligned}\langle(\Lambda_{\gamma_t}-\Lambda_{\gamma_0})f,g\rangle_{\partial\Omega} &= \langle\Lambda_{\gamma_t}f,g\rangle_{\partial\Omega}-\langle f,\Lambda_{\gamma_0}g\rangle_{\partial\Omega}\\ &= \int_\Omega(\gamma_t-\gamma_0)\nabla u_t\cdot\nabla v_0\,dx.\end{aligned}$$

Consequently

$$\langle(\frac{1}{t}\Lambda_{\gamma_t}-\Lambda_{\gamma_0})f,g\rangle_{\partial\Omega}=\int_\Omega(\dot\gamma(0)+\varepsilon(t))\nabla u_t\cdot\nabla v_0\,dx. \tag{4.2}$$

Using the bounds for γ_t, we have

$$\|u_t\|_{H^1(\Omega)}\le C(E,\Omega)\,\|f\|_{H^{1/2}(\partial\Omega)}\,.$$

Also, since u_t-u_0 solves

$$\begin{aligned}\operatorname{div}(\gamma_t\nabla(u_t-u_0)) &= -\operatorname{div}((\gamma_t-\gamma_0)\nabla u_0)\text{ in }\Omega,\\ u_t-u_0|_{\partial\Omega} &= 0,\end{aligned}$$

we have

$$\begin{aligned}\|u_t-u_0\|_{H^1(\Omega)} &\le C(E,\Omega)\,\|\operatorname{div}((\gamma_t-\gamma_0)\nabla u_0)\|_{H^{-1}(\Omega)}\\ &\le C(E,\Omega)\,\|\gamma_t-\gamma_0\|_{L^\infty(\Omega)}\,\|\nabla u_0\|_{L^2(\Omega)}\\ &\le C(E,\Omega)\,\|\gamma_t-\gamma_0\|_{L^\infty(\Omega)}\,\|f\|_{H^{1/2}(\Omega)}\,.\end{aligned}$$

Since $\|\gamma_t-\gamma_0\|_{L^\infty(\Omega)}\le Ct$, we may combine these facts and take the limit in (4.2) as $t\to 0$ to obtain

$$\langle D\Lambda_{\gamma(0)}[\dot\gamma(0)]f,g\rangle_{\partial\Omega}=\int_\Omega\dot\gamma(0)\nabla u_0\cdot\nabla v_0\,dx.$$

This defines $D\Lambda_{\gamma(0)}[\dot\gamma(0)]f$ weakly as an element of $H^{-1/2}(\partial\Omega)$, since

$$\begin{aligned}\left|\langle D\Lambda_{\gamma(0)}[\dot\gamma(0)]f,g\rangle_{\partial\Omega}\right| &\le \|\dot\gamma(0)\|_{L^\infty(\Omega)}\,\|\nabla u_0\|_{L^2(\Omega)}\,\|\nabla v_0\|_{L^2(\Omega)}\\ &\le \|\dot\gamma(0)\|_{L^\infty(\Omega)}\,\|f\|_{H^{1/2}(\partial\Omega)}\,\|g\|_{H^{1/2}(\partial\Omega)}\,.\end{aligned}$$

It also follows that $D\Lambda_{\gamma(0)}[\dot\gamma(0)]$ is bounded from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. Finally, $D\Lambda_{\gamma(0)}[\dot\gamma(0)]$ is the Fréchet derivative since by (4.2) one has

$$\begin{aligned}&\langle\left[\frac{1}{t}(\Lambda_{\gamma_t}-\Lambda_{\gamma_0})-D\Lambda_{\gamma(0)}[\dot\gamma(0)]\right]f,g\rangle_{\partial\Omega}\\ &=\int_\Omega\dot\gamma(0)(\nabla u_t-\nabla u_0)\cdot\nabla v_0\,dx+\int_\Omega\varepsilon(t)\nabla u_t\cdot\nabla v_0\,dx.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\langle\left[(\frac{1}{t}\Lambda_{\gamma_t}-\Lambda_{\gamma_0})-D\Lambda_{\gamma(0)}[\dot\gamma(0)]\right]f,g\rangle_{\partial\Omega}\right|\\&\le C(\|\nabla u_t-\nabla u_0\|_{L^2(\Omega)}\|v_0\|_{H^1(\Omega)}+\|\varepsilon(t)\|_{L^\infty(\Omega)}\|u_t\|_{H^1(\Omega)}\|v_0\|_{H^1(\Omega)})\\&\le C(t+\|\varepsilon(t)\|_{L^\infty(\Omega)})\|f\|_{H^{1/2}(\partial\Omega)}\|g\|_{H^{1/2}(\partial\Omega)}\,.\end{aligned}$$

The result follows by taking $\gamma(t)=\gamma+t\delta\gamma$. □

Proof. of Theorem 4.1 By Theorem 4.2, the equation

$$D\Lambda_\gamma[\delta\gamma]=0$$

is equivalent to
(4.3)

$$\int_\Omega \delta\gamma\nabla u_1\cdot\nabla u_2\,dx=0 \qquad \text{for all } u_j\in H^1(\Omega) \text{ obeying } \operatorname{div}(\gamma\nabla u_j)=0.$$

If we further restrict to $\gamma=1$ then (4.3) must hold for every pair of harmonic functions u_1 and u_2. A natural set of choices for u_1 and u_2 are exponentials $e^{x\cdot\zeta}$ with $\zeta\cdot\zeta=0$. Substituting

$$u_1=e^{x\cdot\zeta_1},\qquad u_2=e^{x\cdot\zeta_2}$$

with $\zeta_j\cdot\zeta_j=0$, into (4.3) gives

$$\zeta_1\cdot\zeta_2\int_\Omega e^{x\cdot(\zeta_1+\zeta_2)}\delta\gamma\,dx=0$$

By (4.1), we may choose $\zeta_1=\frac{1}{2}(\eta+ik)$ and $\zeta_2=\frac{1}{2}(-\eta+ik)$ with any $k,\eta\in\mathbb{R}^n$ for which $|k|=|\eta|$ and $k\perp\eta$. Then

$$\zeta_1+\zeta_2=ik \qquad\text{and}\qquad \zeta_1\cdot\zeta_2=-\tfrac{1}{2}|k|^2$$

so that $D\Lambda_1[\delta\gamma]=0$ implies that

$$|k|^2\int_\Omega e^{ix\cdot k}\delta\gamma\,dx=0$$

which, in turn, implies that the Fourier transform $(\widehat{\chi_\Omega\delta\gamma})(k)$ vanishes for every nonzero k. Here χ_Ω denotes the characteristic function of the set Ω. However, $\chi_\Omega\delta\gamma$ is an element of $L^2(\mathbb{R}^n)$, so that $\widehat{\chi_\Omega\delta\gamma}$ is in $L^2(\mathbb{R}^n)$ and therefore cannot be supported at a single point. As a consequence

$$\chi_\Omega\delta\gamma=0$$

which proves that $D\Lambda_1[\,\cdot\,]$ is injective. □

4.2. Complex geometrical optics solutions: first proof

The approach that we will use to prove identifiability in Section 4.3 is based on exponential solutions that are perturbations of those for the Laplacian. These solutions have many names, including exponentially growing solutions, Faddeev type solutions, and Sylvester-Uhlmann type solutions. We will call these solutions *complex geometrical optics (CGO) solutions*, since they are a complex phase analogue of the standard geometrical optics solutions for wave equations.

In this section, we will construct CGO solutions to the Schrödinger equation

$$(-\Delta + q)u = 0 \quad \text{in } \Omega.$$

The potential q is assumed to be in $L^\infty(\Omega)$. To motivate the construction, first let $q = 0$. We have seen that there are solutions to the equation $-\Delta u = 0$ having the form of a complex exponential at frequency $\zeta \in \mathbb{C}^n$,

$$u(x) = e^{i\zeta\cdot x}, \quad \zeta \cdot \zeta = 0.$$

Now suppose q is nonzero. The function $u = e^{i\zeta\cdot x}$ is not an exact solution of $(-\Delta + q)u = 0$ anymore, but we can find solutions which resemble complex exponentials. These are the CGO solutions, which have the form

$$u(x) = e^{i\zeta\cdot x}(1 + r(x, \zeta)). \tag{4.4}$$

Here r is a *correction term* which is needed to convert the approximate solution $e^{i\zeta\cdot x}$ to an exact solution.

In fact, we are interested in solutions in the *asymptotic limit* as $|\zeta| \to \infty$. This follows the principle that it is usually not possible to obtain explicit formulas for solutions to variable coefficient equations, but in suitable asymptotic limits explicit expressions for solutions may exist.

The next theorem is the main result on the existence of CGO solutions. Note that the constant function $a \equiv 1$ always satisfies the transport equation $\zeta \cdot \nabla a = 0$, so as a special case one obtains the solutions $u = e^{i\zeta\cdot x}(1 + r)$ mentioned above.

Theorem 4.3. *(CGO solutions) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $q \in L^\infty(\Omega)$. There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max(C_0 \|q\|_{L^\infty(\Omega)}, 1)$, and for any function $a \in H^2(\Omega)$ satisfying*

$$\zeta \cdot \nabla a = 0 \quad \text{in } \Omega,$$

the equation $(-\Delta + q)u = 0$ in Ω has a solution $u \in H^2(\Omega)$ of the form

$$u(x) = e^{i\zeta\cdot x}(a + r), \tag{4.5}$$

where $r \in H^2(\Omega)$ satisfies

$$\|r\|_{H^k(\Omega)} \leq C_0 \, |\zeta|^{k-1} \, \|(-\Delta + q)a\|_{L^2(\Omega)} \, , \qquad k = 0, 1, 2.$$

We note that (4.4) is a solution of $(-\Delta + q)u = 0$ if and only if

$$e^{-i\zeta\cdot x}(-\Delta + q)e^{i\zeta\cdot x}(1 + r) = 0. \tag{4.6}$$

It will be convenient to conjugate the exponentials $e^{i\zeta\cdot x}$ into the Laplacian. By this we mean that

$$e^{-i\zeta\cdot x} D_j(e^{i\zeta\cdot x} v) = (D_j + \zeta_j)v,$$
$$e^{-i\zeta\cdot x} D^2(e^{i\zeta\cdot x} v) = (D + \zeta)^2 v = (D^2 + 2\zeta \cdot D)v.$$

We can rewrite (4.6) as

$$(D^2 + 2\zeta \cdot D + q)(1 + r) = 0.$$

This implies the following equation for r:

$$(D^2 + 2\zeta \cdot D + q)r = -q. \tag{4.7}$$

The solvability of (4.7) is the most important step in the construction of CGO solutions. We proceed in several steps.

4.2.1. Basic estimate. We first consider the free case in which there is no potential on the left hand side of (4.7).

Theorem 4.4. *There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, and for any $f \in L^2(\Omega)$, the equation*

$$(D^2 + 2\zeta \cdot D)r = f \quad \text{in } \Omega \tag{4.8}$$

has a solution $r \in H^2(\Omega)$ satisfying

$$\|r\|_{H^k(\Omega)} \leq C_0 \, |\zeta|^{k-1} \, \|f\|_{L^2(\Omega)} \, , \quad k = 0, 1, 2.$$

Note in particular that

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|} \, \|f\|_{L^2(\Omega)} \, .$$

If $\|f\|_{L^2(\Omega)}$ is uniformly bounded in ζ, this shows that $r \to 0$ in $L^2(\Omega)$ as $|\zeta| \to \infty$. Accordingly, the correction term r in the CGO solution (4.4) will be very small for large ζ, and the solution (4.4) will look like the complex exponential $e^{i\zeta\cdot x}$ then.

The idea of the proof is that (4.7) is a linear equation with constant coefficients, so one can try to solve it by the Fourier transform. Since $(D_j u)\,\hat{}\,(\xi) = \xi_j \hat{u}(\xi)$, the Fourier transformed equation is

$$(\xi^2 + 2\zeta \cdot \xi)\hat{r}(\xi) = \hat{f}(\xi).$$

We would like to divide by $\xi^2 + 2\zeta \cdot \xi$ and use the inverse Fourier transform to get a solution r. However, the symbol $\xi^2 + 2\zeta \cdot \xi$ vanishes for some $\xi \in \mathbb{R}^n$, and the division cannot be done directly.

It turns out that we can divide by the symbol if we use Fourier series in a large cube instead of the Fourier transform, and moreover take the Fourier coefficients in a shifted lattice instead of the usual integer coordinate lattice.

Proof of Theorem 4.4. 1. Write $\zeta = s(\omega_1 + i\omega_2)$ where $s = |\zeta|/\sqrt{2}$ and ω_1 and ω_2 are orthogonal unit vectors in $\mathbb{R}^n$. By rotating coordinates in a suitable way, we can assume that $\omega_1 = e_1$ and $\omega_2 = e_2$ (the first and second coordinate vectors). Thus we need to solve the equation

$$(D^2 + 2s(D_1 + iD_2))r = f.$$

2. We assume for simplicity that Ω is contained in the cube $Q = (-\pi, \pi)^n$. Extend f by zero outside Ω into Q, which gives a function in $L^2(Q)$ also denoted by f. We need to solve

$$(D^2 + 2s(D_1 + iD_2))r = f \quad \text{in } Q. \tag{4.9}$$

Let $w_k(x) = e^{i(k+\frac{1}{2}e_2)\cdot x}$ for $k \in \mathbb{Z}^n$. That is, we consider Fourier series in the lattice $\mathbb{Z}^n + \frac{1}{2}e_2$. Writing

$$(u, v) = (2\pi)^{-n} \int_Q u\bar{v}\, dx, \quad u, v \in L^2(Q),$$

we see that $(w_k, w_l) = 0$ if $k \neq l$ and $(w_k, w_k) = 1$, so $\{w_k\}$ is an orthonormal set in $L^2(Q)$. It is also complete: if $v \in L^2(Q)$ and $(v, w_k) = 0$ for all $k \in \mathbb{Z}^n$ then $(ve^{-\frac{1}{2}ix_2}, e^{ik\cdot x}) = 0$ for all $k \in \mathbb{Z}^n$, which implies $v = 0$.

3. Hilbert space theory gives that f can be written as the series $f = \sum_{k\in\mathbb{Z}^n} f_k w_k$, where $f_k = (f, w_k)$ and $\|f\|^2_{L^2(Q)} = \sum_{k\in\mathbb{Z}^n} |f_k|^2$. Seeking also r in the form $r = \sum_{k\in\mathbb{Z}^n} r_k w_k$, and using that

$$Dw_k = (k + \frac{1}{2}e_2)w_k,$$

the equation (4.9) implies that the Fourier coefficients should satisfy

$$p_k r_k = f_k, \quad k \in \mathbb{Z}^n,$$

where

$$p_k := (k + \frac{1}{2}e_2)^2 + 2s(k_1 + i(k_2 + \frac{1}{2})).$$

Note that $\mathrm{Im}(p_k) = 2s(k_2 + \frac{1}{2})$ is never zero for $k \in \mathbb{Z}^n$, which was the reason for considering the shifted lattice. We define

$$r_k := \frac{1}{p_k} f_k$$

and

$$r^{(N)} := \sum_{k \in \mathbb{Z}^n, |k| \leq N} r_k w_k.$$

The Fourier coefficients satisfy

$$|r_k| \leq \frac{1}{|p_k|} |f_k| \leq \frac{1}{|\mathrm{Im}(p_k)|} |f_k| \leq \frac{1}{\left|2s(k_2 + \frac{1}{2})\right|} |f_k| \leq \frac{1}{s} |f_k| .$$

Now $(r^{(N)})$ is a Cauchy sequence in $L^2(Q)$, since for $M \geq N$

$$\|r_M - r_N\|_{L^2(Q)} = \Big(\sum_{N < |k| \leq M} |r_k|^2 \Big)^{1/2} \leq \frac{1}{s} \Big(\sum_{N < |k| \leq M} |f_k|^2 \Big)^{1/2}$$

and the last quantity can be made arbitrarily small if M and N are large. This shows that $r^{(N)} \to r$ in $L^2(Q)$, where $r \in L^2(Q)$ satisfies

$$r = \sum_{k \in \mathbb{Z}^n} r_k w_k, \quad \|r\|_{L^2(Q)} \leq \frac{1}{s} \|f\|_{L^2(Q)} .$$

4. We next show that $r \in H^2(Q)$. Note that

$$D_a r^{(N)} = \sum_{|k| \leq N} (k + \frac{1}{2} e_2)_a r_k w_k, \quad D_a D_b r^{(N)} = \sum_{|k| \leq N} (k + \frac{1}{2} e_2)_a (k + \frac{1}{2} e_2)_b r_k w_k$$

We claim that for $a, b = 1, \ldots, n$ and for $k \in \mathbb{Z}^n$,

$$\left|(k + \frac{1}{2} e_2)_a r_k\right| \leq 4 |f_k| , \qquad \left|(k + \frac{1}{2} e_2)_a (k + \frac{1}{2} e_2)_b r_k\right| \leq 16 s |f_k| . \tag{4.10}$$

Consider two cases: if $\left|k + \frac{1}{2} e_2\right| \leq 4s$ (the small frequency case) we have

$$\left|(k + \frac{1}{2} e_2)_a r_k\right| \leq \frac{4s}{2s |k_2 + 1/2|} |f_k| \leq 4 |f_k|$$

and

$$\left|(k + \frac{1}{2} e_2)_a (k + \frac{1}{2} e_2)_b r_k\right| \leq \frac{(4s)^2}{2s |k_2 + 1/2|} \leq 16 s |f_k| .$$

If $\left|k + \frac{1}{2} e_2\right| \geq 4s$ (the large frequency case) then

$$\left| \left|k + \frac{1}{2} e_2\right|^2 + 2 s k_1 \right| \geq \left|k + \frac{1}{2} e_2\right|^2 - 2s \left|k + \frac{1}{2} e_2\right| \geq \frac{1}{2} \left|k + \frac{1}{2} e_2\right|^2$$

which implies

$$\left|(k + \frac{1}{2} e_2)_a r_k\right| \leq \frac{\left|k + \frac{1}{2} e_2\right|}{\frac{1}{2} \left|k + \frac{1}{2} e_2\right|^2} |f_k| \leq \frac{1}{2s} |f_k|$$

and

$$\left|(k + \frac{1}{2} e_2)_a (k + \frac{1}{2} e_2)_b r_k\right| \leq \frac{\left|k + \frac{1}{2} e_2\right|^2}{\frac{1}{2} \left|k + \frac{1}{2} e_2\right|^2} |f_k| \leq 2 |f_k| .$$

This proves (4.10). The estimates (4.10) imply that $D_a r^{(N)}$ and $D_a D_b r^{(N)}$ are Cauchy sequences in $L^2(Q)$, and thus converge to some v_a and v_{ab} in $L^2(Q)$. Further, the weak derivatives of r satisfy

$$D_a r = v_a, \quad D_a D_b r = v_{ab}.$$

To see this, let $\varphi \in C_c^\infty(Q)$ and note that

$$\begin{aligned} -\int_Q r D_a \varphi \, dx &= -\lim_{N\to\infty} \int_\Omega r^{(N)} D_a \varphi \, dx \\ &= \lim_{N\to\infty} \int_\Omega (D_a r^{(N)}) \varphi \, dx \\ &= \int_\Omega v_a \varphi \, dx. \end{aligned}$$

The proof for $D_a D_b r$ is analogous.

5. We have proved that $r \in H^2(Q)$ and that

$$D_a r = \sum_{k\in\mathbb{Z}^n} (k + \frac{1}{2} e_2)_a r_k w_k, \quad D_a D_b r = \sum_{k\in\mathbb{Z}^n} (k + \frac{1}{2} e_2)_a (k + \frac{1}{2} e_2)_b r_k w_k.$$

It immediately follows that r solves (4.9), and the norm estimates for r follow from (4.10) and the Parseval identity. □

4.2.2. Basic estimate with potential. Now we consider the solution of (4.7) in the presence of a nonzero potential. It will be convenient to give a name to the solution operator in the free case.

Definition 4.5. Let $\zeta \in \mathbb{C}^n$ satisfy $\zeta \cdot \zeta = 0$ and $|\zeta|$ sufficiently large. We denote by G_ζ the solution operator

$$G_\zeta : L^2(\Omega) \to H^2(\Omega), \quad f \mapsto r,$$

where r is the solution to $(D^2 + 2\zeta \cdot D) r = f$ provided by Theorem 4.4.

Theorem 4.6. *Let $q \in L^\infty(\Omega)$. There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max(C_0 \|q\|_{L^\infty(\Omega)}, 1)$, and for any $f \in L^2(\Omega)$, the equation*

$$(D^2 + 2\zeta \cdot D + q) r = f \quad \text{in } \Omega \tag{4.11}$$

has a solution $r \in H^2(\Omega)$ satisfying

$$\|r\|_{H^k(\Omega)} \leq C_0 |\zeta|^{k-1} \|f\|_{L^2(\Omega)}, \quad k = 0, 1, 2.$$

Proof. If one has $q = 0$, a solution would be given by $r = G_\zeta f$. Here q may be nonzero, so we try a solution of the form

$$r := G_\zeta \tilde{f}, \tag{4.12}$$

where $\tilde{f} \in L^2(\Omega)$ is a function to be determined. Inserting (4.12) in the equation (4.11), and using that $(D^2 + 2\zeta \cdot D)G_\zeta = I$, we see that $\tilde{f}$ should satisfy

$$(I + qG_\zeta)\tilde{f} = f \quad \text{in } \Omega. \tag{4.13}$$

By Theorem 4.4, we have the norm estimate

$$\|qG_\zeta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{C_0 \|q\|_{L^\infty(\Omega)}}{|\zeta|}.$$

If $|\zeta| \geq \max(2C_0 \|q\|_{L^\infty(\Omega)}, 1)$ then

$$\|qG_\zeta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{1}{2}.$$

It follows that $I + qG_\zeta$ is an invertible operator on $L^2(\Omega)$, and the inverse is given by a Neumann series

$$(I + qG_\zeta)^{-1} = \sum_{j=0}^{\infty} (-qG_\zeta)^j.$$

The equation (4.13) has a solution

$$\tilde{f} := (I + qG_\zeta)^{-1} f.$$

The definition (4.12) for r implies

$$(D^2 + 2\zeta \cdot D + q)r = \tilde{f} + qG_\zeta \tilde{f} = (I + qG_\zeta)\tilde{f} = f,$$

and r indeed solves the equation (4.11). Since $\left\|(I + qG_\zeta)^{-1}\right\|_{L^2(\Omega)\to L^2(\Omega)} \leq 2$, we have $\left\|\tilde{f}\right\|_{L^2(\Omega)} \leq 2\|f\|_{L^2(\Omega)}$. The norm estimates in Theorem 4.4 imply the desired estimates for r, if we replace C_0 by $2C_0$. □

It is now easy to prove the main result on the existence of CGO solutions.

Proof. of Theorem 4.3 The function (4.5) is a solution of $(-\Delta + q)u = 0$ if and only if

$$e^{-i\zeta\cdot x}(-\Delta + q)e^{i\zeta\cdot x}(a + r) = 0.$$

As in the beginning of this section, we conjugate the exponentials into the derivatives and rewrite (4.6) as

$$(D^2 + 2\zeta \cdot D + q)(a + r) = 0.$$

Since $\zeta \cdot Da = 0$, this implies the following equation for r:

$$(D^2 + 2\zeta \cdot D + q)r = -(D^2 + q)a.$$

Theorem 4.6 guarantees the existence of a solution r satisfying the norm estimates above. Then (4.5) is the required solution to $(-\Delta + q)u = 0$ in Ω. □

Exercise 4.7. (Small first order perturbations) Prove the analogue of Theorem 4.6 for the equation

$$(D^2 + 2\zeta \cdot D + 2A \cdot (D + \zeta) + q)r = f \quad \text{in } \Omega$$

where $A \in L^\infty(\Omega\,; \mathbb{C}^n)$, $q \in L^\infty(\Omega)$, and $\|A\|_{L^\infty(\Omega)}$ is sufficiently small (depending on Ω and n).

Exercise 4.8. (CGO solutions for small first order perturbations) Prove the analogue of Theorem 4.3 for the equation

$$(-\Delta + 2A \cdot D + q)u = 0 \quad \text{in } \Omega$$

where $A \in L^\infty(\Omega\,; \mathbb{C}^n)$, $q \in L^\infty(\Omega)$, and $\|A\|_{L^\infty(\Omega)}$ is sufficiently small. In this case, the transport equation for a is

$$\zeta \cdot (D + A)a = 0 \quad \text{in } \Omega.$$

Exercise 4.9. (H^k to H^{k+2} mapping properties) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^k boundary. Show that there is a constant C_0 such that for any $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, there is a solution $r \in H^{k+2}(\Omega)$ of the equation

$$(D^2 + 2\zeta \cdot D)r = f \quad \text{in } \Omega$$

satisfying

$$\|r\|_{H^{k+l}(\Omega)} \leq C_0 \left|\zeta\right|^{l-1} \|f\|_{H^k(\Omega)}\,, \quad l = 0, 1, 2.$$

(Hint: use the fact that there is a continuous extension operator $H^k(\Omega) \to H^k_{comp}(\mathbb{R}^n)$.)

Exercise 4.10. (Additional decay for the H^1 norm) If f is a *fixed* function in $L^2(\Omega)$, show that for any $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$ there is a solution $r = r(\,\cdot\,; \zeta) \in H^2(\Omega)$ of the equation

$$(D^2 + 2\zeta \cdot D)r = f \quad \text{in } \Omega$$

satisfying

$$\lim_{|\zeta| \to \infty} \|r(\,\cdot\,; \zeta)\|_{H^1(\Omega)} = 0.$$

(Hint: decompose f into a smooth part and a small part.)

4.3. Interior uniqueness

In the first part of this section, we use the special solutions constructed in 4.2 together with the boundary identifiability result of 4.1 to prove a global identifiability result for dimension $n \geq 3$. This result is originally due to Sylvester and Uhlmann ([S-U II]). The case $n = 2$ will be considered in **??**. In the second part of this section, we extend the main ideas of the proof of the identifiability result in order to establish a result concerning the stable dependence of the conductivity on the boundary measurements. The main identifiability result is

Theorem 4.11. *(Interior uniqueness for Calderón problem) Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with C^2 boundary, and let γ_1, γ_2 be positive functions in $C^2(\overline{\Omega})$. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

We will obtain Theorem 4.11 as a consequence of the analogous theorem for the Schrödinger equation. We have seen in that the Dirichlet problem for the Schrödinger equation need not always have a unique solution, and the DN map may not exist. It is quite natural to use the Cauchy data set, introduced already in Definition 2.68:

$$C_q = \left\{ \ (u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) \ \middle| \ u \in H^1(\Omega), (-\Delta + q)u = 0 \text{ in } \Omega \ \right\}.$$

Here, the normal derivative of a solution of $(-\Delta+q)u = 0$ in Ω is interpreted as an element of $H^{-1/2}(\partial\Omega)$ as in Problem 2.67, and the Cauchy data set is a subset of $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$.

Theorem 4.12. *(Interior uniqueness for Schrödinger equation) Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set, and let $q_1, q_2 \in L^\infty(\Omega)$. If*

$$C_{q_1} = C_{q_2},$$

then

$$q_1 = q_2.$$

If 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω, the Cauchy data set C_q is just the graph of the DN map Λ_q (Problem 2.69). Therefore, the previous theorem has an immediate corollary for the DN map:

Theorem 4.13. *(Interior uniqueness for Schrödinger equation) Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set, let $q_1, q_2 \in L^\infty(\Omega)$, and assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q_1$ or $-\Delta + q_2$ in Ω. If*

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

Let us now give the proofs of these results.

Proof of Theorem 4.12. Since $C_{q_1} = C_{q_2}$, we know from Problem 2.73 that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0 \tag{4.14}$$

for *any* $H^1(\Omega)$ solutions u_j to the equations $(-\Delta + q_j)u_j = 0$, $j = 1, 2$. (In the case of DN maps, so that $\Lambda_{q_1} = \Lambda_{q_2}$, the same relation was proved in

Theorem 2.72.) Thus, to prove that $q_1 = q_2$, it is enough to establish that products $u_1 u_2$ of such solutions are dense in $L^1(\Omega)$.

Fix $\xi \in \mathbb{R}^n$. We would like to choose the solutions in such a way that $u_1 u_2$ is close to $e^{ix\cdot\xi}$, since the functions $e^{ix\cdot\xi}$ form a dense set. We begin by taking unit vectors ω_1 and ω_2 in $\mathbb{R}^n$ such that $\{\omega_1, \omega_2, \xi\}$ is an orthogonal set (here we need that $n \geq 3$). Let

$$\zeta = s(\omega_1 + i\omega_2),$$

so that $\zeta \cdot \zeta = 0$. By Theorem 4.3, if s is sufficiently large there exist H^1 solutions u_1 and u_2 which satisfy $(-\Delta + q_j)u_j = 0$, and which are of the form

$$u_1 = e^{i\zeta\cdot x}(e^{ix\cdot\xi} + r_1),$$
$$u_2 = e^{-i\zeta\cdot x}(1 + r_2),$$

where $\|r_j\|_{L^2(\Omega)} \leq C/s$ for $j = 1, 2$. For the first solution we chose $a = e^{ix\cdot\xi}$ which satisfies $\zeta \cdot \nabla a = (\zeta \cdot \xi)e^{ix\cdot\xi} = 0$ by orthogonality, and for the second solution we chose a to be constant.

Inserting these solutions in (4.14), we obtain

$$\int_\Omega (q_1 - q_2)(e^{ix\cdot\xi} + r_1)(1 + r_2)\, dx = 0. \tag{4.15}$$

In this identity, only r_1 and r_2 depend on s. Since the L^2 norms of r_1 and r_2 are bounded by C/s, it is possible to take the limit as $s \to \infty$ in (4.15), and by Cauchy-Schwarz the terms involving r_1 and r_2 will vanish. This shows that

$$\int_\Omega (q_1 - q_2)e^{ix\cdot\xi}\, dx = 0.$$

This holds for every $\xi \in \mathbb{R}^n$. If $\tilde{q}$ is the function in $L^1(\mathbb{R}^n)$ which is equal to $q_1 - q_2$ in Ω and vanishes outside Ω, the last identity implies that the Fourier transform of $\tilde{q}$ vanishes for every frequency $\xi \in \mathbb{R}^n$. Consequently $\tilde{q} = 0$, and $q_1 = q_2$ in Ω. □

Proof. of Theorem 4.11 Since $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, Theorems 3.3 and 3.17 imply that $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$ and $\partial_\nu\gamma_1|_{\partial\Omega} = \partial_\nu\gamma_2|_{\partial\Omega}$. Also, part (b) of Theorem 2.74 guarantees that

$$\Lambda_{q_j} f = \gamma_j^{-1/2}\Lambda_{\gamma_j}\big(\gamma_j^{-1/2} f\big) + \tfrac{1}{2}\gamma_j^{-1}(\partial_\nu\gamma_j)f\big|_{\partial\Omega}$$

for all f, where q_1 and q_2 are defined by

$$q_j = \frac{\Delta\gamma_j^{1/2}}{\gamma_j^{1/2}}.$$

Thus we have

$$\Lambda_{q_1} = \Lambda_{q_2},$$

and Theorem 4.13 implies that $q_1 = q_2$. Consequently, the function

$$v = \log\big(\frac{\gamma_1}{\gamma_2}\big) = 2\big[\log \gamma_1^{\frac{1}{2}} - \log \gamma_2^{\frac{1}{2}}\big]$$

satisfies

$$\begin{aligned} \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} \nabla v\big) &= 2\nabla \cdot \big[\gamma_2^{\frac{1}{2}} \nabla \gamma_1^{\frac{1}{2}} - \gamma_1^{\frac{1}{2}} \nabla \gamma_2^{\frac{1}{2}}\big] \\ &= 2(\gamma_1\gamma_2)^{\frac{1}{2}}(q_2 - q_1) = 0 \\ v\big|_{\partial\Omega} &= 0 \end{aligned}$$

Internal remark 1.

$$\begin{aligned} \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} \nabla v\big) &= \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} 2\nabla \log\big(\frac{\gamma_1^{1/2}}{\gamma_2^{1/2}}\big)\big) \\ &= \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} 2\big(\gamma_1^{-1/2}\nabla\gamma_1^{1/2} - \gamma_2^{-1/2}\nabla\gamma_2^{1/2}\big)\big) \\ &= 2\nabla \cdot \big(\gamma_2^{1/2}\nabla\gamma_1^{1/2} - \gamma_1^{1/2}\nabla\gamma_2^{1/2}\big) \\ &= 2\big(\nabla\gamma_2^{1/2} \cdot \nabla\gamma_1^{1/2} + \gamma_2^{1/2}\Delta\gamma_1^{1/2} \\ &- \nabla\gamma_1^{1/2} \cdot \nabla\gamma_2^{1/2} - \gamma_1^{1/2}\Delta\gamma_2^{1/2}\big) \\ &= 2\big(\gamma_2^{1/2}\Delta\gamma_1^{1/2} - \gamma_1^{1/2}\Delta\gamma_2^{1/2}\big) \\ &= 2(\gamma_1\gamma_2)^{\frac{1}{2}}(-q_1 + q_2) \end{aligned}$$

and hence

$$v \equiv 0 \text{ in } \Omega$$

i.e., $\gamma_1 = \gamma_2$ in Ω. □

4.4. Stability

A somewhat more carefully crafted version of the uniqueness proof can be used to prove the stable dependence of γ on Λ_γ. By stability, or stable dependence, as opposed to continuous dependence, we mean that, under the hypothesis of an *à priori* bound for γ_1 and γ_2 (or q_1 and q_2) in a high norm, we can estimate the difference, $\gamma_1 - \gamma_2$ (or $q_1 - q_2$), in a lower norm in terms of the difference of the Dirichlet– to Neumann–data maps (or the Cauchy data). The stable dependence results presented here are, except for minor modifications, due to Alessandrini ([Al2]). To measure the distance between the Dirichlet– to Neumann–data maps we use the operator norm for bounded operators between $H^{1/2}$ and $H^{-1/2}$. To measure the distance between the spaces of Cauchy data we use

$$\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) = \max\Big\{ \sup_{(f,g)\in\mathcal{C}_{q_1}} \inf_{(\tilde f,\tilde g)\in\mathcal{C}_{q_2}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}}, \\ \sup_{(f,g)\in\mathcal{C}_{q_2}} \inf_{(\tilde f,\tilde g)\in\mathcal{C}_{q_1}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}}\Big\}$$

The norm on the space $H^{1/2}(\Omega)\oplus H^{-1/2}(\Omega)$ is defined by the expression

$$\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}} = \big(\|f\|^2_{\frac12,\Omega} + \|g\|^2_{-\frac12,\Omega}\big)^{1/2}$$

It is not difficult to see that if the spaces $\mathcal{C}_{q_j}$ are both graphs of corresponding Dirichlet– to Neumann–data maps Λ_{q_j}, then one has the estimates

(4.16)

$$\frac{\|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12}}{\sqrt{1+\|\Lambda_{q_1}\|^2_{\frac12,-\frac12}}\sqrt{1+\|\Lambda_{q_2}\|^2_{\frac12,-\frac12}}} \le \operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) \le \|\Lambda_{q_1}-\Lambda_{q_2}\|_{\frac12,-\frac12}$$

Exercise 4.14. Prove (4.16).

We first show

Proposition 4.15. *Suppose that $\frac{n}{2} < s \in \mathbb{N}$, $n \ge 3$ and*

$$\|q_j\|_{s,\Omega} \le M \tag{4.17}$$

then there exists $C = C(M)$ and $0<\sigma=\sigma(n)<1$ such that

$$\|q_1-q_2\|_{-1,\Omega} \le C\Big(\big|\log\big\{dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2})\big\}\big|^{-\sigma} + dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2})\Big) \tag{4.18}$$

Proof. Our point of departure is the identity in Problem 2.73, which states that

$$\int_\Omega (q_1-q_2)u_1u_2\,dx = -\int_{\partial\Omega}\big(u_2\frac{\partial u_1}{\partial\nu} - u_1\frac{\partial u_2}{\partial\nu}\big)\,dS$$

for all $u_1,u_2\in H^1(\Omega)$ obeying $(\Delta+q_1)u_1=0$ and $(\Delta+q_2)u_2=0$. If (f,g) is an arbitrary element of $\mathcal{C}_{q_1}$ then there exists a function $v\in H^1(\Omega)$ obeying

$$\begin{aligned} &\Delta v + q_1 v = 0 && \text{in } \Omega\\ &v = f \text{ and } \frac{\partial v}{\partial\nu} = g && \text{on } \partial\Omega \end{aligned}$$

so that

$$\begin{aligned} 0 = \int_\Omega (q_1-q_1)u_1 v\; d^nx &= -\int_{\partial\Omega}\big(v\frac{\partial u_1}{\partial\nu} - u_1\frac{\partial v}{\partial\nu}\big)\,d\sigma\\ &= -\int_{\partial\Omega}\big(f\frac{\partial u_1}{\partial\nu} - u_1 g\big)\,d\sigma \end{aligned}$$

and

$$\int_\Omega (q_1 - q_2) u_1 u_2 \; d^n x = - \int_{\partial\Omega} \Big[\frac{\partial u_1}{\partial \nu}(u_2 - f) - \big(\frac{\partial u_2}{\partial \nu} - g\big) u_1 \Big] \; d^n x$$

We continue with

(4.19)

$$\begin{aligned}\Big| \int_\Omega (q_1 - q_2) u_1 u_2 \; d^n x \Big| &\le \Big\| \frac{\partial u_1}{\partial \nu} \Big\|_{-\frac12, \partial\Omega} \| u_2 - f \|_{\frac12, \partial\Omega} + \| u_1 \|_{\frac12, \partial\Omega} \Big\| \frac{\partial u_2}{\partial \nu} - g \Big\|_{-\frac12, \partial\Omega} \\ &\le \Big\| \big(u_1, \frac{\partial u_1}{\partial \nu}\big) \Big\|_{H^{\frac12} \oplus H^{-\frac12}} \; \Big\| \big(u_2 - f, \frac{\partial u_2}{\partial \nu} - g\big) \Big\|_{H^{\frac12} \oplus H^{-\frac12}}\end{aligned}$$

As this is true for all $(f, g) \in \mathcal{C}_{q_1}$,

(4.20)

$$\begin{aligned}\Big| \int_\Omega (q_1 - q_2) u_1 u_2 \; d^n x \Big| &\le \Big\| \big(u_1, \frac{\partial u_1}{\partial \nu}\big) \Big\|_{H^{\frac12} \oplus H^{-\frac12}} \inf_{(f,g) \in \mathcal{C}_{q_1}} \Big\| \big(u_2 - f, \frac{\partial u_2}{\partial \nu} - g\big) \Big\|_{H^{\frac12} \oplus H^{-\frac12}} \\ &\le \Big\| \big(u_1, \frac{\partial u_1}{\partial \nu}\big) \Big\|_{H^{\frac12} \oplus H^{-\frac12}} \cdot \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \; \cdot \; \Big\| \big(u_2, \frac{\partial u_2}{\partial \nu}\big) \Big\|_{H^{\frac12} \oplus H^{-\frac12}}\end{aligned}$$

We remark in passing that, if $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$ are actually the graphs of Dirichlet–to Neumann–data maps Λ_{q_1} and Λ_{q_2}, then (4.20) implies

$$\begin{aligned}&\Big| \int_\Omega (q_1 - q_2) u_1 u_2 \; d^n x \Big| \\ &\quad \le \| u_1 \|_{\frac12, \partial\Omega} \cdot \Big(1 + \| \Lambda_{q_1} \|^2_{\frac12, -\frac12} \Big)^{\frac12} \cdot \| \Lambda_{q_1} - \Lambda_{q_2} \|_{\frac12, -\frac12} \cdot \| u_2 \|_{\frac12, \partial\Omega} \Big(1 + \| \Lambda_{q_2} \|^2_{\frac12, -\frac12} \Big)^{\frac12}\end{aligned}$$

Our next step is to choose u_1 and u_2 to be the solutions produced in Theorem 4.3. That is

$$u_j = e^{x \cdot \zeta_j} \big(1 + \psi_j(x, \zeta_j) \big) \tag{4.21}$$

with

$$\begin{aligned}\zeta_1 &= l + i\big(\frac{k}{2} + m\big) \\ \zeta_2 &= -l + i\big(\frac{k}{2} - m\big)\end{aligned}$$

where k is arbitrary and l and m satisfy the requirements

(4.22)

$$l \cdot k = l \cdot m = k \cdot m = 0 \quad |m|^2 = |l|^2 - \frac{|k|^2}{4} > 0 \quad |l| > \frac{1}{\epsilon} \max_{j=1,2} \big\| (1 + |x|^2)^{\frac12} q_j \big\|_{L^\infty}$$

The functions ψ_j satisfy the estimates

$$\| \psi_j \|_{L^2(\Omega)} \le \frac{C}{|\zeta_j|} \| q_j \|_{L^2(\Omega)} \qquad \text{and } \| \psi_j \|_{1,\Omega} \le C \| q_j \|_{L^2(\Omega)} \tag{4.23}$$

Since $u_j \in H^1(\Omega)$ are solutions to $\Delta u_j + q_j u_j = 0$ (with q_j bounded in L^∞) it follows, by Problem 2.67 and Problem **??**, that

$$\|\frac{\partial u_j}{\partial \nu}\|_{-\frac{1}{2},\partial\Omega} \leq C\|u_j\|_{1,\Omega}$$

Using Theorem **??**, Lemma **??** and (4.23) we now get

$$\big\|\big(u_1, \frac{\partial u_1}{\partial \nu}\big)\big\|_{H^{\frac{1}{2}}\oplus H^{-\frac{1}{2}}} \leq C\|u_1\|_{1,\Omega} \leq C\|e^{x\cdot\zeta_1}\|_{C^1(\Omega)}\,\|1+\psi_1\|_{1,\Omega} \leq C|\zeta_1|e^{|\zeta_1|D}$$

where D denotes the constant $D = \sup_{x\in\Omega}|x|$ and we have increased the value of the constant C a few times. Thus, for any fixed $D_* > D$

$$\big\|\big(u_1, \frac{\partial u_1}{\partial \nu}\big)\big\|_{H^{\frac{1}{2}}\oplus H^{-\frac{1}{2}}} \leq Ce^{D_*|\zeta_1|} \tag{4.24}$$

and similarly

$$\big\|\big(u_2, \frac{\partial u_2}{\partial \nu}\big)\big\|_{H^{\frac{1}{2}}\oplus H^{-\frac{1}{2}}} \leq Ce^{D_*|\zeta_2|} \tag{4.25}$$

Let r denote the parameter $r = \big(\frac{|k|^2}{4} + |m|^2 + |l|^2\big)^{\frac{1}{2}} - |k|$. In terms of r we have that $|\zeta_1| = |\zeta_2| = |k| + r$. The parameter r must be sufficiently large, *i.e.*,

$$r \geq C \gg 1$$

but is otherwise free. A combination of (4.20)–(4.25) now yields

$$\begin{aligned}
&\Big|\int_\Omega (q_1 - q_2)e^{ix\cdot k}\, d^n x\Big| \\
&\qquad \leq Ce^{2D_*(|k|+r)}\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + \int_\Omega |q_1 - q_2|\,\big|\psi_1 \\
&\qquad + \psi_2 + \psi_1\psi_2\big|\, d^n x \\
&\qquad \leq Ce^{2D_*(|k|+r)}\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + \|q_1 - q_2\|_{L^2(\Omega)}\big[\|\psi_1\|_{L^2(\Omega)} + \|\psi_2\|_{L^2(\Omega)}\big] \\
&\qquad\qquad + \|q_1 - q_2\|_{L^\infty(\Omega)}\|\psi_1\|_{L^2(\Omega)}\|\psi_2\|_{L^2(\Omega)}
\end{aligned}$$

or, by use of (4.23), (4.17) and the Sobolev imbedding result, Problem **??**,

$$\big|(\tilde{q}_1 - \tilde{q}_2)\widehat{\ }(k)\big| \leq C\Big(e^{2D_*(|k|+r)}\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + (M+1)^3\frac{1}{|k|+r}\Big)$$

where, as before, $\tilde{q}_j$ denotes the extension of q_j by zero outside Ω. We therefore have

(4.26)
$$
\begin{aligned}
&\mathrm{v}\tilde{q}_1 - \tilde{q}_2\mathrm{v}^{2-1,n}\\
&= \int_{\mathbb{R}^n} \left|(\tilde{q}_1 - \tilde{q}_2)\hat{\ }(k)\right|^2 (1+|k|^2)^{-1}\, d^n k\\
&\leq \int_{|k|<\rho} |(\tilde{q}_1 - \tilde{q}_2)\hat{\ }(k)|^2 (1+|k|^2)^{-1}\, d^n k + \int_{|k|>\rho} \left|(\tilde{q}_1 - \tilde{q}_2)\hat{\ }(k)\right|^2 (1+\rho^2)^{-1}\, d^n k\\
&\leq C\rho^n \Big(e^{4D_*(\rho+r)} \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^2 + \frac{1}{r^2}\Big) + \frac{1}{1+\rho^2}\|\tilde{q}_1 - \tilde{q}_2\|^2_{L^2(\Omega)}\\
&\leq C\rho^n e^{4D_*(\rho+r)} \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^2 + C\frac{\rho^n}{r^2} + \frac{C}{\rho^2}
\end{aligned}
$$

In order to make the last two terms in the final expression of (4.26) small and of the same magnitude ($\frac{1}{\rho^2}$), we choose

$$r = \rho^{\frac{n+2}{2}}, \quad \text{for } \rho \gg 1$$

With this choice we also have $r > \rho$. For the first term in the last line of (4.26) we get

$$\rho^n e^{4D_*(\rho+r)} \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^2 \leq C e^{Kr} \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^2 \tag{4.27}$$

uniformly in ρ, for any fixed constant $K > 8D_*$. If we now choose

$$\rho = \Big(\frac{1}{K}\big|\log\{\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}\big|\Big)^{\frac{2}{n+2}}$$

then

$$r = \frac{1}{K}\big|\log\{\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}\big|$$

and therefore

$$e^{Kr} = \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^{-1} \qquad \text{for } \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < 1 \tag{4.28}$$

A combination of the estimates (4.27) and (4.28) gives

$$\rho^n e^{4D_*(\rho+r)} \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})^2 \leq C\ \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \tag{4.29}$$

provided $\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < 1$. Insertion of (4.29) and the definition of ρ (and r) into the last line of (4.26) yields the estimate

(4.30)
$$
\begin{aligned}
\|q_1 - q_2\|^2_{-1,\Omega} \leq \mathrm{v}\tilde{q}_1 - \tilde{q}_2\mathrm{v}^2_{-1,n} &\leq C\left(\big|\log\{\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}\big|^{-\frac{4}{n+2}} + \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\right)\\
&\leq C\big|\log\{\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}\big|^{-\frac{4}{n+2}}
\end{aligned}
$$

for $\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < \frac{1}{2}$. This gives (4.18) with $\sigma = \frac{2}{n+2}$ when $\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) < \frac{1}{2}$. The estimate (4.18) is trivially satisfied for $\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \geq \frac{1}{2}$ because of the assumption (4.17). This completes the proof of Proposition 4.15. $\square$

We now proceed to transform Proposition 4.15 into an analogous result for the conductivity problem. As we saw in the proof of the interior identifiability theorem, the proof of the interior stable dependence result makes use of the continuous dependence result (Theorem 4.62) for the boundary values. Among other things the proof depends on the following lemma.

Lemma 4.16. *Suppose that $\frac{n}{2} < s \in \mathbb{N}$ and that γ_1 and γ_2 are isotropic conductivities on $\Omega \subset \mathbb{R}^n$ satisfying Hypothesis 4.40 and*

(i) $\frac{1}{E} \leq \gamma_j \leq E$

(ii) $\|\gamma_j\|_{s+2,\Omega} \leq E$.

Let q_1 and q_2 denote the potentials defined by

$$q_j = -\frac{\Delta \gamma_j^{1/2}}{\gamma_j^{1/2}} \tag{4.31}$$

There exists $C = C(\Omega, E, n, s)$ and $0 < \sigma = \sigma(s) < 1$ such that

(4.32)

$$dist(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},-\frac{1}{2}} \leq C \left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\sigma}_{\frac{1}{2},-\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}} \right)$$

Proof. The first inequality of (4.32 comes directly from (4.16. Since q_j are related to the conductivities γ_j by means of (4.31 it follows from Theorem 2.74 that the $\mathcal{C}_{q_j}$ are graphs of the corresponding Dirichlet– to Neumann–data maps and that, by Problem **??**,

$$\Lambda_{q_j}\phi = \frac{1}{\gamma_j^{1/2}} \Big(\Lambda_{\gamma_j} \big(\gamma_j^{-\frac{1}{2}} \phi \big) + \frac{\partial \gamma_j^{1/2}}{\partial\, \nu} \phi\big|_{\partial\Omega} \Big) \qquad \forall \phi \in H^{\frac{1}{2}}(\partial\Omega)$$

so that

$$\begin{aligned}
&\big\|(\Lambda_{q_1} - \Lambda_{q_2})\phi\big\|_{-\frac{1}{2},\partial\Omega} \\
&\quad \leq C \Big\| \gamma_1^{-\frac{1}{2}} - \gamma_2^{-\frac{1}{2}} \Big\|_{C^1(\partial\Omega)} \Big\| \Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi) + \frac{\partial \gamma_1^{1/2}}{\partial\, \nu}\phi \Big\|_{-\frac{1}{2},\partial\Omega} \\
&\qquad + C \Big\| \gamma_2^{-\frac{1}{2}} \Big\|_{C^1(\partial\Omega)} \Big(\Big\| \Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi) - \Lambda_{\gamma_2}(\gamma_2^{-\frac{1}{2}}\phi) \Big\|_{-\frac{1}{2},\partial\Omega} \\
&\qquad\qquad + \Big\| \frac{\partial \gamma_1^{1/2}}{\partial\, \nu} - \frac{\partial \gamma_2^{1/2}}{\partial\, \nu} \Big\|_{C^0(\partial\Omega)} \|\phi\|_{L^2(\partial\Omega)} \Big)
\end{aligned}$$

Assumptions (i) and (ii) provide, via the Sobolev imbedding bound of Problem **??**, bounds on $\sup_\Omega \frac{1}{\gamma_1^{1/2}}$, $\sup_\Omega \frac{1}{\gamma_2^{1/2}}$, $\sup_\Omega \gamma_1$, $\sup_\Omega \gamma_2$, $\sup_\Omega |\nabla\gamma_1|$ and $\sup_\Omega |\nabla\gamma_2|$ that depend only on Ω and E. Since $\gamma_1, \gamma_2 \in C^\infty(\overline{\Omega})$, these bounds continue to $\partial\Omega$. As $\big\|\Lambda_{\gamma_1}\big\|_{\frac{1}{2},-\frac{1}{2}}$ and $\big\|\Lambda_{\gamma_2}\big\|_{\frac{1}{2},-\frac{1}{2}}$ are bounded by Theorem 2.64 (with constants depending only on E and Ω by Remark **??** and multiplication by $\gamma_1^{-1/2}$ or $\gamma_2^{-1/2}$ is a bounded map on $H^{1/2}(\partial\Omega)$, by Problem **??**,

(4.33)

$$\|(\Lambda_{q_1}-\Lambda_{q_2})\phi\|_{-\frac{1}{2},\partial\Omega} \leq C\Big(\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)}\|\phi\|_{\frac{1}{2},\partial\Omega}+\big\|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi)-\Lambda_{\gamma_2}(\gamma_2^{-\frac{1}{2}}\phi)\big\|_{-\frac{1}{2},\partial\Omega}\Big)$$

In a similar fashion we may also bound

$$\begin{aligned}\big\|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi)-\Lambda_{\gamma_2}(\gamma_2^{-\frac{1}{2}}\phi)\big\|_{-\frac{1}{2},\partial\Omega} &\leq \big\|\Lambda_{\gamma_1}(\gamma_1^{-\frac{1}{2}}\phi-\gamma_2^{-\frac{1}{2}}\phi)\big\|_{-\frac{1}{2},\partial\Omega}+\big\|(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})(\gamma_2^{-\frac{1}{2}}\phi)\big\|_{-\frac{1}{2},\partial\Omega}\\ &\leq C\Big(\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)}+\big\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\big\|_{\frac{1}{2},-\frac{1}{2}}\Big)\|\phi\|_{\frac{1}{2},\partial\Omega}\end{aligned}$$

Insertion of this into (4.33) yields

(4.34)

$$\big\|(\Lambda_{q_1}-\Lambda_{q_2})\phi\big\|_{-\frac{1}{2},\partial\Omega} \leq C\Big(\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)}+\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}\Big)\|\phi\|_{\frac{1}{2},\partial\Omega}$$

Since $s-\frac{1}{2} > \frac{n}{2}-\frac{1}{2} = \frac{n-1}{2}$ we may use Sobolev's imbedding theorem, Problem **??**, and the logarithmic convexity of the Sobolev norms, Problem **??**, to obtain

$$\begin{aligned}\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)} &\leq C\|\gamma_1-\gamma_2\|_{s+\frac{1}{2},\partial\Omega}\\ &\leq C\big\|\gamma_1-\gamma_2\big\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}}\big\|\gamma_1-\gamma_2\big\|_{s+\frac{3}{2},\partial\Omega}^{\frac{2s+1}{2s+3}}\\ &\leq C\big\|\gamma_1-\gamma_2\big\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}}\end{aligned}$$

Internal remark 2. *We need the index* $s+\frac{3}{2}$ *in* $\big\|\gamma_1-\gamma_2\big\|_{s+\frac{3}{2},\partial\Omega}^{\frac{2s+1}{2s+3}}$ *to be strictly bigger than* $s+\frac{1}{2}$ *in order to give a nonzero power of* $\big\|\gamma_1-\gamma_2\big\|_{L^2(\partial\Omega)}$. *This leads to the index in hypothesis (ii) being strictly bigger than* $s+1$. *To obtain the last inequality we have also used the trace estimate (Theorem* **??***)*

$$\|\gamma_1-\gamma_2\|_{s+\frac{3}{2},\partial\Omega} \leq C\|\gamma_1-\gamma_2\|_{s+2,\Omega}$$

It now follows from the first part, (4.59), of the continuous dependence result on the boundary that

$$\|\gamma_1-\gamma_2\|_{C^1(\partial\Omega)} \leq C\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}^{\frac{2}{2s+3}} \tag{4.35}$$

After insertion of (4.35) into (4.34) we obtain the desired estimate with $\sigma = \frac{2}{2s+3}$ □

The stable dependence result for the conductivity problem is

Theorem 4.17. *Suppose that* $\frac{n}{2} < s \in \mathbb{N}$, $n \geq 3$, *and that* γ_1 *and* γ_2 *are isotropic conductivities on* $\Omega \subset \mathbb{R}^n$ *satisfying Hypothesis 4.40 and*

(i) $1/E \leq \gamma_j \leq E$

(ii) $\|\gamma_j\|_{s+2,\Omega} \leq E$.

Then there exist $C = C(\Omega, E, n, s)$ *and* $0 < \sigma = \sigma(n,s) < 1$ *such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\Big\{\big|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}\big|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}\Big\} \tag{4.36}$$

Proof. In light of the hypothesis (i) it clearly suffices to prove the estimate (4.36) for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$ smaller than any strictly positive constant. The last term in right hand side of (4.36) is there to render the estimate trivially satisfied for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$ larger than the constant. By (4.32), we can choose the constant small enough that $\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$ is also smaller than any desired strictly positive constant.

Consider the function

$$v = \log\big(\frac{\gamma_1}{\gamma_2}\big) = \log(\gamma_1) - \log(\gamma_2)$$

This function obeys the boundary value problem

$$\begin{aligned} \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} \nabla v\big) &= 2(\gamma_1\gamma_2)^{\frac{1}{2}}(q_2 - q_1) \qquad \text{in } \Omega \\ v\big|_{\partial\Omega} &= \log\gamma_1 - \log\gamma_2 \end{aligned}$$

Internal remark 3.

$$\begin{aligned} \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} \nabla v\big) &= \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} 2\nabla \log\big(\frac{\gamma_1^{1/2}}{\gamma_2^{1/2}}\big)\big) \\ &= \nabla \cdot \big((\gamma_1\gamma_2)^{\frac{1}{2}} 2\big(\gamma_1^{-1/2}\nabla\gamma_1^{1/2} - \gamma_2^{-1/2}\nabla\gamma_2^{1/2}\big)\big) \\ &= 2\nabla \cdot \big(\gamma_2^{1/2}\nabla\gamma_1^{1/2} - \gamma_1^{1/2}\nabla\gamma_2^{1/2}\big) \\ &= 2\big(\nabla\gamma_2^{1/2}\cdot\nabla\gamma_1^{1/2} + \gamma_2^{1/2}\Delta\gamma_1^{1/2} - \nabla\gamma_1^{1/2}\cdot\nabla\gamma_2^{1/2} - \gamma_1^{1/2}\Delta\gamma_2^{1/2}\big) \\ &= 2\big(\gamma_2^{1/2}\Delta\gamma_1^{1/2} - \gamma_1^{1/2}\Delta\gamma_2^{1/2}\big) \\ &= 2(\gamma_1\gamma_2)^{\frac{1}{2}}(-q_1 + q_2) \end{aligned}$$

with the q_1 and q_2 defined in (4.31), and hence, by Theorem **??** (and Remark **??**),

$$\|\log\gamma_1 - \log\gamma_2\|_{1,\Omega} = \|v\|_{1,\Omega} \leq C\big(\|q_1 - q_2\|_{-1,\Omega} + \|\log\gamma_1 - \log\gamma_2\|_{\frac{1}{2},\partial\Omega}\big)\mathbf{??}$$

Now

$$\log\gamma_1 - \log\gamma_2 = \left[\int_0^1 \frac{dt}{t\gamma_1 + (1-t)\gamma_2}\right]\cdot(\gamma_1-\gamma_2)$$
$$\nabla\log\gamma_1 - \nabla\log\gamma_2 = \frac{1}{\gamma_1}\nabla\gamma_1 - \frac{1}{\gamma_2}\nabla\gamma_2 = \frac{1}{\gamma_1}\big[\nabla\gamma_1-\nabla\gamma_2\big] + \frac{\gamma_2-\gamma_1}{\gamma_1\gamma_2}\nabla\gamma_2$$

and

$$\gamma_1-\gamma_2 = \left[\int_0^1 e^{t\log\gamma_1+(1-t)\log\gamma_2}\ dt\right]\cdot(\log\gamma_1-\log\gamma_2)$$
$$\nabla\gamma_1-\nabla\gamma_2 = \gamma_1\nabla\log\gamma_1 - \gamma_2\nabla\log\gamma_2 = \gamma_1\big[\nabla\log\gamma_1-\nabla\log\gamma_2\big] + \frac{\gamma_1-\gamma_2}{\gamma_2}\nabla\gamma_2$$

By hypothesis (i), $\frac{1}{E}\le\gamma_j\le E$. By hypothesis (ii) and the Sobolev imbedding theorem, Problem **??**, $|\nabla\gamma_j|\le CE$. It follows that there is a constant c, depending only on n, Ω and E, such that

$$\|\gamma_1-\gamma_2\|_{1,\Omega} \le c\|\log\gamma_1-\log\gamma_2\|_{1,\Omega}$$
$$\|\log\gamma_1-\log\gamma_2\|_{1,\partial\Omega} \le c\|\gamma_1-\gamma_2\|_{1,\partial\Omega}$$

Since $\|\log\gamma_1-\log\gamma_2\|_{\frac12,\partial\Omega}\le\|\log\gamma_1-\log\gamma_2\|_{1,\partial\Omega}$, (**??**) translates into

$$\|\gamma_1-\gamma_2\|_{1,\Omega}\le C\big(\|q_1-q_2\|_{-1,\Omega}+\|\gamma_1-\gamma_2\|_{1,\partial\Omega}\big) \tag{4.37}$$

A combination of the estimates (4.18) and (4.32) gives that for some $0<\sigma_1,\sigma_2<1$

$$\begin{aligned}\|q_1-q_2\|_{-1,\Omega} &\le C\left|\log\{\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})\}\right|^{-\sigma_1}\\ &\le C\left|\log\Big\{\big\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\big\|^{\sigma_2}_{\frac12,-\frac12}\Big\}\right|^{-\sigma_1}\\ &\le C\left|\log\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac12,-\frac12}\right|^{-\sigma_1}\end{aligned} \tag{4.38}$$

for $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac12,-\frac12}$ sufficiently small. In view of Sobolev's imbedding theorem and the logarithmic convexity of the Sobolev norms, we have

$$\begin{aligned}\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)} &\le C\|\gamma_1-\gamma_2\|_{s,\Omega}\\ &\le C\big\|\gamma_1-\gamma_2\big\|^{\frac{s-1}{s+1}}_{s+2,\Omega}\ \big\|\gamma_1-\gamma_2\big\|^{\frac{2}{s+1}}_{1,\Omega}\\ &\le C\big\|\gamma_1-\gamma_2\big\|^{\frac{2}{s+1}}_{1,\Omega}\end{aligned}$$

and

$$\begin{aligned}\|\gamma_1-\gamma_2\|_{1,\partial\Omega} &\le C\big\|\gamma_1-\gamma_2\big\|_{L^2(\partial\Omega)}^{\frac{2s+1}{2s+3}}\big\|\gamma_1-\gamma_2\big\|_{s+\frac{3}{2},\partial\Omega}^{\frac{2}{2s+3}}\\ &\le C\big\|\gamma_1-\gamma_2\big\|_{L^2(\partial\Omega)}^{\frac{2s+1}{2s+3}}\big\|\gamma_1-\gamma_2\big\|_{s+2,\Omega}^{\frac{2}{2s+3}}\\ &\le C\big\|\gamma_1-\gamma_2\big\|_{L^\infty(\partial\Omega)}^{\frac{2s+1}{2s+3}}\end{aligned}$$

Together with (4.37) and (4.38) these two estimates give

(4.39)

$$\begin{aligned}\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)} &\le C\Big(\|q_1-q_2\|_{-1,\Omega}+\|\gamma_1-\gamma_2\|_{1,\partial\Omega}\Big)^{\frac{2}{s+1}}\\ &\le C\Big(\big|\log\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac12,-\frac12}\big|^{-\sigma_1}+\big\|\gamma_1-\gamma_2\big\|_{L^\infty(\partial\Omega)}^{\frac{2s+1}{2s+3}}\Big)^{\frac{2}{s+1}}\end{aligned}$$

for $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac12,-\frac12}$ sufficiently small. By combination with the boundary continuous dependence result (Theorem 4.62) the estimate (4.39) becomes

$$\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)} \le C\big|\log\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac12,-\frac12}\big|^{-\frac{2\sigma_1}{s+1}}$$

for $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_{\frac12,-\frac12}$ sufficiently small. This completes the proof of the theorem. □

Internal remark 4. *We need the $s+2$ because of Lemma 4.16.*

4.5. Complex geometrical optics solutions: second proof

The construction of complex geometrical optics solutions in Theorem 4.3 was based on considering Fourier series in a slightly shifted lattice, which avoided the problem of dividing by symbols having zeros. This construction was sufficient for the interior uniqueness and stability results in the previous sections. However, to obtain a reconstruction procedure for determining a conductivity for a DN map, it is useful to give another proof of the existence of CGO solutions. This proof is valid in $\mathbb{R}^n$ instead of just bounded domains, and it comes with a reasonable uniqueness notion for the CGO solutions upon fixing a decay condition at infinity. These additional properties will also be crucial in the inverse scattering problems in 7.CHscattering.

To construct the solutions we shall make use of the following norms, defined for any $u\in C_0^\infty(\mathbb{R}^n)$ and any $-\infty<\delta<\infty$:

$$\|u\|_{L^2_\delta}=\left(\int_{\mathbb{R}^n}\big(1+|x|^2\big)^\delta|u|^2\,d^nx\right)^{1/2}$$

The space L^2_δ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{L^2_\delta}$. When we say that $u=e^{x\cdot\zeta}\big(1+\psi(x,\zeta)\big)$, with $\zeta\cdot\zeta=0$

and $\psi \in L^2_\delta$, *solves* $\Delta u + qu = 0$, *we mean that* ψ *is a weak solution of* $\Delta\psi + 2\zeta \cdot \nabla\psi = -q - q\psi$. *The latter means that*

$$\langle (\Delta - 2\zeta \cdot \nabla)\varphi, \psi \rangle_{L^2(\mathbb{R}^n)} = -\langle \varphi, q + q\psi \rangle_{L^2(\mathbb{R}^n)}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. *We have encountered weak derivatives before, in part (b) of Proposition* **??**. *The main theorem in this section is:*

Theorem 4.18. *Let* $-1 < \delta < 0$. *There exists* $\epsilon = \epsilon(\delta)$ *and* $C = C(\delta)$ *such that, for every* $q \in L^2_{\delta+1}$ *with* $\left(1 + |x|^2\right)^{1/2} q \in L^\infty$ *and every* $\zeta \in \mathbb{C}^n$ *satisfying*

$$\zeta \cdot \zeta = 0 \qquad \text{and} \qquad \frac{\|(1 + |x|^2)^{1/2} q\|_{L^\infty} + 1}{|\zeta|} \le \epsilon$$

there exists a unique solution to

$$\Delta u + qu = 0 \quad \text{ in } \mathbb{R}^n$$

of the form

$$u = e^{x\cdot\zeta}\big(1 + \psi(x, \zeta)\big)$$

with $\psi(x, \zeta) \in L^2_\delta$. *Furthermore,*

$$\|\psi\|_{L^2_\delta} \le \frac{C}{|\zeta|}\|q\|_{L^2_{\delta+1}}$$

This theorem has a counterpart for the conductivity problem, which is obtained by invoking the correspondence of Theorem 2.74 between the Schrödinger equation and the conductivity equation. The statement is

Theorem 4.19. *Let* $-1 < \delta < 0$. *There exists* $\epsilon = \epsilon(\delta)$ *and* $C = C(\delta)$ *such that, for every positive* γ *with* $\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \in L^2_{\delta+1}$, $\left(1 + |x|^2\right)^{1/2} \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \in L^\infty$ *and every* $\zeta \in \mathbb{C}^n$ *satisfying*

$$\zeta \cdot \zeta = 0 \qquad \text{and} \qquad \frac{\left\|\left(1 + |x|^2\right)^{1/2} \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}\right\|_{L^\infty} + 1}{|\zeta|} \le \epsilon$$

there exists a unique solution to

$$L_\gamma u = 0$$

of the form

$$u = \gamma^{-1/2} e^{x\cdot\zeta}\big(1 + \psi(x, \zeta)\big)$$

with $\psi(x, \zeta) \in L^2_\delta$. *Furthermore,*

$$\|\psi(x, \zeta)\|_{L^2_\delta} \le \frac{C}{|\zeta|}\left\|\frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}\right\|_{L^2_{\delta+1}}$$

Most of the work necessary for the proof of Theorem 4.18 is associated with establishing the following proposition.

Proposition 4.20. *Suppose that $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$, $|\zeta| \geq c > 0$ and $f \in L^2_{\delta+1}$ with $-1 < \delta < 0$. Then there exists a unique $\varphi \in L^2_\delta$ such that*

$$(\Delta + 2\zeta \cdot \nabla)\varphi = f\mathbf{??}$$

weakly. Moreover,

$$\|\varphi\|_{L^2_\delta} \leq \frac{C(\delta, c)}{|\zeta|}\|f\|_{L^2_{\delta+1}}$$

We postpone the proof of this proposition to the end of this section, instead we first show how it may be applied for the

Proof. of Theorem 4.18 We seek u of the form

$$u = e^{x\cdot\zeta}(1 + \psi)$$

satisfying

$$(\Delta + q)\{e^{x\cdot\zeta}(1 + \psi)\} = 0$$

or

$$\Delta\psi + 2\zeta \cdot \nabla\psi = -q - q\psi \tag{4.40}$$

To solve (4.40), we define

$$\psi_{-1} = 1$$

and we recursively define ψ_j by

$$(\Delta + 2\zeta \cdot \nabla)\psi_j = -q\psi_{j-1} \quad \text{for } j \geq 0 \tag{4.41}$$

Then, formally,

$$\psi := \sum_{j=0}^{\infty} \psi_j \tag{4.42}$$

obeys

$$\Delta\psi + 2\zeta\cdot\nabla\psi = \sum_{j=0}^{\infty}(\Delta + 2\zeta\cdot\nabla)\psi_j = -\sum_{j=0}^{\infty} q\psi_{j-1} = -q\psi_{-1} - \sum_{j=0}^{\infty} q\psi_j = -q - q\psi$$

and so is the desired solution. It needs to be proved that the functions ψ_j, $j \geq 0$, are well defined, and that the series (4.42) converges appropriately. We may without loss of generality restrict our attention to $\epsilon < 1$, so that we only consider ζ for which $|\zeta| \geq 1$. Since $q \in L^2_{\delta+1}$ and $\psi_{-1} = 1$ it follows from Proposition 4.20 that there exists a unique $\psi_0 \in L^2_\delta$ that solves (4.41) with $j = 0$. This ψ_0 satisfies

$$\|\psi_0\|_{L^2_\delta} \leq \frac{C(\delta)}{|\zeta|}\|q\|_{L^2_{\delta+1}} \tag{4.43}$$

If v is an element in L^2_δ, then the fact that $\left(1+|x|^2\right)^{1/2}q$ is in L^∞ immediately implies that qv is in $L^2_{\delta+1}$ with the estimate

$$\|qv\|_{L^2_{\delta+1}} \leq \left\|\left(1+|x|^2\right)^{1/2}q\right\|_{L^\infty}\|v\|_{L^2_\delta}\textbf{??}$$

Using this observation in conjunction with Proposition 4.20 we conclude that if ψ_{j-1} is in L^2_δ then there exists a unique solution, ψ_j, to (4.41) in L^2_δ and this solution satisfies

$$\|\psi_j\|_{L^2_\delta} \leq \frac{C(\delta)}{|\zeta|}\|q\psi_{j-1}\|_{L^2_{\delta+1}} \leq \left(\frac{C(\delta)\|(1+|x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|}\right)\|\psi_{j-1}\|_{L^2_\delta}\textbf{??}$$

An induction argument based on the estimates (4.43) and (**??**) now gives that ψ_j, $j \geq 0$, are all elements of L^2_δ and satisfy the estimates

$$\|\psi_j\|_{L^2_\delta} \leq \frac{C(\delta)}{|\zeta|}\theta^j\|q\|_{L^2_{\delta+1}} \qquad \text{with} \qquad \theta = \frac{C(\delta)\|(1+|x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|}$$

By selecting ϵ sufficiently small that $\theta < 1/2$, we now obtain that the series (4.42) is convergent, in L^2_δ, with the bound

$$\|\psi\|_{L^2_\delta} \leq 2\frac{C(\delta)}{|\zeta|}\|q\|_{L^2_{\delta+1}}$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $(\Delta - 2\zeta\cdot\nabla)\varphi \in L^2_{-\delta}$ so that $\sum_{j=0}^\infty \langle(\Delta - 2\zeta\cdot\nabla)\varphi, \psi_j\rangle$ converges to $\langle(\Delta - 2\zeta\cdot\nabla)\varphi, \psi\rangle$. By (**??**), the series $\sum_{j=0}^\infty q\psi_j$ converges in $L^2_{1+\delta}$ to $q\psi$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi \in L^2_{-1-\delta}$ so that $\sum_{j=0}^\infty \langle\varphi, q\psi_j\rangle$ converges to $\langle\varphi, q\psi\rangle$. This completes the proof of the existence part of Theorem 4.18.

To verify the uniqueness of the solution ψ (and therefore of u), suppose that

$$\Delta\psi + 2\zeta\cdot\nabla\psi = -q - q\psi$$

and

$$\Delta\tilde{\psi} + 2\zeta\cdot\nabla\tilde{\psi} = -q - q\tilde{\psi}$$

with ψ and $\tilde{\psi} \in L^2_\delta$. Then

$$\Delta(\tilde{\psi} - \psi) + 2\zeta\cdot\nabla(\tilde{\psi} - \psi) = q(\psi - \tilde{\psi})$$

so that, according to Proposition 4.20 and (**??**)

$$\|\tilde{\psi} - \psi\|_{L^2_\delta} \leq \frac{C\|(1+|x|^2)^{1/2}q\|_{L^\infty}}{|\zeta|}\|\tilde{\psi} - \psi\|_{L^2_\delta} \leq \frac{1}{2}\|\tilde{\psi} - \psi\|_{L^2_\delta}$$

which can only occur if

$$\|\tilde{\psi} - \psi\|_{L^2_\delta} = 0$$

□

Proof. of Theorem 4.3 Define

$$\tilde{q} = \begin{cases} \text{q} & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

We may apply Theorem 4.18 to $\tilde{q}$, say with $\delta = -\frac{1}{2}$. In this way we obtain the existence of a solution to $\Delta u + \tilde{q}u = 0$ in $\mathbb{R}^n$ (and therefore a solution to $\Delta u + qu = 0$ in Ω) of the form $u = e^{x\cdot\zeta}\big(1 + \psi(x,\zeta)\big)$ with

$$\|\psi\|_{L^2(\Omega)} \leq c_1\|\psi\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{c_2}{|\zeta|}\|\tilde{q}\|_{L^2_{\delta+1}(\mathbb{R}^n)} \leq \frac{C}{|\zeta|}\|q\|_{L^2(\Omega)}\mathbf{??}$$

Similarly, for any $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta\psi \in L^2(\mathbb{R}^n)$ and obeys

$$\Delta(\eta\psi) = -\eta\big(2\zeta\cdot\nabla\psi + \tilde{q} + q\psi\big) + 2\nabla\eta\cdot\nabla\psi + (\Delta\eta)\psi \in H^{-1}(\mathbb{R}^n)$$

Hence $\eta\psi \in H^1(\mathbb{R}^n)$ and $\psi \in H^1(\Omega)$. That $u \in H^2(\Omega)$ follows from Proposition **??**, since $\Delta u = -\tilde{q}u \in L^2(\Omega')$ for all bounded open subsets $\Omega' \subset \mathbb{R}^n$. So it only remains to prove the estimate concerning the H^1 norm of ψ.

From equation (4.40), we get that

$$\Delta\psi = -2\zeta\cdot\nabla\psi - \tilde{q} - \tilde{q}\psi \qquad \text{in } \mathbb{R}^n$$

and the interior estimate of Proposition **??** thus gives

$$\|\psi\|_{1,\Omega} \leq C\big(\|2\zeta\cdot\nabla\psi + \tilde{q} + \tilde{q}\psi\|_{-1,\Omega'} + \|\psi\|_{L^2(\Omega')}\big)\mathbf{??}$$

for $\Omega \subset\subset \Omega'$. On the other hand, we also have

(4.44)

$$\begin{aligned}\|2\zeta\cdot\nabla\psi + \tilde{q} + \tilde{q}\psi\|_{-1,\Omega'} &\leq 2n|\zeta|\|\psi\|_{L^2(\Omega')} + \|\tilde{q}\|_{L^2(\Omega')} + \|\tilde{q}\psi\|_{L^2(\Omega')} \\ &\leq 2n|\zeta|\|\psi\|_{L^2(\Omega')} + \|q\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)}\|\psi\|_{L^2(\Omega')}\end{aligned}$$

and

$$\|\psi\|_{L^2(\Omega')} \leq \frac{C}{|\zeta|}\|\tilde{q}\|_{L^2(\Omega')} = \frac{C}{|\zeta|}\|q\|_{L^2(\Omega)}\mathbf{??}$$

The estimate (**??**) is obtained by replacing Ω by Ω' in the estimate (**??**) (the constant C changes). A combination of (**??**)-(**??**) yields

$$\|\psi\|_{1,\Omega} \leq C\left(\|q\|_{L^2(\Omega)} + \frac{\|q\|_{L^\infty(\Omega)}\|q\|_{L^2(\Omega)}}{|\zeta|} + \frac{\|q\|_{L^2(\Omega)}}{|\zeta|}\right)$$

and since the assumption on $|\zeta|$ implies that $\frac{1}{|\zeta|} \leq 1$ and $\frac{1}{|\zeta|}\|q\|_{L^\infty(\Omega)} \leq 1$, we immediately get

$$\|\psi\|_{1,\Omega} \leq C\|q\|_{L^2(\Omega)}$$

as desired. $\square$

We now return to the

Proof. of Proposition 4.20 We first prove uniqueness. If $w \in \mathcal{S}(\mathbb{R}^n)$ and

$$\Delta w + 2\zeta \cdot \nabla w = 0$$

Fourier transformation gives

$$(-|k|^2 + 2\zeta \cdot ik)\,\hat{w}(k) = 0\textbf{??}$$

As this equation is invariant under rotations, we may assume, without loss of generality, that the real part of ζ is in the positive e_1 direction and the imaginary part of ζ is in the span of e_1 and e_2 with negative e_2 component. By (4.1), the real and imaginary parts of ζ must be mutually perpendicular and of the same length, so that

$$\zeta = se_1 - ise_2 \qquad \text{with} \qquad s = \frac{|\zeta|}{\sqrt{2}}$$

in which case (**??**) is equivalent to

$$\left[-\left(k_1^2 + (k_2 - s)^2 + k_3^2 \cdots + k_n^2 - s^2\right) + 2isk_1\right]\ \hat{w} = 0\textbf{??}$$

Let

$$\mathcal{M}(s) = \left\{\ k \in \mathbb{R}^n\ \middle|\ k_1 = 0,\ k_1^2 + (k_2 - s)^2 + k_3^2 \cdots + k_n^2 = s^2\ \right\}$$

denote the codimension 2 sphere which arises as the intersection of the plane $k_1 = 0$ and the n-1 dimensional sphere with center se_2 and radius s. The content of (**??**) is that $\hat{w}$ is supported on $\mathcal{M}(s)$ and so must vanish.

Now let $w \in L^2_\delta$ be any weak solution to $\Delta w + 2\zeta \cdot \nabla w = 0$. To show that $w = 0$, it suffices to show that $\langle w, \varphi\rangle = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. To do so, we approximate w by $w_\varepsilon \in L^2(\mathbb{R}^n)$. Let

$$\chi \in C^\infty([0,\infty)) \qquad \text{with} \qquad \operatorname{supp}\chi \subset [0,1) \qquad \text{and} \qquad \int_{\mathbb{R}^n} \chi(|k|^2)\,\frac{d^nk}{(2\pi)^n} = 1$$

and

$$\beta(x) = \int_{\mathbb{R}^n} e^{ik\cdot x}\chi(|k|^2)\,\frac{d^nk}{(2\pi)^n}$$

Then

$$w_\varepsilon(x) = \beta(\varepsilon x)w(x) \in L^2(\mathbb{R}^n)$$

As $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi_\varepsilon(x) = \beta(\varepsilon x)\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$ and

$$\lim_{\varepsilon\searrow 0}\left(1 + |x|^2\right)^{-\delta/2}\varphi_\varepsilon(x) = \left(1 + |x|^2\right)^{-\delta/2}\varphi(x) \qquad \text{in } L^2(\mathbb{R}^n)$$

by the Lebesgue dominated convergence theorem. Consequently

$$\langle w, \varphi\rangle = \lim_{\varepsilon\searrow 0}\langle w, \varphi_\varepsilon\rangle = \lim_{\varepsilon\searrow 0}\langle w_\varepsilon, \varphi\rangle = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n} \hat{w}_\varepsilon(k)\overline{\hat{\varphi}(k)}\,\frac{d^nk}{(2\pi)^n}$$

Observe that $\left(\frac{1}{\varepsilon^n}\hat{\beta}\left(\frac{k}{\varepsilon}\right)\right)^{\vee}(x) = \beta(\varepsilon x)$. By Problem 4.24, below,

$$\operatorname{supp}\hat{w}_\varepsilon \subset N_\varepsilon\big(\mathcal{M}(s)\big) = \left\{\ k\ \middle|\ \operatorname{dist}(k, \mathcal{M}(s)) \le \varepsilon\ \right\}$$

so that

$$|\langle w,\varphi\rangle| \le \limsup_{\varepsilon\searrow 0}\ \varepsilon\left(\int_{N_\varepsilon}|\hat{w}_\varepsilon(k)|^2\frac{d^nk}{(2\pi)^n}\right)^{1/2}\left(\frac{1}{\varepsilon^2}\int_{N_\epsilon}|\hat{\varphi}(k)|^2\frac{d^nk}{(2\pi)^n}\right)^{1/2}$$

As $\hat{\varphi}$ is smooth and $\frac{1}{\varepsilon^2}$(volume of N_ϵ) converges to a constant times the surface area of $\mathcal{M}(s)$,

$$|\langle w,\varphi\rangle| \le C\left(\limsup_{\varepsilon\searrow 0}\ \varepsilon\|\hat{w}_\varepsilon\|_{L^2}\right)\left(\int_{\mathcal{M}(s)}|\hat{\varphi}(k)|^2 d^{n-2}\sigma(k)\right)^{1/2} \tag{4.45}$$

Moreover,

$$\frac{1}{(2\pi)^n}\|\hat{w}_\epsilon\|^2_{L^2} = \|w_\epsilon\|^2_{L^2} = \int_{\mathbb{R}^n}|\beta(\varepsilon x)|^2\ |w(x)|^2\ d^nx \le \|w\|^2_{L^2_\delta}\sup_x \beta(\varepsilon x)^2\big(1+|x|^2\big)^{-\delta}$$

As $\beta\in\mathcal{S}(\mathbb{R}^n)$ and $\delta<0$

$$\begin{aligned}\|\hat{w}_\epsilon\|^2_{L^2} &\le C\ \|w\|^2_{L^2_\delta}\ \sup_x\big(1+\varepsilon^2|x|^2\big)^{\delta}\big(1+|x|^2\big)^{-\delta} = C\ \|w\|^2_{L^2_\delta}\ \sup_x\left[\frac{1+|x|^2}{1+\varepsilon^2|x|^2}\right]^{|\delta|}\\ &\le C\ \varepsilon^{2\delta}\ \|w\|^2_{L^2_\delta}\end{aligned}$$

Returning to (4.45)

$$|\langle w,\varphi\rangle| \le C\limsup_{\epsilon\searrow 0}\ (\varepsilon\cdot\varepsilon^{\delta})\ \|w\|_{L^2_\delta}\ \left(\int_{\mathcal{M}(s)}|\hat{\varphi}(k)|^2 d^{n-1}\sigma(k)\right)^{1/2}$$

Since $\delta>-1$, it therefore follows that

$$\langle w,\varphi\rangle = 0$$

for every $\varphi\in\mathcal{S}$, so that $w=0$.

We turn to proving existence of a solution to (**??**). Suppose for now that $f\in\mathcal{S}(\mathbb{R}^n)$ and define

$$\hat{w}(k) = \frac{\hat{f}(k)}{-|k|^2+2i\zeta\cdot k}$$

We shall prove that w is well defined and satisfies the estimate

$$\|w\|_{L^2_\delta} \le \frac{C}{|\zeta|}\|f\|_{L^2_{\delta+1}}$$

Once this estimate is established we can dispense with the assumption that $f\in\mathcal{S}(\mathbb{R}^n)$ by continuity. As we did in the uniqueness proof, we may assume that

$$\zeta = s(e_1-ie_2) \qquad \text{with} \qquad s=\frac{|\zeta|}{\sqrt{2}}$$

and therefore

$$-|k|^2+2i\zeta\cdot k = -\big[k_1^2+(k_2-s)^2+k_3^2\cdots+k_n^2-s^2\big]+2isk_1 = P(k,s)$$

Since the polynomial $P(k,s)$ is homogeneous of degree two,

$$P(k,s) = s^2 P(k/s, 1)$$

As before we denote

$$N_r\big(\mathcal{M}(s)\big) = \left\{\ k \in \mathbb{R}^n\ \middle|\ \mathrm{dist}(k, \mathcal{M}(s)) \le r\ \right\}$$

Every point $p \in \mathcal{M}(s)$ obeys $p_1 = 0$ and $|p - se_2| = s$. Hence for all $k \in \mathbb{R}^n$ and $p \in \mathcal{M}(s)$ we have that $|k-p| \ge |k_1|$ and $|k-p| \ge \big||k - se_2| - |p - se_2|\big| = \big||k-se_2| - s\big|$ which implies that $\mathrm{dist}(k, \mathcal{M}(s)) \ge |k_1|$ and $\mathrm{dist}(k, \mathcal{M}(s)) \ge \big||k - se_2| - s\big|$. As a result, if $k \in N_{s/2n}\big(\mathcal{M}(s)\big)$, then $|k_1| \le \frac{s}{2n}$ and $|k - se_2| \ge s - \frac{s}{2n}$ so that at least one component of $k - se_2$ must be at least $\frac{1}{\sqrt{n}}\big(s - \frac{s}{2n}\big) > \frac{s}{2n}$. Consequently,

$$\begin{aligned}
\mathcal{O}_1(s) &= \mathbb{R}^n \setminus N_{s/2n}\big(\mathcal{M}(s)\big) \\
\mathcal{O}_2(s) &= \big\{|k_2 - s| > \frac{s}{2n}\big\} \cap N_s\big(\mathcal{M}(s)\big)^\circ \\
\mathcal{O}_j(s) &= \big\{|k_j| > \frac{s}{2n}\big\} \cap N_s\big(\mathcal{M}(s)\big)^\circ \qquad \text{for } j > 2
\end{aligned}$$

is an open cover of $\mathbb{R}^n$. The singularity of $\frac{\hat{f}(k)}{P(k,s)}$ on $\mathcal{M}(s)$ has been excluded from $\mathcal{O}_1(s)$. The remaining sets $\mathcal{O}_2(s), \cdots, \mathcal{O}_n(s)$ cover $N_{s/2n}\big(\mathcal{M}(s)\big) \subset N_s\big(\mathcal{M}(s)\big)^\circ$ with the j^{th} component of $|k - se_2|$ being relatively large on $\mathcal{O}_j$. It is useful to note that $\mathcal{M}(s) = s\mathcal{M}(1)$ and that $\mathcal{O}_j(s) = s\mathcal{O}_j(1)$. Let $\chi_j(k)$ be a partition of unity subordinate to this open cover, so that

$$\hat{w}(k) = \sum_{j=1}^{n} \frac{\chi_j(k)\hat{f}(k)}{P(k,s)} = \sum_{j=1}^{n} \hat{w}_j(k)$$

Since $\mathcal{O}_1(1)$ is bounded away from $\mathcal{M}(1)$ and since $P(k,1) \to \infty$ as $|k| \to \infty$ there exists a constant c such that

$$|P(k,1)| \ge c > 0 \qquad \forall k \in \mathcal{O}_1(1)$$

For $k \in \mathcal{O}_1(s)$ this leads to the estimate

$$|P(k,s)| = s^2 |P(k/s, 1)| \ge cs^2$$

so that

$$\|w_1\|_{L^2_\delta} \le \|w_1\|_{L^2} \le \frac{1}{cs^2}\|f\|_{L^2} \le \frac{1}{cs^2}\|f\|_{L^2_{\delta+1}} \tag{4.46}$$

Here we have used the assumptions that $\delta < 0$ and $\delta + 1 > 0$. Since our hypothesis guarantees that $|\zeta| = \sqrt{2}s$ is greater than some $c > 0$, (4.46) gives the desired estimate for w_1.

To estimate each w_j, with $j = 2, \cdots, n$, we first introduce new coordinates in $\mathcal{O}_j(s)$ by

$$
\begin{aligned}
\eta_1 &= 2k_1 \\
\eta_\ell &= k_\ell \qquad \text{for } \ell \neq 1,\ j \\
\eta_j &= \frac{k_1^2 + (k_2 - s)^2 + k_3^2 + \cdots + k_n^2 - s^2}{s}
\end{aligned}
\tag{4.47}
$$

In terms of these new coordinates

$$\hat{w}_j(\eta) = \frac{\chi_j(k)\hat{f}(k)}{s(-\eta_j + i\eta_1)}$$

Since

$$\frac{\partial\, \eta_\ell}{\partial k_m} = \begin{cases} 2 & \text{if } \ell = m = 1 \\ 1 & \text{if } \ell = m,\ \ell \neq 1, j \\ 0 & \text{if } \ell \neq m,\ \ell \neq j \\ \frac{2k_m}{s} & \text{if } \ell = j,\ m \neq 2 \\ \frac{2(k_2 - s)}{s} & \text{if } \ell = j,\ m = 2 \end{cases}$$

the Jacobian of this coordinate transformation on $\mathcal{O}_j(s)$ is

$$\big|\det\big[\frac{\partial \eta}{\partial k}\big]\big| = \begin{cases} \frac{4|k_j|}{s} & if j \neq 2 \\ \frac{4|k_2 - s|}{s} & if j = 2 \end{cases}$$

which is bounded above by 8 and below by $\frac{2}{n}$ on $\mathcal{O}_j(s)$, $j = 2, ..., n$ for all s. At this point we shall make use of the following three results, the proofs of which will be given later.

Lemma 4.21. *For each $j = 2, \ \cdots, \ n$, the map Z_j defined by*

$$(Z_j f)(x) = \big(\frac{\hat{f}}{-k_j + ik_1}\big)^{\vee}(x) \qquad f \in \mathcal{S}(\mathbb{R}^n)$$

has a unique continuous extension to a bounded linear operator from $L^2_{\delta+1}$ to L^2_δ. For each $f \in L^2_{\delta+1}$, $Z_j f$ is a weak solution to $(\partial_{x_1} + i\partial_{x_j})w = f$. That is,

$$\big\langle \big(- \partial_{x_1} + i\partial_{x_j}\big)\varphi, Z_j f\big\rangle = \langle \varphi, f\rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in L^2_{\delta+1}$.

Lemma 4.22. *For any $\chi \in C_0^\infty(\mathbb{R}^n)$ and any $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\big\|\big(\chi(k)\hat{f}(k)\big)^{\vee}\big\|_{L^2_{\delta+1}} \leq C\|f\|_{L^2_{\delta+1}}$$

where the constant C depends on χ, but is independent of f.

Lemma 4.23. *Let $\mathcal{O}$ and $\mathcal{O}'$ be open subsets of $\mathbb{R}^n$ and $\mathcal{C}$ be a compact subset of $\mathcal{O}'$. Let $\hat{f} \in C_0^\infty(\mathcal{C})$ and let Ψ be a smooth diffeomorphism from $\mathcal{O}$ to $\mathcal{O}'$. Then,*

$$\big\|\big(\hat{f}\circ\Psi^{-1}\big)^\vee\big\|_{L^2_\delta} \le C\|f\|_{L^2_\delta} \qquad \big\|\big(\hat{f}\circ\Psi\big)^\vee\big\|_{L^2_{\delta+1}} \le C\|f\|_{L^2_{\delta+1}}$$

The constant C depends on Ψ and $\mathcal{C}$, but is independent of f.

The proof of Proposition 4.20 now proceeds as follows. If $\Psi(\eta)$ is the inverse map of the change of coordinates (4.47), then

$$\hat{w}_j(k) = \frac{\chi_j(k)\hat{f}(k)}{P(k,s)} = \frac{1}{s}\frac{(\chi_j\hat{f})\circ\Psi}{-\eta_j + i\eta_1}\circ\Psi^{-1}(k)$$

Set

$$g_j(x) = \big[\big(\chi_j\hat{f}\,\big)\circ\Psi\big]^\vee(x) \qquad h_j = \big[\chi_j\hat{f}\,\big]^\vee(x)$$

In this notation

$$\hat{w}_j = \frac{1}{s}\widehat{Z_j g_j}\circ\Psi^{-1}$$

Using, in order, Lemma 4.23, Lemma 4.21, Lemma 4.23 and Lemma 4.22, we obtain that

$$\|w_j\|_{L^2_\delta} \le \frac{c_1}{s}\|Z_j g_j\|_{L^2_\delta} \le \frac{c_2}{s}\|g_j\|_{L^2_{\delta+1}} \le \frac{c_3}{s}\|h_j\|_{L^2_{\delta+1}} \le \frac{c_4}{s}\|f_j\|_{L^2_{\delta+1}}$$

Recalling that $s = \frac{|\zeta|}{\sqrt{2}}$ and invoking the formula $w = \sum_{j=1}^n w_j$ completes the proof of Proposition 4.20. □

Exercise 4.24. Let $w \in L^2_\delta$ be any weak solution to $\Delta w + 2\zeta\cdot\nabla w = 0$. Let

$$\chi \in C^\infty([0,\infty)) \qquad \text{with} \qquad \operatorname{supp}\chi \subset [0,1) \qquad \text{and} \qquad \int_{\mathbb{R}^n}\chi(|k|^2)\,\frac{d^nk}{(2\pi)^n} = 1$$

and

$$w_\varepsilon(x) = \beta(\varepsilon x)w(x) \qquad \text{where} \qquad \beta(x) = \int_{\mathbb{R}^n} e^{ik\cdot x}\chi(|k|^2)\,\frac{d^nk}{(2\pi)^n}$$

(a) (a) Prove that if the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ vanishes in $N_\varepsilon\big(\mathcal{M}(s)\big) = \big\{\, k \,\big|\, \operatorname{dist}(k,\mathcal{M}(s)) \le \varepsilon \,\big\}$, then there is a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\beta(\varepsilon x)\varphi(x) = \Delta\psi - 2\bar{\zeta}\cdot\nabla\psi$$

(b) (b) Prove that $\hat{w}_\varepsilon(k)$ is supported in $N_\varepsilon\big(\mathcal{M}(s)\big)$.

It still remains to prove the three auxiliary Lemmas 4.21–4.23. If we note that $\mathrm{v}f\mathrm{v}_{s,n} = \frac{1}{(2\pi)^{n/2}}\|\hat{f}\|_{L^2_s(\mathbb{R}^n)}$, then Lemmas 4.22 and 4.23 merely state the well–known facts that multiplication by smooth, compactly supported functions and composition with smooth diffeomorphisms are bounded operators on $H^s(\mathbb{R}^n)$. The former is Lemma **??**. The latter is

Exercise 4.25. Let $s \in \mathbb{R}$. Let $\mathcal{O}$ and $\mathcal{O}'$ be open subsets of $\mathbb{R}^n$ and $\mathcal{C}$ be a compact subset of $\mathcal{O}'$. Let Ψ be a smooth diffeomorphism from $\mathcal{O}$ to $\mathcal{O}'$. Prove that there is a constant C, depending only on Ψ, s, $\mathcal{O}$ and $\mathcal{C}$, such that

$$\mathrm{v} u \circ \Psi^{-1} \mathrm{v}_{s,n} \leq C \mathrm{v} u \mathrm{v}_{s,n}$$

for all $u \in C_0^\infty(\mathcal{C})$.

It thus only remains to give the

Proof. of Lemma 4.21 To prove Lemma 4.21, it clearly suffices to consider a single value of the index j, like $j = 2$. We furthermore claim that it suffices to prove the estimate $\|Z_2 f\|_{L^2_\delta} \leq C\|f\|_{L^2_{\delta+1}}$ in $\mathbb{R}^2$. To see this we note that

$$\begin{aligned} \|u\|^2_{L^2_\delta(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1 + |x|^2)^\delta |u(x)|^2 \; d^n x \\ &\leq \int_{\mathbb{R}^n} (1 + x_1^2 + x_2^2)^\delta |u(x)|^2 \; d^n x \end{aligned}$$

since $\delta < 0$. Therefore

$$\|Z_2 f\|^2_{L^2_\delta(\mathbb{R}^n)} \leq \int dx_3 \cdots dx_n \; \|Z_2 f(\,\cdot\,, \cdot\,, x_3, \cdots, x_n)\|^2_{L^2_\delta(\mathbb{R}^2)} \tag{4.48}$$

Here we use the fact that $(Z_2 f)(x_1, x_2, \ldots x_n) = [Z_2 f(\cdot, \tilde{x})](x_1, x_2)$, i.e., we use that $\tilde{x} = (x_3, \cdots, x_n)$ may be treated as parameters untouched by Z_2. At the same time

$$\begin{aligned} \|f\|^2_{L^2_{1+\delta}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1 + |x|^2)^{1+\delta} |f(x)|^2 \; d^n x \\ &\geq \int_{\mathbb{R}^n} (1 + x_1^2 + x_2^2)^{1+\delta} |f(x)|^2 \; d^n x \end{aligned}$$

since $1 + \delta > 0$. Therefore

$$\|f\|^2_{L^2_{\delta+1}(\mathbb{R}^n)} \geq \int dx_3 \ldots dx_n \; \|f(\,\cdot\,, \cdot\,, x_3, \cdots, x_n)\|^2_{L^2_\delta(\mathbb{R}^2)} \tag{4.49}$$

The estimates (4.48) and (4.49) immediately imply that it suffices to prove the estimate $\|Z_2 f\|_{L^2_\delta} \leq C\|f\|_{L^2_{\delta+1}}$ in two dimensions. This latter estimate is a consequence of the following lemma with $p = 2$.

We now prove

$$\big\langle \big(- \partial_{x_1} + i\partial_{x_j} \big)\varphi, Z_j f \big\rangle = \langle \varphi, f \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in L^2_{\delta+1}$, assuming the boundedness of the map $Z_j : L^2_{\delta+1} \to L^2_\delta$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $\big(- \partial_{x_1} + i\partial_{x_j} \big)\varphi \in L^2_{-\delta}$

and $\varphi \in L^2_{-\delta-1}$. So, by continuity, it suffices to consider $f \in \mathcal{S}(\mathbb{R}^n)$. For $\varphi, f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\langle \varphi, f \rangle &= \frac{1}{(2\pi)^n} \langle \hat{\varphi}(k), \hat{f}(k) \rangle = \frac{1}{(2\pi)^n} \left\langle (-k_j - ik_1)\hat{\varphi}(k), \frac{\hat{f}(k)}{-k_j + ik_1} \right\rangle \\
&= \frac{1}{(2\pi)^n} \Big\langle \big((-\partial_{x_1} + i\partial_{x_j})\varphi\big)\widehat{\ }(k), \frac{\hat{f}(k)}{-k_j + ik_1} \Big\rangle \\
&= \big\langle (-\partial_{x_1} + i\partial_{x_j})\varphi, Z_j f \big\rangle
\end{aligned}$$

The formula $\frac{1}{(2\pi)^n} \big\langle \hat{\psi}(k), \hat{g}(k) \big\rangle = \langle \psi, g \rangle$ is usually first proven for $\psi, g \in \mathcal{S}(\mathbb{R}^n)$. But, by the Lebesgue dominated convergence theorem, it extends to $\hat{g} \in L^1(\mathbb{R}^n)$, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ and $\hat{g} \in L^1(\mathbb{R}^n)$ implies $g \in L^\infty(\mathbb{R}^n)$. $\square$

Lemma 4.26. *Define Z by*

$$(Zf)(u_1, u_2) := \int_{\mathbb{R}^2} \frac{1}{-(u_2 - v_2) + i(u_1 - v_1)} f(v_1, v_2)\, d^2v \qquad \textit{for } f \in \mathcal{S}(\mathbb{R}^2)$$

(a) (a) Then Zf is bounded from $L^p_{\delta+1}(\mathbb{R}^2)$ to $L^p_\delta(\mathbb{R}^2)$ provided $p > 1$ and $-\frac{2}{p} < \delta < 1 - \frac{2}{p}$. The space L^p_δ consists of the functions

$$\left\{\ u \ \middle|\ \ (1 + |x|^2)^{\delta/2} u \in L^p(\mathbb{R}^n)\ \right\}$$

equipped with the norm $\|u\|_{L^p_\delta} = \|(1 + |x|^2)^{\delta/2} u\|_{L^p(\mathbb{R}^n)}$.

(b) (b) Furthermore

$$(Zf)(u_1, u_2) = -2\pi i \big(\frac{\hat{f}}{-k_2 + ik_1}\big)^\vee (u_1, u_2) \qquad \textit{for all } f \in \mathcal{S}(\mathbb{R}^2)$$

Proof. f (a) Since the spaces $L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ are dual, provided $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$, the same is true for $L^q_{-\delta}$ and L^p_δ. As a result, it suffices to verify the estimate $|\,\langle Zf, g\rangle\,| \le C \|f\|_{L^p_{\delta+1}} \|g\|_{L^q_{-\delta}}$ for all $g \in L^q_{-\delta}$. We have

$$\begin{aligned}
|\,\langle Zf, g\rangle\,| &= \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\overline{g(u)} f(v)}{-(u_2 - v_2) + i(u_1 - v_1)}\ dudv \right| \\
&\le \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Big(|g(u)|(1 + |u|^2)^{\beta/2}(1 + |v|^2)^{-\alpha/2}\Big) \cdot \Big(|f(v)|(1 + |u|^2)^{-\beta/2}(1 + |v|^2)^{\alpha/2}\Big)}{|u - v|^{1/p} \cdot |u - v|^{1/q}}\ dudv
\end{aligned}$$

where $\alpha > 0$ and $\beta > 0$ will be chosen later. Employing Hölder's inequality,

$$
\begin{aligned}
|\langle Zf, g\rangle| \le& \left(\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2}\frac{(1+|u|^2)^{-p\beta/2}(1+|v|^2)^{p(\alpha-\delta-1)/2}}{|u-v|}du\right\}(1+|v|^2)^{p(\delta+1)/2}|f(v)|^p dv\right)^{1/p}\\
&\times\left(\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2}\frac{(1+|u|^2)^{q(\beta+\delta)/2}(1+|v|^2)^{-q\alpha/2}}{|u-v|}dv\right\}(1+|u|^2)^{-q\delta/2}|g(u)|^q du\right)^{1/q}\\
\le& C\|f\|_{L^p_{\delta+1}}\cdot\|g\|_{L^q_{-\delta}}
\end{aligned}
$$

provided the constant

$$
\begin{aligned}
C =&\left(\sup_v\int_{\mathbb{R}^2}\frac{(1+|u|^2)^{-p\beta/2}(1+|v|^2)^{p(\alpha-\delta-1)/2}}{|u-v|}du\right)^{1/p}\\
&\qquad\times\left(\sup_u\int_{\mathbb{R}^2}\frac{(1+|u|^2)^{q(\beta+\delta)/2}(1+|v|^2)^{-q\alpha/2}}{|u-v|}dv\right)^{1/q}
\end{aligned}
$$

is finite.

Since $1+|x|^2 \le (1+|x|)^2 \le 2(1+|x|^2)$ it suffices to check that

$$
\sup_v\int_{\mathbb{R}^2}\frac{(1+|u|)^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)}}{|u-v|}du \quad\text{and}\quad \sup_u\int_{\mathbb{R}^2}\frac{(1+|u|)^{q(\beta+\delta)}(1+|v|)^{-q\alpha}}{|u-v|}dv
$$

are finite, for appropriate choices of α, β, p, q and δ, which we now do. We impose the constraint that $\delta = \alpha - \beta - \frac{1}{q}$, which implies that

$$
\begin{aligned}
p(\alpha-\delta-1) &= p\big(\beta+\frac{1}{q}-1\big) = p\big(\beta-\frac{1}{p}\big) = p\beta - 1\\
q(\beta+\delta) &= q\big(\alpha-\frac{1}{q}\big) = q\alpha - 1
\end{aligned}
$$

For each fixed v with $|v| \ge 1$, let R be a rotation chosen so that $v = |v|Re_1$. Making the change of variables $u = |v|Rw$, we see that the integral

$$
\begin{aligned}
\int_{\mathbb{R}^2}\frac{(1+|u|)^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)}}{|u-v|}d^2u &= (1+|v|)^{p(\alpha-\delta-1)}|v|\int_{\mathbb{R}^2}\frac{(1+|v||w|)^{-p\beta}}{|w-e_1|}dw\\
&\le (1+|v|)^{p(\alpha-\delta-1)}|v|\int_{\mathbb{R}^2}\frac{(|v||w|)^{-p\beta}}{|w-e_1|}dw\\
&= (1+|v|)^{p\beta-1}|v|^{1-p\beta}\int_{\mathbb{R}^2}\frac{1}{|w-e_1|\,|w|^{p\beta}}d^2w
\end{aligned}
$$

converges and is bounded uniformly for $|v| \ge 1$ if $1 < p\beta < 2$. For each fixed v with $|v| < 1$, we make the change of variables $u = w + v$ and use the bound $1+|w+v| \geq c(1+|w|)$ to see that the integral

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{(1+|u|)^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)}}{|u-v|} du &= (1+|v|)^{p(\alpha-\delta-1)} \int_{\mathbb{R}^2} \frac{(1+|w+v|)^{-p\beta}}{|w|} dw \\ &\leq c^{-p\beta}(1+|v|)^{p(\alpha-\delta-1)} \int_{\mathbb{R}^2} \frac{1}{|w|(1+|w|)^{p\beta}} dw\end{aligned}$$

again converges and is bounded uniformly for $|v| \leq 1$ if $p\beta > 1$. Similarly, for each fixed u with $|u| \geq 1$, let R be a rotation chosen so that $u = |u|Re_1$. Making the change of variables $v = |u|Rw$, we see that the integral

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{(1+|u|)^{q(\beta+\delta)}(1+|v|)^{-q\alpha}}{|u-v|} d^2v &= (1+|u|)^{q(\beta+\delta)}|u| \int_{\mathbb{R}^2} \frac{(1+|u||w|)^{-q\alpha}}{|w-e_1|} dw \\ &\leq (1+|u|)^{q(\beta+\delta)}|u| \int_{\mathbb{R}^2} \frac{(|u||w|)^{-q\alpha}}{|w-e_1|} dw \\ &= (1+|u|)^{q\alpha-1}|u|^{1-q\alpha} \int_{\mathbb{R}^2} \frac{1}{|w-e_1|\,|w|^{q\alpha}} dw\end{aligned}$$

converges and is bounded uniformly for $|u| \geq 1$ if $1 < q\alpha < 2$. For each fixed u with $|u| < 1$, we make the change of variables $v = w + u$ and use the bound $1 + |w+u| \geq c(1+|w|)$ to see that the integral

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{(1+|u|)^{q(\beta+\delta)}(1+|v|)^{-q\alpha}}{|u-v|} dv &= (1+|u|)^{q(\beta+\delta)} \int_{\mathbb{R}^2} \frac{(1+|w+u|)^{-q\alpha}}{|w|} dw \\ &\leq c^{-q\alpha}(1+|u|)^{q(\beta+\delta)} \int_{\mathbb{R}^2} \frac{1}{|w|(1+|w|)^{q\alpha}} dw\end{aligned}$$

again converges and is bounded uniformly for $|u| \leq 1$ if $q\alpha > 1$.

Thus, in order to guarantee that C is finite, it suffices to require that

$$\frac{1}{p} < \beta < \frac{2}{p} \qquad \text{and} \qquad \frac{1}{q} < \alpha < \frac{2}{q} \tag{4.50}$$

with

$$\delta = \alpha - \beta - \frac{1}{q} \tag{4.51}$$

As α and β run over the region (4.50), $-\beta$ runs over $-\frac{2}{p} < -\beta < -\frac{1}{p}$ and $\alpha - \beta - \frac{1}{q}$ runs over

$$-\frac{2}{p} = \frac{1}{q} - \frac{2}{p} - \frac{1}{q} < \alpha - \beta - \frac{1}{q} < \frac{2}{q} - \frac{1}{p} - \frac{1}{q} = 1 - \frac{2}{p}$$

Thus, if $p > 1$ and δ satisfies

$$-\frac{2}{p} < \delta < 1 - \frac{2}{p}$$

then it is always possible to select α and β such that (4.50) and (4.51) are satisfied. This completes the proof of Lemma 4.26 and consequently the proof of Lemma 4.21.

(b) For $f \in \mathcal{S}(\mathbb{R}^n)$, both

(4.52)

$$\int_{\mathbb{R}^2} \frac{1}{-(u_2 - v_2) + i(u_1 - v_1)} f(v_1, v_2)\, d^2v \qquad \text{and} \qquad -2\pi i \Big(\frac{\hat{f}}{-k_2 + ik_1}\Big)^\vee (u_1, u_2)$$

are bounded continuous functions. To show that they are equal, it suffices to show that they have the same inner products with all $g \in \mathcal{S}(\mathbb{R}^n)$. This follows from Problem 4.27, below.

$\square$

Internal remark 5. *Here is the justification for "This follows from Problem 4.27, below." For $f, g \in \mathcal{S}(\mathbb{R}^n)$, the Lebesgue dominated convergence theorem gives*

$$\begin{aligned}
&\int_{\mathbb{R}^4} g(u_1, u_2) \frac{1}{-(u_2 - v_2) + i(u_1 - v_1)} f(v_1, v_2)\ d^2u d^2v \\
&\qquad = \lim_{R\to\infty} \int_{\mathbb{R}^4} g(u_1, u_2) \frac{\chi(|u - v| < R)}{-(u_2 - v_2) + i(u_1 - v_1)} f(v_1, v_2)\ d^2u d^2v \\
&\qquad = \lim_{R\to\infty} \int_{\mathbb{R}^2} \hat{g}(k_1, k_2) C_R(k) \hat{f}(k_1, k_2)\ d^2k
\end{aligned}$$

where $C_R(k)$ is the Fourier transform of $\frac{\chi(|x|<R)}{-x_2+ix_1}$. By Problem 4.27, the integrand is bounded by the L^1 function $\hat{g}(k_1, k_2) \frac{4\pi}{|-k_2+ik_1|} \hat{f}(k_1, k_2)$ and approaches $\hat{g}(k_1, k_2) \frac{-2\pi i}{-k_2+ik_1} \hat{f}(k_1, k_2)$ pointwise as $R \to \infty$. So the claim follows by the Lebesgue dominated convergence theorem.

Exercise 4.27. (a) Prove that

$$\int_{|x|\le R} \frac{e^{-ik\cdot x}}{-x_2 + ix_1} d^2x = \frac{2i}{-k_2 + ik_1} \int_0^\pi d\theta\ \big[e^{-i|k|R\cos\theta} - 1\big]$$

(b) Prove that

$$\Big| \int_{|x|\le R} \frac{e^{-ik\cdot x}}{-x_2 + ix_1} d^2x \Big| \le \frac{4\pi}{|-k_2 + ik_1|} \qquad \text{and}$$

$$\lim_{R\to\infty} \int_{|x|\le R} \frac{e^{-ik\cdot x}}{-x_2 + ix_1} d^2x = \frac{-2\pi i}{-k_2 + ik_1}$$

for all $k \neq 0$.

4.6. Complex geometrical optics solutions: third proof

4.7. Reconstruction

Earlier we proved a uniqueness result in the Calderón problem, stating that if two positive conductivities $\gamma_1, \gamma_2 \in C^2(\overline{\Omega})$ satisfy $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then necessarily $\gamma_1 = \gamma_2$ in Ω. The proof was not constructive and did not give a procedure to determine γ from Λ_γ. In this section we will give a constructive proof that results in a reconstruction procedure for this inverse problem.

Theorem 4.28. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with smooth boundary and suppose that $\gamma \in C^2(\overline{\Omega})$ is a positive function. From the knowledge of the map*

$$\Lambda_\gamma : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$$

it is possible to determine γ in Ω in a constructive way.

As before, this result will be obtained as a consequence of a reconstruction procedure for the inverse problem for a Schrödinger equation.

Theorem 4.29. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with smooth boundary and suppose that $q \in L^\infty(\Omega)$. Assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. From the knowledge of the map*

$$\Lambda_q : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$$

it is possible to determine q in Ω in a constructive way.

Let us give an outline of the proof. It is very similar to the uniqueness proof earlier, and relies on complex geometrical optics solutions $u_\zeta(x) = e^{\zeta \cdot x}(1 + r(x,\zeta))$ for $\zeta \in \mathbb{C}^n$ such that $\zeta \cdot \zeta = 0$ and $|\zeta|$ is large. Even though the original problem is stated in the domain Ω, we extend q by zero outside of Ω and consider the solutions u_ζ in $\mathbb{R}^n$. It is important that the solution u_ζ is unique as long as $r(x,\zeta)$ satisfies a decay condition as $|x| \to \infty$.

It will be possible to characterize the boundary value $u_\zeta|_{\partial\Omega}$ as the unique solution $f \in H^{3/2}(\partial\Omega)$ of the following integral equation on the boundary:

$$(\mathrm{Id} + S_\zeta(\Lambda_q - \Lambda_0))f = e^{\zeta \cdot x} \qquad \text{on } \partial\Omega.$$

Here S_ζ is a modified single layer potential depending on the complex vector ζ. The point is that the operator on the left hand side only depends on the data Λ_q and other known quantities, so one can compute $u_\zeta|_{\partial\Omega}$ from the boundary data by solving this integral equation. Using these functions in a suitable integral identity and taking a limit as $|\zeta| \to \infty$ allows to recover the Fourier transform of q.

Before going to the proof of Theorem 4.29, let us see how Theorem 4.28 follows from it.

Proof. Proof of Theorem 4.28 Suppose that one is given the map $\Lambda_\gamma : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$. If q is defined by $q = \Delta\gamma^{1/2}/\gamma^{1/2}$, it was proved in Theorem ??? that

$$\Lambda_q f = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2}f) + \frac{1}{2}\gamma^{-1}(\partial_\nu\gamma)f|_{\partial\Omega}.$$

From Theorem ???, we know that from the knowledge of Λ_γ it is possible to reconstruct the boundary value $\gamma|_{\partial\Omega}$ and the normal derivative $\partial_\nu\gamma|_{\partial\Omega}$. Thus, we have access to the map $\Lambda_q : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$.

By Theorem 4.29, we can determine q constructively from this information. Consider the unique weak solution $v \in H^1(\Omega)$ of the equation

$$(-\Delta + q)v = 0 \qquad \text{in } \Omega$$

with boundary value $v|_{\partial\Omega} = \gamma^{1/2}|_{\partial\Omega}$. Since the coefficient q and the boundary value are known, we can compute the solution v. But the function $\gamma^{1/2}$ solves this Dirichlet problem, so we also know $v = \gamma^{1/2}$. This determines γ in Ω. $\square$

Assume that $q \in L^\infty(\Omega)$ is such that 0 is not a Dirichlet eigenvalue of $-\Delta+q$ in Ω. The objective is to reconstruct q in Ω from the knowledge of Λ_q. The first step is to extend q by zero into $\mathbb{R}^n$, with the extension also denoted by q. We then have the following complex geometrical optics solutions.

Proposition 4.30. *Let $q \in L^\infty_{comp}(\mathbb{R}^n)$, and fix δ with $-1 < \delta < 0$. There exists $C = C(\delta, q) > 0$ such that for any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq C$, there exists a unique solution*

$$u(x) = u_\zeta(x) = e^{\zeta\cdot x}(1 + r(x,\zeta))$$

of the equation $(-\Delta + q)u = 0$ in $\mathbb{R}^n$ where $r(\,\cdot\,,\zeta) \in L^2_\delta(\mathbb{R}^n)$. Moreover, $u \in H^2_{loc}(\mathbb{R}^n)$, and one has

$$\|r\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|}.$$

In the uniqueness proof of the inverse problem for the Schrödinger equation, we began from the assumption $\Lambda_{q_1} = \Lambda_{q_2}$ and used the integral identity

$$\langle(\Lambda_{q_1} - \Lambda_{q_2})(u_1|_{\partial\Omega}), u_2|_{\partial\Omega}\rangle_{\partial\Omega} = \int_\Omega (q_1 - q_2)u_1u_2\,dx$$

where u_j are solutions of $(-\Delta + q_j)u_j = 0$ in Ω. We then chose complex geometrical optics solutions u_j such that $u_1u_2 \approx e^{ix\cdot\xi}$ for large values of the complex vector. Since the left hand side of the identity is zero, this resulted in the vanishing of the Fourier transform of $q_1 - q_2$.

In the reconstruction problem we are instead given the DN map Λ_q for an unknown potential q, and the objective is to determine the Fourier transform of q from this information. This will be achieved by comparing Λ_q to Λ_0, the DN map with zero potential. The next result shows how this is precisely done.

Proposition 4.31. *Let $q \in L^\infty(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. Let also $\xi \in \mathbb{R}^n$ with $\xi \neq 0$. If $s > 0$ is sufficiently large, there exist $\zeta_j = \zeta_j(s, \xi) \in \mathbb{C}^n$ with $\zeta_j \cdot \zeta_j = 0$ and $|\zeta_j| = s$ for $j = 1, 2$, such that*

$$\lim_{s\to\infty} \langle (\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial\Omega}), e^{\zeta_2 \cdot x}|_{\partial\Omega}\rangle_{\partial\Omega} = \int_\Omega q e^{ix\cdot\xi}\, dx.$$

Here u_{ζ_1} is the solution of $(-\Delta + q)u = 0$ in $\mathbb{R}^n$ given in Proposition 4.30.

Proof. Let $\alpha, \beta \in \mathbb{R}^n$ be such that $\{\alpha, \beta, \xi/|\xi|\}$ is an orthonormal set in $\mathbb{R}^n$. We define complex vectors

$$\zeta_1 = \frac{s}{\sqrt{2}}(\alpha + i(\frac{\xi}{2} + \sqrt{\frac{s^2}{2} - \frac{\xi^2}{4}}\beta)),$$
$$\zeta_2 = \frac{s}{\sqrt{2}}(-\alpha + i(\frac{\xi}{2} - \sqrt{\frac{s^2}{2} - \frac{\xi^2}{4}}\beta)).$$

By using the fact that $\alpha, \beta, \xi/|\xi|$ are orthonormal, it follows that $\zeta_j \cdot \zeta_j = 0$ and $|\zeta_j| = s$. The main point in the choice of ζ_1 and ζ_2 is that

$$\zeta_1 + \zeta_2 = i\xi.$$

Let u_{ζ_1} be the solution of $(-\Delta + q)u = 0$ in $\mathbb{R}^n$ in Proposition 4.30, and note that $e^{\zeta_2\cdot x}$ solves the same equation with zero potential, that is, $\Delta(e^{\zeta_2 \cdot x}) = 0$. The integral identity in Theorem ??? implies that

$$\begin{aligned}\langle (\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial\Omega}), e^{\zeta_2\cdot x}|_{\partial\Omega}\rangle_{\partial\Omega} &= \int_\Omega q u_{\zeta_1} e^{\zeta_2\cdot x}\, dx \\ &= \int_\Omega q e^{(\zeta_1+\zeta_2)\cdot x}(1 + r(x, \zeta_1))\, dx \\ &= \int_\Omega q e^{ix\cdot\xi}(1 + r(x, \zeta_1))\, dx.\end{aligned}$$

Since $\|r(\,\cdot\,, \zeta_1)\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C}{s}$, the result follows by taking the limit as $s \to \infty$. $\square$

From Proposition 4.31 we see that the Fourier transform of q at nonzero frequencies ξ can be recovered from the map Λ_q, as long as the boundary value $u_{\zeta_1}|_{\partial\Omega}$ of the solution in Proposition 4.30 can be somehow determined *from Λ_q. Since q is compactly supported, its Fourier transform is continuous and would therefore be determined also at $\xi = 0$.*

The determination of $u_\zeta|_{\partial\Omega}$ from Λ_q will require certain facts on layer potentials. This may be motivated as follows.

Motivation 4.32. Since q is extended by zero outside of Ω, the equation $(-\Delta + q)u_\zeta = 0$ in $\mathbb{R}^n$ implies

$$-\Delta u_\zeta = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

Thus u_ζ is harmonic in the exterior domain $\mathbb{R}^n \setminus \overline{\Omega}$. Note that this equation does not involve the unknown potential q. Writing $u_\zeta = e^{\zeta \cdot x} + v_\zeta$, we see that also the correction term v_ζ is harmonic in the exterior domain. Further, the function $e^{-\zeta \cdot x} v_\zeta$ satisfies a decay condition at infinity by construction.

It is known that harmonic functions in the exterior domain satisfying certain decay conditions can be represented in terms of single layer potentials, which are integral operators mapping functions on $\partial\Omega$ to functions in $\mathbb{R}^n$. In our case we will have

$$v_\zeta|_{\mathbb{R}^n \setminus \overline{\Omega}} = S_\zeta h|_{\mathbb{R}^n \setminus \overline{\Omega}}$$

for some function $h \in H^{1/2}(\partial\Omega)$, where S_ζ is a modified (or Faddeev type) single layer potential that differs from the standard one by factors of $e^{\zeta \cdot x}$ in its integral kernel. It will turn out that $h = -(\Lambda_q - \Lambda_0)(u_\zeta|_{\partial\Omega})$, and collecting these facts gives that

$$u_\zeta = e^{\zeta \cdot x} - S_\zeta(\Lambda_q - \Lambda_0)(u_\zeta|_{\partial\Omega}) \quad \text{in } \Omega_+.$$

The integral equation characterizing $u_\zeta|_{\partial\Omega}$ in terms of Λ_q follows by restricting the last identity on $\partial\Omega$.

Let us begin the development of the required layer potentials. We decompose $\mathbb{R}^n$ in three disjoint parts,

$$\mathbb{R}^n = \Omega_- \cup \Gamma \cup \Omega_+,$$

where $\Omega_- = \Omega$ is the interior domain, $\Gamma = \partial\Omega$ is the boundary of Ω, and $\Omega_+ = \mathbb{R}^n \setminus \overline{\Omega}$ is the exterior domain.

Consider the trace operator on Γ,

$$\gamma : H^1(\mathbb{R}^n) \to H^{1/2}(\Gamma), \quad \gamma u = u|_{\partial\Omega},$$

and the corresponding trace operators from the interior and exterior,

$$\gamma_- : H^1(\Omega) \to H^{1/2}(\Gamma), \quad \gamma_- u = u|_\Gamma,$$
$$\gamma_+ : H^1(\Omega_+) \to H^{1/2}(\Gamma), \quad \gamma_+ u = u|_\Gamma.$$

Since taking traces is a local operation near Γ, the operators $\gamma_\pm$ can also be applied to functions that are in $H^1(U \cap \Omega_\pm)$ where U is some open set containing Γ. If u is a function that is H^1 is a full neighborhood U of Γ, it follows that

$$\gamma_- u = \gamma_+ u = \gamma u.$$

If u is a H^2 function in $U \cap \Omega_\pm$ where U is some neighborhood of Γ, we denote by $(\partial_\nu u)_\pm$ the normal derivative of u from the interior or exterior. Note that in this case $(\partial_\nu u)_\pm$ is in $H^{1/2}(\Gamma)$.

The standard single layer potential on $\mathbb{R}^n$ will be obtained from the fundamental solution of the Laplacian, given by the Newtonian potential in the next problem.

Exercise 4.33. Let $n \geq 3$, and let

$$k_0(x) = c_n \, |x|^{2-n}, \quad x \in \mathbb{R}^n.$$

Show that this function gives rise to a convolution operator

$$K_0 : L^2_{comp}(\mathbb{R}^n) \to H^2_{loc}(\mathbb{R}^n), \quad K_0 f(x) = \int_{\mathbb{R}^n} k_0(x-y) f(y)\, dy$$

with the property that

$$-\Delta K_0 f = f, \qquad f \in L^2_{comp}(\mathbb{R}^n).$$

Proof. 1. Let F_1 be a compact set in $\mathbb{R}^n$, and let $f \in L^2(\mathbb{R}^n)$ with $\mathrm{supp}\,(f) = F_2$ compact. We will show that

$$\|K_0 f\|_{L^2(F_1)} \leq C_{F_1,F_2} \, \|f\|_{L^2(F_2)} \,. \tag{4.53}$$

This proves that $K_0 f \in L^2_{loc}$ whenever $f \in L^2_{comp}$. Define

$$F = \{x - y\,;\, x \in F_2, y \in F_1\}.$$

Then also F is a compact set, and we have

$$K_0 f(x) = \int_{\mathbb{R}^n} \chi_F(x-y) k_0(x-y) f(y)\, dy, \qquad x \in F_1.$$

Since $\chi_F k_0 \in L^1(\mathbb{R}^n)$, Young's inequality for convolutions (Lemma ???) implies that

$$\|K_0 f\|_{L^2(F_1)} \leq \|\chi_F k_0\|_{L^1(\mathbb{R}^n)} \, \|f\|_{L^2(F_2)} \leq C \, \|f\|_{L^2(F_2)} \,.$$

2. We next show that

$$-\Delta K_0 f = f, \qquad f \in C_c^\infty(\mathbb{R}^n). \tag{4.54}$$

Do the details...

3. The next step is to show that

$$-\Delta K_0 f = f, \qquad f \in L^2_{comp}(\mathbb{R}^n). \tag{4.55}$$

Let $f \in L^2(\mathbb{R}^n)$ with $\mathrm{supp}\,(f)$ compact. Choose a sequence $(f_j) \subset C_c^\infty(\mathbb{R}^n)$ so that $f_j \to f$ in $L^2(\mathbb{R}^n)$ and $\mathrm{supp}\,(f_j)$ lies in a fixed compact set F_2 for each j. By (4.53), we have

$$K_0 f_j \to K_0 f \qquad \text{in } L^2_{loc}(\mathbb{R}^n).$$

Then also

$$-\Delta K_0 f_j \to -\Delta K_0 f \qquad \text{in } H^{-2}_{loc}(\mathbb{R}^n).$$

Moreover, we have already seen that $-\Delta K_0 f_j = f_j$, so also

$$-\Delta K_0 f_j \to f \qquad \text{in } L^2(\mathbb{R}^n).$$

Uniqueness of limits implies (4.55).

4. It remains to show that $K_0 f \in H^2_{loc}$ whenever $f \in L^2_{comp}$. By the previous arguments, $u = K_0 f$ satisfies

$$-\Delta u = f \in L^2_{comp}, \quad u \in L^2_{loc}.$$

Interior elliptic regularity (Theorem ???) readily implies that $u \in H^2_{loc}$. $\square$

Definition 4.34. The standard single layer potential on $\mathbb{R}^n$ is defined as the operator

$$S_0 = K_0 \gamma^* : H_{-1/2}(\Gamma) \to H^1_{loc}(\mathbb{R}^n).$$

Proposition 4.35. *Let $\zeta \in \mathbb{C}^n$ satisfy $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, and let $-1 < \delta < 0$. There is a linear operator*

$$G_\zeta : L^2_{\delta+1}(\mathbb{R}^n) \to H^2_\delta(\mathbb{R}^n)$$

such that for any $f \in L^2_{\delta+1}(\mathbb{R}^n)$, the function $u = G_\zeta f$ is the unique solution in $L^2_\delta(\mathbb{R}^n)$ of the equation $e^{-\zeta\cdot x}(-\Delta)(e^{\zeta\cdot x}u) = f$ in $\mathbb{R}^n$. One has the norm bounds

$$\begin{aligned}
\|G_\zeta f\|_{L^2_\delta} &\leq \frac{C}{|\zeta|}\,\|f\|_{L^2_{\delta+1}}\,,\\
\|G_\zeta f\|_{H^1_\delta} &\leq C\,\|f\|_{L^2_{\delta+1}}\,,\\
\|G_\zeta f\|_{H^2_\delta} &\leq C\,|\zeta|\,\|f\|_{L^2_{\delta+1}}\,.
\end{aligned}$$

Proof. Follows from Theorem ???. $\square$

We will obtain the modified single layer potential S_ζ from the following inverse operator K_ζ of the Laplacian.

Proposition 4.36. *Let $\zeta \in \mathbb{C}^n$ satisfy $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, and let $-1 < \delta < 0$. The operator*

$$K_\zeta : L^2_{comp}(\mathbb{R}^n) \to H^2_{loc}(\mathbb{R}^n), \quad K_\zeta f = e^{\zeta\cdot x} G_\zeta(e^{-\zeta\cdot x} f)$$

satisfies

$$-\Delta K_\zeta f = f, \qquad \text{in } f \in L^2_{comp}(\mathbb{R}^n).$$

Proof. This follows directly from Proposition 4.35. If $f \in L^2_{comp}$, then $e^{-\zeta\cdot x} f \in L^2_{comp}$ and thus $G_\zeta(e^{-\zeta\cdot x} f) \in H^2_\delta$ whenever $-1 < \delta < 0$. It follows that $K_\zeta f \in H^2_{loc}$ for any $f \in L^2_{comp}$. Also, if $u = G_\zeta(e^{-\zeta\cdot x} f)$, we have $e^{-\zeta\cdot x}(-\Delta)(e^{\zeta\cdot x} u) = e^{-\zeta\cdot x} f$, showin that $-\Delta K_\zeta f = f$. □

The next result shows that the operator K_ζ differs from the usual fundamental solution K_0 of the Laplacian by a smoothing operator.

Proposition 4.37. *Let $\zeta \in \mathbb{C}^n$ satisfy $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, and let $-1 < \delta < 0$. Then*

$$K_\zeta = K_0 + R_\zeta$$

where R_ζ is an operator satisfying for any $k \geq 0$

$$R_\zeta : H^{-k}_{comp}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n).$$

There is a function $r_\zeta \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$R_\zeta f(x) = \int_{\mathbb{R}^n} r_\zeta(x,y) f(y)\, dy, \qquad f \in L^1_{comp}(\mathbb{R}^n).$$

Proof. By the mapping properties of K_ζ and K_0, we may define

$$R_\zeta = K_\zeta - K_0 : L^2_{comp}(\mathbb{R}^n) \to H^2_{loc}(\mathbb{R}^n).$$

Since both K_ζ and K_0 are right inverses of the Laplacian, we also have

$$-\Delta R_\zeta f = 0, \qquad f \in L^2_{comp}(\mathbb{R}^n).$$

Elliptic regularity (Theorem ???) implies that $R_\zeta f \in H^k_{loc}(\mathbb{R}^n)$ whenever $f \in L^2_{comp}(\mathbb{R}^n)$, and for any bounded open sets $U, V \subseteq \mathbb{R}^n$ one has the estimate

$$\|R_\zeta f\|_{H^k(U)} \leq C_U \|f\|_{L^2(V)}, \qquad f \in L^2(\mathbb{R}^n),\ \operatorname{supp}(f) \subseteq V.$$

Let now φ and ψ be any functions in $C^\infty_c(\mathbb{R}^n)$.

$$\varphi R_\zeta \psi : L^2(\mathbb{R}^n) \to H^k(\mathbb{R}^n).$$

□

The next result considers solutions of the equation $(-\Delta + q)u = 0$ in $\mathbb{R}^n$, where q vanishes outside Ω, having the form

$$u = u_0 + e^{\zeta\cdot x} r$$

where u_0 is any harmonic function in $H^2_{\mathrm{loc}}(\mathbb{R}^n)$, and $r \in H^1_\delta(\mathbb{R}^n)$. We will later take $u_0 = e^{\zeta\cdot x}$, but the following equivalences work for any harmonic function u_0.

Proposition 4.38. *Let $q \in L^\infty(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω, let $\zeta \in \mathbb{C}^n$ satisfy $\zeta \cdot \zeta = 0$ with $|\zeta|$ sufficiently large, and let $-1 < \delta < 0$. Further, let $u_0 \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $\Delta u_0 = 0$ in $\mathbb{R}^n$. Consider the following problems:*

$$\text{(DE)} \begin{cases} (-\Delta + q)u = 0 \text{ in } \mathbb{R}^n \\ e^{-\zeta\cdot x}(u - u_0) \in H^1_\delta(\mathbb{R}^n), \end{cases}$$

$$\text{(IE)} \begin{cases} u + K_\zeta(qu) = u_0 \text{ in } \mathbb{R}^n \\ u \in H^2_{loc}(\mathbb{R}^n), \end{cases}$$

$$\text{(EP)} \begin{cases} \text{i)} \quad \Delta u = 0 \text{ in } \Omega_+ \\ \text{ii)} \quad u = \tilde{u}|_{\Omega_+} \text{ for some } \tilde{u} \in H^2_{\mathrm{loc}}(\mathbb{R}^n) \\ \text{iii)} \quad e^{-\zeta\cdot x}(u - u_0)|_{\Omega_+} = \tilde{r}|_{\Omega_+} \text{ for some } \tilde{r} \in H^1_\delta(\mathbb{R}^n) \\ \text{iv)} \quad (\partial_\nu u)_+ = \Lambda_q(\gamma_+ u) \text{ on } \Gamma, \end{cases}$$

$$\text{(BE)} \begin{cases} (\mathrm{Id} + \gamma S_\zeta(\Lambda_q - \Lambda_0))f = u_0 \text{ on } \Gamma \\ f \in H^{3/2}(\Gamma). \end{cases}$$

Each of these problems has a unique solution. Further, these problems are equivalent in the sense that u solves (DE) iff u solves (IE), if u solves (DE) then $u|_{\Omega_+}$ solves (EP), if u solves (EP) then there is a solution $\tilde{u}$ of (DE) with $\tilde{u}|_{\Omega_+} = u$, if u solves (DE) then $f = u|_\Gamma$ solves (BE), and finally if f solves (BE) then there is a solution u of (DE) with $u|_\Gamma = f$.

Proof. The function $u = u_0 + e^{\zeta\cdot x} r$ solves $(-\Delta + q)u = 0$ in $\mathbb{R}^n$ if and only if

$$e^{-\zeta\cdot x}(-\Delta + q)e^{\zeta\cdot x} r = -e^{-\zeta\cdot x} q u_0 \quad \text{in } \mathbb{R}^n.$$

The right hand side is in $L^2_c(\mathbb{R}^n)$, so by Proposition 4.35 there is a unique solution $r \in H^1_\delta(\mathbb{R}^n)$ where $-1 < \delta < 0$ if $|\zeta|$ is sufficiently large. This proves that (DE) has a unique solution. It remains to prove that all four problems are equivalent in the sense described above.

(DE) $\implies$ (IE): Assume u solves (DE). Then $u = u_0 + e^{\zeta\cdot x} r$ where $r \in H^1_\delta(\mathbb{R}^n)$, and

$$e^{-\zeta\cdot x}(-\Delta + q)e^{\zeta\cdot x} r = -e^{-\zeta\cdot x} q u_0 \quad \text{in } \mathbb{R}^n.$$

By Proposition 4.35 we have $r = G_\zeta v \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$ where v satisfies

$$v + qr = -e^{-\zeta\cdot x} q u_0.$$

Since q is compactly supported in $\mathbb{R}^n$ also $v = -q(r + e^{-\zeta\cdot x} u_0)$ is compactly supported. Thus we may apply G_ζ to both sides of the last identity to obtain

$$r + G_\zeta(qr) = -G_\zeta(e^{-\zeta\cdot x} q u_0).$$

Multiplying by $e^{\zeta\cdot x}$ and adding u_0 to both sides gives (IE).

(IE) $\implies$ (DE): Assume u solves (IE). Then the function $r = e^{-\zeta\cdot x}(u - u_0)$ satisfies

$$r = -G_\zeta(e^{-\zeta\cdot x}qu). \tag{4.56}$$

This shows that $r \in H^1_\delta(\mathbb{R}^n)$, and (DE) follows by applying $-\Delta$ to both sides of (IE).

(DE) $\implies$ (EP): Let $\tilde{u}$ solve (DE), and define $u = \tilde{u}|_{\Omega_+}$. Clearly properties i), ii) and iii) of (EP) are valid. We need to show iv). Since $\tilde{u}$ solves the equation $(-\Delta + q)\tilde{u} = 0$ in Ω, we have

$$(\partial_\nu u)_+ = \partial_\nu \tilde{u}|_\Gamma = \Lambda_q(\tilde{u}|_\Gamma) = \Lambda_q(\gamma_+ u).$$

(EP) $\implies$ (DE): Suppose u solves (EP). Define $v \in H^2(\Omega)$ as the unique solution of the equation $(-\Delta + q)v = 0$ in Ω with $v|_\Gamma = \gamma_+ u|_\Gamma$, and define

$$\tilde{u}(x) = \begin{cases} v(x), & x \in \Omega, \\ u(x), & x \in \Omega_+. \end{cases}$$

Then $\gamma_- \tilde{u}|_\Gamma = \gamma_+ \tilde{u}|_\Gamma$ and

$$(\partial_\nu \tilde{u})_-|_\Gamma = \Lambda_q(\gamma_+ u|_\Gamma) = (\partial_\nu \tilde{u})_+|_\Gamma$$

by (EP) iv). It follows that $\tilde{u} \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $(-\Delta + q)\tilde{u} = 0$ in $\mathbb{R}^n$. Further, $e^{-\zeta\cdot x}(\tilde{u} - u_0) \in H^1_\delta(\mathbb{R}^n)$ by (EP) iii).

(DE) $\implies$ (BE): Let u solve (DE), and let $f = u|_\Gamma$. We fix a point $x \in \Omega_+$ and let $v(y) = K_\zeta(x, y)$ where $y \in \Omega$. This is a smooth function in Ω by Lemma ???.

Now Green's theorem implies

$$\int_\Gamma (u\partial_\nu v - v\partial_\nu u)\, dS = \int_\Omega (u\Delta v - v\Delta u)\, dy.$$

By (DE) we have $\Delta u = qu$ and $\partial_\nu u|_\Gamma = \Lambda_q f$. Using the properties in Lemma ??? we obtain

$$\int_\Gamma u\partial_\nu v\, dS - S_\zeta \Lambda_q f(x) = -K_\zeta(qu)(x),$$

which is valid for $x \in \Omega_+$. The function v is harmonic in Ω, hence $\partial_\nu v|_\Gamma = \Lambda_0(v|_\Gamma)$. The symmetry of Λ_0 implies

$$\int_\Gamma u\partial_\nu v\, dS = \int_\Gamma u\Lambda_0(v|_\Gamma)\, dS = \int_\Gamma \Lambda_0(u|_\Gamma)v\, dS = S_\zeta \Lambda_0 f(x).$$

We obtain

$$S_\zeta(\Lambda_q - \Lambda_0)f = K_\zeta(qu) \quad \text{in } \Omega_+. \tag{4.57}$$

Adding u to both sides, using the fact that u solves (IE), and taking traces on Γ gives (BE).

(BE) $\implies$ (EP): Let f solve (BE). We define a function $\tilde{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$\tilde{u} = u_0 - S_\zeta(\Lambda_q - \Lambda_0)f.$$

This function is harmonic in $\mathbb{R}^n \setminus \Gamma$ by Lemma ???, and $\tilde{u}|_\Gamma = f$ by using (BE). The jump relation for S_ζ implies that on Γ

$$(\partial_\nu \tilde{u})_- - (\partial_\nu \tilde{u})_+ = -(\Lambda_q - \Lambda_0)f.$$

But $(\partial_\nu \tilde{u})_- = \Lambda_0 f$, so we have $(\partial_\nu \tilde{u})_+ = \Lambda_q(\gamma_+ \tilde{u})$. Therefore $\tilde{u}|_{\Omega_+}$ satisfies (EP) i) and iv). Also (EP) ii) is valid by mapping properties of S_ζ.

To prove (EP) iii) it is sufficient to show that for any $h \in H^{1/2}(\Gamma)$,

$$e^{\zeta \cdot x} S_\zeta h|_{\Omega_+} = w|_{\Omega_+} \quad \text{for some } w \in H^1_\delta(\mathbb{R}^n).$$

Formally one has $e^{\zeta \cdot x} S_\zeta h = G_\zeta e^{\zeta \cdot x} \gamma^* h$ where G_ζ maps $L^2_c(\mathbb{R}^n)$ to $H^1_\delta(\mathbb{R}^n)$. However, we have not proved that G_ζ has good mapping properties on negative order Sobolev spaces. □

Finally, let us verify that the boundary integral equation (BE) in Proposition 4.38 is indeed Fredholm.

Proposition 4.39. *The operator*

$$\gamma S_\zeta(\Lambda_q - \Lambda_0) : H^{3/2}(\Gamma) \to H^{3/2}(\Gamma)$$

is compact.

Proof. Let $f \in H^{3/2}(\Gamma)$, and let $u = P_q f$ where $P_q : H^{3/2}(\Gamma) \to H^2(\Omega)$ is the Poisson operator mapping h_0 to v_0 where $(-\Delta + q)v_0 = 0$ in Ω and $v_0|_\Gamma = h_0$. The exact same argument leading to (4.57) in the proof of Proposition 4.38 shows that

$$S_\zeta(\Lambda_q - \Lambda_0)f = K_\zeta(qEJu) \quad \text{in } \Omega_+$$

where $E : L^2(\Omega) \to L^2(\mathbb{R}^n)$ is extension by zero and $J : H^2(\Omega) \to L^2(\Omega)$ is the natural inclusion. Taking traces on Γ, we obtain the factorization

$$\gamma S_\zeta(\Lambda_q - \Lambda_0) = \gamma K_\zeta q E J P_q.$$

The result follows since on the right hand side J is compact and all other operators are bounded. □

4.8. Old problems for $n = 2$

Hypothesis 4.40. There is a function[1]As in **??**, we have made this assumption stronger than necessary for simplicity of notation. It is easy to generalize the results of this section to $\gamma \in C^\ell(\overline{\Omega})$ for suitable ℓ. $\gamma \in C^\infty(\overline{\Omega})$ such that

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the conductivity at $x \in \Omega$ is $\gamma(x)\mathbb{1}$ where $\mathbb{1}$ is the $n \times n$ identity matrix. We also use the notation $\gamma(x)$ to represent the matrix valued function $\gamma(x)\mathbb{1}$.

The following problems provide an introduction to the complex derivative operators

$$\bar\partial = \tfrac{1}{2}\big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\big) \qquad \text{and} \qquad \partial = \tfrac{1}{2}\big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\big)$$

The differential operator $\bar\partial = \frac{1}{2}\big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\big)$ corresponds, upon Fourier transforming, to multiplication by $\frac{1}{2}(ik_1 - k_2)$. By (4.52), convolution by $2\frac{1}{-2\pi i}\frac{1}{-x_2+ix_1} = \frac{1}{\pi}\frac{1}{x_1+ix_2}$ provides an inverse to that differential operator. Similarly, convolution by $\frac{1}{\pi}\frac{1}{x_1-ix_2}$ provides an inverse to the differential operator $\partial = \frac{1}{2}\big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\big)$. So we define, for various classes of functions f, to be made precise below,

$$\partial^{-1} f(z) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{1}{\bar z - \bar\zeta} f(\zeta)d\mu(\zeta) \qquad \bar\partial^{-1} f(z) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{1}{z - \zeta} f(\zeta)d\mu(\zeta)$$

where $d\mu$ is Lebesgue measure on $\mathbb{R}^2$.

Exercise 4.41. Let Ω be an open subset of $\mathbb{R}^2$ and let $f \in L^1(\mathbb{R}^2)$ vanish in Ω. Prove that $\bar\partial^{-1} f(z)$ and $\overline{\partial^{-1} f(z)}$ are well–defined and analytic for $z \in \Omega$.

Exercise 4.42. Prove that if $f \in C_0^1(\mathbb{R}^2)$, then

$$\partial^{-1}\partial f = f \qquad \text{and} \qquad \bar\partial^{-1}\bar\partial f = f$$

Many of the regularity properties of ∂^{-1} and $\bar\partial^{-1}$ will be stated in terms of the norms

$$\|f\|_{C^\epsilon} = \|f\|_{L^\infty} + |f|_{C^\epsilon} \qquad \text{where} \qquad |f|_{C^\epsilon} = \sup_{z \ne w} \frac{|u(z) - u(w)|}{|z - w|^\epsilon}$$

for the space $C^\epsilon(\mathbb{R}^2)$, with $0 < \epsilon < 1$. Then $C^{1+\epsilon}(\mathbb{R}^2)$ is the collection of functions for which the norm $\|u\|_{L^\infty(\mathbb{R}^2)} + \|\nabla u\|_{C^\epsilon(\mathbb{R}^2)}$ is finite. More generally, if $n \in \mathbb{N}_0$, $C^{n+\epsilon}(\mathbb{R}^2)$ is the collection of functions for which the norm $\sum_{\substack{\alpha \in \mathbb{N}_0^2 \\ |\alpha| < n}} \|\partial^\alpha u\|_{L^\infty(\mathbb{R}^2)} + \sum_{\substack{\alpha \in \mathbb{N}_0^2 \\ |\alpha| = n}} \|\partial^\alpha u\|_{C^\epsilon(\mathbb{R}^2)}$ is finite. The next few problems concern $C^\epsilon(\mathbb{R}^2)$.

Exercise 4.43. Let $0 < \epsilon < 1$. Prove that it $f, g \in C^\epsilon(\mathbb{R}^n)$, then $fg \in C^\epsilon(\mathbb{R}^n)$ and

$$\|fg\|_{C^\epsilon(\mathbb{R}^n)} \le \|f\|_{C^\epsilon(\mathbb{R}^n)}\|g\|_{C^\epsilon(\mathbb{R}^n)}$$

Exercise 4.44. Let $0 < \epsilon < 1$. Prove that it $f \in C^1(\mathbb{R}^n)$ is bounded with bounded first partial derivatives, then $f \in C^\epsilon(\mathbb{R}^n)$ and

$$\|f\|_{C^\epsilon(\mathbb{R}^n)} \le \|f\|_{L^\infty}^{1-\epsilon}\big(\|f\|_{L^\infty}^\epsilon + 2\|\nabla f\|_{L^\infty}^\epsilon\big)$$

Exercise 4.45. Let $0 < \epsilon < 1$.

(a) Let the Fourier transform $\hat{f}$ of $f \in L^1(\mathbb{R}^n)$ obey $\big(1+|k|^\epsilon\big)\hat{f}(k) \in L^1(\mathbb{R}^n)$. Prove that f has a representative in $C^\epsilon(\mathbb{R}^n)$ with

$$\|f\|_{C^\epsilon(\mathbb{R}^n)} \le \big\|\big(1+|k|^\epsilon\big)\hat{f}(k)\big\|_{L^1}$$

(b) Let $f(x) \in C^\epsilon(\mathbb{R}^n)$ vanish for $|x| > R$. Prove that there is a constant $C(R, n)$, depending only on R and n such that

$$\big|\hat{f}(k)\big| \le \frac{C(R,n)}{1+|k|^\epsilon}\|f\|_{C^\epsilon}$$

(c) Let $f(x) \in C^\epsilon(\mathbb{R}^n)$ vanish for $|x| > R$. Prove that if $0 < s < \epsilon$, then $f \in H^s(\mathbb{R}^n)$ and that there is a constant C, depending only on R, n and $\epsilon - s$ such that

$$\mathrm{v}f\mathrm{v}_s \le C\|f\|_{C^\epsilon}$$

Now here is a problem which collects together some regularity properties of ∂^{-1} and $\bar\partial^{-1}$. Part (c) also contains the result that $\partial\partial^{-1}f = \bar\partial\bar\partial^{-1}f = f$, at least for compactly supported $f \in C^\epsilon(\mathbb{R}^2)$ with $\epsilon > 0$. Recall that we have already shown, in Problem 4.42, that $\partial^{-1}\partial f = \bar\partial^{-1}\bar\partial f = f$ for $f \in C_0^1(\mathbb{R}^2)$.

Exercise 4.46. Let $0 < \epsilon < 1$ and K be any compact subset of $\mathbb{R}^2$.

(a) Prove that there is a constant $C(K, \epsilon)$ such that if $f \in L^\infty(\mathbb{R}^2)$ is supported in K, then $\partial^{-1}f, \bar\partial^{-1}f \in C^\epsilon(\mathbb{R}^2)$ and

$$\|\partial^{-1}f\|_{C^\epsilon(\mathbb{R}^2)}\,,\ \|\bar\partial^{-1}f\|_{C^\epsilon(\mathbb{R}^2)} \le C(K,\epsilon)\|f\|_{L^\infty}$$

(b) Let $n \in \mathbb{N}$. Prove that there is a constant $C(K, n, \epsilon)$ such that if $f \in C^n(\mathbb{R}^2)$ is supported in K, then $\partial^{-1}f, \bar\partial^{-1}f \in C^{n+\epsilon}(\mathbb{R}^2)$,

$$\partial^\alpha\partial^{-1}f(z) = \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{1}{\bar z-\bar\zeta}\partial^\alpha f(\zeta)\,d\mu(\zeta) \qquad \partial^\alpha\bar\partial^{-1}f(z) = \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{1}{z-\zeta}\partial^\alpha f(\zeta)\,d\mu(\zeta)$$

for all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \le n$, and

$$\|\partial^{-1}f\|_{C^{n+\epsilon}(\mathbb{R}^2)}\,,\ \|\bar\partial^{-1}f\|_{C^{n+\epsilon}(\mathbb{R}^2)} \le C(K,n,\epsilon)\|f\|_{C^n(\mathbb{R}^2)}$$

(c) Let $f \in C^\epsilon(\mathbb{R}^2)$ be supported in K and let $\chi \in C_0^\infty(\mathbb{R}^2)$ be identically one on K. Prove that, for each $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 1$, the first order partial derivatives $\partial^\alpha\partial^{-1}f$ and $\partial^\alpha\bar\partial^{-1}f$ exist and

$$\partial^\alpha \partial^{-1} f(z) = -\frac{(-i)^{\alpha_2}}{\pi} \int_{\mathbb{R}^2} \frac{1}{(\bar{z}-\bar{\zeta})^2} \chi(\zeta)\big[f(\zeta) - f(z)\big]\, d\mu(\zeta) + f(z)\,\partial^\alpha \partial^{-1}\chi(z)$$
$$\partial^\alpha \bar{\partial}^{-1} f(z) = -\frac{i^{\alpha_2}}{\pi} \int_{\mathbb{R}^2} \frac{1}{(z-\zeta)^2} \chi(\zeta)\big[f(\zeta) - f(z)\big]\, d\mu(\zeta) + f(z)\,\partial^\alpha \bar{\partial}^{-1}\chi(z)$$

Prove furthermore that

$$\partial\partial^{-1} f(z) = f(z) \qquad \text{and} \qquad \bar{\partial}\bar{\partial}^{-1} f(z) = f(z)$$

(d) Let $n \in \mathbb{N}_0$ and $0 < \epsilon' < \epsilon$. Prove that there is a constant $C(K, n, \epsilon, \epsilon')$ such that if $f \in C^{n+\epsilon}(\mathbb{R}^2)$ is supported in K, then $\partial^{-1} f, \bar{\partial}^{-1} f \in C^{n+1+\epsilon'}(\mathbb{R}^2)$ and

$$\|\partial^{-1} f\|_{C^{n+1+\epsilon'}(\mathbb{R}^2)}\, ,\ \|\bar{\partial}^{-1} f\|_{C^{n+1+\epsilon'}(\mathbb{R}^2)} \le C(K, n, \epsilon, \epsilon') \|f\|_{C^{n+\epsilon}(\mathbb{R}^2)}$$

We have just seen that if $f \in C_0^\epsilon(\mathbb{R}^2)$, then $\partial^{-1} f$ and $\bar{\partial}^{-1} f$ are differentiable. If we are willing to accept weak derivatives, we can relax the conditions on f. Recall that, for any $1 \le p \le \infty$ and any $-\infty < \delta < \infty$, the space $L^p_\delta(\mathbb{R}^n)$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{L^p_\delta} = \big\| \big(1 + |x|^2\big)^{\delta/2} u \big\|_{L^p}$$

and that

Definition 4.47. Let $\alpha \in \mathbb{N}_0^n$ and let $f, g \in L^1_\delta(\mathbb{R}^n)$ for some $\delta \in \mathbb{R}$. Then g is said to be the α^{th} weak (or distributional) derivative of f if

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We persist in writing $g = \partial^\alpha f$.

Exercise 4.48. Let $1 \le p < q \le \infty$ and $\delta, \delta' \in \mathbb{R}$ with $\delta' < \delta - \frac{n}{p} \frac{q-p}{q}$ (when $q = \infty$, $\delta' < \delta - \frac{n}{p}$). Prove that if $f \in L^q_\delta(\mathbb{R}^n)$, then $f \in L^p_{\delta'}(\mathbb{R}^n)$ and $\|f\|_{L^p_{\delta'}} \le C \|f\|_{L^q_\delta}$ for some constant C that depends only on δ, δ', p, q and n.

Exercise 4.49. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$ and let ∂^α refer to the α^{th} weak derivative. Let $f, u, v \in L^1_\delta(\mathbb{R}^n)$ for some $\delta \in \mathbb{R}$.

(a) Prove that if $\partial^\alpha f = u$ and $\partial^\alpha f = v$, then $u = v$.

(b) Prove that if f is continuously differentiable and the α^{th} classical derivative equals u, then $\partial^\alpha f = u$.

Exercise 4.50. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$ and let ∂^α refer to the α^{th} weak derivative. Let $\delta \in \mathbb{R}$ and $f, u \in L^1_\delta(\mathbb{R}^n)$. Suppose that $\{f_j\}_{j \in \mathbb{N}}$ is a sequence in $L^1_\delta(\mathbb{R}^n)$ such that f_j converges to f in $L^1_\delta(\mathbb{R}^n)$ and $\partial^\alpha f_j$ converges to u in $L^1_\delta(\mathbb{R}^n)$. Prove that $\partial^\alpha f = u$.

Exercise 4.51. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 1$ and let ∂^α refer to the α^{th} weak derivative. Let $f \in L^1_\delta(\mathbb{R}^n)$ for some $\delta \in \mathbb{R}$.

(a) Let ψ be once continuously differentiable with polynomially bounded derivatives. Prove that $\partial^\alpha(\psi f) = \psi \partial^\alpha f + (\partial^\alpha \psi) f$.

(b) Let $\psi \in C_0^\infty(\mathbb{R}^n)$. Prove that $\partial^\alpha(\psi * f) = \psi * (\partial^\alpha f)$.

(c) Let $\psi : \mathbb{R} \to \mathbb{R}$ be once continuously differentiable. Suppose that f is continuous. Suppose further that there are monotone increasing functions $\Psi, F : [0, \infty)$ such that $|\psi(t)|, |\psi'(t)| \leq \Psi(|t|)$, $|f(x)| \leq F(|x|)$ and $\Psi \circ F$ is polynomially bounded. Prove that $\partial^\alpha(\psi \circ f) = (\psi' \circ f)\partial^\alpha f$.

We will later need to consider $\bar{\partial}^{-1} f$ where f only decays sufficiently quickly at infinity to lie in $L^2(\mathbb{R}^2)$. As $\frac{1}{z-\zeta}$ does not decay quickly enough as $\zeta \to \infty$ to be in L^2, $\bar{\partial}^{-1} f$, as currently defined, will not converge. Fortunately, the inverse of $\bar{\partial}$ is only defined up to an additive constant. Replacing the kernel $\frac{1}{z-\zeta}$ in the definition of $\bar{\partial}^{-1} f$ by $\frac{1}{z-\zeta} + \frac{1}{\zeta} = \frac{z}{(z-\zeta)\zeta}$ only adds a constant (i.e. a z–independent term) to $\bar{\partial}^{-1} f$ but still increases the decay rate at infinity from $\frac{1}{\zeta}$, which is not square integrable to $\frac{z}{\zeta^2}$, which is square integrable. Unfortunately it also introduces a new singularity at $\zeta = 0$. We can eliminate the singularity by replacing $\frac{1}{\zeta}$ by $\frac{\chi(\zeta)}{\zeta}$ where χ is any C^∞ function that vanishes for $|\zeta| < 1$ and is identically one for $|\zeta| \geq 2$. To distinguish the new inverse for $\bar{\partial}$ from the already defined $\bar{\partial}^{-1} f$, we denote it

$$\bar{\mathrm{f}}^{-1} f(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \Big(\frac{1}{z-\zeta} + \frac{\chi(\zeta)}{\zeta}\Big) f(\zeta)\, d\mu(\zeta)$$

Exercise 4.52. The purpose of this problem is to start providing some intuition concerning the behaviour of $\bar{\mathrm{f}}^{-1} f(z)$. Define

$$D(z,\zeta) = \frac{1}{z-\zeta} + \frac{\chi(\zeta)}{\zeta} \qquad S_\sigma(\zeta) = \begin{cases} \frac{1}{|\zeta|^\sigma} & \text{if } |\zeta| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$L_\lambda(\zeta) = \begin{cases} \frac{1}{|\zeta|^\lambda} & \text{if } |\zeta| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Observe that $S_\sigma \in L^p(\mathbb{R}^2)$ if and only if $\sigma < \frac{2}{p}$ and that $L_\lambda \in L^p(\mathbb{R}^2)$ if and only if $\lambda > \frac{2}{p}$. Assume that $0 < \sigma < 2$ and that $\lambda > 0$. Prove that there are constants C_σ and C_λ such that, if $|z| \le 1$, then

$$\int_{\mathbb{R}^2} |D(z,\zeta)| S_\sigma(\zeta)\ d\mu(\zeta) \le C_\sigma \begin{cases} \frac{1}{|z|^{\sigma-1}} & \text{if } 1 < \sigma < 2 \\ \ln \frac{1}{|z|} & \text{if } \sigma = 1 \end{cases}$$

$$\int_{\mathbb{R}^2} |D(z,\zeta)| L_\lambda(\zeta)\ d\mu(\zeta) \le C_\lambda |z|$$

and if $|z| \ge 2$, then

$$\int_{\mathbb{R}^2} |D(z,\zeta)| S_\sigma(\zeta)\ d\mu(\zeta) \le C_\sigma \frac{1}{|z|}$$

$$\int_{\mathbb{R}^2} |D(z,\zeta)| L_\lambda(\zeta)\ d\mu(\zeta) \le C_\lambda \begin{cases} |z|^{1-\lambda} & \text{if } 0 < \lambda < 1 \\ \ln |z| & \text{if } \lambda = 1 \\ 1 & \text{if } \lambda > 1 \end{cases}$$

Exercise 4.53. Let $\langle X, \mu \rangle$ and $\langle Y, \nu \rangle$ be measure spaces and let $k(x,y) = k_1(x,y) k_2(x,y)$ be a measurable function on $X \times Y$. Set

$$L = \sup_{x \in X} \left\{ \int_Y |k_1(x,y)|^2\ d\nu(y) \right\}^{1/2}$$

$$R = \sup_{y \in Y} \left\{ \int_X |k_2(x,y)|^2\ d\mu(x) \right\}^{1/2}$$

Prove that, if $L < \infty$ and $R < \infty$, then the map

$$(Kf)(x) = \int_Y k(x,y) f(y)\ d\nu(y)$$

is a bounded linear operator from $L^2(Y,\nu)$ to $L^2(X,\mu)$ with operator norm $\|K\| \le LR$.

Exercise 4.54. Let

$$D(z,\zeta) = \frac{1}{z-\zeta} + \frac{\chi(\zeta)}{\zeta}$$

and set

$$L_{\alpha_1,\beta_1} = \sup_{z \in \mathbb{R}^2} \left\{ \int_{\mathbb{R}^2} \left(1+|z|^2\right)^{\alpha_1 \delta} |D(z,\zeta)|^{2\beta_1}\ d\mu(\zeta) \right\}^{1/2}$$

$$R_{\alpha_2,\beta_2} = \sup_{\zeta \in \mathbb{R}^2} \left\{ \int_{\mathbb{R}^2} \left(1+|z|^2\right)^{\alpha_2 \delta} |D(z,\zeta)|^{2\beta_2}\ d\mu(z) \right\}^{1/2}$$

Prove that L_{α_1,β_1} and R_{α_2,β_2} are finite if $\frac{1}{2} < \beta_1 < 1$, $\beta_2 < 1$, $\alpha_1(-\delta) \ge 1-\beta_1$ and $\alpha_2(-\delta) > 1$.

Exercise 4.55. Let $2 < p < \infty$, $\varepsilon < 1 - \frac{2}{p}$ and let K be any compact subset of $\mathbb{R}^2$. Prove that there is a constant $C(K, \epsilon, p)$ such that if $f \in L^p(\mathbb{R}^2)$ is supported in K, then $\partial^{-1} f, \bar{\partial}^{-1} f \in C^\epsilon(\mathbb{R}^2)$ and

$$\|\partial^{-1} f\|_{C^\epsilon(\mathbb{R}^2)} \,,\, \|\bar{\partial}^{-1} f\|_{C^\epsilon(\mathbb{R}^2)} \le C(K, \epsilon)\|f\|_{L^p(\mathbb{R}^2)}$$

Exercise 4.56. Let, for each sufficiently small $h \in \mathbb{C}$, A_h be a bounded linear operator on the Banach space $\mathcal{B}$. Suppose that

- $\mathbb{1} - A_0$ has a bounded inverse on $\mathcal{B}$
- $\lim_{h \to 0} \|A_h - A_0\| = 0$
- for each $f \in \mathcal{B}$, the map $h \mapsto A_h f$ is differentiable at $h = 0$ in $\mathcal{B}$.

Prove that $\mathbb{1} - A_h$ has a bounded inverse on $\mathcal{B}$ for all sufficiently small h and that for each $f \in \mathcal{B}$, the map $h \mapsto (\mathbb{1} - A_h)^{-1} f$ is differentiable at $h = 0$ in $\mathcal{B}$, with the derivative being $-(\mathbb{1} - A_0)^{-1} A_0' (\mathbb{1} - A_0)^{-1} f$.

Exercise 4.57. Let $f \in L^1(\mathbb{R}^n)$. Prove that

$$\lim_{r \to 0+} \sup_{c \in \mathbb{R}^n} \int_{B_r(c)} |f(x)| \, d^n x = 0$$

where $B_r(c)$ is the ball of radius r centred on c.

Exercise 4.58. Let $u \in H^1(\Omega)$ where Ω is a convex, bounded, open subset of $\mathbb{R}^2$ with smooth boundary. Let S_1 and S_2 be two measurable subsets of Ω. Prove that

$$\big|(u)_{S_1} - (u)_{S_2}\big| \le \sqrt{\pi}\,(\mathrm{diam}\Omega)^2 \Big(\frac{1}{|S_1|} + \frac{1}{|S_2|}\Big) \|\nabla u\|_{L^2(\Omega)}$$

Exercise 4.59. Let $\delta \in \mathbb{R}$. Let $\chi \in C_0^\infty(\mathbb{R}^2)$, $u \in L^2_{loc} \cap L^1_\delta$ and $g \in L^1_\delta$. Assume further that u has a weak derivative $\bar{\partial} u \in L^1_\delta$ and that g is continuous and has a weak derivative $\bar{\partial} g \in L^2_{loc}$. Prove that

$$\bar{\partial}\big(\chi u e^{-g}\big) = u e^{-g} \bar{\partial}\chi + \chi e^{-g} \bar{\partial} u - \chi u e^{-g} \bar{\partial} g$$

Exercise 4.60. Let $\varphi(x)$ be a smooth nonnegative function that vanishes for $|x| > 1$ and that is normalized by $\int \varphi(x)\, d\mu(x) = 1$. Let $f \in C^\epsilon(\mathbb{R}^2)$ and set, for $0 < t \le 1$, $f_t = \varphi_t * f$ where $\varphi_t(x) = t^{-2}\varphi\big(\frac{x}{t}\big)$. Prove that

$$\begin{aligned} \|f - f_t\|_{L^\infty} &\le |f|_{C^\epsilon} t^\epsilon \\ |f_t|_{C^\epsilon} &\le |f|_{C^\epsilon} \\ \Big\|\frac{\partial^\alpha f_t}{\partial x^\alpha}\Big\|_{L^\infty} &\le C_\alpha |f|_{C^\epsilon} t^{\epsilon - |\alpha|} \qquad \text{if } |\alpha| \ge 1 \end{aligned}$$

4.9. Identification of Boundary Values of Isotropic Conductivities

Theorem 4.61. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and suppose that γ_1 and γ_2 are isotropic conductivities in Ω obeying Hypothesis 4.40. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then, for any integer $\ell \geq 0$,

$$\Big(\frac{\partial}{\partial\nu}\Big)^\ell \gamma_1 = \Big(\frac{\partial}{\partial\nu}\Big)^\ell \gamma_2 \qquad \text{on } \partial\Omega \tag{4.58}$$

Theorem 4.62. *Suppose that γ_0 and γ_1 are isotropic conductivities on $\Omega \subset \mathbb{R}^n$ satisfying Hypothesis 4.40 and*

(i) $1/E \leq \gamma_i \leq E$

(ii) $\|\gamma_i\|_{C^2(\overline{\Omega})} \leq E$,

Given any $0 < \sigma < \frac{1}{n+3}$ there exists $C = C(\Omega, E, n, \sigma)$ such that

$$\big\|\gamma_1 - \gamma_2\big\|_{L^\infty(\partial\Omega)} \leq C\big\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\big\|_{\frac{1}{2},-\frac{1}{2}} \tag{4.59}$$

and

$$\Big\|\frac{\partial\gamma_1}{\partial\nu} - \frac{\partial\gamma_2}{\partial\nu}\Big\|_{L^\infty(\partial\Omega)} \leq C\big\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\big\|^\sigma_{\frac{1}{2},-\frac{1}{2}} \tag{4.60}$$

Chapter 5

The Calderón problem in the plane

In this chapter we discuss the Calderón problem in two dimensions. The arguments presented here have a somewhat different flavor compared to the case $n \geq 3$, and will rely heavily on complex analysis. We will give the proof of Astala and Päivärinta which allows to treat bounded measurable conductivities. This will involve the theory of quasiconformal mappings, a generalization of the standard theory of analytic functions.

In this chapter we will denote the conductivity by σ instead of γ. This is customary in the two-dimensional results and emphasizes the fact that the conductivity has to be real valued. Let us state more precisely the result that will be proved.

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and suppose that σ_1 and σ_2 are two positive functions in $L^\infty(\Omega)$. If*

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$$

then

$$\sigma_1 = \sigma_2.$$

We will begin with a reduction. If $B \subset \mathbb{R}^2$ is an open ball centered at the origin such that $\overline{\Omega} \subset B$, we define new conductivities

$$\tilde{\sigma}_j(x) = \begin{cases} \sigma_j(x) & \text{if } x \in \Omega, \\ 1 & \text{if } x \in B \setminus \Omega. \end{cases}$$

Then $\tilde{\sigma}_j$ are positive functions in $L^\infty(B)$, and the condition $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ on $\partial\Omega$ implies that $\Lambda_{\tilde{\sigma}_1} = \Lambda_{\tilde{\sigma}_2}$ on ∂B. (The extension as constant indicates why it is useful to be able to work with bounded measurable conductivities.) By a simple rescaling, it is sufficient to prove Theorem 5.1 when the domain is the unit disc $\mathbb{D}$. We will assume throughout this chapter that $\Omega = \mathbb{D}$.

Exercise 5.2. Fill in the details for the reduction to $\Omega = \mathbb{D}$.

5.1. Complex derivatives

This section contains a brief review of the complex derivatives ∂ and $\bar{\partial}$. Readers who are somewhat familiar with these topics may skip this section for the time being and return whenever needed.

If $x = (x_1, x_2) \in \mathbb{R}^2$ we write $z = x_1 + ix_2$ and identify $\mathbb{R}^2$ with $\mathbb{C}$ in this way. The complex derivatives are defined by

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \quad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

Let $\Omega \subset \mathbb{R}^2$ be an open set, and let $f : \Omega \to \mathbb{C}$ be a C^1 function. Write $f = u + iv$ where $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$. Recall that f is said to be analytic in Ω if it satisfies the Cauchy-Riemann equations

$$\partial_1 u = \partial_2 v, \quad \partial_2 u = -\partial_1 v.$$

Recall also that any analytic function is C^∞. The $\bar{\partial}$ operator satisfies

$$\bar{\partial} f = \frac{1}{2}(\partial_1 + i\partial_2)(u + iv) = \frac{1}{2}(\partial_1 u - \partial_2 v + i(\partial_1 v + \partial_2 u).$$

This immediately implies that the $\bar{\partial}$ operator characterizes analytic functions.

Lemma 5.3. *f is analytic in Ω if and only if $\bar{\partial} f = 0$ in Ω.*

Another fact to note is that the Laplacian $\Delta = \partial_1^2 + \partial_2^2$ factors in terms of the complex derivatives as

$$\Delta = 4\partial\bar{\partial}.$$

Thus, if $f = u + iv$ is analytic then $\Delta f = 4\partial(\bar{\partial} f) = 0$. Since u and v are real valued it follows that $\Delta u = \Delta v = 0$, that is, the real and imaginary parts of an analytic function are harmonic.

The next result establishes the existence of conjugate harmonic functions.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set, and let $u : \Omega \to \mathbb{R}$ be a C^∞ function with $\Delta u = 0$. There exists a C^∞ function $v : \Omega \to \mathbb{R}$, unique up to an additive constant, such that $\Delta v = 0$ and $f = u + iv$ is analytic.*

Exercise 5.5. Let $f, g : \mathbb{R}^2 \to \mathbb{C}$ be C^1 functions.

(a) Suppose that $f(x,y) = F(x+iy)$ with $F : \mathbb{C} \to \mathbb{C}$ analytic and $g(x,y) = G(x-iy)$ with $G : \mathbb{C} \to \mathbb{C}$ analytic. Prove that

$$\begin{aligned} \partial f(x,y) &= F'(x+iy), & \bar\partial f(x,y) &= 0, \\ \partial g(x,y) &= 0, & \bar\partial g(x,y) &= G'(x-iy). \end{aligned}$$

Prove conversely that, if $\bar\partial f = 0$, then there is an analytic function $F(z)$ such that $f(x,y) = F(x+iy)$ and if $\partial g = 0$, then there is an analytic function $G(z)$ such that $g(x,y) = G(x-iy)$.

(b) Prove that

$$\partial f \bar\partial g + \bar\partial f \partial g = \tfrac{1}{2} \nabla f \cdot \nabla g.$$

Prove that, if f is C^2, then

$$4\partial\bar\partial f = 4\bar\partial\partial f = \Delta f.$$

Prove that, if f is C^2, then $\Delta f = 0$ if and only if there are analytic functions F and G such that $f(x,y) = F(x+iy) + G(x-iy)$.

(c) Prove that

$$\begin{aligned} \partial(fg) &= f\partial g + g\partial f, & \partial(f \circ g) &= (\partial f) \circ g \; \partial g + (\bar\partial f) \circ g \; \partial \bar g, \\ \bar\partial(fg) &= f\bar\partial g + g\bar\partial f, & \bar\partial(f \circ g) &= (\partial f) \circ g \; \bar\partial g + (\bar\partial f) \circ g \; \bar\partial \bar g. \end{aligned}$$

(d) Prove that

$$\begin{aligned} \overline{(\bar\partial f)(z)} &= \partial\big(\overline{f(z)}\big), & \partial\big(f(\bar z)\big) &= (\bar\partial f)(\bar z), & \partial\big(\overline{f(\bar z)}\big) &= \overline{(\partial f)(\bar z)}, \\ \overline{(\partial f)(z)} &= \bar\partial\big(\overline{f(z)}\big), & \bar\partial\big(f(\bar z)\big) &= (\partial f)(\bar z), & \bar\partial\big(\overline{f(\bar z)}\big) &= \overline{(\bar\partial f)(\bar z)}. \end{aligned}$$

Exercise 5.6. Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set (this means that Ω is connected and every closed curve in Ω can be continuously deformed to a point, or equivalently that Ω and $S^2 \backslash \Omega$ are connected where $S^2 = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere). Let $F : \Omega \to \mathbb{R}^2$ be a C^∞ vector field whose curl vanishes,

$$\partial_1 F_2 - \partial_2 F_1 = 0 \quad \text{in } \Omega.$$

Show that $F = \nabla p$ for some C^∞ function $p : \Omega \to \mathbb{R}$.

Exercise 5.7. Prove Lemma 5.4.

Exercise 5.8. Let Ω be a bounded open subset of $\mathbb{R}^2$ with smooth boundary. Denote by (ν_1, ν_2) the unit outer normal to $\partial\Omega$. Give $\partial\Omega$ the standard orientation. That is, when you walk along $\partial\Omega$ in the positive direction, ν is on your right hand side.

(a) Let each component of the vector field (f_1, f_2) be in $C^1(\overline{\Omega})$. Prove that

$$\int_\Omega \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right] d^2x = \int_{\partial\Omega} \left[\nu_1 f_1 + \nu_2 f_2\right] ds$$

where s is arc length.

(b) Let $f \in C^1(\overline{\Omega})$. Prove that

$$\int_\Omega \partial f \; d^2x = \int_{\partial\Omega} \nu f \; ds \qquad \int_\Omega \bar{\partial} f \; d^2x = \int_{\partial\Omega} \bar{\nu} f \; ds$$

where $\nu = \frac{1}{2}(\nu_1 - i\nu_2)$ and $\bar{\nu} = \frac{1}{2}(\nu_1 + i\nu_2)$.

Exercise 5.9. Let Ω be a bounded, open, simply connected subset of $\mathbb{R}^2$ with smooth boundary. Let each component of the vector field (f_1, f_2) be in $C^1(\overline{\Omega})$. Recall that if $\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$, then there is a function $g \in C^2(\overline{\Omega})$ such that $f_1 = \frac{\partial g}{\partial x_1}$ and $f_2 = \frac{\partial g}{\partial x_2}$. Prove that if $\bar{\partial} f_1 = \partial f_2$, then there is a function $g \in C^2(\overline{\Omega})$ such that $f_1 = \partial g$ and $f_2 = \bar{\partial} g$.

5.2. Reduction to Beltrami equation

In the first step of the proof, we reduce the conductivity equation $\nabla \cdot \sigma \nabla u = 0$ to a certain first order equation involving the complex derivatives,

$$\bar{\partial} f = \mu \overline{\partial f}.$$

In this section we will also show that the DN map Λ_σ for the conductivity equation uniquely determines a corresponding boundary map for this first order equation, namely the μ-Hilbert transform $\mathcal{H}_\mu$.

Proposition 5.10. *Let $u \in H^1(\mathbb{D})$ be a real valued solution of*

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \mathbb{D}.$$

There exists a real valued $v \in H^1(\mathbb{D})$, unique up to an additive constant, such that $f = u + iv$ satisfies the equation

$$\bar{\partial} f = \mu \overline{\partial f} \quad \text{in } \mathbb{D}$$

where

$$\mu = \frac{1-\sigma}{1+\sigma}.$$

Conversely, if $f = u + iv \in H^1(\mathbb{D})$ satisfies (5.10) *where $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_{L^\infty(\mathbb{D})} < 1$, then*

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \mathbb{D}, \quad \nabla \cdot \sigma^{-1} \nabla v = 0 \text{ in } \mathbb{D}$$

where

$$\sigma = \frac{1-\mu}{1+\mu}.$$

Proof. Suppose that $\nabla\cdot\sigma\nabla u = 0$, and define the vector field $F = (-\sigma\partial_2 u, \sigma\partial_1 u)$. Then

$$\partial_1 F_2 - \partial_2 F_1 = 0.$$

It follows from Problem 5.11 below that there is a real valued $v \in H^1(\mathbb{D})$, unique up to an additive constant, such that

$$(\partial_1 v, \partial_2 v) = (-\sigma\partial_2 u, \sigma\partial_1 u).$$

Let $f = u + iv$. Then

$$\begin{aligned}\bar\partial f &= \frac{1}{2}(\partial_1 u - \partial_2 v + i(\partial_2 u + \partial_1 v)) = \frac{1}{2}(1-\sigma)(\partial_1 u + i\partial_2 u),\\ \mu\overline{\partial f} &= \frac{1-\sigma}{1+\sigma}\frac{1}{2}(\partial_1 u + \partial_2 v + i(\partial_2 u - \partial_1 v)) = \frac{1}{2}(1-\sigma)(\partial_1 u + i\partial_2 u).\end{aligned}$$

This shows (5.10).

For the converse direction, if $f = u + iv$ solves (5.10) and if $\sigma = \frac{1-\mu}{1+\mu}$, then

$$\bar\partial u + i\bar\partial v = \bar\partial f = \tfrac{1-\sigma}{1+\sigma}\overline{\partial f} = \tfrac{1-\sigma}{1+\sigma}(\bar\partial u - i\bar\partial v)$$

and thus

$$(1+\sigma)(\bar\partial u + i\bar\partial v) = (1-\sigma)(\bar\partial u - i\bar\partial v).$$

This shows that $2i\bar\partial v + 2\sigma\bar\partial u = 0$, and therefore

$$(\partial_1 v, \partial_2 v) = (-\sigma\partial_2 u, \sigma\partial_1 u).$$

It follows that

$$\partial_1(\sigma\partial_1 u) + \partial_2(\sigma\partial_2 u) = \partial_1\partial_2 v - \partial_2\partial_1 v = 0$$

and

$$\partial_1(\sigma^{-1}\partial_1 v) + \partial_2(\sigma^{-1}\partial_2 v) = -\partial_1\partial_2 u + \partial_2\partial_1 u = 0.$$

The last computations are easy to justify in the weak sense, which proves the result. $\square$

Exercise 5.11. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, and let $F : \Omega \to \mathbb{R}^2$ be a vector field with components in $L^2(\Omega)$ whose curl vanishes in the weak sense,

$$\partial_1 F_2 - \partial_2 F_1 = 0 \quad \text{in } \Omega.$$

Show that $F = \nabla p$ for some real valued function $p \in H^1(\Omega)$.

Exercise 5.12. Justify in the weak sense the computations in the end of the proof in Proposition 5.10.

Several remarks are in order. The first order equation (5.10) is called a *Beltrami equation*. There is a one-to-one correspondence between solutions of $\nabla\cdot\sigma\nabla u = 0$ and of $\bar\partial f = \mu\overline{\partial f}$, where σ and μ are related by the formulas in Proposition 5.10. It is very important that the complex coefficient μ satisfies

$\|\mu\|_{L^\infty} < 1$; this is the condition for uniform ellipticity of the Beltrami equation and ensures that the corresponding conductivity stays positive and bounded.

Solutions of $\nabla \cdot \sigma \nabla u = 0$ are called *σ-harmonic functions*. The function v constructed in Proposition 5.10 is called a *σ-harmonic conjugate* of u. The situation is symmetric so that the σ-harmonic conjugate v of u is itself a $1/\sigma$-harmonic function.

In a sense, the σ-harmonic conjugate "completes" the original σ-harmonic function u into a "μ-analytic" function $f = u + iv$. This generalizes the fact that a harmonic function u can be "completed" into an analytic function $f = u + iv$ by using the harmonic conjugate v (this is also obtained from Proposition 5.10 in the special case where $\sigma = 1$, so that $\mu = 0$ and the Beltrami equation is just $\bar{\partial} f = 0$). The theory of analytic functions (solutions of $\bar{\partial} f = 0$) provides strong complex analysis tools to the study of harmonic functions (solutions of $\Delta u = 0$). In the same way, the point of view of the Beltrami equation provides powerful tools (now based on the theory of quasiconformal mappings) to the analysis of σ-harmonic functions and also the Calderón problem. For completeness, we will give the definition of quasiconformal mappings in the end of the section.

Above we have reduced the study of solutions of $\nabla \cdot \sigma \nabla u = 0$ to the Beltrami equation $\bar{\partial} f = \mu \overline{\partial f}$. Now we make a similar reduction on the level of boundary measurements. Given a σ-harmonic function $u \in H^1(\mathbb{D})$, we specify a unique σ-harmonic conjugate $v \in H^1(\mathbb{D})$ by requiring that

$$\int_{\partial \mathbb{D}} v \, dS = 0.$$

If $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_{L^\infty(\mathbb{D})} < 1$, we define the μ-Hilbert transform

$$\mathcal{H}_\mu : H^{1/2}(\partial \mathbb{D}; \mathbb{R}) \to H^{1/2}(\partial \mathbb{D}; \mathbb{R}), \quad u|_{\partial \mathbb{D}} \mapsto v|_{\partial \mathbb{D}}$$

where $f = u + iv$ solves $\bar{\partial} f = \mu \overline{\partial f}$ in $\mathbb{D}$ and $\int_{\partial \mathbb{D}} v \, dS = 0$.

Exercise 5.13. Show that $\mathcal{H}_\mu$ is a well-defined bounded linear map on $H^{1/2}(\partial \mathbb{D}; \mathbb{R})$.

Proposition 5.14. *Knowledge of Λ_σ determines the operators $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$, and $\Lambda_{1/\sigma}$. Further, one has the identity*

$$\mathcal{H}_{-\mu} \mathcal{H}_\mu g = \mathcal{H}_\mu \mathcal{H}_{-\mu} g = -g$$

for any $g \in H^{1/2}(\partial \mathbb{D})$ with $\int_{\partial \mathbb{D}} g \, dS = 0$.

Proof. 1. Let $g \in H^{1/2}(\partial \mathbb{D})$, let $u \in H^1(\mathbb{D})$ satisfy $\nabla \cdot \sigma \nabla u = 0$ with $u|_{\partial \mathbb{D}} = g$, and let $\varphi \in H^1(\mathbb{D})$. We have

$$\langle \Lambda_\sigma g, \varphi \rangle = \int_{\mathbb{D}} \sigma \nabla u \cdot \nabla \varphi \, dx.$$

Letting v be the σ-harmonic conjugate of u with $\int_{\partial\mathbb{D}} v\,dS = 0$, so that $(\partial_1 v, \partial_2 v) = (-\sigma\partial_2 u, \sigma\partial_1 u)$, we have

$$\langle \Lambda_\sigma g, \varphi\rangle = \int_{\mathbb{D}} (\partial_2 v, -\partial_1 v)\cdot\nabla\varphi\,dx = \int_{\partial\mathbb{D}} ((-\nu_2, \nu_1)\cdot\nabla v)\varphi\,dS.$$

This shows that

$$\Lambda_\sigma = \partial_T \mathcal{H}_\mu$$

in the weak sense, where $\partial_T = (-\nu_2, \nu_1)\cdot\nabla$ is the tangential derivative along $\partial\mathbb{D}$.

2. We have recovered the tangential derivative of $\mathcal{H}_\mu$ from the DN map, and it is enough to prove that any function $h \in H^{1/2}(\mathbb{D})$ whose integral over $\partial\mathbb{D}$ vanishes is uniquely determined by $\partial_T h$. This can be done by using Fourier coefficients (see Problem 5.15 below): one expands h in Fourier series

$$h(e^{i\theta}) = \sum_{m=-\infty}^{\infty} \hat{h}(m)e^{im\theta}$$

where the Fourier coefficients are given by

$$\hat{h}(m) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}h(e^{i\theta})\,d\theta.$$

The Fourier coefficients of $\partial_T h$ are given by

$$(\partial_T h)\,\hat{}\,(m) = im\hat{h}(m).$$

One has $\hat{h}(0) = 0$ since the integral of h over $\partial\mathbb{D}$ vanishes. Consequently

$$h(e^{i\theta}) = \sum_{m=-\infty, m\neq 0}^{\infty} \frac{(\partial_T h)\,\hat{}\,(m)}{im}e^{im\theta}.$$

We have proved that $\Lambda_\sigma g = \partial_T \mathcal{H}_\mu g$ determines $\mathcal{H}_\mu g$.

3. The identity (5.14) is proved in Problem 5.16 below. Assuming this identity, we see that $\mathcal{H}_{\pm\mu}$ are bijective operators on the space of functions in $H^{1/2}(\partial D)$ whose integral vanishes. Therefore

$$\mathcal{H}_{-\mu} = -(\mathcal{H}_\mu)^{-1}$$

for such functions. Since $\mathcal{H}_{\pm\mu}(c) = 0$ for any constant c, the operator $\mathcal{H}_\mu$ determines $\mathcal{H}_{-\mu}$. Noting that

$$-\mu = \frac{\sigma-1}{\sigma+1} = \frac{1-1/\sigma}{1+1/\sigma}$$

we have $\Lambda_{1/\sigma} = \partial_T\mathcal{H}_{-\mu}$. Thus the operator Λ_σ indeed determines $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$ and $\Lambda_{1/\sigma}$. $\square$

Exercise 5.15. If $s \geq 0$, show that

$$H^s(\partial\mathbb{D}) = \left\{ \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \,;\, \sum_{m=-\infty}^{\infty} (1+|m|)^{2s} \, |a_m|^2 < \infty \right\}.$$

Extend this characterization to all $s \in \mathbb{R}$, and show that any element $h \in H^s(\partial\mathbb{D})$ can be written as the Fourier series

$$h = \sum \hat{h}(m) e^{im\theta}, \quad \hat{h}(m) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} e^{-im\theta} h(e^{i\theta}) \, d\theta$$

with suitable interpretations for the sum and the integral. Show that

$$(\partial_T h) \; \hat{} \; (m) = im\hat{h}(m).$$

Exercise 5.16. Prove the identity (5.14) for any $g \in H^{1/2}(\partial\mathbb{D})$ whose integral over $\partial\mathbb{D}$ vanishes.

We now explain how the previous reduction is related to quasiconformal mappings. To do this, we first need to give a definition of quasiconformal mappings. Let $\Omega \subset \mathbb{C}$ be a bounded open set, and let $f \in W^{1,1}_{loc}(\Omega)$. Then the first partial derivatives of f exist at almost every point in Ω, and at these points we can define the directional derivative

$$\partial_\alpha f(z) = \lim_{r \to 0} \frac{f(z + re^{i\alpha}) - f(z)}{r} = \cos(\alpha)\partial_1 f(z) + \sin(\alpha)\partial_2 f(z).$$

For small r, we consider the image of the circle $\alpha \mapsto z + re^{i\alpha}$ under the map f. The value $|\partial_\alpha f(z)|$ measures how much f distorts an infinitesimal circle at z in direction α. The next problem expresses this in terms of the complex derivatives.

Exercise 5.17. Show that

$$\partial_\alpha f(z) = \bar{\partial} f(z) e^{-2i\alpha} + \partial f(z).$$

Show that $\partial_\alpha f(z)$ is independent of α if and only if $\bar{\partial} f(z) = 0$.

Recall that a map $f : \Omega \to \Omega'$ between two open sets in $\mathbb{C}$ is called *conformal* if it is analytic and bijective onto its image (this implies that the derivative of f is nonvanishing in Ω). The previous problem can be interpreted so that conformal functions map infinitesimal circles to infinitesimal circles: if $\bar{\partial} f = 0$ then

$$f(z + re^{i\alpha}) \approx f(z) + \partial_\alpha f(z) re^{i\alpha} = f(z) + \partial f(z) re^{i\alpha}$$

This is, in a sense, a very strong requirement. For instance, it follows from the Schwarz lemma in complex analysis that

$$f : \mathbb{D} \to \mathbb{D} \text{ conformal} \implies f \text{ is a Möbius map, } f(z) = \tfrac{az+b}{cz+d}.$$

Quasiconformal mappings relax this requirement, and require that f maps infinitesimal circles to infinitesimal ellipses with uniformly bounded eccentricity. Thus, instead of the condition that $|\partial_\alpha f(z)|$ is independent of α, the condition is

$$\max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha f(z)| \quad \text{for a.e. } z \in \Omega,$$

where K is a uniform constant with $1 \leq K < \infty$. The precise definition is as follows.

Definition 5.18. Let $1 \leq K < \infty$. A mapping $f \in W^{1,2}_{loc}(\Omega)$ is called K-*quasiregular* if it is orientation preserving (in the sense that its Jacobian $|\partial f|^2 - |\bar\partial f|^2$ is nonnegative almost everywhere) and if the condition (5.2) holds. If in addition f is a homeomorphism onto its image, then f is called K-*quasiconformal.*

Quasiconformal mappings form a much larger class of mappings than the conformal ones. For instance, compare the result of the next problem to the fact mentioned above that any conformal map $\mathbb{D} \to \mathbb{D}$ must be a Möbius map:

Exercise 5.19. Show that any C^1 orientation preserving diffeomorphism $f : \Omega \to \Omega'$, where Ω and Ω' are open sets containing $\overline{\mathbb{D}}$, restricts to a quasiconformal map $\mathbb{D} \to \mathbb{D}$.

Despite being a much larger class than conformal mappings, quasiconformal mappings still have a powerful and well established theory with many applications in elliptic PDE (both linear and nonlinear), conformal geometry, complex dynamics, and inverse problems. It is remarkable that quasiconformal maps can also be characterized as the solutions of a PDE, the Beltrami equation, showing that the inequality (5.2) is in effect equivalent with a partial differential equation. The next problem discusses this equivalence for mappings with C^1 regularity.

Exercise 5.20. Let $f : \Omega \to \Omega'$ be a C^1 orientation preserving diffeomorphism between two open subsets of $\mathbb{C}$. Show that f is quasiconformal if and only if

$$\bar\partial f = \mu \partial f \quad \text{in } \Omega$$

for some $\mu \in L^\infty(\Omega)$ with $\|\mu\|_{L^\infty(\Omega)} < 1$.

Note that the Beltrami equation in (5.10) differs from this Beltrami equation in the theory of quasiconformal mappings by having $\overline{\partial f}$ instead of ∂f on the right-hand side. However, one can sometimes make a reduction: (5.10) can be written formally as

$$\bar\partial f = \tilde\mu \partial f, \quad \tilde\mu = \mu \frac{\overline{\partial f}}{\partial f}.$$

The last step can often be made rigorous, since in many cases of interest one has $\partial f \neq 0$ almost everywhere.

5.3. Cauchy and Beurling transforms

In the previous section, we made a reduction from the conductivity equation

$$\nabla \cdot \sigma \nabla u = 0$$

into the Beltrami equation

$$\bar{\partial} f = \mu \overline{\partial f}.$$

In the study of the Calderón problem in the plane, it will be very useful to know about properties of solutions of various Beltrami equations. The simplest such equation is the $\bar{\partial}$-equation

$$\bar{\partial} u = f \quad \text{in } \mathbb{R}^2.$$

We will construct a solution operator P, called the *Cauchy transform* (sometimes also *solid Cauchy transform*), such that $u = Pf$ will be a unique solution of this equation in suitable function classes. In fact, we will prove the following result. If $K \subset \mathbb{R}^2$ is a compact set write

$$L^p_K(\mathbb{R}^2) := \{f \in L^p(\mathbb{R}^2)\,;\, \mathrm{supp}\,(f) \subset K\}.$$

Proposition 5.21. *Let $p > 2$. There is a linear operator*

$$P : L^p_{comp}(\mathbb{R}^2) \to W^{1,p}(\mathbb{R}^2)$$

such that for any $f \in L^p_{comp}(\mathbb{R}^2)$ the function $u = Pf$ is the unique solution in $W^{1,p}(\mathbb{R}^2)$ of the equation

$$\bar{\partial} u = f \quad \text{in } \mathbb{R}^2.$$

If $K \subset \mathbb{R}^2$ is a compact set then there is a constant $C > 0$ such that

$$\|Pf\|_{W^{1,p}(\mathbb{R}^2)} \leq C\,\|f\|_{L^p(\mathbb{R}^2)}\,, \quad f \in L^p_K(\mathbb{R}^2).$$

For later purposes, we also record a result for the Cauchy transform acting on functions that are not compactly supported, but rather lie in the space

$$L^{2\pm}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2)\,;\, f \in L^{2+\varepsilon}(\mathbb{R}^2) \cap f \in L^{2-\varepsilon}(\mathbb{R}^2) \text{ for some } \varepsilon > 0\}.$$

Proposition 5.22. *The Cauchy transform extends as a linear operator*

$$P : L^{2\pm}(\mathbb{R}^2) \to C_0(\mathbb{R}^2).$$

The other main result in this section concerns a somewhat more general Beltrami type equation, given by

$$\bar{\partial} u - \nu \partial u = f \quad \text{in } \mathbb{R}^2.$$

Here we assume that $\nu \in L^\infty_{comp}(\mathbb{R}^2)$ satisfies the ellipticity condition $\|\nu\|_{L^\infty} < 1$. We can try to solve this equation by treating it as a perturbation of the $\bar\partial$-equation and by looking for a solution in the form Pw. Since one should have $\bar\partial P w = w$, this equation reduces to

$$(I - \nu S)w = f \quad \text{in } \mathbb{R}^2$$

where S is the *Beurling operator* (or *Beurling-Ahlfors operator*)

$$S = \partial P.$$

This operator turns out to be bounded on L^p spaces, and the ellipticity condition $\|\nu\|_{L^\infty} < 1$ allows to solve the equation $(I - \nu S)w = f$ at least for p close to 2. The Beurling operator has the important property that it intertwines the ∂ and $\bar\partial$ operators.

Proposition 5.23. *Let $1 < p < \infty$. There is a bounded linear operator*

$$S : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$$

such that for any $f \in W^{1,1}_{loc}(\mathbb{R}^2)$ with $\bar\partial f \in L^p(\mathbb{R}^2)$,

$$S\bar\partial f = \partial f.$$

If $w \in L^q_{comp}(\mathbb{R}^2)$ with $q > 2$ one also has

$$Sw = \partial P w.$$

The norm of this operator satisfies

$$\|S\|_{L^2 \to L^2} = \lim_{p \to 2} \|S\|_{L^p \to L^p} = 1.$$

The rest of this section is devoted to proving Propositions 5.21 and 5.23. Let us first consider solving the equation

$$\bar\partial u = f \quad \text{in } \mathbb{R}^2.$$

This is a linear partial differential equation with constant coefficients, and we can formally solve it by Fourier analysis. Fourier transforming this equation leads to

$$\frac{i}{2}(\xi_1 + i\xi_2)\hat{u} = \hat{f}$$

and formally dividing by the symbol shows that the solution u should be given by

$$u = F^{-1}\left\{\frac{i}{2}\frac{1}{\xi_1 + i\xi_2}\hat{f}\right\}.$$

Since the Fourier transform maps convolutions to products, this formally implies that $u = K * f$, where

$$K = F^{-1}\left\{\frac{i}{2}\frac{1}{\xi_1 + i\xi_2}\right\}.$$

It is fortunate that this inverse Fourier transform can be computed explicitly.

Proposition 5.24. *Writing $z = x_1 + ix_2$, we have*
$$K = \frac{1}{\pi z}.$$

Proof. Writing
$$\frac{1}{\xi_1 + i\xi_2} = \chi_{\{|\xi|\leq 1\}} \frac{1}{\xi_1 + i\xi_2} + \chi_{\{|\xi|>1\}} \frac{1}{\xi_1 + i\xi_2} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2),$$
we see that $\frac{1}{\xi_1+i\xi_2}$ is a tempered distribution and hence its inverse Fourier transform K is also a tempered distribution. We first give a heuristic argument that allows to guess what K should be. Formally
$$K(x) = (2\pi)^{-2} \frac{2}{i} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \frac{1}{\xi_1 + i\xi_2} \, d\xi.$$
This expression has the following property under dilations: a change of variables $\xi = \lambda^{-1}\eta$ shows that
$$K(\lambda x) = (2\pi)^{-2} \frac{2}{i} \int_{\mathbb{R}^2} e^{ix\cdot\lambda\xi} \frac{1}{\xi_1 + i\xi_2} \, d\xi = \lambda^{-1} K(x), \quad \lambda > 0.$$
Also, if $R_\theta = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$ is the rotation matrix that corresponds to multiplication by $e^{i\theta}$ in the complex notation, a change of variables $\xi = R_\theta \eta$ gives
$$K(e^{i\theta} z) = (2\pi)^{-2} \frac{2}{i} \int_{\mathbb{R}^2} e^{iR_\theta x\cdot\xi} \frac{1}{\xi_1 + i\xi_2} \, d\xi = e^{-i\theta} K(z).$$
Based on these symmetries, one expects that
$$K(re^{i\theta}) = r^{-1} K(e^{i\theta}) = r^{-1} e^{-i\theta} K(1).$$
Since $K(1)$ is just some constant c, we make the guess that
$$K(z) = c\frac{1}{z}.$$
We will now prove that this guess is correct, and we also determine the constant c.

A. □

Issue. [Note: much of Proposition 5.21 is already proved in the earlier version of the 2D result (Section 5.4 in the book). One could use all the material and exercises from there that can be used. The $L^p_{comp} \to W^{1,p}$ bound follows from convolution estimates and properties of the kernel.]

Issue. [Note: Proposition 5.23 is proved by noting that the kernel of ∂P gives rise to a Calderón-Zygmund operator, and by appealing to their L^p boundedness. The property $S\bar{\partial}f = \partial f$ for f in this class follows by using convolution approximations. The norm bound on L^2 is true because on the Fourier side the operator acts by

multiplication by a unimodular function, and the L^p statement follows from the Riesz-Thorin interpolation theorem.]

5.4. Existence and uniqueness of CGO solutions

We are ready to introduce the complex geometrical optics solutions that will be used to resolve the Calderón problem in the plane. Recall that in dimensions $n \geq 3$, we employed exponentially growing solutions to the Schrödinger equation $(-\Delta+q)u = 0$ which resembled the harmonic exponentials depending on a vector $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$,

$$u_0(x) = e^{\rho \cdot x}, \quad \Delta u_0 = 0.$$

In the case $n = 2$ we have reduced the conductivity equation to a Beltrami equation. The complex geometrical optics solutions to the Beltrami equation will be based on the analytic exponentials depending on a complex number $k \in \mathbb{C}$,

$$f_0(z) = e^{ikz}, \quad \bar{\partial} f_0 = 0.$$

The next proposition is the basic existence and uniqueness result for such solutions. Their behavior for large values of k is a subtle issue and will be considered only later.

Proposition 5.25. *Let $\mu \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$ with $\|\mu\|_{L^\infty} < 1$. For any $k \in \mathbb{C}$, there is a unique solution $f_\mu = f_\mu(\,\cdot\,, k)$ of the equation*

$$\bar{\partial} f = \mu \overline{\partial f} \quad \text{in } \mathbb{R}^2$$

having the form

$$f_\mu(z,k) = e^{ikz}(1 + \eta_\mu(z,k))$$

where

$$\eta_\mu(\,\cdot\,,k) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2),$$
$$\eta_\mu(z,k) = O(\frac{1}{z}) \text{ as } |z| \to \infty.$$

Further, $\eta_\mu(\,\cdot\,,k) \in W^{1,p}(\mathbb{R}^2)$ for some $p > 2$. In the case $k = 0$ one has $f_\mu(z,0) = 1$.

This result is ultimately a consequence of a strong form of Liouville's theorem. The usual form of this theorem states that any analytic function which is uniformly bounded on $\mathbb{R}^2$ is constant. It is convenient for later purposes to define the space of continuous functions vanishing at infinity,

$$C_0(\mathbb{R}^2) = \{f : \mathbb{R}^2 \to \mathbb{C} \text{ continuous}\,;\ f(z) \to 0 \text{ as } |z| \to \infty\}.$$

It follows from the definition that any function in $C_0(\mathbb{R}^2)$ is uniformly bounded on $\mathbb{R}^2$.

Exercise 5.26. Prove the following forms of Liouville's theorem:

(a) Any bounded analytic function on $\mathbb{R}^2$ is constant.

(b) Any analytic function in $C_0(\mathbb{R}^2)$ is identically zero.

(c) Any function $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ satisfying $\bar\partial f = 0$ and $f(z) = o(1)$ as $|z| \to \infty$ is identically zero. (The last condition means that

$$\lim_{R\to\infty} \text{ess sup}_{|z|\geq R} |f(z)| = 0.)$$

The following is the strong form that we will use.

Lemma 5.27. *Let $\nu \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$ with $\|\nu\|_{L^\infty} < 1$, and assume that $\alpha \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$. Let also $p > 2$ be such that*

$$\|\nu\|_{L^\infty} \|S\|_{L^p\to L^p} < 1.$$

Then for any $f \in L^p_{\mathrm{comp}}(\mathbb{R}^2)$, the equation

$$\bar\partial g - \nu\overline{\partial g} + \alpha\bar g = f \quad \text{in } \mathbb{R}^2$$

has a unique solution $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ with $g(z) = o(1)$ as $|z| \to \infty$. Moreover, g is in $W^{1,p}(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$ and $g(z) = O(1/z)$ as $|z| \to \infty$. If f is supported in a fixed compact set K, then

$$\|g\|_{W^{1,p}(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}$$

for some constant C independent of f.

At this point, Proposition 5.25 follows easily. In the proof and at many points below, we will use the notation

$$e_k(z) := e^{i(kz+\overline{kz})}.$$

Proof of Proposition 5.25. Note that

$$\bar\partial(e^{ikz}v) = e^{ikz}\bar\partial v, \qquad \partial(e^{ikz}v) = e^{ikz}(\partial + ik)v.$$

Inserting the form $f = e^{ikz}(1+\eta)$ into $\bar\partial f = \mu\overline{\partial f}$, we have the equivalences

$$\begin{aligned}
&\bar\partial f = \mu\overline{\partial f} \\
&\Leftrightarrow e^{ikz}\bar\partial\eta = \mu e^{-i\overline{kz}}(\overline{\partial\eta} - i\bar k\bar\eta - i\bar k) \\
&\Leftrightarrow \bar\partial\eta - \mu e_{-k}\overline{\partial\eta} + i\bar k\mu e_{-k}\bar\eta = -i\bar k\mu e_{-k}.
\end{aligned}$$

Notice that $|e_k(z)| = 1$. Therefore the coefficients of the last equation are in $L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$ with $\|\mu e_{-k}\|_{L^\infty} < 1$, and the right hand side is in $L^q_{\mathrm{comp}}(\mathbb{R}^2)$ for any $q \geq 1$. Fix $p > 2$ such that

$$\|\mu e_{-k}\|_{L^\infty} \|S\|_{L^p\to L^p} < 1.$$

By Lemma 5.27 the last equation for η has a unique solution with $\eta \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and $\eta(z) = O(\frac{1}{z})$ as $|z| \to \infty$, and further $\eta \in W^{1,p}(\mathbb{R}^2)$. If $k = 0$ the right hand side is 0, and then the unique solution is $\eta(z,0) = 0$ showing that $f(z,0) = 1$. $\square$

To warm up for the proof of Lemma 5.27, we first give a slightly simpler result for a related equation.

Lemma 5.28. *Let $\nu \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$ with $\|\nu\|_{L^\infty} < 1$, and assume that $\alpha \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$. Let also $p > 2$ be such that*

$$\|\nu\|_{L^\infty} \|S\|_{L^p \to L^p} < 1.$$

Then for any $f \in L^p_{\mathrm{comp}}(\mathbb{R}^2)$, the equation

$$\bar\partial g - \nu \partial g + \alpha g = f \quad \text{in } \mathbb{R}^2.$$

has a unique solution $g \in W^{1,p} \cap C_0(\mathbb{R}^2)$.

Proof. 1. The first step is to reduce (5.28) to the case $\alpha = 0$. To do this, we find a function $\beta \in W^{1,p}(\mathbb{R}^2)$ satisfying

$$\bar\partial \beta - \nu \partial \beta = \alpha.$$

A solution is given by $\beta = Pw$, provided that $w \in L^p_{\mathrm{comp}}(\mathbb{R}^2)$ satisfies

$$w - \nu S w = \alpha.$$

This has the solution $w = (\mathrm{Id} - \nu S)^{-1}\alpha$, where $\mathrm{Id} - \nu S$ is invertible on L^p by the assumption. By writing $w = \alpha + \nu S(\mathrm{Id} - \nu S)^{-1}\alpha$ we see that $w \in L^p_{\mathrm{comp}}$, and consequently $\beta \in W^{1,p}(\mathbb{R}^2)$ as required.

2. To show the existence of a solution, note that the equation (5.28) is equivalent with

$$(\bar\partial - \nu\partial)(e^\beta g) = e^\beta f.$$

We try $e^\beta g = Pw$. This solves the last equation provided that w satisfies

$$w - \nu S w = e^\beta f.$$

Since $\beta \in W^{1,p} \subset L^\infty$, it follows that $e^\beta f \in L^p_{\mathrm{comp}}$ and the function $w = (\mathrm{Id} - \nu S)^{-1}(e^\beta f) \in L^p_{\mathrm{comp}}$ satisfies the equation required of w. Then

$$g = e^{-\beta} P w$$

is a solution of (5.28) with $g \in W^{1,p} \cap C_0(\mathbb{R}^2)$.

3. For uniqueness, assume that $g \in W^{1,p} \cap C_0(\mathbb{R}^2)$ solves

$$\bar\partial g - \nu \partial g + \alpha g = 0.$$

Choosing β as in Step 2, this is equivalent with

$$(\bar\partial - \nu\partial)(e^\beta g) = 0.$$

Since $e^\beta g \in W^{1,p}$, the function $w = \bar\partial(e^\beta g) \in L^p$ satisfies $e^\beta g = Pw$. Consequently

$$w - \nu S w = 0.$$

Now $\mathrm{Id} - \nu S$ is invertible on L^p so $w = 0$. Thus $\bar\partial(e^\beta g) = 0$, and since $e^\beta g \in C_0$ we obtain $e^\beta g = 0$ from Liouville's theorem (Problem 5.26(c)). $\square$

The proof of Lemma 5.27 is similar to the previous proof, the main difference being the appearance of complex conjugates of g and ∂g.

Proof of Lemma 5.27. 1. We begin by proving uniqueness of solutions. Suppose that $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, $g(z) = o(1)$ as $|z| \to \infty$, and

$$\bar\partial g - \nu \overline{\partial g} + \alpha \overline{g} = 0.$$

Define

$$\tilde\nu(z) = \begin{cases} \nu \frac{\overline{\partial g}}{\partial g}, & \partial g(z) \neq 0, \\ 0, & \partial g(z) = 0, \end{cases} \qquad \tilde\alpha(z) = \begin{cases} \alpha \frac{\bar g}{g}, & g(z) \neq 0, \\ 0, & g(z) = 0. \end{cases}$$

Then $\tilde\nu, \tilde\alpha \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$. These functions are measurable since g and ∂g are. This may be seen for instance by writing

$$\frac{\bar g}{g}\chi_{\{g\neq 0\}} = \lim_{\varepsilon\to 0} \frac{\bar g^2}{|g|^2+\varepsilon}\chi_{\{g\neq 0\}}.$$

Note also that

$$\|\tilde\nu S\|_{L^p\to L^p} \leq \|\nu S\|_{L^p\to L^p} < 1.$$

Then g is a solution of

$$\bar\partial g - \tilde\nu \partial g + \tilde\alpha g = 0.$$

Choosing $\beta \in W^{1,p}(\mathbb{R}^2)$ such that $\bar\partial\beta - \tilde\nu\partial\beta = \tilde\alpha$ as in Step 1 of the proof of Lemma 5.28 and writing $h = e^\beta g$, we reduce matters to the equation

$$\bar\partial h - \tilde\nu \partial h = 0.$$

2. It is sufficient to prove that any solution $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ of (5.4) satisfying $h(z) = o(1)$ as $|z| \to \infty$ is identically zero. From (5.4) we see that $\bar\partial h \in L^2_{\mathrm{comp}}(\mathbb{R}^2)$ and consequently $\partial h = S\bar\partial h$. It follows that $w = \bar\partial h$ satisfies

$$w - \tilde\nu S w = 0.$$

Since $\mathrm{Id} - \tilde\nu S$ is invertible on L^2, it follows that $w = 0$. Thus $\bar\partial h = 0$, and Liouville's theorem (Problem 5.26(c)) shows that $h = 0$.

3. It remains to prove existence of solutions. Because of the $\bar g$ term the reduction to the case $\alpha = 0$ is not immediately available. However, we can use the uniqueness of solutions combined with the Fredholm alternative to get the desired result. Write $C : g \mapsto \bar g$ for the complex conjugation operator. We need to solve

$$(\bar\partial - \nu C\partial + \alpha C)g = f.$$

Looking for a solution in the form $g = Pw$ for $w \in L^p_{\mathrm{comp}}$, it follows that w should satisfy

$$(\mathrm{Id} - \nu CS + \alpha CP)w = f.$$

Since $\|\nu CS\|_{L^p \to L^p} < 1$, the real-linear operator $\mathrm{Id} - \nu CS$ has the bounded inverse

$$(\mathrm{Id} - \nu CS)^{-1} = \sum_{j=0}^{\infty} (\nu CS)^j.$$

The equation (5.4) for w is equivalent with

$$(\mathrm{Id} + R)w = (\mathrm{Id} - \nu CS)^{-1} f$$

where

$$R = (\mathrm{Id} - \nu CS)^{-1} \alpha CP.$$

4. Let $K \subset \mathbb{R}^2$ be a compact set containing the supports of ν, α, and f. Since P is bounded $L^p_K(\mathbb{R}^2) \to W^{1,p}(\mathbb{R}^2)$ and multiplication by a compactly supported function is a compact operator from $W^{1,p}(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ (this is a consequence of compact Sobolev embedding), it follows that R is a compact real-linear operator on $L^p_K(\mathbb{R}^2)$. By the Fredholm alternative, (5.4) has a solution $w \in L^p_K(\mathbb{R}^2)$ if and only if $\mathrm{Id} + R$ is injective on $L^p_K(\mathbb{R}^2)$. But by tracing back the steps above, any solution $w \in L^p_K(\mathbb{R}^2)$ of $(\mathrm{Id} + R)w = 0$ gives rise to $h = Pw \in W^{1,p} \cap C_0$ satisfying

$$\bar{\partial} h - \nu \overline{\partial h} + \alpha \bar{h} = 0.$$

Thus $h = 0$, and also $w = \bar{\partial} h = 0$. This shows injectivity of $\mathrm{Id} + R$ and solvability of (5.4), which implies existence of a solution $g \in W^{1,p}(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$ to the original equation. The Fredholm alternative also implies that the inverse of $\mathrm{Id} + R$ is bounded on $L^p_K(\mathbb{R}^2)$, and the norm bound for g follows from (5.4) since

$$g = P(\mathrm{Id} + R)^{-1} (\mathrm{Id} - \nu CS)^{-1} f.$$

5. The final step is to prove that any solution $g \in C_0(\mathbb{R}^2)$ satisfies $g(z) = O(1/z)$ as $|z| \to \infty$. Since $\bar{\partial} g = 0$ outside some large disk with radius R, the function $h(z) = g(1/z)$ is analytic in $B \setminus \{0\}$ where $B = B(0, 1/R)$. By the condition $g \in C_0(\mathbb{R}^2)$, we have that $h(z) \to 0$ as $z \to 0$. The removable singularities theorem in complex analysis implies that h is analytic in B with $h(0) = 0$, and consequently we may write $h(z) = zv(z)$ for some function v analytic in B. Thus $g(z) = v(1/z)/z$ for $|z|$ large, showing that $g(z) = O(1/z)$ as $|z| \to \infty$. $\square$

Exercise 5.29. Prove the removable singularities theorem used in the previous proof: if w is analytic and uniformly bounded in $\mathbb{D} \setminus \{0\}$, then w has a unique extension as an analytic function into $\mathbb{D}$.

We also give a variant of Lemma 5.27 that will be used later.

Lemma 5.30. *Let $\nu \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$ with $\|\nu\|_{L^\infty} < 1$, and assume that $\alpha \in L^\infty_{\mathrm{comp}}(\mathbb{R}^2)$. Let also $p > 2$ be such that*

$$\|\nu\|_{L^\infty} \|S\|_{L^p \to L^p} < 1.$$

If $g \in W^{1,2}_{loc}(\mathbb{R}^2)$ solves the equation

$$\bar\partial g - \nu \overline{\partial g} + \alpha \bar g = 0 \quad \text{in } \mathbb{R}^2,$$

then g is continuous. If additionally g is bounded, then $g = Ce^\beta$ where C is a constant and $\beta \in W^{1,p}(\mathbb{R}^2)$.

Exercise 5.31. Prove Lemma 5.30 by modifying the proof of Lemma 5.27.

5.5. Basic properties of CGO solutions

Using Proposition 5.25, we can easily show one basic property of the solutions f_μ: their values in $\mathbb{R}^2 \setminus \mathbb{D}$ are uniquely determined by the DN map. This will be an important step in the solution of the Calderón problem.

Proposition 5.32. *If $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then*

$$f_{\pm\mu_1}(z,k) = f_{\pm\mu_2}(z,k) \quad \text{for } z \in \mathbb{R}^2 \setminus \mathbb{D}.$$

Proof. Since $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, we obtain from Proposition 5.14 that

$$\mathcal{H}_{\pm\mu_1} = \mathcal{H}_{\pm\mu_2}.$$

It is enough to consider the solutions f_{μ_j}. Decompose $f_{\mu_j} = f_{\mu_j}(\,\cdot\,,k)$ into real and imaginary parts as

$$f_{\mu_j} = u_j + iv_j.$$

We wish to define a function

$$\tilde f(z) = \begin{cases} \tilde u(z) + i\tilde v(z), & z \in \mathbb{D}, \\ u_2(z) + iv_2(z), & z \in \mathbb{R}^2 \setminus \mathbb{D}, \end{cases}$$

such that $\tilde f$ solves $\bar\partial \tilde f = \mu_1 \overline{\partial \tilde f}$ in $\mathbb{R}^2$.

Let first $\tilde u$ be the unique $W^{1,2}(\mathbb{D})$ solution of

$$\nabla \cdot \sigma_1 \nabla \tilde u = 0 \text{ in } \mathbb{D}, \quad \tilde u|_{\partial\mathbb{D}} = u_2|_{\partial\mathbb{D}}.$$

Since $v_2|_{\mathbb{D}}$ is a σ-harmonic conjugate of $u_2|_{\mathbb{D}}$ in $\mathbb{D}$, it follows that for some constant c_0 one has

$$v_2|_{\partial\mathbb{D}} = \mathcal{H}_{\mu_2}(u_2|_{\partial\mathbb{D}}) + c_0.$$

Let $\tilde v \in W^{1,2}(\mathbb{D})$ be a σ_1-harmonic conjugate of $\tilde u$ in $\mathbb{D}$. This is unique up to an additive constant, and we fix this constant by requiring that

$$\int_{\partial\mathbb{D}} (\tilde v - c_0)\, dS = 0.$$

Then

$$\begin{aligned}\tilde{v}|_{\partial\mathbb{D}} &= (\tilde{v} - c_0)|_{\partial\mathbb{D}} + c_0 \\ &= \mathcal{H}_{\mu_1}(\tilde{u}|_{\partial\mathbb{D}}) + c_0 \\ &= \mathcal{H}_{\mu_2}(\tilde{u}|_{\partial\mathbb{D}}) + c_0 \\ &= \mathcal{H}_{\mu_2}(u_2|_{\partial\mathbb{D}}) + c_0 \\ &= v_2|_{\partial\mathbb{D}}.\end{aligned}$$

Now define $\tilde{f}$ by (5.5). Since $\tilde{u} + i\tilde{v}|_{\partial\mathbb{D}} = u_2 + iv_2|_{\partial\mathbb{D}}$, the function $\tilde{f}$ is in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$. Then

$$\bar{\partial}\tilde{f} = \mu_1\overline{\partial\tilde{f}} \quad \text{in } \mathbb{R}^2$$

since $\tilde{f}$ satisfies this equation both in $\mathbb{D}$ and $\mathbb{R}^2 \setminus \mathbb{D}$. Also, one has

$$\tilde{f} = e^{ikz}(1 + \tilde{\eta})$$

where $\tilde{\eta} = e^{-ikz}\tilde{f} - 1$ is in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$. But since $\tilde{f} = f_{\mu_2} = e^{ikz}(1 + \eta_2)$ in $\mathbb{R}^2 \setminus \mathbb{D}$ where $\eta_2(z) = O(1/z)$ for $|z|$ large, it follows that $\tilde{\eta}(z) = O(1/z)$ for $|z|$ large. We can now invoke the uniqueness part of Proposition 5.25 to conclude that

$$\tilde{f} = f_{\mu_1} \quad \text{in } \mathbb{R}^2.$$

This shows that $f_{\mu_1} = f_{\mu_2}$ in $\mathbb{R}^2 \setminus \mathbb{D}$. □

Another property of the solutions f_μ that follows quite easily is their smoothness with respect to the parameter k. This will be used in the next section when deriving the $\partial_{\bar{k}}$ equation for the scattering transform. Recall that we write $f_\mu(z,k) = e^{ikz}(1 + \eta_\mu(z,k))$ where $\eta_\mu(\,\cdot\,,k) \in W^{1,p}(\mathbb{R}^2)$ for some $p > 2$.

Proposition 5.33. *For some $p > 2$, $k \mapsto \eta_\mu(\,\cdot\,,k)$ is a C^∞ map from $\mathbb{C}$ into $W^{1,p}(\mathbb{R}^2)$ and $k \mapsto f_\mu(\,\cdot\,,k)$ is a C^∞ map from $\mathbb{C}$ into $W^{1,p}_{loc}(\mathbb{R}^2)$.*

The statement for $F : k \mapsto \eta_\mu(\,\cdot\,,k)$ means that all Frechet derivatives $(D^mF)_k$ exist at each $k \in \mathbb{C}$, and the maps $k \mapsto (D^mF)_k$ are continuous $\mathbb{C} \to L(\mathbb{C}^m, W^{1,p}(\mathbb{R}^2))$. The proof is an easy consequence of the next result.

Lemma 5.34. *If $k \in \mathbb{C}$ define*

$$L_\mu(k) : W^{1,p}(\mathbb{R}^2) \to W^{1,p}(\mathbb{R}^2), \quad L_\mu(k)g = P(\mu\overline{\partial(e_k g)}).$$

(a) The map $k \mapsto e_{\pm k}|_B$ is a C^∞ map from $\mathbb{C}$ to $W^{1,\infty}(B)$ for any bounded open set $B \subset \mathbb{R}^2$.

(b) The map $k \mapsto L_\mu(k)$ is a C^∞ map from $\mathbb{C}$ to $L(W^{1,p}(\mathbb{R}^2))$.

(c) For each k the map $Id - L_\mu$ is bounded and invertible on $W^{1,p}(\mathbb{R}^2)$, and $k \mapsto (Id - L_\mu)^{-1}$ is a C^∞ map from $\mathbb{C}$ to $L(W^{1,p}(\mathbb{R}^2))$.

Exercise 5.35. If X, Y, Z are normed spaces and $F : U \to Y$ is C^k (resp. C^∞) in some open set $U \subset X$, and if $A : Y \to Z$ is a bounded linear map, then $A \circ F$ is C^k (resp. C^∞) and $D^m(A \circ F) = A \circ D^m F$.

Exercise 5.36. If X, Y, Z are normed spaces and $F : U \to V$, $G : V \to Z$ are C^k (resp. C^∞) maps where $U \subset X$, $V \subset Y$ are open sets, then $G \circ F : U \to Z$ is C^k (resp. C^∞) and

$$D(G \circ F)_x = DG_{F(x)} \circ DF_x.$$

Exercise 5.37. If X is a Banach space and U is the set of invertible elements of $L(X)$, show that the map $U \to L(X), A \mapsto A^{-1}$ is C^∞.

Proof of Proposition 5.33. Note that the solutions $f = e^{ikz}(1+\eta)$ given in Proposition 5.25 are characterized by

$$\begin{aligned}\bar\partial f = \mu\overline{\partial f} &\iff e^{ikz}\bar\partial\eta = \mu\overline{\partial(e^{ikz}\eta)} - \mu i\bar k e^{-i\overline{kz}} \\ &\iff \bar\partial\eta = \mu\overline{\partial(e_k\eta)} - \mu i\bar k e_{-k}.\end{aligned}$$

Since $\eta \in W^{1,p}(\mathbb{R}^2)$ and the right hand side is in $L^p_{comp}(\mathbb{R}^2)$ for some $p > 2$, it follows that

$$\eta = P(\mu\overline{\partial(e_k\eta)}) - i\bar k P(\mu e_{-k}).$$

Equivalently, $(I - L_\mu(k))\eta = -i\bar k P(\mu e_{-k})$. By Lemma 5.34, we have

$$\eta_\mu(\,\cdot\,, k) = -i\bar k (I - L_\mu(k))^{-1}(P(\mu e_{-k})).$$

□

Proof of Lemma 5.34. (a) Let $F : k \mapsto e_k|_B$ from $\mathbb{C}$ to $W^{1,\infty}(B)$. If $t \in \mathbb{R}$, the Taylor expansion of $f(t) = e^{it}$ implies that

$$e^{it} = 1 + it + \ldots + \frac{i^m}{m!}t^m + R_{m+1}(t), \quad R_{m+1}(t) = \int_0^t e^{is}(t-s)^m\,ds.$$

Since $e_{k+h} = e_k e_h$ for $k, h \in \mathbb{C}$, we have

$$e_{k+h} = e_k\left[1 + i(hz + \overline{hz}) + \ldots + \frac{i^m}{m!}(hz + \overline{hz})^m\right] + e_k R_{m+1}(hz + \overline{hz}).$$

The remainder term satisfies

$$\left\|e_k R_{m+1}(hz + \overline{hz})\right\|_{W^{1,\infty}(B)} \le C_{m,B,k}\,|h|^{m+1}.$$

(b) Let $B = B(0,R)$ be a ball containing the support of μ. Define

$$(DL_\mu)_k[h]g = P(\mu C\partial(e_k(i(hz + \overline{hz}))g)).$$

We have

$$(L_\mu(k+h) - L_\mu(k) - (DL_\mu)_k[h])g = P(\mu C\partial(e_k R_2(hz + \overline{hz}))).$$

We leave it as an exercise to check that the higher derivatives also exist and $k \mapsto L_\mu(k)$ is a C^∞ map.

(b) It follows from (a) that $I-L_\mu$ is bounded on $W^{1,p}(\mathbb{R}^2)$. To show that it is invertible, we need to show that the equation $(I-L_\mu)g = f$ has a unique solution $g \in W^{1,p}(\mathbb{R}^2)$ for any $f \in W^{1,p}(\mathbb{R}^2)$, and that $\|g\|_{W^{1,p}} \leq C\,\|f\|_{W^{1,p}}$ for some constant C independent of f.

Writing $g = f + h$ where $h \in W^{1,p}(\mathbb{R}^2)$, the equation is equivalent with

$$(I - L_\mu)h = L_\mu f.$$

We write $L_\mu w = P(\mu e_k \overline{\partial w} + \mu(\bar\partial e_k)\bar w)$. Since μ is compactly supported and $h, f \in W^{1,p}(\mathbb{R}^2)$, we may take $\bar\partial$ of the earlier equation and obtain the equivalent equation

$$\bar\partial h - \mu e_k \overline{\partial h} - \mu(\bar\partial e_k)\bar h = \mu e_k \overline{\partial f} + \mu(\bar\partial e_k)\bar f.$$

The right hand side is in $L^p_{comp}(\mathbb{R}^2)$ and the coefficients μe_k, $\mu(\bar\partial e_k)$ are in $L^\infty_{comp}(\mathbb{R}^2)$ with $\|\mu e_k\|_{L^\infty} < 1$. Thus Lemma 5.27 applies, and it follows that the last equation has a unique solution $h \in W^{1,p}(\mathbb{R}^2)$ for any $f \in W^{1,p}(\mathbb{R}^2)$ with the norm bound

$$\|h\|_{W^{1,p}} \leq C\left\|\mu e_k \overline{\partial f} + \mu(\bar\partial e_k)\bar f\right\|_{L^p} \leq C\,\|f\|_{W^{1,p}}\,.$$

The invertibility of $I - L_\mu$ on $W^{1,p}(\mathbb{R}^2)$ follows. □

Exercise 5.38. Complete the details of the proof of part (b) in Lemma 5.34.

5.6. Scattering transform

In the previous sections we have made a reduction from the conductivity equation $\nabla \cdot \sigma \nabla u = 0$ into a Beltrami equation $\bar\partial f = \mu \overline{\partial f}$, where μ and σ are related by

$$\mu = \frac{1-\sigma}{1+\sigma}, \quad -\mu = \frac{1-1/\sigma}{1+1/\sigma}.$$

Note the symmetry between $\pm\mu$ and $\sigma^{\pm 1}$; from this point on it is convenient to consider both μ and $-\mu$ (or σ and $1/\sigma$) simultaneously.

We have also defined the μ-Hilbert transform $\mathcal{H}_\mu$ and constructed CGO solutions f_μ to the Beltrami equation, and have proved the following implications:

$$\begin{aligned} &\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \\ &\implies \mathcal{H}_{\pm\mu_1} = \mathcal{H}_{\pm\mu_2} \\ &\implies f_{\pm\mu_1}|_{\mathbb{C}\setminus\mathbb{D}} = f_{\pm\mu_2}|_{\mathbb{C}\setminus\mathbb{D}}. \end{aligned}$$

The final step in the proof is to show that $f_{\pm\mu_1} = f_{\pm\mu_2}$ also in $\mathbb{D}$. From this fact, one can conclude (at least formally) that

$$\mu_1|_{\mathbb{D}} = \frac{\bar{\partial} f_{\mu_1}}{\overline{\partial f_{\mu_1}}}\Big|_{\mathbb{D}} = \frac{\bar{\partial} f_{\mu_2}}{\overline{\partial f_{\mu_2}}}\Big|_{\mathbb{D}} = \mu_2|_{\mathbb{D}}.$$

Using the relationship between σ and μ, this would immediately imply that $\sigma_1 = \sigma_2$ in $\mathbb{D}$.

The proof that $f_{\pm\mu_1}|_{\mathbb{D}} = f_{\pm\mu_2}|_{\mathbb{D}}$ is involved, and it relies on the fact that the solutions $f_{\pm\mu} = f_{\pm\mu}(z,k)$ satisfy a $\bar{\partial}$-type equation also with respect to the complex variable k. In fact, the $\partial_{\bar{k}}$ equation for these solutions turns out to be in a sense simpler than the $\partial_{\bar{z}}$ equation $\partial_{\bar{z}} f_{\pm\mu} = \pm\mu\overline{\partial_z f_{\pm\mu}}$ that we have studied before.

To state the $\partial_{\bar{k}}$ equations, it is convenient to switch from solutions of the Beltrami equation back to the original conductivity equation. From Proposition 5.10 and the relation (5.6) we know that $\mathrm{Re}(f_\mu)$ and $\mathrm{Im}(f_{-\mu})$ are solutions of $\nabla \cdot \sigma \nabla u = 0$ in $\mathbb{R}^2$, and $\mathrm{Im}(f_\mu)$ and $\mathrm{Re}(f_{-\mu})$ are solutions of $\nabla \cdot (1/\sigma)\nabla u = 0$ in $\mathbb{R}^2$. We define two complex valued functions

$$u_\sigma := \mathrm{Re}(f_\mu) + i\,\mathrm{Im}(f_{-\mu}),$$
$$u_{1/\sigma} := \mathrm{Re}(f_{-\mu}) + i\,\mathrm{Im}(f_\mu).$$

Note that the pair $(u_\sigma, u_{1/\sigma})$ uniquely determines the pair $(f_\mu, f_{-\mu})$, and vice versa.

Proposition 5.39. *Let a denote either σ or $1/\sigma$. For any $k \in \mathbb{C}$ the function $u_a(z,k)$ is the unique complex valued solution of*

$$\nabla \cdot a \nabla u_a(\,\cdot\,, k) = 0 \text{ in } \mathbb{R}^2$$

having the form

$$u_a(z,k) = e^{ikz}(1 + r_a(z,k))$$

where $r_a(\,\cdot\,,k) \in W^{1,2}_{loc}(\mathbb{R}^2)$ and $r_a(z,k) = o(1)$ as $z \to \infty$. Further, u_a is C^∞ with respect to the k variable, and for any $z \in \mathbb{C}$ it satisfies the $\partial_{\bar{k}}$-equation

$$\partial_{\bar{k}} u_a(z,\,\cdot\,) = -i\tau_a(\,\cdot\,)\overline{u_a(z,\,\cdot\,)} \text{ in } \mathbb{R}^2$$

where $\tau_a(k)$ is a complex function in $\mathbb{R}^2$.

The coefficient $\tau_a(k)$ deserves a special name. We will explain the reason for this terminology in the end of this section.

Definition 5.40. The function $\tau_\sigma(k)$ is called the *scattering transform* or *nonlinear Fourier transform* of a positive L^∞ conductivity σ.

A number of basic properties of $\tau_\sigma(k)$ for a positive L^∞ conductivity σ, with $\sigma = 1$ in $\mathbb{R}^2 \setminus \mathbb{D}$, are given in the next proposition.

Proposition 5.41. *The scattering transform is a C^∞ function with respect to k, and it satisfies* **[Note: formatting]**

$$|\tau_\sigma(k)| \le 1 \text{ for any } k \in \mathbb{C},$$

$$\tau_{1/\sigma}(k) = -\tau_\sigma(k).$$

It is determined from the special solutions $f_{\pm\mu}$ by **[Note: formatting]**

$$\overline{\tau_\sigma(k)} = \frac{1}{2}(b_1^\mu(k) - b_1^{-\mu}(k)),$$

$$b_1^{\pm\mu}(k) := \lim_{z\to\infty} z(e^{-ikz} f_{\pm\mu}(z,k) - 1), \quad k \in \mathbb{C}.$$

In particular, if $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then $\tau_{\sigma_1} = \tau_{\sigma_2}$.

We will begin the proof of Propositions 5.39 and 5.41. Recall that

$$\eta_\mu(z,k) = e^{-ikz} f_\mu(z,k) - 1.$$

The equation $\bar\partial f_\mu = \mu\overline{\partial f_\mu}$, together with the fact that $\mu = 0$ in $\mathbb{R}^2 \setminus \mathbb{D}$, implies that

$$\partial_{\bar z}\eta_\mu(\,\cdot\,,k) = 0 \ \text{ in } \mathbb{R}^2 \setminus \mathbb{D}.$$

Since also $\eta_\mu(z,k) = O(1/z)$ as $z \to \infty$, the function $z \mapsto \eta_\mu(1/z,k)$ is an analytic function in $\mathbb{D}$ vanishing at 0 and continuous up to $\partial\mathbb{D}$. It follows that

$$\eta_\mu(1/z,k) = \sum_{m=1}^\infty b_m^\mu(k) z^k, \quad |z| \le 1,$$

where $b_m^\mu(k)$ is obtained as the Fourier coefficient

$$b_m^\mu(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\eta_\mu(e^{-i\theta},k)\,d\theta.$$

A similar argument works for $\eta_{-\mu}$, and we have

$$\eta_{\pm\mu}(z,k) = \sum_{m=1}^\infty b_m^{\pm\mu}(k) z^{-m}, \quad |z| \ge 1.$$

Proof of Proposition 5.39. It is enough to consider u_σ. Since $f_\mu = e^{ikz}(1+\eta_\mu)$, we have

$$\mathrm{Re}(f_\mu) = \frac{1}{2}e^{ikz}(1 + \eta_\mu + e_{-k}(1+\overline{\eta_\mu})),$$

$$\mathrm{Im}(f_{-\mu}) = \frac{1}{2i}e^{ikz}(1 + \eta_{-\mu} - e_{-k}(1+\overline{\eta_{-\mu}})).$$

Consequently

$$u_\sigma = \mathrm{Re}(f_\mu) + i\,\mathrm{Im}(f_{-\mu}) = e^{ikz}\left(1 + \frac{\eta_\mu + \eta_{-\mu}}{2} + e_{-k}\frac{\bar\eta_\mu - \bar\eta_{-\mu}}{2}\right).$$

Thus u_σ is a solution of $\nabla\cdot\sigma\nabla u = 0$ in $\mathbb{R}^2$ of the form $e^{ikz}(1+r_\sigma)$ where $r_\sigma(z,k)$ has the required properties. We leave the uniqueness of solutions of this type as an exercise. (Note that a slightly weaker statement, namely the uniqueness of the pair $(u_\sigma, u_{1/\sigma})$ satisfying conductivity equations for σ and $1/\sigma$ and having asymptotics as above, would reduce easily to the uniqueness of f_μ and $f_{-\mu}$. Also this weaker fact would be sufficient for completing the proof of the Calderón problem.)

Proposition 5.33 and (5.6) show that u_σ is C^∞ in k. Computing the $\partial_{\bar k}$ derivative gives

$$\partial_{\bar k} u_\sigma = e^{ikz}\left[-i\bar z e_{-k}\frac{\bar\eta_\mu - \bar\eta_{-\mu}}{2} + \partial_{\bar k}\left(\frac{\eta_\mu+\eta_{-\mu}}{2}\right) + e_{-k}\partial_{\bar k}\left(\frac{\bar\eta_\mu - \bar\eta_{-\mu}}{2}\right)\right].$$

The expression (5.6) implies that

$$\bar z\frac{\bar\eta_\mu - \bar\eta_{-\mu}}{2} = \frac{\overline{b_1^\mu(k) - b_1^{-\mu}(k)}}{2} + \tilde r(z,k)$$

where $\tilde r(\,\cdot\,,k)\in W^{1,2}_{loc}(\mathbb{R}^2)$ and $\tilde r = O(1/z)$ as $z\to\infty$. Defining $\tau_\sigma(k)$ by

$$\overline{\tau_\sigma(k)} = \frac{1}{2}(b_1^\mu(k) - b_1^{-\mu}(k)),$$

we obtain

$$\partial_{\bar k} u_\sigma = e^{-i\overline{kz}}\left[-i\tau_\sigma(k) + \hat r(z,k)\right]$$

with $\hat r(\,\cdot\,,k)\in W^{1,2}_{loc}(\mathbb{R}^2)$ and $\hat r = O(1/z)$ as $z\to\infty$.

On the other hand, since u_σ solves $\nabla\cdot\sigma\nabla u_\sigma = 0$ in $\mathbb{R}^2$ and since it is smooth in k, also the function $\partial_{\bar k} u_\sigma$ solves the same equation. Thus we have two solutions of this equation,

$$\overline{\partial_{\bar k} u_\sigma} = e^{ikz}\left[i\overline{\tau_\sigma(k)} + o(1)\right],$$
$$i\overline{\tau_\sigma(k)}u_\sigma = e^{ikz}\left[i\overline{\tau_\sigma(k)} + o(1)\right]$$

as $z\to\infty$. The uniqueness statement for these solutions implies that they have to be equal, thus resulting in the $\partial_{\bar k}$ equation

$$\partial_{\bar k} u_\sigma(z,\,\cdot\,) = -i\tau_\sigma(\,\cdot\,)\overline{u_\sigma(z,\,\cdot\,)}.$$

□

Exercise 5.42. Show the uniqueness statement for the complex geometrical optics solutions in Proposition 5.39.

For the proof of the bound (5.41) for the scattering transform, we first record a number of useful properties of the functions

$$M_{\pm\mu}(z,k) := e^{-ikz} f_{\pm\mu}.$$

We will make use the next elementary fact and also the classical Schwarz lemma from complex analysis.

Exercise 5.43. If $z, w \in \mathbb{C}$, $w \neq 0$, and $\mathrm{Re}(z/w) > 0$, then one has $z+w \neq 0$ and
$$\left|\frac{z-w}{z+w}\right| < 1.$$

Exercise 5.44. Prove the Schwarz lemma: if h is analytic in $\mathbb{D}$ with $h(0) = 0$ and $|h(z)| \leq 1$ in $\mathbb{D}$, then $|h(z)| \leq |z|$ in $\mathbb{D}$.

Lemma 5.45. *The function $M_{\pm\mu}$ is nonvanishing. Moreover,*
$$M_{\pm\mu}(z,k) = e^{\beta_\pm(z,k)}$$
where, for some $p > 2$, $\beta_\pm(\,\cdot\,,k) \in W^{1,p}(\mathbb{R}^2)$ for each k. One also has
$$\mathrm{Re}\left(\frac{M_\mu}{M_{-\mu}}\right) > 0,$$
so that $M_\mu + M_{-\mu} \neq 0$ everywhere and
$$\left|\frac{M_\mu - M_{-\mu}}{M_\mu + M_{-\mu}}\right| < 1, \quad \left|\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right| < 1.$$
If $|z| \geq 1$ then
$$\left|\frac{M_\mu - M_{-\mu}}{M_\mu + M_{-\mu}}\right| \leq \frac{1}{|z|}, \quad \left|\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right| \leq \frac{1}{|z|}.$$

Proof. The Beltrami equation for f_μ immediately implies that
$$\bar\partial M_\mu = \mu e_{-k}\overline{\partial M_\mu} - i\bar k \mu e_{-k}\overline{M_\mu} \text{ in } \mathbb{R}^2.$$
Since $M_\mu = 1 + \eta_\mu$ where $\eta_\mu(\,\cdot\,,k) \in W^{1,p}(\mathbb{R}^2)$, the function $M_\mu(\,\cdot\,,k)$ is in $W^{1,2}_{loc}$ and bounded for any $k \in \mathbb{C}$. Lemma 5.30 shows that $M_\mu = Ce^\beta$ for some constant C and some $\beta \in W^{1,p}(\mathbb{R}^2)$ for some $p > 2$. Since $M_\mu \to 1$ as $z \to \infty$, we have $C = 1$. The same proof works for $M_{-\mu}$.

To show the positivity of $\mathrm{Re}(M_\mu/M_{-\mu})$, assume on the contrary that the real part is nonpositive at some point in $\mathbb{C}$. Since $M_\mu/M_{-\mu}$ is continuous and has limit 1 as $z \to \infty$, it is not possible that $\mathrm{Re}(M_\mu/M_{-\mu}) < 0$ everywhere. Thus, $\mathrm{Re}(M_\mu/M_{-\mu}) = 0$ for some $z_0 \in \mathbb{C}$, so there is $t \in \mathbb{R}$ with
$$M_\mu(z_0,k) = itM_{-\mu}(z_0,k).$$
Writing $h(z,k) = M_\mu(z,k) - itM_{-\mu}(z,k)$, the equations for $M_{\pm\mu}$ show that
$$\bar\partial h = \mu e_k\overline{\partial h} - i\bar k \mu e_{-k}\bar h \text{ in } \mathbb{R}^2.$$
Now h is bounded, so by Lemma 5.30 we have $h = Ce^\beta$. Using that $h(z_0) = 0$ we must have $C = 0$, so $M_\mu = itM_{-\mu}$. This contradicts the fact that the limit of $M_{\pm\mu}$ as $z \to \infty$ is 1.

We have proved that $\mathrm{Re}(M_\mu/M_{-\mu}) > 0$. Consequently $M_\mu + M_{-\mu}$ is nonvanishing, and the conditions (5.45) follow from Problem 5.43. If $|z| \geq 1$, define

$$m(z,k) := \frac{M_\mu - M_{-\mu}}{M_\mu + M_{-\mu}}.$$

By (5.45) we have

$$|m(z,k)| < 1.$$

The function $h(z) := m(1/z, k)$ is then analytic in $\mathbb{D}$, satisfies $h(0) = 0$ by the asymptotics for $M_\mu = 1 + \eta_\mu$, and $|h(z)| < 1$ in $\overline{\mathbb{D}}$. The Schwarz lemma (Problem 5.44) implies that $|h(z)| \leq |z|$. Consequently $|m(z,k)| \leq 1/|z|$, which shows the bound (5.45). $\square$

Proof of Proposition 5.41. In the proof of Proposition 5.39 we defined $\tau_\sigma(k)$ so that (5.41) is satisfied, and the expression (5.41) follows immediately from (5.6) and the definition of $f_{\pm\mu}$. Since Proposition 5.32 shows that Λ_σ determines $f_{\pm\mu}$ in $\mathbb{R}^2 \setminus \mathbb{D}$, we see from (5.41) that Λ_σ determines $b_1^{\pm}(k)$ and thus $\tau_\sigma(k)$ for all k. The property (5.41) is a direct consequence of (5.41) since the conductivity $1/\sigma$ corresponds to Beltrami coefficient $-\mu$. Also, the fact that τ_σ is smooth with respect to k follows since $b_1^{\pm\mu}$ are smooth, using the Fourier coefficient definition (5.6) and Proposition 5.33.

It remains to prove the bound (5.41). By (5.41), (5.41) the scattering transform can be expressed in terms of the functions $M_{\pm\mu}$ as

$$\tau_\sigma(k) = \lim_{z\to\infty} z\frac{M_\mu - M_{-\mu}}{2} = \lim_{z\to\infty} z\frac{M_\mu - M_{-\mu}}{M_\mu + M_{-\mu}}.$$

Here we used that $M_{\pm\mu} \to 1$ as $z \to \infty$. Now the bound (5.41) follows from (5.45). $\square$

To explain the origin of the terminology for $\tau_\sigma(k)$, we describe briefly and in a formal way a one-dimensional scattering problem where similar concepts appear. (The multidimensional case will be discussed in Chapter X.)

Consider an infinite string whose displacement at time t is given by the function $u(x,t)$. If the string is homogeneous, the displacement function solves the wave equation

$$(\partial_t^2 - \partial_x^2)u = 0.$$

A particular solution is given by the one-dimensional plane wave

$$u_0(t,x) = \delta(t-x).$$

This is a Dirac delta function moving to the right, whose peak at time t is at the point $x = t$.

Consider now an inhomogeneous string, whose displacement function solves the perturbed wave equation

$$(\partial_t^2 - \partial_x^2 + q)u = 0.$$

The inhomogeneity is given by the potential $q = q(x) \in C_c^\infty(\mathbb{R})$, and we assume that $\operatorname{supp}(q) \subset [-R, R]$. The free plane wave $u_0(x,t)$ solves this equation if $t < -R$, and there is a unique solution $u_q(x,t)$ satisfying

$$(\partial_t^2 - \partial_x^2 + q)u_q = 0, \quad u_q = u_0 \text{ when } t < -R.$$

The function u_q is called the (incoming) distorted plane wave corresponding to u_0.

To describe the distorted plane waves in more detail, we take Fourier transforms with respect to t in (5.6). Writing

$$\psi_q(x,k) := \int_{-\infty}^{\infty} e^{-ikt} u_q(x,t)\,dt,$$
$$\psi_0(x,k) := \int_{-\infty}^{\infty} e^{-ikt} u_0(x,t)\,dt = e^{-ikx},$$

this results in the equation

$$(-\partial_x^2 - k^2 + q)\psi_q = 0 \text{ in } \mathbb{R}.$$

The incoming condition $u_q - u_0 = 0$ for $t < -R$ is transformed into analyticity of $\psi_q - \psi_0$ for $\operatorname{Im}(k) < 0$, and the unique solution $\psi_q(x,k)$ satisfying the analyticity condition can be explicitly written in terms of resolvent operators.

The functions ψ_0 and ψ_q are also called plane waves (even though they are the time Fourier transforms of actual solutions of the wave equation). They are of fundamental importance in scattering theory, and can be used to parametrize the generalized eigenfunctions of the operators $-\partial_x^2$ and $-\partial_x^2 + q$ and also to study various scattering measurements. The family of exponentials $\psi_0(x,k) = e^{-ikx}$ is related to the usual Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\psi_0(x,k)\,dx,$$

while the family $\psi_q(x,k)$ gives rise to a distorted Fourier transform

$$\mathcal{F}_q\{f\}(k) := \int_{-\infty}^{\infty} f(x)\psi_q(x,k)\,dx.$$

Counterparts of the inversion and Plancherel formulas are valid also for the distorted transform $\mathcal{F}_q$.

Finally, we discuss an analogy between the scattering transform $\tau_\sigma(k)$ and the above concepts. Consider the inverse backscattering problem of determining the potential q from the measurements

$$\lim_{x\to-\infty} u_q(x,t), \quad t\in\mathbb{R}.$$

This corresponds to sending a free plane wave from $x=-\infty$ and then measuring the response also at $x=-\infty$. From these measurements one can determine $\lim_{x\to-\infty}\psi_q(x,k)$ for any k by taking the Fourier transform in time. Simple properties of the resolvent give the asymptotics

$$\psi_q(x,k)=\psi_0(x,k)+\frac{1}{2ik}e^{-ikx}\int_{-\infty}^{\infty} e^{-iky}q(y)\psi_q(y,k)\,dy, \quad x<-R.$$

Consequently, one determines the backscattering transform of q,

$$B_q(k):=\int_{-\infty}^{\infty} e^{-iky}q(y)\psi_q(y,k)\,dy, \quad k\in\mathbb{R}.$$

Note that in the Born approximation $\psi_q\approx\psi_0$, one recovers the usual Fourier transform

$$B_q(k)\approx\int_{-\infty}^{\infty} e^{-2iky}q(y)\,dy.$$

We have shown that backscattering measurements for the equation $(-\partial_x^2-k^2+q)\psi=0$ determine a nonlinear Fourier transform of q, via the special solutions ψ_q, by

$$B_q(k)=\lim_{x\to-\infty} 2ike^{ikx}(\psi_q(x,k)-\psi_0(x,k)), \quad k\in\mathbb{R}.$$

Similarly, boundary measurements for the two-dimensional conductivity equation $\nabla\cdot\sigma\nabla u=0$ (or equivalently the Beltrami equation $\bar\partial f=\mu\overline{\partial f}$) determine, via the special CGO solutions $f_{\pm\mu}$, the nonlinear Fourier transform of σ,

$$b_1^{\pm\mu}(k)=\lim_{z\to\infty} ze^{-ikz}(f_{\pm\mu}(z,k)-f_0(z,k)), \quad k\in\mathbb{C},$$

$$\tau_\sigma(k)=\frac{1}{2}\overline{(b_1^{\mu}(k)-b_1^{-\mu}(k))}.$$

This analogy motivates calling τ_σ the scattering transform or nonlinear Fourier transform.

5.7. Uniqueness for C^2 conductivities

In this section we show that in the case of C^2 conductivities, one can finish the uniqueness proof of the Calderón problem. The more difficult case of bounded measurable conductivities will be dealt with in the following sections.

Theorem 5.46. *Let σ_1,σ_2 be two positive functions in $C^2(\overline{\mathbb{D}})$ with $\sigma_1=\sigma_2=1$ near $\partial\mathbb{D}$. If*

$$\Lambda_{\sigma_1}=\Lambda_{\sigma_2}$$

then

$$\sigma_1 = \sigma_2.$$

After a standard reduction, the above theorem implies an analogous result on any smooth domain.

Exercise 5.47. If Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary and $\sigma_1, \sigma_2 \in C^2(\overline{\Omega})$ are positive functions satisfying $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, show that $\sigma_1 = \sigma_2$.

Assume that σ_j are extended as functions in $C^2(\mathbb{R}^2)$ so that $\sigma_j = 1$ outside of $\mathbb{D}$. Even for bounded measurable conductivities, up to this point we have shown that

$$\begin{aligned}
&\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \\
&\implies \mathcal{H}_{\pm\mu_1} = \mathcal{H}_{\pm\mu_2} \\
&\implies f_{\pm\mu_1}|_{\mathbb{C}\setminus\mathbb{D}} = f_{\pm\mu_2}|_{\mathbb{C}\setminus\mathbb{D}} \\
&\implies \tau_{\sigma_1} = \tau_{\sigma_2}
\end{aligned}$$

where $\tau_\sigma(k)$ is the scattering transform appearing as a coefficient in the $\partial_{\bar{k}}$ equation (where $z \in \mathbb{C}$ is fixed)

$$\partial_{\bar{k}} u_\sigma(z, \,\cdot\,) = -i\tau_\sigma(\,\cdot\,)\overline{u_\sigma(z, \,\cdot\,)} \quad \text{in } \mathbb{R}^2.$$

Here u_σ is the unique solution of $\nabla\cdot\sigma\nabla u = 0$ in $\mathbb{R}^2$ of the form $u = e^{ikz}(1+r)$ where $r \in W^{1,2}_{loc}(\mathbb{R}^2)$ and $r = o(1)$ as $z \to \infty$.

In particular, writing

$$\tau(k) = \tau_{\sigma_1}(k) = \tau_{\sigma_2}(k),$$

the functions u_{σ_1} and u_{σ_2} both solve the same $\partial_{\bar{k}}$ equation:

$$\begin{aligned}
\partial_{\bar{k}} u_{\sigma_1}(z, \,\cdot\,) &= -i\tau(\,\cdot\,)\overline{u_{\sigma_1}(z, \,\cdot\,)} \quad \text{in } \mathbb{R}^2, \\
\partial_{\bar{k}} u_{\sigma_2}(z, \,\cdot\,) &= -i\tau(\,\cdot\,)\overline{u_{\sigma_2}(z, \,\cdot\,)} \quad \text{in } \mathbb{R}^2.
\end{aligned}$$

Using that $u_\sigma = e^{ikz}(1 + r_\sigma)$, the equations become

$$\begin{aligned}
\partial_{\bar{k}} r_{\sigma_1}(z,k) &= -i\tau(k)e_{-k}(z)\overline{r_{\sigma_1}(z,k)} - i\tau(k)e_{-k}(z) \quad \text{for } k \in \mathbb{R}^2, \\
\partial_{\bar{k}} r_{\sigma_2}(z,k) &= -i\tau(k)e_{-k}(z)\overline{r_{\sigma_2}(z,k)} - i\tau(k)e_{-k}(z) \quad \text{for } k \in \mathbb{R}^2.
\end{aligned}$$

Under the assumption that σ_j are C^2 functions, we can show that for any fixed $z \in \mathbb{C}$ solutions to this equation are unique. We need the following Liouville type result, which differs from Lemma 5.27 (the case $\nu = 0$) by involving a coefficient α that is not compactly supported but that lies in the space

$$L^{2\pm}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2)\,;\, f \in L^{2+\varepsilon}(\mathbb{R}^2) \cap L^{2-\varepsilon}(\mathbb{R}^2) \text{ for some } \varepsilon > 0\}.$$

Proposition 5.48. *Let $\alpha \in L^{2\pm}(\mathbb{R}^2)$. If $g \in W^{1,2}_{loc}(\mathbb{R}^2)$ is a bounded solution of*

$$\bar\partial g = \alpha \bar g \quad \text{in } \mathbb{R}^2,$$

then $g = Ce^{\beta}$ for some constant C and some $\beta \in C_0(\mathbb{R}^2)$. If further $g(z) \to 0$ as $z \to \infty$, then $g = 0$.

Proof. We make the same reduction as in the proof of Lemma 5.27: define

$$\tilde\alpha(z) = \begin{cases} \alpha \frac{\bar g}{g}, & g(z) \neq 0, \\ 0, & g(z) = 0 \end{cases}$$

so $\tilde\alpha \in L^{2\pm}(\mathbb{R}^2)$ and g solves

$$\bar\partial g - \tilde\alpha g = 0 \quad \text{in } \mathbb{R}^2.$$

We wish to find an integrating factor β such that

$$\bar\partial \beta = \tilde\alpha \quad \text{in } \mathbb{R}^2.$$

If $\tilde\alpha$ were in $L^p_{comp}(\mathbb{R}^2)$ for some $p > 2$, we could use Proposition 5.21 and take $\beta = P\tilde\alpha$. However, it is not hard to see that the Cauchy transform P is also well defined on $L^{2\pm}(\mathbb{R}^2)$: decompose

$$\tilde\alpha = \chi\tilde\alpha + (1-\chi)\tilde\alpha$$

where χ is the characteristic function of the unit disc. Then $\chi\tilde\alpha \in L^p_{comp}(\mathbb{R}^2)$ for some $p > 2$, and $\beta_1 = P(\chi\tilde\alpha)$ is in $W^{1,p}(\mathbb{R}^2)$. Further, $(1-\chi)\tilde\alpha \in L^q(\mathbb{R}^2)$ for some $q < 2$, and the function

$$\beta_2 = \frac{1}{\pi z} * (1-\chi)\tilde\alpha$$

is in $C_0(\mathbb{R}^2)$ as the convolution of functions in L^q and $L^{q'}$. Then $\beta = \beta_1 + \beta_2 \in C_0(\mathbb{R}^2)$ satisfies $\bar\partial\beta = \tilde\alpha$ in the weak sense. We have

$$\bar\partial(e^{-\beta} g) = 0$$

in the weak sense. Since $e^{-\beta} g$ is bounded, the Liouville theorem implies that $g = Ce^{\beta}$. Further, if $g \to 0$ as $z \to \infty$, it follows that $g = 0$. $\square$

We also need certain boundedness and decay conditions for r_σ and τ_σ.

Proposition 5.49. *Let $\sigma \in C^2(\mathbb{R}^2)$ with $\sigma = 1$ for $|z| > 1$. Then, for fixed $z \in \mathbb{C}$,*

$$|r_\sigma(z, \cdot\,)| \leq C_z,$$
$$\lim_{k\to\infty} r_\sigma(z,k) = \sigma^{-1/2}(z) - 1.$$

Also, there is $\varepsilon > 0$ such that

$$|\tau_\sigma(k)| \leq C(1+|k|)^{-1-\varepsilon}, \quad k \in \mathbb{R}^2.$$

Given this result, we can complete the proof of uniqueness for C^2 conductivities.

Proof of Theorem 5.46. We have seen that $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ implies $\tau_{\sigma_1} = \tau_{\sigma_2}$. By the discussion in the beginning of this section, and writing $\tau = \tau_{\sigma_1} = \tau_{\sigma_2}$, for any fixed $z \in \mathbb{C}$ the function $r = r_{\sigma_1} - r_{\sigma_2}$ solves the equation

$$\partial_{\bar{k}} r(z,k) = -i\tau(k) e_{-k}(z) \overline{r(z,k)} \quad \text{for } k \in \mathbb{R}^2.$$

By Propositions 5.39 and 5.41, the functions $r(z,\,\cdot\,)$ and $\tau(\,\cdot\,)$ are C^∞. Then by Proposition 5.49 we have $r(z,\,\cdot\,) \in W^{1,2}_{loc}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and $\tau \in L^{2\pm}(\mathbb{R}^2)$. The Liouville theorem, Proposition 5.48, shows that

$$r(z,k) = C(z) e^{\beta(z,k)}$$

where $\beta(z,\,\cdot\,) \in C_0(\mathbb{R}^2)$.

We will evaluate the last condition at $k = 0$. Recall that

$$f_{\pm\mu_j}(z,0) = 1$$

and thus $u_{\sigma_j}(z,0) = 1$ so $r_{\sigma_j}(z,0) = 0$. Thus $C(z) = 0$ for all $z \in \mathbb{C}$. But also

$$0 = \lim_{k\to\infty} C(z) e^{\beta(z,k)} = \lim_{k\to\infty} (r_{\sigma_1}(z,k) - r_{\sigma_2}(z,k)) = \sigma_1^{-1/2}(z) - \sigma_2^{-1/2}(z).$$

This shows that $\sigma_1 = \sigma_2$. □

It remains to show the estimates in Proposition 5.49. This is most conveniently done by reducing the conductivity equation to the Schrödinger equation, as we did in the uniqueness proof for the Calderón problem in three and higher dimensions. Recall that

$$\nabla \cdot \sigma \nabla u = 0 \iff (-\Delta + q_\sigma)\psi = 0$$

where $u = \sigma^{-1/2}\psi$, and the potential q_σ is given by

$$q_\sigma = \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}.$$

Proposition 5.50. *Let $\sigma \in C^2(\mathbb{R}^2)$ with $\sigma = 1$ for $|z| > 1$. For any $k \in \mathbb{C}$, the equation*

$$(-\Delta + q_\sigma)\psi = 0 \quad \textit{in } \mathbb{R}^2$$

has a unique solution $\psi_\sigma(z,k) = e^{ikz}(1 + s_\sigma(z,k))$ with $s_\sigma \in W^{1,2}_{loc}(\mathbb{R}^2)$ and $s_\sigma(z,k) \to 0$ as $z \to \infty$. The solution ψ_σ is related to u_σ by

$$\psi_\sigma = \sigma^{1/2} u_\sigma, \quad s_\sigma = \sigma^{1/2} - 1 + \sigma^{1/2} r_\sigma.$$

Proof. This follows immediately from Proposition 5.39: if u_σ is the solution of $\nabla \cdot (\sigma \nabla u) = 0$ in $\mathbb{R}^2$ with $u_\sigma = e^{ikz}(1 + r_\sigma)$, then $\psi_\sigma = \sigma^{1/2} u_\sigma$ solves $(-\Delta + q_\sigma)\psi_\sigma = 0$ in $\mathbb{R}^2$ and

$$\psi_\sigma = e^{ikz}(\sigma^{1/2} + \sigma^{1/2} r_\sigma) = e^{ikz}(1 + s_\sigma)$$

where s_σ is as above. For uniqueness, if $\psi = e^{ikz}(1+s)$ and $\tilde{\psi} = e^{ikz}(1+\tilde{s})$ are two solutions of $(-\Delta + q_\sigma)v = 0$ with $s, \tilde{s} = o(1)$ as $z \to \infty$, then $u = \sigma^{-1/2}\psi$ and $\tilde{u} = \sigma^{-1/2}\tilde{\psi}$ are two solutions of $\nabla \cdot \sigma \nabla v = 0$ and

$$u = e^{ikz}(1 + (\sigma^{-1/2} - 1 + \sigma^{-1/2} s)), \quad \tilde{u} = e^{ikz}(1 + (\sigma^{-1/2} - 1 + \sigma^{-1/2}\tilde{s})).$$

The uniqueness part of Proposition 5.39 implies that $s = \tilde{s}$. $\square$

We will next show that the correction term $s_\sigma(z,k)$ in the Schrödinger solution goes to zero as $k \to \infty$, when z is kept fixed. To do this, we give a representation of s_σ in terms an inverse of the conjugated Laplacian. Note that

$$e^{-ikz}\Delta(e^{ikz}v) = 4\bar{\partial} e^{-ikz}\partial(e^{ikz}v) = 4\bar{\partial}(\partial + ik)v.$$

Proposition 5.51. *Let $1 < p < 2$. The equation*

$$\bar{\partial} u = f$$

has a unique solution $u \in L^{p^*}(\mathbb{R}^2)$ for any $f \in L^p(\mathbb{R}^2)$, and

$$\|u\|_{L^{p*}} \leq C_p \|f\|_{L^p}.$$

Proposition 5.52. *Let $k \in \mathbb{C}$ and let $1 < p < 2$. The equation*

$$(\partial + ik)u = f$$

has a unique solution $u \in L^{p^*}(\mathbb{R}^2)$ for any $f \in L^p(\mathbb{R}^2)$, and

$$\|u\|_{L^{p*}} \leq C_p \|f\|_{L^p}.$$

Further, if $f \in L^{p^*}$ and $\partial f \in L^p$, the solution is of the form

$$u = \frac{1}{ik}(f - (\partial + ik)^{-1}\partial f).$$

Proposition 5.53. *Let $k \in \mathbb{C} \setminus \{0\}$ and let $1 < p < 2$. For any $f \in L^p(\mathbb{R}^2)$ the equation*

$$e^{-ikz}\Delta(e^{ikz}u) = f$$

has a unique solution $u \in W^{1,p^*}(\mathbb{R}^2)$. Further,

$$\|u\|_{L^{p*}} \leq \frac{C}{|k|}\|f\|_{L^p}, \quad \|u\|_{W^{1,p*}} \leq C\|f\|_{L^p}.$$

Proof. The equation reads

$$4\bar{\partial}(\partial + ik)u = f.$$

Choose

$$v = \bar{\partial}^{-1}(f/4).$$

Then $\|v\|_{L^{p^*}} \leq C\,\|f\|_{L^p}$. Now choose u to solve

$$(\partial + ik)u = v.$$

We have

$$u = \frac{1}{ik}(v - (\partial + ik)^{-1}\partial v) = \frac{1}{4ik}(\bar{\partial}^{-1}f - (\partial + ik)^{-1}Sf).$$

□

Proposition 5.54. *If $2 < q < \infty$, one has $\psi = e^{ikz}(1+s)$ where $s \in W^{1,q}$ and*

$$s = G_k(I - q_\sigma G_k)^{-1}q_\sigma.$$

Further,

$$\|s\|_{L^q} \leq \frac{C}{|k|}, \quad \|s\|_{W^{1,q}} \leq C$$

and for any $\varepsilon > 0$

$$\|s(\,\cdot\,,k)\|_{L^\infty} \leq C(1+|k|)^{-1+\varepsilon}.$$

Proposition 5.55.

$$\partial_{\bar{k}}\psi_\sigma = \frac{1}{4\pi\bar{k}}t(k)\overline{\psi_\sigma}$$

where

$$t(k) = \int_{\mathbb{R}^2} e^{i\overline{kz}}q_\sigma(z)\psi_\sigma(z,k)\,dm(z).$$

Proof.

$$\partial_{\bar{k}}(G_k f) = -\frac{1}{4\pi\bar{k}}\hat{f}(k)e_{-k}.$$

□

$$-i\tau_\sigma(k) = \frac{1}{4\pi\bar{k}}t(k).$$

5.8. Topological methods

Recall the $\partial_{\bar{k}}$ equation for the solutions u_σ,

$$\partial_{\bar{k}} u_\sigma(z,k) = -i\tau_\sigma(k)\overline{u_\sigma(z,k)} \quad \text{in } \mathbb{R}^2.$$

We would like to conclude that solutions to this equation are unique. However, in the case where σ is only bounded and measurable, the solutions u_σ will not have sufficient decay properties as $k \to \infty$ to have uniqueness. The logarithm of u_σ will have some decay properties, and we will eventually use nonlinear $\partial_{\bar{k}}$ equations for the logarithms to obtain a uniqueness statement.

Topological methods will be the main tool for dealing with the nonlinear partial differential equations that arise in this process. The first result is a simple surjectivity statement.

Lemma 5.56. *Let $\lambda \in \mathbb{C} \setminus \{0\}$, and let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function such that*

$$F(z) = \lambda z + z\varepsilon(z) \quad \text{when } |z| > 1, \quad \varepsilon(z) \to 0 \text{ as } z \to \infty.$$

Then F is surjective.

Intuitively, Lemma 5.56 should be true since for large $r > 0$ the curves $t \mapsto F(re^{it})$ look like circles with large radius, and for very small $r > 0$ the same curves are close to the point $F(0)$. By continuously deforming these curves with large r into the curves for small r, one should pass through any given point of $\mathbb{C}$ (the point $F(0)$ is obtained as the limit when $r \to 0$).

The second result is a version of the argument principle for certain solutions of $\bar{\partial}F + \kappa F = 0$ where $\kappa \in L^\infty(\mathbb{R}^2)$. It shows that one has some control of the zeros of F even if there is no decay for κ at infinity.

Lemma 5.57. *Let $\lambda \in \mathbb{C} \setminus \{0\}$, and let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function such that*

$$F(z) = \lambda z + z\varepsilon(z) \quad \text{when } |z| > 1, \quad \varepsilon(z) \to 0 \text{ as } z \to \infty.$$

If additionally $F \in W^{1,p}_{loc}(\mathbb{C})$ for some $p > 2$ and for some $C > 0$

$$\left|\bar{\partial}F\right| \leq C\,|F| \quad \text{almost everywhere in } \mathbb{C},$$

then F has exactly one zero in $\mathbb{C}$.

Let us give some definitions to prepare for the proofs.

Definition 5.58. Let Ω be a connected open subset of $\mathbb{C}$. A *curve* in Ω is a continuous map $\gamma : [0,1] \to \Omega$. The curve is closed if $\gamma(0) = \gamma(1)$, and the image of the curve is

$$\gamma^* := \gamma([0,1]).$$

Definition 5.59. Let Ω be a connected open subset of $\mathbb{C}$. Two closed curves γ_0 and γ_1 in Ω are *homotopic* in Ω if there is a continuous map $H : [0,1] \times [0,1] \to \Omega$ satisfying for all $s, t \in [0,1]$

$$H(0,t) = \gamma_0(t), \quad H(1,t) = \gamma_1(t),$$
$$H(s,0) = H(s,1).$$

Writing $\gamma_s(t) := H(s,t)$, we say that curve γ_0 is continuously deformed into γ_1 through the family $\{\gamma_s\}_{s\in[0,1]}$.

We also need a way of detecting if a given point is "inside" or "outside" a closed curve. This is given by the index of a point.

Definition 5.60. Let γ be a closed C^1 curve in $\mathbb{C}$. The *index* (or *winding number*) of a point $z \in \mathbb{C} \setminus \gamma^*$ relative to the curve γ is defined as

$$\mathrm{Ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{w-z}\, dw.$$

The next set of problems contains some basic facts about the index required for the proof of Lemma 5.56.

Exercise 5.61. If $\gamma(t) = Re^{2\pi i t}$ for $t \in [0,1]$, show that $\mathrm{Ind}_\gamma(z) = 1$ when $|z| < R$ and $\mathrm{Ind}_\gamma(z) = 0$ for $|z| > R$.

Exercise 5.62. Prove that the map $z \mapsto \mathrm{Ind}_\gamma(z)$ assumes only integer values, is constant on the connected components of $\mathbb{C} \setminus \gamma^*$, and is equal to 0 on the unbounded component.

Exercise 5.63. If $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ are closed C^1 curves and if $z \in \mathbb{C}$ is such that

$$|\gamma_0(t) - \gamma_1(t)| < |\gamma_0(t) - z|, \quad t \in [0,1],$$

show that $\mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z)$.

Exercise 5.64. Prove that if Ω is a connected open subset of $\mathbb{C}$ and if γ_0 and γ_1 are closed C^1 curves homotopic to each other in Ω, then

$$\mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z), \quad z \notin \Omega.$$

Exercise 5.65. Show that $\mathrm{Ind}_\gamma(z)$ is well defined for any closed curve γ (which may not be C^1) by

$$\mathrm{Ind}_\gamma(z) := \lim_{j\to\infty} \mathrm{Ind}_{\gamma_j}(z)$$

where γ_j are trigonometric polynomials that approximate γ in the $L^\infty([0,1])$ norm. Show also that the results of Problems 5.62 and 5.64 remain valid for continuous curves.

Proof of Lemma 5.56. Given $w_0 \in \mathbb{C}$, it is required to show that $F(z_0) = w_0$ for some $z_0 \in \mathbb{C}$. Replacing F by $F - w_0$, we may assume that $w_0 = 0$. We argue by contradiction and suppose that $F(z) \neq 0$ for all $z \in \mathbb{C}$. Since F is continuous, the curves

$$\gamma_0(t) = F(\delta e^{it}), \quad \gamma_1(t) = F(Re^{it})$$

are homotopic in $\mathbb{C} \setminus \{0\}$ for any $\delta, R > 0$. It follows that

$$\mathrm{Ind}_{\gamma_0}(0) = \mathrm{Ind}_{\gamma_1}(0).$$

Now, using that $F(0) \neq 0$, we may find some small $\delta > 0$ so that $F(\partial B(0,\delta)) \subset B(F(0), |F(0)|/2)$. Then 0 is in the unbounded component of $\mathbb{C} \setminus \gamma_0^*$, and

$$\mathrm{Ind}_{\gamma_0}(0) = 0.$$

However, using the assumption on F we also see that $F(\partial B(0,R))$ is homotopic to the circle $\partial B(0,R)$ in $\mathbb{C} \setminus \{0\}$ for some $R > 0$ sufficiently large. This implies that

$$\mathrm{Ind}_{\gamma_1}(0) = \mathrm{Ind}_{\partial B(0,R)}(0) = 1.$$

We have reached a contradiction. $\square$

The next problems contain the proof of Lemma 5.57.

Exercise 5.66. Prove a version of the argument principle: if F is a holomorphic function in a ball $B(0,R)$, if $\gamma(t) = re^{it}$ for some $r < R$, and if F has no zeros on γ^*, then the number of zeros N of F in $B(0,r)$ (counted with multiplicities) is equal to

$$N = \frac{1}{2\pi i} \int_\gamma \frac{F'(z)}{F(z)}\,dz = \mathrm{Ind}_{F(\gamma)}(0).$$

Exercise 5.67. Prove another version of the argument principle: if F is a holomorphic function in $\{|z| < R\}$ that is continuous on $\{|z| \leq R\}$, and if the curve $t \mapsto F(Re^{2\pi it})$ is homotopic to $t \mapsto Re^{2\pi it}$ in $\mathbb{C} \setminus \{0\}$, then the function F has exactly one zero in $\{|z| < R\}$.

Exercise 5.68. Prove Lemma 5.57 by filling in the details of the following outline: reduce the inequality $|\bar\partial F| \leq C\,|F|$ to the equation $\bar\partial F + \kappa F = 0$ for some $\kappa \in L^\infty$, let $\beta \in W^{1,p}(\mathbb{R}^2)$ with $p > 2$ satisfy $\bar\partial\beta = \kappa\chi_B$ where χ_B is the characteristic function of a suitable large disc B, and apply the argument principle to the function $e^\beta F$ that is holomorphic in B.

5.9. Uniqueness for bounded measurable conductivities

In this section we consider the case of bounded measurable conductivities, and give the proof of Theorem 5.1 modulo the fundamental subexponential growth estimate for the complex geometrical optics solutions. Assume that $\sigma_1, \sigma_2 \in L^\infty(\mathbb{D})$ are positive functions such that $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$. In Proposition 5.39 we have constructed CGO solutions u_{σ_j} to the corresponding conductivity equations. Since these solutions are uniquely determined by $f_{\pm\mu_j}$ (where μ_j is the complex coefficient related to σ_j), Proposition 5.32 implies that

$$u_{\sigma_1}(z,k) = u_{\sigma_2}(z,k), \quad z \in \mathbb{R}^2 \setminus \mathbb{D}, \quad k \in \mathbb{C}.$$

The next proposition gives a similar result in the interior of $\mathbb{D}$.

Proposition 5.69. *If $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then*

$$u_{\sigma_1}(z,k) = u_{\sigma_2}(z,k), \quad z, k \in \mathbb{C}.$$

The solution of the Calderón problem is an immediate consequence of this result and a basic fact on quasiregular mappings.

Proposition 5.70. *Let $\Omega \subset \mathbb{R}^2$ be a connected open set. If $f \in W^{1,2}_{loc}(\mathbb{R}^2)$ satisfies*

$$\bar\partial f = \mu \overline{\partial f} \quad \text{in } \Omega$$

where $\mu \in L^\infty(\Omega)$ and $\|\mu\|_{L^\infty(\Omega)} < 1$, then $\partial f \neq 0$ almost everywhere in Ω unless f is constant.

Proof. See Astala-Iwaniec-Martin. □

In the case where $\mu \in C^1(\overline{\Omega})$ is real valued (corresponding to a C^1 conductivity), we can prove a slightly weaker result that would still be sufficient for completing the uniqueness proof.

Proposition 5.71. *Let $\Omega \subset \mathbb{R}^2$ be a connected open set. If $f \in W^{1,2}_{loc}(\mathbb{R}^2)$ satisfies*

$$\bar\partial f = \mu \overline{\partial f} \quad \text{in } \Omega$$

where $\mu \in C^1(\overline{\Omega})$ is real valued and $\|\mu\|_{L^\infty(\Omega)} < 1$, then $\partial f \neq 0$ in a dense subset of Ω unless f is constant.

Proof. Let $S = \{x \in \Omega\,;\, \partial f(x) = 0\}$. If $\Omega \setminus S$ is not dense, there is a point $x_0 \in \Omega$ and a ball B centered at x_0 such that $\partial f|_B = 0$. Then also $\bar\partial f|_B = 0$, so the Jacobian matrix of f vanishes a.e. on B. Then f is a constant map on B, and also $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are constant on B. But u and v satisfy the conductivity equations

$$\nabla \cdot \sigma \nabla u = 0, \quad \nabla \cdot \sigma^{-1} \nabla v = 0$$

where

$$\sigma = \frac{1-\mu}{1+\mu}.$$

By unique continuation, it follows that u and v must be constant in Ω, and consequently f is a constant map in Ω. $\square$

Proof of Theorem 5.1. Since $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ implies $\Lambda_{1/\sigma_1} = \Lambda_{1/\sigma_2}$, Proposition 5.69 is valid also with σ_j replaced by $1/\sigma_j$. The pair $(f_{\mu_j}, f_{-\mu_j})$ is uniquely determined by the pair $(u_{\sigma_j}, u_{1/\sigma_j})$, so it follows that

$$f_{\mu_1}(z,k) = f_{\mu_2}(z,k), \quad z, k \in \mathbb{C}.$$

The Beltrami equations $\bar\partial f_{\mu_j} = \mu_j \overline{\partial f_{\mu_j}}$ imply that

$$\mu_1 = \frac{\bar\partial f_{\mu_1}}{\overline{\partial f_{\mu_1}}} = \frac{\bar\partial f_{\mu_2}}{\overline{\partial f_{\mu_2}}} = \mu_2$$

at all those points of $\mathbb{D}$ where ∂f_{μ_1} and ∂f_{μ_2} are nonzero. By Proposition 5.71 this is true almost everywhere. It follows that $\mu_1 = \mu_2$ almost everywhere in $\mathbb{D}$, and since

$$\sigma_1 = \frac{1-\mu_1}{1+\mu_1}, \quad \sigma_2 = \frac{1-\mu_2}{1+\mu_2}$$

we see that also $\sigma_1 = \sigma_2$ almost everywhere in $\mathbb{D}$. $\square$

We now focus on proving Proposition 5.69. Due to Proposition 5.39, the solutions u_{σ_j} satisfy the $\partial_{\bar k}$ equations

$$\partial_{\bar k} u_{\sigma_j}(z, \cdot\,) = -i\tau_{\sigma_j}(\,\cdot\,)\overline{u_{\sigma_j}(z, \cdot\,)}.$$

Also, from the fact that $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ and from Proposition 5.41 we know that

$$\tau_{\sigma_1}(k) = \tau_{\sigma_2}(k), \quad k \in \mathbb{C}.$$

Let $\tau(k) := \tau_{\sigma_1}(k) = \tau_{\sigma_2}(k)$. It follows that for fixed $z \in \mathbb{C}$, both $u_{\sigma_1}(z, \cdot\,)$ and $u_{\sigma_2}(z, \cdot\,)$ are solutions of the equation

$$\partial_{\bar k} u(k) = -i\tau(k)\overline{u(k)}.$$

If both the coefficient τ and the solution u (or the function $e^{-ikz}u$) would have suitable decay properties as $k \to \infty$, solutions to this equation would be unique and one would obtain that $u_{\sigma_1} = u_{\sigma_2}$ as required. (Lemma 5.27 is one example of such a uniqueness result.)

It turns out that when the conductivities are only bounded and measurable, one cannot expect any decay with respect to k in the solutions u_σ. However, one can prove that at least $e^{-ikz}u_\sigma(z,k)$ does not grow exponentially in k. The proof is deferred to the next section.

Proposition 5.72. *We have*

$$u_\sigma(z,k) = e^{\delta_\sigma(z,k)}$$

where, for fixed $k \in \mathbb{C}$, one has $\delta_\sigma(\,\cdot\,,k) \in W^{1,p}_{loc}(\mathbb{R}^2)$ for some $p > 2$ and

$$\delta_\sigma(z,k) = ikz + v_k(z), \quad v_k(z) \to 0 \text{ as } z \to \infty.$$

Also, for fixed $z \in \mathbb{C}$, the function $\delta_\sigma(z,\,\cdot\,)$ is C^∞ and

$$\delta_\sigma(z,k) = ikz + k\varepsilon_z(k), \quad \varepsilon_z(k) \to 0 \text{ as } k \to \infty.$$

Not discouraged by the lack of decay with respect to k in the solution u_σ, we will try to exploit the decay in its logarithm δ_σ. The $\partial_{\bar{k}}$ equations (5.9) for u_{σ_j} imply the following nonlinear equations for δ_{σ_j}:

$$\begin{aligned}\partial_{\bar{k}}\delta_{\sigma_j}(z,k) &= -i\tau(k)e^{\overline{\delta_{\sigma_j}(z,k)} - \delta_{\sigma_j}(z,k)} \\ &= -i\tau(k)e^{-2i\,\mathrm{Im}(\delta_{\sigma_j}(z,k))}.\end{aligned}$$

This shows that we have two families of solutions $\{\delta_{\sigma_j}(z,\,\cdot\,)\}_{z\in\mathbb{C}}$ to the same nonlinear $\partial_{\bar{k}}$ equation. Moreover, the two families have the same asymptotics as $k \to \infty$ by Proposition 5.72. This turns out to be sufficient for uniqueness.

Proof of Proposition 5.69. To have $u_{\sigma_1}(z,k) = u_{\sigma_2}(z,k)$ for all $z, k \in \mathbb{C}$, it is enough to show that

$$\delta_{\sigma_1}(z,k) = \delta_{\sigma_2}(z,k), \quad z,k \in \mathbb{C}.$$

We generalize the setup a little bit, and consider the function

$$F_k(z,w) := \delta_{\sigma_1}(z,k) - \delta_{\sigma_2}(w,k), \quad z,w,k \in \mathbb{C}.$$

It is enough to prove that F_k vanishes on the diagonal for any k.

If $k = 0$, we know by Proposition 5.25 that $f_{\pm\mu}(z,0) = 1$. This implies that $u_\sigma(z,0) = 1$, so the logarithm satisfies $\delta_\sigma(z,0) = 0$. This proves that

$$F_0(z,z) = 0, \quad z \in \mathbb{C}.$$

Let now $k_0 \neq 0$, and suppose that $z_0 \in \mathbb{C}$ is fixed. We do the proof that $F_{k_0}(z_0,z_0) = 0$ in two steps. The first step is to show that

$$F_{k_0}(z_0,w_0) = 0 \quad \text{for some } w_0 \in \mathbb{C}.$$

To see this, it is enough to observe that by Proposition 5.72 and Lemma 5.56 the map $w \mapsto \delta_{\sigma_2}(w,k_0)$ is a surjective map from $\mathbb{C}$ to $\mathbb{C}$. This shows that there exists some $w_0 \in \mathbb{C}$ such that $\delta_{\sigma_2}(w_0,k_0) = \delta_{\sigma_1}(z_0,k_0)$, which gives (5.9).

The second step is to show that for any $z, w \in \mathbb{C}$, one has

$$F_{k_0}(z,w) = 0 \implies z = w.$$

Assume that $F_{k_0}(z,w) = 0$, and note that the map $k \mapsto F_k(z,w)$ satisfies the $\partial_{\bar{k}}$ equation

$$\partial_{\bar{k}} F_k(z,w) = -i\tau(k)(e^{-2i\,\mathrm{Im}(\delta_{\sigma_1}(z,k))} - e^{-2i\,\mathrm{Im}(\delta_{\sigma_2}(w,k))}).$$

We now use the bound $|\tau(k)| \leq 1$ from Proposition 5.41 and the elementary fact that

$$\left|e^{is} - e^{it}\right| \leq |s-t|, \quad s,t \in \mathbb{R}.$$

Taking absolute values in the previous equation for $F_k(z,w)$ yields

$$\begin{aligned} |\partial_{\bar{k}} F_k(z,w)| &\leq 2\,|\mathrm{Im}(\delta_{\sigma_1}(z,k)) - \mathrm{Im}(\delta_{\sigma_2}(w,k))| \\ &\leq 2\,|F_k(z,w)|\,. \end{aligned}$$

We also know from Proposition 5.72 that

$$F_k(z,w) = i(z-w)k + k\tilde{\varepsilon}(k), \quad \tilde{\varepsilon}(k) \to 0 \text{ as } k \to \infty.$$

If $z \neq w$, it follows from Lemma 5.57 that the map $k \mapsto F_k(z,w)$ has only one zero. However, we always have $F_0(z,w) = 0$, which contradicts the assumption that $F_{k_0}(z,w) = 0$ where $k_0 \neq 0$. It must follow that $z = w$, finishing the proof of (5.9).

Finally, combining (5.9) and (5.9) yields $F_{k_0}(z_0, z_0) = 0$. Since $k_0 \neq 0$ and $z_0 \in \mathbb{C}$ were arbitrary, this concludes the proof that F_k vanishes on the diagonal. □

5.10. Subexponential growth

To complete the uniqueness result for bounded measurable conductivities, it remains to prove Proposition 5.72 from the previous section concerning subexponential growth of $e^{-ikz}u_\sigma$ with respect to k. To prove this, it will be useful to go back to the Beltrami equation and the solutions f_μ. This makes it possible to make efficient use of the theory of quasiconformal mappings. In the following we will use some facts from this theory without proof, but we will give a precise reference each time we do so.

In order to pass from growth properties of solutions of Beltrami equations back to the conductivity equation, it is useful to generalize the setup slightly. If $\lambda \in \partial\mathbb{D}$, we denote by $f_{\lambda\mu}$ the solution of the Beltrami equation

$$\bar{\partial} f_{\lambda\mu} = \lambda\mu\overline{\partial f_\mu} \quad \text{in } \mathbb{R}^2$$

satisfying $f_{\lambda\mu}(\,\cdot\,,k) \in W^{1,2}_{loc}(\mathbb{R}^2)$ and

$$\begin{aligned} f_{\lambda\mu}(z,k) &= e^{ikz}(1 + \eta_{\lambda\mu}(z,k)), \\ \eta_{\lambda\mu}(z,k) &= O(1/z) \text{ as } z \to \infty. \end{aligned}$$

Here $\mu = (1-\sigma)/(1+\sigma)$ as before. Precisely the same proof as in Proposition 5.25 shows that such solutions exist, are unique, and satisfy $\eta_{\lambda\mu}(\,\cdot\,,k) \in W^{1,p}(\mathbb{R}^2)$ for some $p > 2$. The following result states that the functions $e^{-ikz}f_{\lambda\mu}(z,k)$ have subexponential growth with respect to k.

Proposition 5.73. *For fixed $z \in \mathbb{C}$, we have*

$$f_{\lambda\mu}(z,k) = e^{ik\phi_{\lambda\mu}(z,k)}$$

where $\phi_{\lambda\mu}$ is C^∞ in k and, for any $k \in \mathbb{C}$,

$$\phi_{\lambda\mu}(z,k) = z + v_1(z,k), \quad v_1(\,\cdot\,,k) \in W^{1,p}(\mathbb{R}^2)$$

for some $p > 2$. For any $z \in \mathbb{C}$ the function $\phi_{\lambda\mu}(z,\,\cdot\,)$ satisfies

$$\phi_{\lambda\mu}(z,k) = z + \varepsilon_z(k), \quad \varepsilon_z(k) \to 0 \text{ as } k \to \infty.$$

After some work, Proposition 5.72 will now follow.

Proof of Proposition 5.72. We need to show that

$$u_\sigma = e^{ikz+v(z,k)}$$

where

$$v(z,k) \to 0 \text{ as } z \to \infty, \text{ when } k \text{ isfixed},$$

and

$$v(z,k) = |k|\, o(1) \text{ as } k \to \infty, \text{ when } z \text{ isfixed}.$$

To warm up, let us show that u_σ never vanishes. We argue by contradiction and assume that $u_\sigma(z_0,k_0) = 0$ for some $z_0, k_0 \in \mathbb{C}$. Since

$$u_\sigma(z_0,k_0) = \mathrm{Re}(f_\mu(z_0,k_0)) + i\,\mathrm{Im}(f_{-\mu}(z_0,k_0))$$

it follows that $f_\mu(z_0,k_0) = it$ and $f_{-\mu}(z_0,k_0) = s$ for some $t, s \in \mathbb{R}$. But then

$$1 = \left|\frac{it-s}{it+s}\right| = \left|\frac{f_\mu(z_0,k_0) - f_{-\mu}(z_0,k_0)}{f_\mu(z_0,k_0) + f_{-\mu}(z_0,k_0)}\right|.$$

The last quantity is < 1 by (5.45). This is not possible, so u_σ must be nonvanishing.

More generally, we show that in fact u_σ has a well-defined logarithm. Write

$$\begin{aligned} u_\sigma &= \frac{1}{2}(f_\mu + \overline{f_\mu} + f_{-\mu} - \overline{f_{-\mu}}) \\ &= \frac{1}{2}(f_\mu + f_{-\mu}) \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}} \right) \\ &= f_\mu \frac{f_\mu + f_{-\mu}}{2 f_\mu} \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}} \right) \\ &= f_\mu \left(1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} \right)^{-1} \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}} \right). \end{aligned}$$

If $\mathbb{D}(z, r)$ is the open disc centered at z with radius r, then by (5.45)

$$1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} \in \mathbb{D}(1,1), \quad 1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}} \in \mathbb{D}(1,1).$$

Consider the principal branch of the complex logarithm,

$$\mathrm{Log} : \mathbb{C} \setminus (-\infty, 0] \to \{z \in \mathbb{C}\,;\, |\mathrm{Im}(z)| < \pi\}.$$

Since $z/w \in \mathbb{C} \setminus (-\infty, 0]$ whenever $\mathrm{Re}(z), \mathrm{Re}(w) > 0$, we may define

$$g := \mathrm{Log} \left[\left(1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} \right)^{-1} \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}} \right) \right].$$

Proposition 5.73 shows that $f_\mu(z, k) = e^{ikz + \beta(z,k)}$ where $\beta(\,\cdot\,, k) \in W^{1,p}(\mathbb{R}^2)$ for some $p > 2$, when $k \in \mathbb{C}$ is fixed. We choose

$$\delta_\sigma(z, k) := ikz + \beta(z, k) + g(z, k).$$

It follows that $u_\sigma = e^{\delta_\sigma}$. It is also clear that δ_σ is C^∞ in k, since this is true for β and g by Propositions 5.73 and 5.33.

Let now $k \in \mathbb{C}$ be fixed, and consider the asymptotics as $z \to \infty$. Define

$$v(z, k) := \beta(z, k) + g(z, k).$$

Here $\beta(\,\cdot\,, k) \in W^{1,p}(\mathbb{R}^2)$, and $g(\,\cdot\,, k)$ is in $W^{1,p}_{loc}(\mathbb{R}^2)$ since it is the logarithm of a $W^{1,p}_{loc}(\mathbb{R}^2)$ function. By (5.45) we have

$$g(z, k) = o(1) \quad \text{as } z \to \infty.$$

This shows that $v(\,\cdot\,, k) \in W^{1,p}_{loc}(\mathbb{R}^2)$ with $v(z, k) = o(1)$ as $z \to \infty$.

Finally, let $z \in \mathbb{C}$ be fixed and consider the asymptotics as $k \to \infty$. We have

$$u_\sigma(z, k) = e^{ikz + v(z,k)}$$

and we need to show that

$$v(z, k) = |k|\, o_z(1) \quad \text{as } k \to \infty.$$

Here we write $o_z(1)$ for any quantity converging to 0 as $k \to \infty$.

Using that $v(z,k) = \beta(z,k) + g(z,k)$, we have

$$|\mathrm{Im}(v)| \leq |\beta| + |g| \leq |k|\, o_z(1) + \pi = |k|\, o_z(1).$$

To bound $\mathrm{Re}(v)$, we note that

$$u_\sigma = \frac{1}{2}(f_\mu + \overline{f_\mu} + f_{-\mu} - \overline{f_{-\mu}})$$

which implies

$$e^v = e^{-ikz}u_\sigma = \frac{1}{2}(M_\mu + e_{-k}\overline{M_\mu} + M_{-\mu} - e_{-k}\overline{M_{-\mu}})$$

and consequently, by the triangle inequality,

$$e^{\mathrm{Re}(v)} = |e^v| \leq |M_\mu| + |M_{-\mu}| \leq e^{|k| o_z(1)}.$$

This shows that

$$\mathrm{Re}(v) \leq |k|\, o_z(1).$$

It remains to prove that

$$\mathrm{Re}(v) \geq -\,|k|\, o_z(1),$$

or equivalently,

$$\left| e^{-ikz}u_\sigma \right| \geq e^{-|k| o_z(1)}.$$

This is the only point where we need to use the more general solutions $f_{\lambda\mu}$ with λ not equal to 1. We write

$$u_\sigma = \frac{f_\mu + f_{-\mu}}{2}\left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}}\right) = \frac{f_\mu + f_{-\mu}}{2}\overline{\left[1 + e^{it}\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right]}$$

where $t \in \mathbb{R}$ is chosen so that

$$e^{-it} = \frac{\overline{f_\mu} + \overline{f_{-\mu}}}{f_\mu + f_{-\mu}}.$$

Therefore

$$\left| e^{-ikz}u_\sigma \right| = \left| \frac{M_\mu + M_{-\mu}}{2} \right| \left| 1 + e^{it}\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} \right|.$$

Here

$$\left| \frac{M_\mu + M_{-\mu}}{2} \right| = \left| \frac{1 + M_{-\mu}/M_\mu}{2M_\mu} \right| \geq e^{-|k| o_z(1)}$$

since $\mathrm{Re}(M_{-\mu}/M_\mu) > 0$ by Lemma 5.45 and by Proposition 5.73. The result will follow if we can prove that

$$\inf_{t\in\mathbb{R}} \left| 1 + e^{it}\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} \right| \geq e^{-|k| o_z(1)}.$$

To see the last inequality, note that

$$1 + e^{it}\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} = \frac{2\tilde{f}_t}{f_\mu + f_{-\mu}}$$

where the function

$$\tilde{f}_t := \frac{1}{2}\left[(1+e^{it})f_\mu + (1-e^{it})f_{-\mu}\right]$$

satisfies, by a direct computation, the equation

$$\bar{\partial}\tilde{f}_t = e^{it}\mu\overline{\partial\tilde{f}_t}.$$

The properties of $f_{\pm\mu}$ imply that $\tilde{f}_t$ is a $W^{1,2}_{loc}(\mathbb{R}^2)$ function satisfying $\tilde{f}_t = e^{ikz}(1+O(1/z))$ as $z \to \infty$. The uniqueness of complex geometrical optics solutions, as discussed in the beginning of this section, shows that $\tilde{f}_t = f_{\lambda\mu}$ for $\lambda = e^{it}$. Consequently, by Proposition 5.73,

$$\left|1 + e^{it}\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right| = \left|\frac{2M_{\lambda\mu}}{M_\mu + M_{-\mu}}\right| \geq e^{-|k|o_z(1)}.$$

This concludes the proof. □

Chapter 6

Partial Data

In §**??**, we showed in dimensions $n \geq 3$ that if the boundary measurements for two C^2 conductivities coincide on the whole boundary, then the conductivities are equal. Here we consider the case where measurements are made only on part of the boundary.

The first result that we will prove is due to Isakov. It states that if one knows the Dirichlet-to-Neumann map on a open subset $\Gamma \subset \partial\Omega$ for any Dirichlet data supported in the same set, and if the inaccessible part $\Gamma_0 = \partial\Omega \setminus \Gamma$ is part of a hyperplane, then this data determines the conductivity.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. Assume that $\Omega \subset \{x_n > 0\}$, let $\Gamma_0 = \partial\Omega \cap \{x_n = 0\}$, and let $\Gamma = \partial\Omega \setminus \Gamma_0$. If*

$$\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma \quad \textit{for all } f \in H^{1/2}(\partial\Omega) \textit{ with } \operatorname{supp}(f) \subset \Gamma,$$

then $\gamma_1 = \gamma_2$ in Ω.

The result follows quite easily from the full data results in **??** by using a reflection argument. We will also show a similar theorem where the inaccessible set Γ_0 is part of a sphere.

For more general domains, the first partial result was proved by Bukhgeim and Uhlmann. It involves a unit vector α in $\mathbb{R}^n$ and the subset of the boundary

$$\partial\Omega_{-,\varepsilon} = \left\{ \, x \in \partial\Omega \;\middle|\; \alpha \cdot \nu(x) < \varepsilon \, \right\}$$

The theorem is as follows.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. If $\alpha \in \mathbb{R}^n$*

is a unit vector, if $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$*, and if for some* $\varepsilon > 0$ *one has*

$$\Lambda_{\gamma_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{\gamma_2} f|_{\partial\Omega_{-,\varepsilon}} \quad \textit{for all } f \in H^{1/2}(\partial\Omega),$$

then $\gamma_1 = \gamma_2$ *in* Ω*.*

The proof is based on complex geometrical optics solutions, but requires new elements since we need some control of the solutions on parts of the boundary. The main tool is a weighted norm estimate known as a *Carleman estimate*. This estimate also gives rise to a new construction of complex geometrical optics solutions, which does not involve Fourier analysis.

6.1. Reflection approach

As before, we will obtain the partial data result for the conductivity equation by proving a uniqueness result for the Schrödinger equation.

Theorem 6.3 (Partial data for Schrödinger)**.** *Let* $\Omega \subset \{x_n > 0\}$ *be a bounded open set with smooth boundary, let* $\Gamma_0 = \partial\Omega \cap \{x_n = 0\}$*, and let* $\Gamma = \partial\Omega \setminus \Gamma_0$*. Let* $q_1, q_2 \in L^\infty(\Omega)$ *be such that* 0 *is not a Dirichlet eigenvalue for* $-\Delta + q_j$ *in* Ω*. If one has*

$$\Lambda_{q_1} f|_\Gamma = \Lambda_{q_2} f|_\Gamma \quad \textit{for all } f \in H^{1/2}(\partial\Omega) \textit{ with } \operatorname{supp}(f) \subset \Gamma,$$

then $q_1 = q_2$ *in* Ω*.*

Theorem 6.1 is an immediate consequence of Theorem 6.3 and the unique continuation result proved later in Theorem 6.19.

Proof of Theorem 6.1. Define $q_j = \Delta(\gamma_j^{1/2})/\gamma_j^{1/2}$. By Theorem 2.74 the Dirichlet problem for $-\Delta + q_j$ in Ω is well-posed, and the Dirichlet-to-Neumann maps are related by

$$\Lambda_{q_j} f = \gamma_j^{-1/2} \Lambda_{\gamma_j}(\gamma_j^{-1/2} f) + \frac{1}{2}\gamma_j^{-1}(\partial_\nu \gamma_j) f|_{\partial\Omega}.$$

Recall from the boundary determination results, Theorems 3.3 and 3.17, that the knowledge of Λ_γ on Γ for any f supported in Γ determines $\gamma|_\Gamma$ and $\partial_\nu \gamma|_\Gamma$ uniquely. The assumption

$$\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma \quad \text{for all } f \in H^{1/2}(\partial\Omega) \text{ with } \operatorname{supp}(f) \subset \Gamma,$$

therefore implies that

$$\gamma_1|_\Gamma = \gamma_2|_\Gamma, \quad \partial_\nu \gamma_1|_\Gamma = \partial_\nu \gamma_2|_\Gamma.$$

The expression for Λ_{q_j} above shows that

$$\Lambda_{q_1} f|_\Gamma = \Lambda_{q_2} f|_\Gamma \quad \text{for all } f \in H^{1/2}(\partial\Omega) \text{ with } \operatorname{supp}(f) \subset \Gamma.$$

By Theorem 6.3, we have $q_1 = q_2$ in Ω. Define $q = q_1 = q_2$. Since $q_j = \Delta\gamma_j^{1/2}/\gamma_j^{1/2}$, we have

$$(-\Delta + q)\gamma_j^{1/2} = 0 \qquad \text{in } \Omega.$$

We also have the boundary conditions

$$\gamma_1^{1/2}|_\Gamma = \gamma_2^{1/2}|_\Gamma, \quad \partial_\nu\gamma_1^{1/2}|_\Gamma = \partial_\nu\gamma_2^{1/2}|_\Gamma.$$

By Theorem 6.19, any two solutions of $(-\Delta + q)u = 0$ having the same Cauchy data on an open subset of the boundary must be equal in Ω. This proves that $\gamma_1 = \gamma_2$. $\square$

We move to the proof of Theorem 6.3. In the present setting where the inaccessible part of the boundary is part of the hyperplane $\{x_n = 0\}$, it is natural to use the reflection which takes a complex vector $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ to the vector

$$z^* = (z_1, \ldots, z_{n-1}, -z_n).$$

Proof of Theorem 6.3. Recall from Theorem 2.72 the integral identity

$$\langle(\Lambda_{q_1} - \Lambda_{q_2})(u_1|_{\partial\Omega}), u_2|_{\partial\Omega}\rangle_{\partial\Omega} = \int_\Omega (q_1 - q_2)u_1u_2\,dx$$

valid for any $u_j \in H^1(\Omega)$ with $(-\Delta + q_j)u_j = 0$ in Ω. Assume that $u_1|_{\partial\Omega}$ is supported in Γ. In this case we know that

$$\Lambda_{q_1}(u_1|_{\partial\Omega})|_\Gamma = \Lambda_{q_2}(u_1|_{\partial\Omega})|_\Gamma,$$

and consequently $(\Lambda_{q_1} - \Lambda_{q_2})(u_1|_{\partial\Omega})$ vanishes on Γ. If also $u_2|_{\partial\Omega}$ is supported in Γ, the whole boundary integral is zero. Therefore, our partial data assumption implies that

$$\int_\Omega (q_1 - q_2)u_1u_2\,dx = 0 \tag{6.1}$$

for any $u_j \in H^1(\Omega)$ with $(-\Delta + q_j)u_j = 0$ in Ω and $\operatorname{supp}(u_j|_{\partial\Omega}) \subset \Gamma$.

We now use the reflection idea and consider the open set

$$U = \Omega \cup \operatorname{int}(\Gamma_0) \cup \Omega^*$$

where the interior of Γ_0 is relative to $\{x_n = 0\}$, and

$$\Omega^* = \{x^*\,;\, x \in \Omega\}.$$

We also define the even extensions of the potentials,

$$q_j^e(x) = \begin{cases} q_j(x) & \text{if } x \in \Omega, \\ q_j(x^*) & \text{if } x \in \Omega^*. \end{cases}$$

These are L^∞ functions in U.

If $\zeta_1, \zeta_2 \in \mathbb{C}^n$ satisfy $\zeta_j \cdot \zeta_j = 0$ with $|\zeta_j|$ large enough, Theorem 4.3 shows that there exist $w_j \in H^1(U)$ satisfying

$$(-\Delta + q_j)w_j = 0 \quad \text{in } U,$$

having the form

$$w_j = e^{\zeta_j \cdot x}(1 + \psi_j)$$

where

$$\|\psi_j\|_{L^2(U)} \leq \frac{C}{|\zeta_j|}.$$

In fact, after extending q_j^e by zero to $\mathbb{R}^n \setminus U$, we obtain solutions of this type in $\mathbb{R}^n$ and the functions w_j are obtained just by taking the restrictions to U.

We now define functions $u_j(x) = w_j(x) - w_j^*(x)$ in the original domain Ω, where we write

$$f^*(x) = f(x^*).$$

An easy computation shows that u_j satisfies $(-\Delta + q_j)u_j = 0$ in Ω and $u_j \in H^1(\Omega)$. Since the set $\{x_n = 0\}$ is invariant under reflection, we have $u_j|_{\Gamma_0} = 0$. Thus (6.1) is valid for these choices of u_j.

We wish to compute the product $u_1 u_2$. Since

$$u_j = e^{\zeta_j \cdot x}(1 + \psi_j) - e^{\zeta_j \cdot x^*}(1 + \psi_j^*),$$

we have

$$\begin{aligned} u_1 u_2 = {} & e^{(\zeta_1 + \zeta_2)\cdot x}(1 + \psi_1)(1 + \psi_2) - e^{(\zeta_1^* + \zeta_2)\cdot x}(1 + \psi_1^*)(1 + \psi_2) \\ & - e^{(\zeta_1 + \zeta_2^*)\cdot x}(1 + \psi_1)(1 + \psi_2^*) + e^{(\zeta_1 + \zeta_2)\cdot x^*}(1 + \psi_1^*)(1 + \psi_2^*). \end{aligned}$$

We would like to arrange $u_1 u_2$ to look like $e^{ix\cdot\xi}$ for given $\xi \in \mathbb{R}^n$ when $|\zeta_j|$ are large. In particular, we do not want $u_1 u_2$ to grow exponentially with respect to $|\zeta_j|$. It will be useful to choose ζ_j so that

$$\zeta_j \cdot \zeta_j = 0, \quad \zeta_1 + \zeta_2 = i\xi, \quad \mathrm{Re}(\zeta_1^* + \zeta_2) = \mathrm{Re}(\zeta_1 + \zeta_2^*) = 0.$$

Write $\xi = (\xi', \xi_n)$ where $\xi' = (\xi_1, \ldots, \xi_{n-1})$. We assume for the moment that $|\xi'| > 0$. It will be convenient to use the reflection invariant unit vector

$$v_1 = \frac{1}{|\xi'|}(\xi', 0).$$

Let $v_3 = e_n$ be the nth coordinate vector, and choose some unit vector $v_2 \in \mathbb{R}^n$ such that

$$v_2 \cdot v_1 = v_2 \cdot v_3 = 0.$$

Then $\{v_1, v_2, v_3\}$ are orthogonal unit vectors (here we used that $n \geq 3$). If $\tau > 0$ is large, define the complex vectors

$$\zeta_1 = |\xi| \sqrt{\tau^2 + \frac{1}{4}} v_2 + i \left[\left(\frac{1}{2} |\xi'| + \tau \xi_n \right) v_1 + \left(\frac{1}{2} \xi_n - \tau |\xi'| \right) v_3 \right],$$
$$\zeta_2 = -|\xi| \sqrt{\tau^2 + \frac{1}{4}} v_2 + i \left[\left(\frac{1}{2} |\xi'| - \tau \xi_n \right) v_1 + \left(\frac{1}{2} \xi_n + \tau |\xi'| \right) v_3 \right].$$

It is easy to check that $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$ and $|\zeta_1| = |\zeta_2| = \sqrt{2} |\xi| \sqrt{\tau^2 + 1/4}$. We also have

$$\zeta_1 + \zeta_2 = i(|\xi'| v_1 + \xi_n v_3) = i\xi$$

and, since v_1 and v_2 are reflection invariant,

$$\zeta_1^* + \zeta_2 = i(|\xi'| v_1 + 2\tau |\xi'| v_3),$$
$$\zeta_1 + \zeta_2^* = i(|\xi'| v_1 - 2\tau |\xi'| v_3).$$

Thus ζ_1 and ζ_2 satisfy all the properties mentioned above.

Keeping ξ fixed and using that $\|\psi_j\|_{L^2(\Omega)} \leq C/\tau$ and $\left\|\psi_j^*\right\|_{L^2(\Omega)} \leq C/\tau$, all terms involving the remainder terms ψ_j will be small as $\tau \to \infty$. By (6.1) we have

$$0 = \int_\Omega (q_1 - q_2) u_1 u_2 \, dx = \int_\Omega (q_1 - q_2)(e^{ix\cdot\xi} - e^{ix\cdot\eta_1(\tau)} - e^{ix\cdot\eta_2(\tau)} + e^{ix^*\cdot\xi}) \, dx + o(1)$$

as $\tau \to \infty$, where $\eta_1(\tau) = |\xi'| v_1 + 2\tau |\xi'| v_3$ and $\eta_2(\tau) = |\xi'| v_1 - 2\tau |\xi'| v_3$. Defining $q(x) = q_1(x) - q_2(x)$ when $x \in \Omega$ and $q(x) = 0$ otherwise, we have

$$\int_\Omega q e^{ix\cdot\eta_j(\tau)} \, dx = \hat{q}(-\eta_j(\tau)).$$

Now $q \in L^1(\mathbb{R}^n)$, so $\hat{q}(\eta) \to 0$ as $|\eta| \to \infty$ by the Riemann-Lebesgue lemma (Theorem ???). Since $|\eta_j(\tau)| \to \infty$ as $\tau \to \infty$, we have

$$\lim_{\tau\to\infty} \int_\Omega q e^{ix\cdot\eta_j(\tau)} \, dx = 0.$$

Therefore

$$\int_\Omega (q_1 - q_2)(e^{ix\cdot\xi} + e^{ix^*\cdot\xi}) \, dx = \lim_{\tau\to\infty} \int_\Omega (q_1 - q_2)(e^{ix\cdot\xi} - e^{ix\cdot\eta_1(\tau)} - e^{ix\cdot\eta_2(\tau)} + e^{ix^*\cdot\xi}) \, dx = 0.$$

In the integral involving $e^{ix^*\cdot\xi}$, we can make the change of variables $x \to x^*$ to obtain the following statement for even extensions of q_j in the double domain U:

$$\int_U (q_1^e - q_2^e) e^{ix\cdot\xi} \, dx = 0.$$

Thus $(q_1^e - q_2^e)\hat{\ }(-\xi) = 0$. This is true for any fixed $\xi \in \mathbb{R}^n$ with $|\xi'| > 0$. But since $q_1^e - q_2^e$ is compactly supported in $\mathbb{R}^n$, its Fourier transform is continuous, and we have $(q_1^e - q_2^e)\hat{}(-\xi) = 0$ for any $\xi \in \mathbb{R}^n$. This shows that $q_1 = q_2$. $\square$

Note that in the previous proof, one of the crucial points was the fact that a solution of the Schrödinger equation in the double domain $\Omega \cup \mathrm{int}(\Gamma_0) \cup \Omega^*$ can be reflected from Ω^* to produce a solution of the Schrödinger equation back in Ω. That is, the Schrödinger operator $-\Delta + q$ should be preserved under reflection. This is quite clear for the reflection $(x', x_n) \mapsto (x', -x_n)$, but one can ask if there are other reflection operators that have this property.

This question can be answered by using some differential (or Riemannian) geometry facts as in Chapter ???. However, to keep things simple, we will describe the argument in a self-contained way. Let $F : U \to V$ be a C^∞ bijective map between two open subsets of $\mathbb{R}^n$ (the map F is our reflection operator). We denote the Euclidean Laplacian in V by

$$\Delta_e = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

Here e is the Euclidean metric, corresponding to the identity matrix $e(x) = I = [\delta_{j,k}]_{j,k=1}^n$. We want to compute how Δ_e transforms under the reflection F. To do this, we will "pull back" quantities on V by the map F into quantities on U.

First, define the pullback of a function $v \in C^\infty(V)$ as the function

$$F^* v(x) = (v \circ F)(x), \quad x \in U.$$

Next, define the pullback of the Euclidean metric e on V as the matrix function

$$F^* e(x) = (DF(x))^t DF(x), \quad x \in U.$$

Also, if $g = [g_{jk}(x)]_{j,k=1}^n$ is a positive definite matrix function whose entries are C^∞ functions on U, we define the Laplace-Beltrami operator

$$\Delta_g u = |g|^{-1/2} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left[|g|^{1/2} g^{j,k} \frac{\partial u}{\partial x_k} \right], \quad u \in C^\infty(U).$$

Here $[g^{j,k}]_{j,k=1}^n$ is the inverse matrix of $[g_{j,k}]_{j,k=1}^n$, and $|g| = \det[g_{j,k}]$.

The next problem shows that under a map F, the Euclidean Laplacian Δ_e transforms into the Laplace-Beltrami operator Δ_{F^*e}.

Exercise 6.4. Let $F : U \to V$ be a bijective C^∞ map between open sets of F. If $v \in C^\infty(V)$, show that

$$F^*(\Delta_e v)(x) = (\Delta_{F^*e} F^* v)(x), \quad x \in U.$$

Our question was to find those reflection operators F such that the Euclidean Schrödinger operator $-\Delta_e + q$ transforms into another Euclidean Schrödinger operator $-\Delta_e + \tilde{q}$ under F. This is true when $F^*e = e$, or equivalently when $(DF)^t DF$ is the identity matrix. The reflection $F(x', x_n) = (x', -x_n)$ satisfies this property. However, we also know that it is possible to convert conductivity operators $v \mapsto \mathrm{div}(\gamma\nabla v)$ into Schrödinger operators by choosing $v = \gamma^{-1/2}u$. Thus we could ask to find reflections F such that $F^*e = ce$ for some positive scalar function c. Using the definition of F^*e and taking determinants, the last condition is equivalent with

$$(DF(x))^t DF(x) = \det(DF(x))^{2/n} I.$$

Any C^∞ bijective map satisfying this equation is called a conformal transformation.

It is a theorem of Liouville that any conformal transformation between two open subsets of $\mathbb{R}^n$, $n \geq 3$, is obtained by composing rotations, translations, scalings, reflections $(x', x_n) \mapsto (x', -x_n)$, and Kelvin transforms

$$F(x) = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

In the same way that $(x', x_n) \mapsto (x', -x_n)$ reflects across the hyperplane $\{x_n = 0\}$, the Kelvin transform acts as a reflection across the unit sphere $\{|x| = 1\}$. It also has the property that it maps hyperplanes and spheres in $\mathbb{R}^n \setminus \{0\}$ to hyperplanes and spheres.

Exercise 6.5. Show that F maps the set $\{x \in \mathbb{R}^n \,;\, 0 < |x| < 1\}$ onto $\{x \in \mathbb{R}^n \,;\, |x| > 1\}$ and preserves the set $\{|x| = 1\}$. Show also that $F(F(x)) = x$, and that

$$F^*e(x) = |x|^{-4}\, I.$$

Exercise 6.6. Show that F maps the spherical set

$$\{x = (x', x_n) \in \mathbb{R}^n \,;\, \left|x'\right|^2 + (x_n - 1/2)^2 = (1/2)^2,\ x \neq 0\}$$

onto the hyperplane $\{(x', x_n) \,;\, x_n = 1\}$.

We now give the partial data results for the conductivity and Schrödinger equations for the case where the inaccessible part of the boundary of part of a sphere. The first theorem follows from the second one exactly as in the case where part of the boundary is flat, so we will only prove the Schrödinger case.

Theorem 6.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. Assume that $\Omega \subset B$ for some open ball B in $\mathbb{R}^n$, let $\Gamma_0 = \partial\Omega \cap \partial B$, and let $\Gamma = \partial\Omega \setminus \Gamma_0$. Assume also that $\partial B \setminus \partial\Omega \neq \emptyset$. If*

$$\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma \quad \textit{for all } f \in H^{1/2}(\partial\Omega) \textit{ with } \mathrm{supp}\,(f) \subset \Gamma,$$

then $\gamma_1 = \gamma_2$ *in* Ω.

Theorem 6.8. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with smooth boundary, where* $n \geq 3$, *and let* $q_1, q_2 \in L^\infty(\Omega)$ *be such that* 0 *is not a Dirichlet eigenvalue of* $-\Delta + q_1$ *or* $-\Delta + q_2$ *in* Ω. *Assume that* $\Omega \subset B$ *for some open ball* B *in* $\mathbb{R}^n$, *let* $\Gamma_0 = \partial\Omega \cap \partial B$, *and let* $\Gamma = \partial\Omega \setminus \Gamma_0$. *Assume also that* $\partial B \setminus \partial\Omega \neq \emptyset$. *If*

$$\Lambda_{q_1} f|_\Gamma = \Lambda_{q_2} f|_\Gamma \quad \text{for all } f \in H^{1/2}(\partial\Omega) \text{ with } \operatorname{supp}(f) \subset \Gamma,$$

then $q_1 = q_2$ *in* Ω.

Proof. Since $\partial B \setminus \partial\Omega \neq \emptyset$, there is some $x_0 \in \partial B$ with $x_0 \notin \overline{\Omega}$. Thus there is a small ball centered at x_0 which does not intersect Ω. We may choose coordinates so that $x_0 = 0$ and $B \subset \{x_n > 0\}$. If B has radius r, it follows that

$$B = \{x = (x', x_n)\,;\, |x'|^2 + |x_n - r|^2 < r^2\}.$$

For simplicity we will assume that $r = 1/2$ (the general case can be reduced to this by scaling).

Let $F(x) = x/|x|^2$ be the Kelvin transform as above. Define $\tilde{\Omega} = F^{-1}(\Omega)$, and note that $\tilde{\Omega}$ is a bounded domain with C^∞ boundary such that $F^{-1}(\Gamma_0)$ is contained in the hyperplane $\{(x', x_n)\,;\, x_n = 1\}$ by Problem 6.6. If $u \in H^1(\Omega)$ solves $(-\Delta_e + q)u = 0$ in Ω, then by Problem 6.4, $\tilde{u} = F^* u \in H^1(\tilde{\Omega})$ solves

$$(-\Delta_{F^*e} + F^* q)\tilde{u} = 0 \qquad \text{in } \tilde{\Omega}.$$

Using Problem 6.5, we have

$$\Delta_{F^*e} v = \Delta_{|x|^{-4}e} v = |x|^{2n} \sum_{j=1}^n \partial_j (|x|^{-2n+4}\, \partial_j v).$$

This looks like the conductivity operator $\operatorname{div}(\gamma \nabla v)$ with $\gamma(x) = |x|^{-2n+4}$. The substitution $\tilde{u} = \gamma^{-1/2}\tilde{v} = |x|^{n-2}\,\tilde{v}$ gives, as in Theorem 2.74, that

$$\Delta_{F^*e}(|x|^{n-2}\,\tilde{v}) = |x|^{n+2}\,(\Delta_e - c)\tilde{v}$$

where

$$c(x) = \frac{\Delta(|x|^{2-n})}{|x|^{2-n}}.$$

However, since $|x|^{2-n}$ is harmonic when $n \geq 3$, we have $c = 0$. Combining these facts, we have seen that if $u \in H^1(\Omega)$ solves $(-\Delta_e + q)u = 0$ in Ω, then $\tilde{v} = |x|^{2-n}\, F^* u \in H^1(\tilde{\Omega})$ solves

$$(-\Delta_e + |x|^{-4}\, F^* q)\tilde{v} = 0 \qquad \text{in } \tilde{\Omega}.$$

Tracing back the steps, we see that if $\tilde{v}$ solves the above equation, then $u = F^*(|x|^{n-2}\,\tilde{v})$ solves $(-\Delta_e + q)u = 0$ in Ω (recall that $F(F(x)) = x$). It also follows that the Dirichlet problem for $(-\Delta_e+q)u=0$ in Ω is well-posed if and only if the Dirichlet problem for the corresponding equation for $\tilde{v}$ is well-posed in $\tilde{\Omega}$.

Denote by Λ_q the DN map for the equation $(-\Delta_e+q)u=0$ in Ω, and by $\tilde{\Lambda}_{\tilde{q}}$ the DN map for the equation $(-\Delta_e+\tilde{q})\tilde{v}=0$ in $\tilde{\Omega}$. We will prove that if

$$\Lambda_{q_1} f|_\Gamma = \Lambda_{q_2} f|_\Gamma \quad \text{for all } f\in H^{1/2}(\partial\Omega) \text{ with } \operatorname{supp}(f)\subset\Gamma,$$

then

$$\tilde{\Lambda}_{|x|^{-4}F^*q_1}\tilde{f}|_{\tilde{\Gamma}} = \Lambda_{|x|^{-4}F^*q_2}\tilde{f}|_{\tilde{\Gamma}} \quad \text{for all } \tilde{f}\in H^{1/2}(\partial\tilde{\Omega}) \text{ with } \operatorname{supp}(\tilde{f})\subset\tilde{\Gamma}.$$

Here $\tilde{\Gamma}=F^{-1}(\Gamma)$. Since $\partial\tilde{\Omega}\setminus\tilde{\Gamma}=F^{-1}(\Gamma_0)$ is contained in the hyperplane $\{x_n=1\}$ and since $\tilde{\Omega}$ is contained in $\{x_n>1\}$, this would imply by Theorem 6.3 that $|x|^{-4}F^*q_1=|x|^{-4}F^*q_2$ in $\tilde{\Omega}$, and therefore $q_1=q_2$ in Ω.

Let $\tilde{f}\in H^{1/2}(\partial\tilde{\Omega})$ with $\operatorname{supp}(\tilde{f})\subset\tilde{\Gamma}$, and let $\tilde{v}_j\in H^1(\tilde{\Omega})$ solve

$$(-\Delta_e+|x|^{-4}F^*q_j)\tilde{v}_j=0 \qquad \text{in } \tilde{\Omega}$$

with boundary condition $\tilde{v}_j|_{\partial\tilde{\Omega}}=\tilde{f}$. Define $u_j=F^*(|x|^{n-2}\tilde{v}_j)\in H^1(\Omega)$, so that $(-\Delta_e+q_j)u_j=0$ in Ω with $u_j|_{\partial\Omega}=F^*(|x|^{n-2}\tilde{f})$. Since $\operatorname{supp}(F^*(|x|^{n-2}\tilde{f}))\subset\Gamma$, the assumption on the DN maps Λ_{q_1} and Λ_{q_2} implies that

$$\Lambda_{q_1}f|_\Gamma=\Lambda_{q_2}f|_\Gamma.$$

We now note that, by Theorem 2.72,

$$\langle(\tilde{\Lambda}_{|x|^{-4}F^*q_1}-\tilde{\Lambda}_{|x|^{-4}F^*q_2})\tilde{f},\tilde{f}\rangle_{\partial\tilde{\Omega}}=\int_{\tilde{\Omega}}|x|^{-4}F^*(q_1-q_2)\tilde{v}_1\tilde{v}_2\,dx.$$

Changing variables $x=F^{-1}(y)$ and noting that $\left|\det D(F^{-1})(y)\right|=|y|^{-2n}$, the previous expression becomes

$$\int_\Omega |y|^4(q_1-q_2)(y)\,|y|^{n-2}u_1(y)\,|y|^{n-2}u_2(y)\,|y|^{-2n}\,dy=\int_\Omega(q_1-q_2)u_1u_2\,dy$$
$$=\langle(\Lambda_{q_1}-\Lambda_{q_2})f,f\rangle_{\partial\Omega}.$$

This concludes the proof that $\tilde{\Lambda}_{|x|^{-4}F^*q_1}\tilde{f}|_{\tilde{\Gamma}}=\tilde{\Lambda}_{|x|^{-4}F^*q_2}\tilde{f}|_{\tilde{\Gamma}}$ whenever $\operatorname{supp}(\tilde{f})\subset\tilde{\Gamma}$. $\square$

6.2. Carleman estimate approach

Again, we first consider the Schrödinger equation, $(-\Delta+q)u=0$ in Ω, where $q\in L^\infty(\Omega)$ and $\Omega\subset\mathbb{R}^n$ is a bounded open set with smooth boundary.

Motivation. Recall from Theorem **??? cgo_solvability ???** that in the construction of complex geometrical optics solutions, which depend on a large vector $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$, we needed to solve equations of the form

$$(D^2 + 2\zeta \cdot D + q)r = f \qquad \text{in } \Omega,$$

or written in another way,

$$e^{-i\zeta \cdot x}(-\Delta + q)e^{i\zeta \cdot x} r = f \qquad \text{in } \Omega.$$

In particular, Theorem **??? cgo_solvability ???** shows the existence of a solution and implies the estimate

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|} \|f\|_{L^2(\Omega)} .$$

We write

$$\zeta = \frac{1}{h}(\beta + i\alpha),$$

where α and β are orthogonal unit vectors in $\mathbb{R}^n$, and $h > 0$ is a *small parameter*. The estimate for r may be written as

$$\|r\|_{L^2(\Omega)} \leq C_0 h \left\| e^{\frac{1}{h}\alpha \cdot x}(-\Delta + q)e^{-\frac{1}{h}\alpha \cdot x} r \right\|_{L^2(\Omega)}$$

It is possible to view this as a uniqueness result: if the right hand side is zero, then the solution r also vanishes. It turns out that such a uniqueness result can be proved directly without Fourier analysis, and this is sufficient to imply also existence of a solution.

Remark. We will systematically use a small parameter h instead of a large parameter $|\zeta|$ (these are related by $h = \frac{\sqrt{2}}{|\zeta|}$). This is of course just a matter of convention, but has the benefit of being consistent with *semiclassical calculus* which is a well-developed theory for the analysis of certain asymptotic limits. We will also arrange so that our basic partial derivatives will be hD_j instead of $\frac{\partial}{\partial x_j}$. The usefulness of these choices will hopefully be evident below.

6.2.1. Carleman estimates for test functions. We begin with the simplest Carleman estimate, which is valid for test functions and does not involve boundary terms.

Theorem 6.9. *(Carleman estimate) Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbb{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, we have*

$$\|u\|_{L^2(\Omega)} \leq Ch \left\| e^{\varphi/h}(-\Delta + q)e^{-\varphi/h} u \right\|_{L^2(\Omega)}, \quad u \in C_c^\infty(\Omega).$$

We introduce some notation which will be used in the proof and also later. If $u, v \in L^2(\Omega)$ we write

$$(u|v) = \int_\Omega u\bar{v}\, dx$$
$$\|u\| = (u|u)^{1/2} = \|u\|_{L^2(\Omega)}$$

Consider the semiclassical Laplacian

$$P_0 = -h^2\Delta = (hD)^2$$

and the corresponding Schrödinger operator

$$P = h^2(-\Delta + q) = P_0 + h^2 q$$

The operators conjugated with exponential weights will be denoted by

$$P_{0,\varphi} = e^{\varphi/h} P_0 e^{-\varphi/h}$$
$$P_\varphi = e^{\varphi/h} P e^{-\varphi/h} = P_{0,\varphi} + h^2 q$$

We will also need the concept of adjoints of differential operators. If

$$L = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

is a differential operator in Ω, with $a_\alpha \in W^{|\alpha|,\infty}(\Omega)$ (that is, all partial derivatives up to order $|\alpha|$ are in $L^\infty(\Omega)$), then L^* is the differential operator which satisfies

$$(Lu|v) = (u|L^*v), \qquad u, v \in C_c^\infty(\Omega)$$

For L of the above form, an integration by parts shows that

$$L^*v = \sum_{|\alpha| \leq m} D^\alpha(\overline{a_\alpha(x)} v)$$

Proof of Theorem 6.9. Using the notation above, the desired estimate can be written as

$$h\,\|u\| \leq C\,\|P_\varphi u\|\,, \qquad u \in C_c^\infty(\Omega).$$

First consider the case $q = 0$, that is, the estimate

$$h\,\|u\| \leq C\,\|P_{0,\varphi} u\|\,, \qquad u \in C_c^\infty(\Omega).$$

We need an explicit expression for $P_{0,\varphi}$. On the level of operators, one has

$$e^{\varphi/h} hD_j e^{-\varphi/h} = hD_j + i\partial_j\varphi.$$

Since $\varphi(x) = \alpha \cdot x$ where α is a unit vector, we obtain

$$\begin{aligned} P_{0,\varphi} &= \sum_{j=1}^{n} (e^{\varphi/h} hD_j e^{-\varphi/h})(e^{\varphi/h} hD_j e^{-\varphi/h}) = \sum_{j=1}^{n} (hD_j + i\alpha_j)^2 \\ &= (hD)^2 - 1 + 2i\alpha \cdot hD \end{aligned}$$

The objective is to prove a positive lower bound for

$$\|P_{0,\varphi}u\|^2 = (P_{0,\varphi}u|P_{0,\varphi}u).$$

To this end, we decompose $P_{0,\varphi}$ in a way which is useful for determining which parts in the inner product are positive and which may be negative. Write

$$P_{0,\varphi} = A + iB$$

where $A^* = A$ and $B^* = B$. Here, A and iB are the self-adjoint and skew-adjoint parts of $P_{0,\varphi}$. Since

$$\begin{aligned} P_{0,\varphi}^* &= (e^{\varphi/h} P_0 e^{-\varphi/h})^* = e^{-\varphi/h} P_0 e^{\varphi/h} = P_{0,-\varphi} \\ &= (hD)^2 - 1 - 2i\alpha \cdot hD \end{aligned}$$

we obtain A and B from the formulas (cf. the real and imaginary parts of a complex number)

$$\begin{aligned} A &= \frac{P_{0,\varphi} + P_{0,\varphi}^*}{2} = (hD)^2 - 1 \\ B &= \frac{P_{0,\varphi} - P_{0,\varphi}^*}{2i} = 2\alpha \cdot hD \end{aligned}$$

Now we have

$$\begin{aligned} \|P_{0,\varphi}u\|^2 &= (P_{0,\varphi}u|P_{0,\varphi}u) = ((A+iB)u|(A+iB)u) \\ &= (Au|Au) + (Bu|Bu) + i(Bu|Au) - i(Au|Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u) \end{aligned}$$

where $[A,B] = AB - BA$ is the commutator of A and B. This argument used integration by parts and the fact that $A^* = A$ and $B^* = B$. There are no boundary terms since $u \in C_c^\infty(\Omega)$.

The terms $\|Au\|^2$ and $\|Bu\|^2$ are nonnegative, so the only negative contributions could come from the commutator term. But in our case A and B are constant coefficient differential operators, and these operators always satisfy

$$[A,B] \equiv 0$$

Therefore

$$\|P_{0,\varphi}u\|^2 = \|Au\|^2 + \|Bu\|^2$$

By the Poincaré inequality (see [Ola])[1] In fact, if $\alpha \in \mathbb{R}^n$ is a unit vector, then the proof given in [Ola] implies the following Poincaré inequality in the unbounded strip $S = \{ x \in \mathbb{R}^n \mid a < x \cdot \alpha < b \}$:

$$\|u\|_{L^2(S)} \leq \frac{b-a}{\sqrt{2}} \|\alpha \cdot Du\|_{L^2(S)}, \quad u \in C_c^\infty(S).$$

,

$$\|Bu\| = 2h \|\alpha \cdot Du\| \geq ch \|u\|,$$

where c depends on Ω. This shows that for any $h > 0$, one has

$$h \|u\| \leq C \|P_{0,\varphi} u\|, \qquad u \in C_c^\infty(\Omega).$$

Finally, consider the case where q may be nonzero. The last estimate implies that for $u \in C_c^\infty(\Omega)$, one has

$$\begin{aligned} h \|u\| &\leq C \|P_{0,\varphi} u\| \leq C \left\|(P_{0,\varphi} + h^2 q)u\right\| + C \left\|h^2 q u\right\| \\ &\leq C \|P_\varphi u\| + Ch^2 \|q\|_{L^\infty(\Omega)} \|u\| \end{aligned}$$

Choose h_0 so that $C \|q\|_{L^\infty(\Omega)} h_0 = \frac{1}{2}$, that is,

$$h_0 = \frac{1}{2C \|q\|_{L^\infty(\Omega)}}.$$

Then, if $0 < h \leq h_0$,

$$h \|u\| \leq C \|P_\varphi u\| + \frac{1}{2} h \|u\|.$$

The last term may be absorbed in the left hand side, which completes the proof. $\square$

Exercise 6.10. (H^1 Carleman estimate) Let $\varphi(x) = \alpha \cdot x$ and let $q \in L^\infty(\Omega)$. Show that there are $C > 0$ and $h_0 > 0$ such that for any h with $0 < h \leq h_0$, one has

$$\|u\| + \|hDu\| \leq Ch \left\| e^{\varphi/h}(-\Delta + q)e^{-\varphi/h} u \right\|, \quad u \in C_c^\infty(\Omega).$$

Exercise 6.11. (Large first order perturbations) Let $\varphi(x) = \alpha \cdot x$, let $A = (A_1, \ldots, A_n) \in L^\infty(\Omega\,; \mathbb{R}^n)$ be a vector field, and let $q \in L^\infty(\Omega)$. Show that there are $C > 0$ and $h_0 > 0$ such that for any h with $0 < h \leq h_0$, one has

$$\|u\| + \|hDu\| \leq Ch \left\| e^{\varphi/h}(-\Delta + A \cdot \nabla + q)e^{-\varphi/h} u \right\|, \quad u \in C_c^\infty(\Omega).$$

(Hint: use the convexified weight $\varphi_\varepsilon = \varphi + \frac{h}{\varepsilon}\frac{\varphi^2}{2}$, where $\varepsilon > 0$ is small but fixed.)

[1]1

6.2.2. Complex geometrical optics solutions. Here, we show how the Carleman estimate gives a new method for constructing complex geometrical optics solutions. We first establish an existence result for an inhomogeneous equation, analogous to Theorem **??? cgo_solvability ???** .

Theorem 6.12. *Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbb{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, the equation*

$$e^{\varphi/h}(-\Delta + q)e^{-\varphi/h} r = f \quad \text{in } \Omega$$

has a solution $r \in L^2(\Omega)$ for any $f \in L^2(\Omega)$, satisfying

$$\|r\|_{L^2(\Omega)} \leq Ch \|f\|_{L^2(\Omega)}$$

Remark. With some knowledge of unbounded operators on Hilbert space, the proof is immediate. Consider $P_\varphi^* : L^2(\Omega) \to L^2(\Omega)$ with domain $C_c^\infty(\Omega)$. It is a general fact that

$$\left.\begin{array}{r} T \text{ injective} \\ \text{range of } T \text{ closed} \end{array}\right\} \implies T^* \text{ surjective.}$$

Since the Carleman estimate is valid for P_φ^* one obtains injectivity and closed range for P_φ^*, and thus solvability for P_φ. Below we give a direct proof based on duality and the Hahn–Banach theorem, and also obtain the norm bound.

Proof of Theorem 6.12. Note that $P_\varphi^* = P_{0,-\varphi} + h^2 \bar{q}$. If h_0 is as in Theorem 6.9, for $h \leq h_0$ we have

$$\|u\| \leq \frac{C}{h} \left\|P_\varphi^* u\right\|, \quad u \in C_c^\infty(\Omega)$$

Let $D = P_\varphi^* C_c^\infty(\Omega)$ be a subspace of $L^2(\Omega)$, and consider the linear functional

$$L : D \to \mathbb{C}, \quad L(P_\varphi^* v) = (v|f), \quad \text{for } v \in C_c^\infty(\Omega)$$

This is well defined since any element of D has a unique representation as $P_\varphi^* v$ with $v \in C_c^\infty(\Omega)$, by the Carleman estimate. Also, the Carleman estimate implies

$$\left|L(P_\varphi^* v)\right| \leq \|v\| \, \|f\| \leq \frac{C}{h} \|f\| \left\|P_\varphi^* v\right\|$$

Thus L is a bounded linear functional on D.

The Hahn-Banach theorem ensures that there is a bounded linear functional $\hat{L} : L^2(\Omega) \to \mathbb{C}$ satisfying $\hat{L}|_D = L$ and $\left\|\hat{L}\right\| \leq Ch^{-1} \|f\|$. By the Riesz representation theorem, there is $\tilde{r} \in L^2(\Omega)$ such that

$$\hat{L}(w) = (w|\tilde{r}), \quad w \in L^2(\Omega),$$

and $\|\tilde{r}\| \leq Ch^{-1}\|f\|$. Then, for $v \in C_c^\infty(\Omega)$, by the definition of weak derivatives we have

$$(v|P_\varphi \tilde{r}) = (P_\varphi^* v|\tilde{r}) = \hat{L}(P_\varphi^* v) = L(P_\varphi^* v) = (v|f),$$

which shows that $P_\varphi \tilde{r} = f$ in the weak sense.

Finally, set $r = h^2\tilde{r}$. This satisfies $e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}r = f$ in Ω, and $\|r\| \leq Ch\|f\|$. □

We now give a construction of complex geometrical optics solutions to the equation $(-\Delta + q)u = 0$ in Ω, based on Theorem 6.12. This is slightly more general than the discussion in Chapter 3, and is analogous to the *WKB construction* used in finding geometrical optics solutions for the wave equation.

Our solutions are of the form

$$u = e^{-\frac{1}{h}(\varphi + i\psi)}(a + r). \tag{6.2}$$

Here $h > 0$ is small and $\varphi(x) = \alpha \cdot x$ as before, ψ is a real valued phase function, a is a complex amplitude, and r is a correction term which is small when h is small.

Writing $\rho = \varphi + i\psi$ for the complex phase, using the formula

$$e^{\rho/h} hD_j e^{-\rho/h} = hD_j + i\partial_j \rho$$

which is valid for operators, and inserting (6.2) in the equation, we have

$$\begin{aligned} & (-\Delta + q)u = 0 \\ \Leftrightarrow \quad & e^{\rho/h}((hD)^2 + h^2 q)e^{-\rho/h}(a + r) = 0 \\ \Leftrightarrow \quad & e^{\rho/h}((hD)^2 + h^2 q)e^{-\rho/h} r = -((hD + i\nabla\rho)^2 + h^2 q)a \end{aligned}$$

The last equation may be written as

$$e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}(e^{-i\psi/h} r) = f$$

where

$$f = -e^{-i\psi/h}\Big(-h^{-2}(\nabla\rho)^2 + h^{-1}[2\nabla\rho \cdot \nabla + \Delta\rho] + (-\Delta + q)\Big)a.$$

Now, Theorem 6.12 ensures that one can find a correction term r satisfying $\|r\| \leq Ch$, thus showing the existence of complex geometrical optics solutions, provided that

$$\|f\| \leq C$$

with C independent of h. Looking at the expression for f, we see that it is enough to choose ψ and a in such a way that

$$\begin{aligned} (\nabla\rho)^2 &= 0 \\ 2\nabla\rho \cdot \nabla a + (\Delta\rho)a &= 0 \end{aligned}$$

Since $\varphi(x) = \alpha\cdot x$ with α a unit vector, expanding the square in $(\nabla\rho)^2 = 0$ gives the following equations for ψ:

$$|\nabla\psi|^2 = 1, \qquad \alpha\cdot\nabla\psi = 0.$$

This is an *eikonal equation* (a certain nonlinear first order PDE) for ψ. We obtain one solution by choosing $\psi(x) = \beta\cdot x$ where $\beta\in\mathbb{R}^n$ is a unit vector satisfying $\alpha\cdot\beta = 0$. It would be possible to use other solutions ψ, but this choice is close to the discussion in Chapter **???** **???** .

If $\psi(x) = \beta\cdot x$, then the second equation becomes

$$(\alpha + i\beta)\cdot\nabla a = 0.$$

This is a complex *transport equation* (a first order linear equation) for a, analogous to the equation for a in Theorem **??? thm:cgo_construction ???** . One solution is given by $a\equiv 1$. Again, other choices would be possible.

This ends the construction of complex geometrical optics solutions based on Carleman estimates. There is one additional difference with the analogous result in Theorem **??? thm:cgo_construction ???** : the correction term r given by this argument is only in $L^2(\Omega)$, not in $H^1(\Omega)$. The same is true for the solution u. One can in fact obtain r and u in $H^1(\Omega)$ (and even in $H^2(\Omega)$), but this requires a slightly stronger Carleman estimate and some additional work. Some details for this were given **??? in the exercises and lectures ???** .

6.2.3. Carleman estimates with boundary terms. We will continue by deriving a Carleman estimate for functions which vanish at the boundary but are not compactly supported. The estimate will include terms involving the normal derivative. We will use the notation

$$(u|v)_{\partial\Omega} = \int_{\partial\Omega} u\bar{v}\,dS$$
$$\partial_\nu u = \nabla u\cdot\nu|_{\partial\Omega}$$

and

$$\partial\Omega_\pm = \partial\Omega_\pm(\alpha) = \left\{\ x\in\partial\Omega\ \middle|\ \pm\,\alpha\cdot\nu(x)\geq 0\ \right\}$$

Theorem 6.13. *(Carleman estimate with boundary terms) Let $q\in L^\infty(\Omega)$, let α be a unit vector in $\mathbb{R}^n$, and let $\varphi(x) = \alpha\cdot x$. There exist constants $C>0$ and $h_0>0$ such that whenever $0<h\leq h_0$, we have*

$$\begin{aligned} &-h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + \|u\|^2_{L^2(\Omega)} \\ &\qquad\leq Ch^2\left\|e^{\varphi/h}(-\Delta+q)e^{-\varphi/h}u\right\|^2_{L^2(\Omega)} + Ch((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+}\end{aligned}$$

for any $u\in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$.

Note that the sign of $\alpha\cdot\nu$ on $\partial\Omega_\pm$ ensures that all terms in the Carleman estimate are nonnegative.

Proof. We first claim that

$$ch^2\|u\|^2 - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega} \leq \|P_{0,\varphi}u\|^2 \tag{6.3}$$

for $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$. It is easy to see that this implies the desired estimate in the case $q = 0$.

As in the proof of Theorem 6.9, we decompose

$$P_{0,\varphi} = A + iB$$

where $A = (hD)^2 - 1$ and $B = 2\alpha\cdot hD$, and $A^* = A$, $B^* = B$. Then

$$\begin{aligned}\|P_{0,\varphi}u\|^2 &= (P_{0,\varphi}u|P_{0,\varphi}u) = ((A+iB)u|(A+iB)u)\\ &= \|Au\|^2 + \|Bu\|^2 + i(Bu|Au) - i(Au|Bu)\end{aligned}$$

We wish to integrate by parts to obtain the commutator term involving $i[A,B]$, but this time boundary terms will arise. We have

$$\begin{aligned}i(Bu|(hD)^2u) &= \sum_{j=1}^n i(Bu|(hD_j)^2u)\\ &= \sum_{j=1}^n \Big[i(Bu|\frac{h}{i}\nu_j hD_ju)_{\partial\Omega} + i(hD_jBu|hD_ju)\Big]\\ &= -2h^3(\alpha\cdot\nabla u|\partial_\nu u)_{\partial\Omega} + \sum_{j=1}^n\Big[i(hD_jBu|\frac{h}{i}\nu_ju)_{\partial\Omega} + i((hD_j)^2Bu|u)\Big]\end{aligned}$$

But $u|_{\partial\Omega} = 0$, so the boundary term involving $\frac{h}{i}\nu_j u$ is zero. For the first boundary term we use the decomposition

$$\nabla u|_{\partial\Omega} = (\partial_\nu u)\nu + (\nabla u)_{\tan}$$

where $(\nabla u)_{\tan} := \nabla u - (\nabla u\cdot\nu)\nu|_{\partial\Omega}$ is the tangential part of ∇u, which vanishes since $u|_{\partial\Omega} = 0$. By these facts, we obtain

$$i(Bu|Au) = i(ABu|u) - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega}$$

Similarly, using that $u|_{\partial\Omega} = 0$,

$$\begin{aligned}i(Au|Bu) &= i(Au|2\alpha\cdot\frac{h}{i}\nu u)_{\partial\Omega} + i(BAu|u)\\ &= i(BAu|u)\end{aligned}$$

We have proved that

$$\|P_{0,\varphi}u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u) - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega}$$

Again, since A and B are constant coefficient operators, we have $[A, B] = AB - BA \equiv 0$. The Poincaré inequality gives $\|Bu\| \geq ch\,\|u\|$, which proves (6.3).

Writing (6.3) in a different form, we have

$$\begin{aligned}&-2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + c\,\|u\|^2 \\ &\qquad\leq h^2 \left\| e^{\varphi/h}(-\Delta)e^{-\varphi/h}u \right\|^2 + 2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+}\end{aligned}$$

Adding a potential, it follows that

$$\begin{aligned}&-2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + c\,\|u\|^2 \\ &\qquad\leq h^2 \left\| e^{\varphi/h}(-\Delta+q)e^{-\varphi/h}u \right\|^2 + h^2\,\|q\|_{L^\infty(\Omega)}^2\,\|u\|^2 + 2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+}\end{aligned}$$

Choosing h small enough (depending on $\|q\|_{L^\infty(\Omega)}$), the term involving $\|u\|^2$ on the right can be absorbed to the left hand side. This concludes the proof. $\square$

Exercise 6.14. (Solvability with vanishing data on part of boundary) Show that there are $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, the equation

$$\begin{cases} e^{\varphi/h}(-\Delta+q)e^{-\varphi/h}r = f & \text{in } \Omega \\ r = 0 & \text{on } \partial\Omega_+ \end{cases}$$

has a solution $r \in L^2(\Omega)$ for any $f \in L^2(\Omega)$, with $\|r\| \leq Ch\,\|f\|$. (Hint: use test functions which vanish, along with their normal derivative, on suitable parts of the boundary.)

6.3. Uniqueness with partial data

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary, where $n \geq 3$. If $\alpha \in \mathbb{R}^n$, recall the subsets of the boundary

$$\begin{aligned}\partial\Omega_\pm &= \{\, x \in \partial\Omega \mid \pm\,\alpha\cdot\nu(x) > 0 \,\} \\ \partial\Omega_{-,\varepsilon} &= \{\, x \in \partial\Omega \mid \alpha\cdot\nu(x) < \varepsilon \,\}\end{aligned}$$

Also, let $\partial\Omega_{+,\varepsilon} = \{\, x \in \partial\Omega \mid \alpha\cdot\nu(x) > \varepsilon \,\}$. We first consider a partial data uniqueness result for the Schrödinger equation.

Theorem 6.15. *Let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ are well-posed. If α is a unit vector in $\mathbb{R}^n$ and if*

$$\Lambda_{q_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{q_2} f|_{\partial\Omega_{-,\varepsilon}} \quad \textit{for all } f \in H^{1/2}(\partial\Omega)$$

then $q_1 = q_2$ in Ω.

Given this result, it is easy to prove the corresponding theorem for the conductivity equation.

Proof that Theorem 6.15 implies Theorem 6.2. Define $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$. By Lemma **??? lemma:dn_conductivity_schrodinger ???** , we have the relation

$$\Lambda_{q_j} f = \gamma_j^{-1/2}\Lambda_{\gamma_j}(\gamma_j^{-1/2} f) + \frac{1}{2}\gamma_j^{-1}\frac{\partial\gamma_j}{\partial\nu} f\Big|_{\partial\Omega}.$$

Since $\Lambda_{\gamma_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{\gamma_2} f|_{\partial\Omega_{-,\varepsilon}}$ for all f, boundary determination results (see [Ola]) imply that

$$\gamma_1|_{\partial\Omega_{-,\varepsilon}} = \gamma_2|_{\partial\Omega_{-,\varepsilon}}, \quad \frac{\partial\gamma_1}{\partial\nu}|_{\partial\Omega_{-,\varepsilon}} = \frac{\partial\gamma_2}{\partial\nu}|_{\partial\Omega_{-,\varepsilon}}$$

$$\frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}} \quad \text{in } \Omega.$$

Now also $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$, so the arguments in Section **??? sec:uniqueness_reduction ???** imply that $\gamma_1 = \gamma_2$ in Ω. □

We proceed to the proof of Theorem 6.15. The main tool is the Carleman estimate in Theorem 6.13, which will be applied with the weight $-\varphi$ instead of φ. The estimate then has the form

$$h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+} + \|u\|_{L^2(\Omega)}^2 \leq Ch^2\left\|e^{-\varphi/h}(-\Delta+q)e^{\varphi/h}u\right\|_{L^2(\Omega)}^2 - Ch((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-}$$

with $u \in C^\infty(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$. Choosing $v = e^{\varphi/h}u$ and noting that $v|_{\partial\Omega} = 0$, this may be written as

(6.4)

$$h\left((\alpha\cdot\nu)e^{-\varphi/h}\partial_\nu v|e^{-\varphi/h}\partial_\nu v\right)_{\partial\Omega_+} + \left\|e^{-\varphi/h}v\right\|_{L^2(\Omega)}^2 \leq Ch^2\left\|e^{-\varphi/h}(-\Delta+q)v\right\|_{L^2(\Omega)}^2 - Ch\big((\alpha\cdot\nu)e^{-\varphi/h}\partial_\nu v|e^{-\varphi/h}\partial_\nu v\big)_{\partial\Omega_-}$$

This last estimate is valid for all $v \in H^2 \cap H_0^1(\Omega)$, which follows by an approximation argument (or can be proved directly).

Proof of Theorem 6.15. Recall from Lemma **??? lemma:identity_schrodinger ???** that

$$\int_\Omega (q_1-q_2)u_1u_2\,dx = \big\langle(\Lambda_{q_1}-\Lambda_{q_2})(u_1|_{\partial\Omega})\,,\,u_2|_{\partial\Omega}\big\rangle_{\partial\Omega} \tag{6.5}$$

whenever $u_j \in H^1(\Omega)$ are solutions of $(-\Delta+q_j)u_j = 0$ in Ω. By the assumption on the DN maps, the boundary integral is really over $\partial\Omega_{+,\varepsilon}$. If further $u_1 \in H^2(\Omega)$, then

$$\Lambda_{q_1}(u_1|_{\partial\Omega}) = \partial_\nu u_1|_{\partial\Omega}$$

since $\nabla u_1 \in H^1(\Omega)$ and $\partial_\nu u_1|_{\partial\Omega} = (\operatorname{tr}\nabla u_1)\cdot\nu|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Also,

$$\Lambda_{q_2}(u_1|_{\partial\Omega}) = \partial_\nu \tilde{u}_2|_{\partial\Omega},$$

where $\tilde{u}_2$ solves

$$\begin{cases} (-\Delta + q_2)\,\tilde{u}_2 = 0 & \text{in } \Omega \\ \qquad\qquad \tilde{u}_2 = u_1 & \text{on } \partial\Omega \end{cases}$$

We have $\tilde{u}_2 \in H^2(\Omega)$ since $u_1|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$. Therefore, (6.5) implies

$$\int_\Omega (q_1 - q_2)u_1u_2\,dx = \int_{\partial\Omega_{+,\varepsilon}} \partial_\nu(u_1 - \tilde{u}_2)u_2\,dS$$

for any $u_j \in H^2(\Omega)$ which solve $(-\Delta + q_j)u_j = 0$ in Ω.

Given the unit vector $\alpha \in \mathbb{R}^n$, let $\xi \in \mathbb{R}^n$ be a vector orthogonal to α, and let $\beta \in \mathbb{R}^n$ be a unit vector such that $\{\alpha, \beta, \xi\}$ is an orthogonal triplet. Write $\varphi(x) = \alpha\cdot x$ and $\psi(x) = \beta\cdot x$. Theorem **??? thm:cgo_construction ???** ensures that there exist CGO solutions to $(-\Delta + q_j)u_j = 0$ of the form

$$\begin{aligned} u_1 &= e^{\frac{1}{h}(\varphi+i\psi)}e^{ix\cdot\xi}(1 + r_1) \\ u_2 &= e^{-\frac{1}{h}(\varphi+i\psi)}(1 + r_2), \end{aligned}$$

where $\|r_j\| \le Ch$, $\|\nabla r_j\| \le C$, and $u_j \in H^2(\Omega)$ (the part that $r_j \in H^2(\Omega)$ **??? was in the exercises ???**). Then, writing $u := u_1 - \tilde{u}_2 \in H^2 \cap H_0^1(\Omega)$, we have

$$\int_\Omega e^{ix\dot\xi}(q_1 - q_2)(1 + r_1 + r_2 + r_1r_2)\,dx = \int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u)u_2\,dS. \tag{6.6}$$

By the estimates for r_j, the limit as $h \to 0$ of the left hand side is $\int_\Omega e^{ix\cdot\xi}(q_1 - q_2)\,dx$. We wish to show that the right hand side converges to zero as $h \to 0$.

By Cauchy–Schwarz, one has

(6.7)
$$\begin{aligned} \Big|\int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u)u_2\,dS\Big|^2 &= \Big|\int_{\partial\Omega_{+,\varepsilon}} e^{-\varphi/h}(\partial_\nu u)e^{\varphi/h}u_2\,dS\Big|^2 \\ &\le \Big(\int_{\partial\Omega_{+,\varepsilon}} \Big|e^{-\varphi/h}\partial_\nu u\Big|^2\,dS\Big)\Big(\int_{\partial\Omega_{+,\varepsilon}} \Big|e^{\varphi/h}u_2\Big|^2\,dS\Big). \end{aligned}$$

To use the Carleman estimate, we note that $\varepsilon \le \alpha\cdot\nu$ on $\partial\Omega_{+,\varepsilon}$, By (6.4) applied to u and with potential q_2, and using that $\partial_\nu u|_{\partial\Omega_{-,\varepsilon}} = 0$ by the assumption on DN maps, we obtain for small h that

$$\begin{aligned}\int_{\partial\Omega_{+,\varepsilon}} \left|e^{-\varphi/h}\partial_\nu u\right|^2 &\leq \frac{1}{\varepsilon}\int_{\partial\Omega_{+,\varepsilon}} (\alpha\cdot\nu)\left|e^{-\varphi/h}\partial_\nu u\right|^2 \, dS \\ &\leq \frac{1}{\varepsilon}Ch\left\|e^{-\varphi/h}(-\Delta+q_2)u\right\|^2_{L^2(\Omega)}\end{aligned}$$

The reason for choosing the potential q_2 is that

$$(-\Delta+q_2)u = (-\Delta+q_2)u_1 = (q_2-q_1)u_1$$

Thus, the solution $\tilde{u}_2$ goes away, and we are left with an expression involving only u_1 for which we know exact asymptotics. We have

$$\int_{\partial\Omega_{+,\varepsilon}} \left|e^{-\varphi/h}\partial_\nu u\right|^2 \leq \frac{1}{\varepsilon}Ch\left\|(q_2-q_1)e^{i\psi/h}e^{ix\cdot\xi}(1+r_1)\right\|^2_{L^2(\Omega)} \leq Ch$$

This takes care of the first term on the right hand side of (6.7). For the other term we compute

$$\begin{aligned}\int_{\partial\Omega_{+,\varepsilon}} \left|e^{\varphi/h}u_2\right|^2 \, dS &= \int_{\partial\Omega_{+,\varepsilon}} |1+r_2|^2 \, dS \\ &\leq \frac{1}{2}\int_{\partial\Omega_{+,\varepsilon}} (1+r_2^2)\, dS \leq C(1+\|r_2\|^2_{L^2(\partial\Omega)})\end{aligned}$$

By the trace theorem, $\|r_2\|_{L^2(\partial\Omega)} \leq C\|r_2\|_{H^1(\Omega)} \leq C$. Combining these estimates, we have for small h that

$$\Big|\int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u)u_2 \, dS\Big| \leq C\sqrt{h}.$$

Taking the limit as $h \to 0$ in (6.6), we are left with

$$\int_\Omega e^{ix\cdot\xi}(q_1-q_2)\, dx = 0. \tag{6.8}$$

This is true for all $\xi \in \mathbb{R}^n$ orthogonal to α. However, since the DN maps agree on $\partial\Omega_{-,\varepsilon}(\alpha)$ for a fixed constant $\varepsilon > 0$, they also agree on $\partial\Omega_{-,\varepsilon'}(\alpha')$ for α' sufficiently close to α on the unit sphere and for some smaller constant ε'. Thus, in particular, (6.8) holds for ξ in an open cone in $\mathbb{R}^n$. Writing q for the function which is equal to $q_1 - q_2$ in Ω and which is zero outside of Ω, this implies that the Fourier transform of q vanishes in an open set. But since q is compactly supported, the Fourier transform is analytic by the Paley–Wiener theorem, and this implies that $q \equiv 0$. We have proved that $q_1 \equiv q_2$. □

Exercise 6.16. Let $f : \overline{\Omega} \to \mathbb{C}$ be continuous, where $\Omega \subset \mathbb{R}^n$ is bounded. Show that there is a modulus of continuity ω such that $|f(x) - f(y)| \leq \omega(|x-y|)$.

Exercise 6.17. Assuming the claim in Problem 6.17, determine $\Lambda_\gamma f$.

Exercise 6.18. Let $\gamma \equiv 1$ in the unit disc $\mathbb{D} \subset \mathbb{R}^2$. Show that the solution in $H^1(\mathbb{D})$ to $\nabla \cdot \gamma \nabla u = 0$ in $\mathbb{D}$, with $u|_{\partial\mathbb{D}} = f \in H^{1/2}(\partial\mathbb{D})$, is given by

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) e^{ik\theta}$$

Below, $q \in L^\infty(\Omega)$, and $\varphi(x) = \alpha \cdot x$ where α is a unit vector in $\mathbb{R}^n$.

6.4. Unique continuation

In this section we prove the unique continuation result required in the proof of Theorem 6.1. The uniqueness results below are true for rather general elliptic equations, but for simplicity we restrict our attention to solutions of the Schrödinger equation $(-\Delta + q)u = 0$.

Theorem 6.19. *(Unique continuation from local Cauchy data) Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with smooth boundary, and let $q \in L^\infty(\Omega)$. Assume that Γ is a nonempty open subset of $\partial\Omega$. If $u \in H^2(\Omega)$ satisfies*

$$(-\Delta + q)u = 0 \quad \text{in } \Omega$$

and

$$u|_\Gamma = \partial_\nu u|_\Gamma = 0,$$

then $u = 0$ in Ω.

As an immediate corollary, if two solutions to the Schrödinger equation $(-\Delta + q)u = 0$ have the same Cauchy data on an open subset of the boundary, then the solutions are identical in the whole domain. This is an instance of the unique continuation principle for elliptic equations. We also state two closely related variants. The first one is called weak unique continuation and it concerns uniqueness of solutions which vanish in some ball. (One also has strong unique continuation meaning that any solution vanishing to infinite order at a point in a suitable sense must vanish everywhere, but we will not need this.)

Theorem 6.20. *(Weak unique continuation) Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open set, and let $q \in L^\infty(\Omega)$. If $u \in H^2(\Omega)$ satisfies*

$$(-\Delta + q)u = 0 \qquad \text{in } \Omega$$

and

$$u = 0 \text{ in some ball } B \text{ contained in } \Omega,$$

then $u = 0$ in Ω.

The next variant states that a solution to a Schrödinger equation that vanishes on one side of a hypersurface must vanish also on the other side.

Theorem 6.21. *(Unique continuation across a hypersurface) Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $q \in L^\infty(\Omega)$. Suppose that S is a C^∞ hypersurface such that $\Omega = S_+ \cup S \cup S_-$ where S_+ and S_- denote the two sides of S. If $x_0 \in S$ and if V is an open neighborhood of x_0 in Ω, and if $u \in H^2(V)$ satisfies*

$$(-\Delta + q)u = 0 \qquad \text{in } V$$

and

$$u = 0 \qquad \text{in } V \cap S_+,$$

then $u = 0$ in some neighborhood of x_0.

We will in fact prove Theorem 6.21 and deduce the other unique continuation results from that. The standard tool in the proof is a Carleman estimate. Recall first the simple Carleman estimate from Theorem 6.9, stating that whenever φ is a linear function and $h > 0$ is sufficiently small we have

$$\left\| e^{\varphi/h} u \right\|_{L^2(\Omega)} \leq Ch \left\| e^{\varphi/h}(-\Delta + q)u \right\|_{L^2(\Omega)}, \quad u \in C_c^\infty(\Omega).$$

This is already a sort of uniqueness statement: it implies that any solution $u \in C_c^\infty(\Omega)$ of the equation $(-\Delta + q)u = 0$ in Ω must be identically zero in the whole domain. To obtain Theorem 6.21 we will need a Carleman estimate suitable for proving local uniqueness of solutions, and for this it will be useful to consider more general weight functions than linear ones.

We begin by recalling some notation from the proof of Theorem 6.9. Let $(u|v)$ be the inner product in $L^2(\Omega)$ and $\|u\|$ the corresponding norm, and let $P_0 = (hD)^2$ be the semiclassical Laplacian. If $\psi \in C^\infty(\overline{\Omega})$ is a real valued function, we define the conjugated Laplacian

$$P_{0,\psi} = e^{\psi/h} P_0 e^{-\psi/h}.$$

We also write ψ'' for the Hessian matrix

$$\psi''(x) = [\partial_{x_j x_k} \psi(x)]_{j,k=1}^n.$$

The following is an analogue of Theorem 6.9 for a general weight function. The point is that a Carleman estimate of the type $\|P_{0,\psi} u\| \geq ch \|u\|$ may follow if the weight ψ is chosen so that $(i[A,B]u|u)$ is at least nonnegative. In the case when ψ was a linear function, both A and B were constant coefficient operators and the commutator $i[A,B]$ was identically zero. However, if ψ is convex (meaning that the Hessian ψ'' is positive definite) one obtains a better lower bound.

Theorem 6.22. *(Carleman estimate with general weight) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\psi \in C^\infty(\overline{\Omega})$. Then*

$$P_{0,\psi} = A + iB$$

where A and B are the formally self-adjoint operators

$$A = (hD)^2 - |\nabla\psi|^2\,,$$
$$B = \nabla\psi \circ hD + hD \circ \nabla\psi.$$

If $u \in C_c^\infty(\Omega)$ one has

$$\|P_{0,\psi}u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u,u)$$

where the commutator $i[A,B]$ satisfies

$$(i[A,B]u,u) = 4h(\psi'' hDu, hDu) + 4h((\psi''\nabla\psi\cdot\nabla\psi)u,u) - h^3((\Delta^2\psi)u,u).$$

Proof. The first step is to decompose $P_{0,\psi}$ into self-adjoint and skew-adjoint parts as

$$P_{0,\psi} = A + iB$$

where A and B are the formally self-adjoint operators

$$A = \frac{P_{0,\psi} + P_{0,\psi}^*}{2},$$
$$B = \frac{P_{0,\psi} - P_{0,\psi}^*}{2i}.$$

We have

$$P_{0,\psi} = \sum_{j=1}^n (e^{\psi/h} hD_j e^{-\psi/h})^2 = \sum_{j=1}^n (hD_j + i\partial_j\psi)^2 = (hD)^2 - |\nabla\psi|^2 + i\nabla\psi\circ hD + ihD\circ\nabla\psi,$$

and

$$P_{0,\psi}^* = (e^{\psi/h}P_0e^{-\psi/h})^* = e^{-\psi/h}P_0e^{\psi/h} = (hD)^2 - |\nabla\psi|^2 - i\nabla\psi\circ hD - ihD\circ\nabla\psi.$$

The required expressions for A and B follow.

If $u \in C_c^\infty(\Omega)$ we compute

$$\|P_{0,\psi}u\|^2 = ((A+iB)u|(A+iB)u) = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u).$$

It remains to compute the commutator:

$$\begin{aligned} i[A,B]u &= h\big[((hD)^2 - |\nabla\psi|^2)(2\nabla\psi\cdot\nabla u + (\Delta\psi)u) \\ &\qquad - (2\nabla\psi\cdot\nabla + \Delta\psi)((hD)^2u - |\nabla\psi|^2\,u)\big] \\ &= h\big[2\nabla(hD)^2\psi\cdot\nabla u + 4hD\partial_k\psi\cdot hD\partial_k u + ((hD)^2\Delta\psi)u \\ &\qquad + 2hD\Delta\psi\cdot hDu + 2\nabla\psi\cdot\nabla(|\nabla\psi|^2)u\big] \\ &= h\big[4(\psi''\nabla\psi\cdot\nabla\psi)u - 4h^2\partial_{jk}\psi\partial_{jk}u - 4h^2\nabla\Delta\psi\cdot\nabla u \\ &\qquad - h^2(\Delta^2\psi)u\big]. \end{aligned}$$

Integrating by parts once, using that $u|_{\partial\Omega} = 0$, yields

$$(i[A,B]u,u) = 4h^3(\psi''\nabla u, \nabla u) + 4h((\psi''\nabla\psi\cdot\nabla\psi)u,u) \\ - h^3((\Delta^2\psi)u,u).$$

□

The next result shows that if one starts with any function φ with nonvanishing gradient, the convexified weight function $\psi = e^{\lambda\varphi}$ for λ sufficiently large will have a good Carleman estimate.

Theorem 6.23. *(Carleman estimate with weight $e^{\lambda\varphi}$) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $q \in L^\infty(\Omega)$, and assume that $\varphi \in C^\infty(\overline{\Omega})$ satisfies $\varphi \geq 0$ and $\nabla\varphi \neq 0$ in $\overline{\Omega}$. Let*

$$\psi = e^{\lambda\varphi}.$$

There exist $C_0, \lambda_0, h_0 > 0$ such that whenever $\lambda > \lambda_0$ and $0 < h < h_0$, one has

$$\lambda^2 \|u\| + \lambda \|hDu\| \leq C_0 h^{3/2} \left\| e^{\psi/h}(-\Delta + q)e^{-\psi/h}u \right\|, \qquad u \in H^2_{comp}(\Omega).$$

Proof. In the following, the positive constants c and C are always independent of λ, h and u and they may change from line to line. (We understand that c is small and C may be large.) Since $\psi = e^{\lambda\varphi}$, we have

$$\nabla\psi = \lambda e^{\lambda\varphi}\nabla\varphi, \quad \psi'' = \lambda^2 e^{\lambda\varphi}\nabla\varphi\otimes\nabla\varphi + \lambda e^{\lambda\varphi}\varphi''$$

where $\nabla\varphi\otimes\nabla\varphi$ denotes the matrix $[\partial_j\varphi\partial_k\varphi]_{j,k=1}^n$. Assuming that $\lambda \geq 1$, we also have

$$\left|\Delta^2\psi\right| \leq C\lambda^4 e^{\lambda\varphi}.$$

By Theorem 6.22, we have

$$\|P_{0,\psi}u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u,u)$$

where

$$\begin{aligned}(i[A,B]u,u) &= 4h(\psi'' hDu, hDu) + 4h((\psi''\nabla\psi\cdot\nabla\psi)u,u) - h^3((\Delta^2\psi)u,u) \\ &= 4h\lambda^4(e^{3\lambda\varphi}\left|\nabla\varphi\right|^4 u,u) + 4h\lambda^3(e^{3\lambda\varphi}(\varphi''\nabla\varphi\cdot\nabla\varphi)u,u) - h^3((\Delta^2\psi)u,u) \\ &\quad + 4h\lambda^2(e^{\lambda\varphi}\nabla\varphi\cdot hDu, \nabla\varphi\cdot hDu) + 4h\lambda(e^{\lambda\varphi}\varphi'' hDu, hDu).\end{aligned}$$

Consequently

$$(i[A,B]u,u) \geq 4h\lambda^3(e^{3\lambda\varphi}[\lambda\left|\nabla\varphi\right|^4 - \varphi''\nabla\varphi\cdot\nabla\varphi]u,u) - Ch^3\lambda^4(e^{3\lambda\varphi}u,u) \\ - Ch\lambda(e^{\lambda\varphi}hDu, hDu).$$

We used that $1 \leq e^{\lambda\varphi}$ and that $(e^{\lambda\varphi}\nabla\varphi \cdot hDu, \nabla\varphi \cdot hDu) \geq 0$. Now choose λ so large that $\lambda |\nabla\varphi|^4 - \varphi''\nabla\varphi \cdot \nabla\varphi \geq \lambda |\nabla\varphi|^4 /2$ in $\overline{\Omega}$ (and $\lambda \geq 1$), or

$$\lambda \geq \max\left\{1, 2\sup_{x\in\overline{\Omega}} \frac{\varphi''\nabla\varphi \cdot \nabla\varphi}{|\nabla\varphi|^4}\right\}.$$

This is possible since $\nabla\varphi$ is nonvanishing in $\overline{\Omega}$. If h is chosen sufficiently small (independent of λ), it follows that

$$(i[A, B]u, u) \geq ch\lambda^4(e^{3\lambda\varphi}u, u) - Ch\lambda(e^{\lambda\varphi}hDu, hDu).$$

We have proved the inequality

$$\|P_{0,\psi}u\|^2 \geq \|Au\|^2 + \|Bu\|^2 + ch\lambda^4(e^{3\lambda\varphi}u, u) - Ch\lambda(e^{\lambda\varphi}hDu, hDu).$$

The last negative term can be absorbed in the positive term $\|Au\|^2$ as follows. The argument is elementary but slightly tricky. Write

$$\begin{aligned}(e^{\lambda\varphi}hDu, hDu) &= (e^{\lambda\varphi}(hD)^2u, u) + (hD(e^{\lambda\varphi}) \cdot hDu, u)\\ &= (e^{\lambda\varphi}Au, u) + (e^{\lambda\varphi} |\nabla\psi|^2 u, u) - ih\lambda(e^{\lambda\varphi}\nabla\varphi \cdot hDu, u)\\ &= (Au, e^{\lambda\varphi}u) + \lambda^2(e^{3\lambda\varphi} |\nabla\varphi|^2 u, u) - ih\lambda(e^{\lambda\varphi}\nabla\varphi \cdot hDu, u).\end{aligned}$$

By Young's inequality we have $(Au, e^{\lambda\varphi}u) \leq \frac{1}{\delta}\|Au\|^2 + \frac{\delta}{4}\left\|e^{\lambda\varphi}u\right\|^2$ where $\delta > 0$ is a number to be determined later. We obtain

$$(e^{\lambda\varphi}hDu, hDu) \leq \frac{1}{\delta}\|Au\|^2 + \frac{\delta}{4}\left\|e^{\lambda\varphi}u\right\|^2 + C\lambda^2(e^{3\lambda\varphi}u, u) + Ch\lambda\left\|e^{\lambda\varphi/2}hDu\right\|\left\|e^{\lambda\varphi/2}u\right\|.$$

Multiplying by δ and rearranging, we have

$$\begin{aligned}\|Au\|^2 \geq \delta(e^{\lambda\varphi}hDu, hDu) &- \frac{\delta^2}{4}\left\|e^{\lambda\varphi}u\right\|^2 - C\delta\lambda^2(e^{3\lambda\varphi}u, u)\\ &- Ch\lambda\delta\left\|e^{\lambda\varphi/2}hDu\right\|\left\|e^{\lambda\varphi/2}u\right\|.\end{aligned}$$

Combining the above inequalities gives that

$$\begin{aligned}\|P_{0,\psi}u\|^2 \geq \delta(e^{\lambda\varphi}hDu, hDu) &- \frac{\delta^2}{4}(e^{3\lambda\varphi}u, u) - C\delta\lambda^2(e^{3\lambda\varphi}u, u)\\ &- Ch\lambda\delta\left\|e^{\lambda\varphi/2}hDu\right\|\left\|e^{\lambda\varphi/2}u\right\| + ch\lambda^4(e^{3\lambda\varphi}u, u) - Ch\lambda(e^{\lambda\varphi}hDu, hDu).\end{aligned}$$

We used that $1 \leq e^{\lambda\varphi}$ and $\|Bu\|^2 \geq 0$.

The idea is to choose δ so that the last expression is positive. By inspection, we arrive at the choice

$$\delta = \varepsilon h\lambda^2$$

where ε is a fixed constant independent of h and λ. If ε is chosen sufficiently small, it holds that

$$\|P_{0,\psi}u\|^2 \geq ch\lambda^4(e^{3\lambda\varphi}u, u) + (\varepsilon\lambda - C)h\lambda(e^{\lambda\varphi}hDu, hDu) - C\varepsilon h^2\lambda^3\left\|e^{\lambda\varphi/2}hDu\right\|\left\|e^{\lambda\varphi/2}u\right\|.$$

Choosing λ large enough (only depending on ε and C) gives

$$\|P_{0,\psi}u\|^2 \geq ch\lambda^4(e^{3\lambda\varphi}u,u)+ch\lambda^2(e^{\lambda\varphi}hDu,hDu)-C\varepsilon h^2\lambda^3\left\|e^{\lambda\varphi/2}hDu\right\|\left\|e^{\lambda\varphi/2}u\right\|.$$

Now

$$2\lambda^3\left\|e^{\lambda\varphi/2}hDu\right\|\left\|e^{3\lambda\varphi/2}u\right\| \leq \lambda^4(e^{\lambda\varphi}u,u)+\lambda^2(e^{3\lambda\varphi}hDu,hDu).$$

If h is sufficiently small depending on C and ε, we have

$$\|P_{0,\psi}u\|^2 \geq ch\lambda^4(e^{3\lambda\varphi}u,u)+ch\lambda^2(e^{\lambda\varphi}hDu,hDu)$$

Since $e^{\lambda\varphi} \geq 1$, this implies

$$h\lambda^4\|u\|^2+h\lambda^2\|hDu\|^2 \leq C\|P_{0,\psi}u\|^2$$

and consequently

$$\lambda^2\|u\|+\lambda\|hDu\|^2 \leq Ch^{3/2}\left\|e^{\psi/h}(-\Delta)e^{-\psi/h}u\right\|.$$

Adding the potential q gives

$$\lambda^2\|u\|+\lambda\|hDu\|^2 \leq Ch^{3/2}\left\|e^{\psi/h}(-\Delta+q)e^{-\psi/h}u\right\|+Ch^{3/2}\|u\|.$$

Choosing h so small that $Ch^{3/2} \leq 1/2$ and using that $\lambda \geq 1$ gives the required estimate for $u \in C_c^\infty(\Omega)$. The same estimate is true for $u \in H^2_{comp}(\Omega)$ since any such function can be approximated in the $H^2(\Omega)$ norm by $C_c^\infty(\Omega)$ functions. □

We move now to the proof of unique continuation across a hypersurface. To obtain some intuition into the proof, it is useful to stare at Figure ??? and keep in mind the special case where S is the hypersurface $\{x_n = 0\}$ and $\varphi(x) = x_n$. The ingenious idea of forcing the solution to vanish in a neighborhood of x_0 by using a L^2 estimate with slightly bent exponential weights is originally due to Carleman.

Proof of Theorem 6.21. Let $x_0 \in S$ and let V be a neighborhood of x_0 in Ω. The statement is local, so we may assume that V is a small ball centered at x_0 and that there is a real valued function $\varphi \in C^\infty(\overline{V})$, with $\varphi \neq 0$ in $\overline{V}$, such that

$$\begin{aligned} S\cap V &= \{x \in V\,;\, \varphi(x) = \varphi(x_0)\}, \\ S\cap V_+ &= \{x \in V\,;\, \varphi(x) > \varphi(x_0)\}, \\ S\cap V_- &= \{x \in V\,;\, \varphi(x) < \varphi(x_0)\}. \end{aligned}$$

In fact, φ is just a defining function for the hypersurface S, given near x_0 by $\varphi(x', x_n) = x_n - g(x')$ if S is locally the graph $(x', g(x'))$. By adding a constant we may assume that $\varphi \geq 0$ in $\overline{V}$. Also, write $\{\varphi = \varphi(x_0)\}$ for $S \cap V$, $\{\varphi > \varphi(x_0)\}$ for $S \cap V_+$, and $\{\varphi < \varphi(x_0)\}$ for $S \cap V_-$.

Assume that $u \in H^2(V)$ satisfies $(-\Delta + q)u = 0$ in V and $u = 0$ in $\{\varphi > \varphi(x_0)\}$. We need to show that u vanishes in some neighborhood of x_0. This will be done by applying a Carleman estimate to χu for a suitable cutoff function $\chi \in C_c^\infty(V)$. Given φ, we have seen that the function

$$\psi = e^{\lambda\varphi}$$

for λ large admits a good Carleman estimate and also satisfies

$$\{\psi = \psi(x_0)\} = \{\varphi = \varphi(x_0)\}, \ \{\psi > \psi(x_0)\} = \{\varphi > \varphi(x_0)\}, \ \{\psi < \psi(x_0)\} = \{\varphi < \varphi(x_0)\}.$$

However, to obtain the conclusion that $u = 0$ near x_0 we need to bend the weights a little bit. Define

$$\tilde{\varphi}(x) = \varphi(x) + \frac{1}{2}\left|x - x_0\right|^2.$$

Then $\nabla\tilde{\varphi}(x) = \nabla\varphi(x) + x - x_0$, and by shrinking V if necessary we have $\nabla\tilde{\varphi} \neq 0$ in $\overline{V}$. Also $\tilde{\varphi} \geq 0$, and by Theorem 6.23 the weight

$$\tilde{\psi} = e^{\lambda\tilde{\varphi}}$$

admits for h small (and for λ fixed but sufficiently large) the Carleman estimate

$$\|v\| + \|hDv\| \leq Ch^{3/2}\left\|e^{\tilde{\psi}/h}(-\Delta + q)e^{-\tilde{\psi}/h}v\right\|, \qquad v \in H^2_{comp}(V).$$

Let V' be an open ball centered at x_0 and strictly contained inside V, and choose $\chi \in C_c^\infty(V)$ so that $\chi = 1$ near $\overline{V'}$. The Carleman estimate applied to $v = e^{\tilde{\psi}/h}\chi u$ implies

$$\begin{aligned}
&\left\|e^{\tilde{\psi}/h}\chi u\right\| + \left\|e^{\tilde{\psi}/h}\chi hDu\right\| \\
&\quad = \left\|e^{\tilde{\psi}/h}\chi u\right\| + \left\|hD(e^{\tilde{\psi}/h}\chi u) - hD(e^{\tilde{\psi}/h}\chi)u\right\| \\
&\quad \leq \left\|e^{\tilde{\psi}/h}\chi u\right\| + \left\|hD(e^{\tilde{\psi}/h}\chi u)\right\| + C\left\|e^{\tilde{\psi}/h}\chi u\right\| + h\left\|e^{\tilde{\psi}/h}(\nabla\chi)u\right\| \\
&\quad \leq Ch^{3/2}\left\|e^{\tilde{\psi}/h}(-\Delta + q)(\chi u)\right\| + h\left\|e^{\tilde{\psi}/h}(\nabla\chi)u\right\| \\
&\quad \leq Ch^{3/2}\left\|e^{\tilde{\psi}/h}\chi(-\Delta + q)u\right\| + Ch^{3/2}\left\|e^{\tilde{\psi}/h}[\Delta, \chi]u\right\| + h\left\|e^{\tilde{\psi}/h}(\nabla\chi)u\right\|.
\end{aligned}$$

Since $(-\Delta + q)u = 0$ in V, the first term on the right vanishes. Also, the properties of χ and the fact that $u = 0$ in $\{\psi > \psi(x_0)\}$ show that the functions $[\Delta, \chi]u$ and $(\nabla\chi)u$ are supported in the set

$$W = (V \setminus V') \cap \{\psi \leq \psi(x_0)\}.$$

Using that $\psi < \tilde{\psi}$ in $V \setminus \{x_0\}$, we can find $\varepsilon > 0$ so that

$$W \subset \{x \in V\,;\, \tilde{\psi}(x) < \tilde{\psi}(x_0) - \varepsilon\}.$$

Choose some small ball B centered at x_0 and contained in $V' \setminus \{\tilde{\psi} < \tilde{\psi}(x_0) - \varepsilon\}$. The previous inequality implies

$$\left\| e^{\tilde{\psi}/h} u \right\|_{L^2(B)} \leq Ch \left\| e^{\tilde{\psi}/h} u \right\|_{L^2(W)}.$$

But now $\tilde{\psi} \geq \tilde{\psi}(x_0) - \varepsilon$ in B and $\tilde{\psi} \leq \tilde{\psi}(x_0) - \varepsilon$ in W. This shows that

$$e^{(\tilde{\psi}(x_0)-\varepsilon)/h} \|u\|_{L^2(B)} \leq Che^{(\tilde{\psi}(x_0)-\varepsilon)/h} \|u\|_{L^2(W)}.$$

Canceling the exponentials and letting $h \to 0$ we see that $u = 0$ in B, and so u indeed vanishes in a neighborhood of x_0. $\square$

We can now prove the other unique continuation statements. Weak unique continuation follows easily from Theorem 6.21 by using a connectedness argument. We first prove a special case.

Theorem 6.24. *(Weak unique continuation for concentric balls) Let $B = B(x_0, R)$ be an open ball in $\mathbb{R}^n$, and let $q \in L^\infty(B)$. If $u \in H^2(B)$ satisfies*

$$(-\Delta + q)u = 0 \qquad \text{in } B$$

and

$$u = 0 \text{ in some ball } B(x_0, r_0) \text{ with } r_0 < R,$$

then $u = 0$ in B.

Proof. Let

$$I = \{r \in (0, R)\,;\, u = 0 \text{ in } B(x_0, r)\}.$$

By assumption, I is nonempty. It is closed in $(0, R)$ since whenever u vanishes on $B(x_0, r_j)$ and $r_j \to r$, then u vanishes on $B(x_0, r)$. We will show that I is open, which implies $I = (0, R)$ by connectedness and therefore proves the result.

Suppose $r \in I$, so $u = 0$ in $B(x_0, r)$. Let S be the hypersurface $\partial B(x_0, r)$. We know that $u = 0$ on one side of this hypersurface. Now Theorem 6.21 implies that for any $z \in S$, there is some open ball $B(z, r_z)$ contained in B so that u vanishes in $B(z, r_z)$. Define the open set

$$U = B(x_0, r) \cup \left(\bigcup_{z \in S} B(z, r_z) \right).$$

The distance between the compact set S and the closed set $\overline{B(x_0, R)} \setminus U$ is positive. In particular, there is some $\varepsilon > 0$ such that $u = 0$ in $B(x_0, r + \varepsilon)$. This shows that I is open. $\square$

Proof of Theorem 6.20. Suppose $u \in H^2(\Omega)$ satisfies $(-\Delta + q)u = 0$ in Ω and $u = 0$ in some open ball contained in Ω. Set

$$A = \{x \in \Omega\,;\, u = 0 \text{ in some neighborhood of } x \text{ in } \Omega\}.$$

By assumption, A is a nonempty open subset of Ω. We will show that it is also closed. This implies by connectedness that $A = \Omega$, so indeed u vanishes in Ω as required.

Suppose on the contrary that A is not closed as a subset of Ω. Then there is some point x_0 on the boundary of A relative to Ω, for which $x_0 \notin A$. Choose $r_0 > 0$ so that $B(x_0, r_0) \subset \Omega$ and choose some point $y \in B(x_0, r_0/4)$ with $y \in A$. Since $y \in A$, we know that u vanishes on some ball $B(y, s_0)$ with $s_0 < r_0/2$. By Theorem 6.24, we see that u vanishes in the ball $B(y, r_0/2) \subset \Omega$. But $x_0 \in B(y, r_0/2)$, so u vanishes near x_0. This contradicts the fact that $x_0 \notin A$. $\square$

In turn, unique continuation from Cauchy data on a subset follows from weak unique continuation upon extending the domain slightly near the set where the Cauchy data vanishes.

Proof of Theorem 6.19. Assume that $u \in H^2(\Omega)$, $(-\Delta + q)u = 0$ in Ω, and $u|_\Gamma = \partial_\nu u|_\Gamma = 0$. Choose some $x_0 \in \Gamma$, and choose coordinates $x = (x', x_n)$ so that $x_0 = 0$ and for some $r > 0$,

$$\Omega \cap B(0, r) = \{x \in B(0, r)\,;\, x_n > g(x')\}$$

where $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is a C^∞ function. We extend the domain near x_0 by choosing $\psi \in C_c^\infty(\mathbb{R}^{n-1})$ with $\psi = 0$ for $|x'| \geq r/2$ and $\psi = 1$ for $|x'| \leq r/4$, and by letting

$$\tilde{\Omega} = \Omega \cup \{x \in B(0, r)\,;\, x_n > g(x') - \varepsilon\psi(x')\}.$$

Here $\varepsilon > 0$ is chosen so small that $\{(x', x_n)\,;\, |x'| \leq r/2,\ x_n = g(x') - \varepsilon\psi(x')\}$ is contained in $B(0, r)$. Then $\tilde{\Omega}$ is a bounded connected open set with C^∞ boundary.

Define the function

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \tilde{\Omega} \setminus \Omega. \end{cases}.$$

Then $\tilde{u}|_\Omega \in H^2(\Omega)$ and $\tilde{u}|_{\tilde{\Omega}\setminus\Omega} \in H^2(\tilde{\Omega} \setminus \Omega)$. Since $u|_\Gamma = \partial_\nu u|_\Gamma = 0$, we also have that the traces of $\tilde{u}$ and $\partial_\nu \tilde{u}$ on the interface $\partial\Omega \setminus \partial\tilde{\Omega}$ vanish when taken both from inside and outside Ω.

It follows from Theorem ??? that $\tilde{u} \in H^2(\tilde{\Omega})$. Defining $\tilde{q}(x) = q(x)$ for $x \in \Omega$ and $\tilde{q}(x) = 0$ for $\tilde{\Omega} \setminus \Omega$, one also gets that $(-\Delta + \tilde{q})\tilde{u} = 0$ almost everywhere in $\tilde{\Omega}$. But $\tilde{u} = 0$ in some open ball contained in $\tilde{\Omega} \setminus \overline{\Omega}$, so we know from Theorem 6.20 that $\tilde{u} = 0$ in the connected domain $\tilde{\Omega}$. Thus also $u = 0$. $\square$

Chapter 7

Scattering Theory

This chapter has two main parts. In the first, up to and including §7.3, we give a brief description of scattering theory for a Schrödinger operator. In the second, §7.4, we prove that the scattering amplitude of a Schrödinger operator uniquely determines its potential.

First, fix $\lambda > 0$ and note that, for each $\omega \in S^{n-1}$, the function

$$\psi_0(x,\omega) = e^{i\sqrt{\lambda}\, x\cdot\omega}$$

obeys

$$H_0\psi_0 = (-\Delta - \lambda)\psi_0(\omega) = 0$$

The functions $\psi_0(\,\cdot\,,\omega)$ are the "eigenfunctions" of $-\Delta$ with eigenvalue λ, and are parametrized by the unit sphere S^{n-1} in $\mathbb{R}^n$. The corresponding solution of the time–dependent Schrödinger equation $i\frac{\partial}{\partial t}\psi = -\Delta\psi$, namely $\psi(x,t) = e^{-i\lambda t}e^{i\sqrt{\lambda}\, x\cdot\omega}$, has phase velocity $\sqrt{\lambda}\,\omega$. That is, if you move with $x(t) = x(0) + \sqrt{\lambda}\,\omega\, t$, you always see the same value of $\psi(x,t)$. We think of $\psi_0(x,\omega)$ as an incoming wave in direction ω.

If we consider the perturbed operator

$$H_q\psi_q = (-\Delta - \lambda + q)\psi_q = 0$$

— we shall assume that q is bounded and compactly supported in $\mathbb{R}^n$— then, as we will demonstrate in §7.2, the eigenfunctions $\psi_q(x,\omega)$ are also parametrized by the sphere. They are unique if we insist that they have the form

$$\psi_q(x,\omega) = e^{i\sqrt{\lambda}\, x\cdot\omega} + \varphi_q(x,\omega)$$

with the conditions that $\varphi_q(\,\cdot\,,\omega) \in L^2_\delta$ for some $\delta < -\frac{1}{2}$ and that $\varphi_q(\,\cdot\,,\omega)$ is outgoing. In §**??**, we defined, for any $-\infty < \delta < \infty$, L^2_δ to be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{L^2_\delta} = \left(\int_{\mathbb{R}^n} \left(1+|x|^2\right)^\delta |u|^2 \; d^n x \right)^{1/2}$$

There are several equivalent definitions of outgoing. The one we shall use is that φ_q is in the range of a certain linear operator, G_0, which we will introduce in the next section. In Corollary 7.19, we shall show that, asymptotically for large $|x|$,

$$\psi_q(\lambda, x, \omega) = e^{i\sqrt{\lambda}\, x\cdot\omega} + \frac{a_q(\sqrt{\lambda}, \theta, \omega)}{|x|^{\frac{n-1}{2}}} e^{i\sqrt{\lambda}\,|x|} + O\big(|x|^{-\frac{n-1}{2}-1}\big) \tag{7.1}$$

where $\theta = \frac{x}{|x|}$. We think of the first term as an incoming wave in direction ω and of the second term as an outgoing radially expanding wave that arises when the incoming wave scatters off of the potential q. The amplitude, $a_q(\sqrt{\lambda}, \theta, \omega)$, of this outgoing wave depends on the direction of view, θ, and is called the *scattering amplitude*. We saw a similar setup in §**??**.3.

7.1. Outgoing Solutions to $(-\Delta - \lambda)u = f$

Our first step (Theorem 7.1 and Proposition 7.2) is to solve the constant coefficient equation

$$(-\Delta - \lambda)u = f \in L^2_\delta \qquad \delta > \tfrac{1}{2} \tag{7.2}$$

by employing the Fourier transform and simply writing

$$\hat{u}(k) = \frac{\hat{f}(k)}{|k|^2 - \lambda}$$

If $\operatorname{Im}\lambda \neq 0$, the denominator does not vanish and $\hat{u}$ is an unambiguously defined element of L^2 with

$$\|u\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{n/2}} \|\hat{u}\|_{L^2(\mathbb{R}^n)} \le \frac{1}{(2\pi)^{n/2}|\operatorname{Im}\lambda|} \|\hat{f}\|_{L^2(\mathbb{R}^n)} = \frac{1}{|\operatorname{Im}\lambda|}\|f\|_{L^2(\mathbb{R}^n)}$$

For real nonzero λ we shall define the unique outgoing solution to (7.2) by

$$\hat{u}(k) = \lim_{\epsilon \downarrow 0} \frac{\hat{f}(k)}{|k|^2 - \lambda - i\epsilon} \tag{7.3}$$

Letting ϵ increase to zero in (7.3) would define the unique incoming solution to (7.2). We will not need to consider incoming solutions. We begin with

Theorem 7.1. *Let $\lambda \in \mathbb{C}\backslash\mathbb{R}^+$, $|\lambda| > \epsilon > 0$, $\delta > \frac{1}{2}$ and $n \ge 2$. There exists a unique weak solution $u \in L^2_{-\delta}(\mathbb{R}^n)$ to*

$$(-\Delta - \lambda)u = f \in L^2_\delta(\mathbb{R}^n) \tag{7.4}$$

Moreover, there is a constant $C = C(n, \delta, \epsilon)$ such that

$$\|u\|_{L^2_{-\delta}} \leq \frac{C}{|\lambda|^{1/2}} \|f\|_{L^2_\delta} \tag{7.5}$$

We shall denote the solution operator to (7.4) by

$$u = G_0(\lambda) f$$

In order to extend $G_0(\lambda)$ to nonzero real values of λ we shall prove

Proposition 7.2. *For every $f \in L^2_\delta(\mathbb{R}^n)$ and $g \in L^2_\delta(\mathbb{R}^n)$, the map*

$$\lambda \mapsto \langle G_0(\lambda) f, g \rangle$$

has unique limiting values as λ approaches the real axis from above. Use $\mathcal{O}_{\delta,-\delta}$ to denote the Banach space of bounded linear operators from L^2_δ to $L^2_{-\delta}$. Then $G_0(\lambda)$ extends to a weakly continuous $\mathcal{O}_{\delta,-\delta}$–valued function of λ on $\operatorname{Im} \lambda \geq 0$.

Definition 7.3. For $\lambda \in \mathbb{R}^+ \backslash \{0\}$ and $f \in L^2_\delta$, define

$$G_0(\lambda) f = \lim_{\epsilon \downarrow 0} G_0(\lambda + i\epsilon) f$$

to be the outgoing solution to (7.2). A function $u \in L^2_\delta(\mathbb{R}^n)$ is said to be outgoing if it belongs to the range of $G_0(\lambda)$.

Proof of Theorem 7.1. Let u and f obey (7.4). Then

$$(|k|^2 - \lambda)\hat{u}(k) = \hat{f}(k)$$

and as the denominator does not vanish

$$\hat{u}(k) = \frac{\hat{f}(k)}{|k|^2 - \lambda}$$

and is therefore unique.

We now define a smooth partition of unity $X_0^2(k, \lambda)$, $X_1^2(k, \lambda)$, $\cdots$, $X_n^2(k, \lambda)$ with X_0 supported away from $|k| = \sqrt{|\lambda|}$, the other X_j's supported near $|k| = \sqrt{|\lambda|}$ and, for $1 \leq j \leq n$, $\frac{|k_j|^2}{|k|^2} \geq \frac{1}{2n}$ on the support of X_j. To do so, let $\underset{0}{\approx\!\!\!X}(k)$, $\ldots$, $\underset{n}{\approx\!\!\!X}(k)$ be non-negative smooth functions which take values in $[0, 1]$ and satisfy

$$\underset{0}{\overset{\approx}{X}}(k) = \begin{cases} 0 & \frac{1}{2} \leq |k| \leq \frac{3}{2} \\ 1 & |k| \leq \frac{1}{4} \text{ or } |k| \geq 2 \end{cases}$$

and, for $j = 1, \ldots, n$,

$$\underset{j}{\overset{\approx}{X}}(k) = \begin{cases} 1 & |k_j|^2 \geq \frac{1}{n}|k|^2 \\ 0 & |k_j|^2 \leq \frac{1}{2n}|k|^2 \end{cases}$$

Let

$$\widetilde{X}_0(k,\lambda) = \underset{0}{\widetilde{\widetilde{X}}}\left(k/\sqrt{|\lambda|}\,\right)$$

$$\widetilde{X}_j(k,\lambda) = \left(1 - \widetilde{X}_0(k,\lambda)\right)\underset{j}{\widetilde{\widetilde{X}}}(k) \qquad \text{for } j = 1,\ \ldots,\ n$$

and, finally, define

$$X_k(k,\lambda) = \frac{\widetilde{X}_k(k,\lambda)}{\sqrt{\sum_{j=0}^n (\widetilde{X}_j(k,\lambda))^2}} \qquad \text{for } k = 0,\ \ldots,\ n$$

so that

$$\sum_{j=0}^n X_j^2(k,\lambda) = 1$$

For any $0 \neq k \in \mathbb{R}^n$, we must have $|k_j|^2 \geq \frac{1}{n}|k|^2$ for at least one $1 \leq j \leq n$ so that $\sum_{j=1}^n \underset{j}{\widetilde{\widetilde{X}}}(k)^2 \geq 1$ for all $k \neq 0$ and

$$\sum_{j=0}^n \widetilde{X}_j(k,\lambda)^2 \geq \widetilde{X}_0(k,\lambda)^2 + \left(1 - \widetilde{X}_0(k,\lambda)\right)^2 \geq \frac{1}{4}$$

Combining this with

$$\sup_{\substack{|\lambda|\geq\epsilon \\ k\in\mathbb{R}^n \\ 0\leq j\leq n}} |D_\lambda^m D_k^\alpha \widetilde{X}_j(k,\lambda)| < \infty$$

yields that, for $|\lambda| \geq \epsilon$,

$$|D_\lambda^m D_k^\alpha X_j(k,\lambda)| \leq C(\alpha, m, \epsilon) \tag{7.6}$$

We now write

$$(G_0(\lambda)f)\hat{\ } = q_0\hat{f} + \sum_{j=1}^n q_j m_j \hat{f} \tag{7.7}$$

with

$$q_0 = \frac{X_0}{|k|^2 - \lambda} \tag{7.8}$$

$$q_j = \frac{X_j}{|k| + \sqrt{\lambda}} \qquad \text{for } j = 1, \cdots, n \tag{7.9}$$

$$m_j = \frac{X_j}{|k| - \sqrt{\lambda}} \qquad \text{for } j = 1, \cdots, n \tag{7.10}$$

where we choose $\sqrt{\lambda}$ to have positive imaginary part. In the remainder of the proof, we shall assume that $\operatorname{Re}\sqrt{\lambda}$ is also nonnegative (as is the case for $\operatorname{Im}\lambda \geq 0$), so that the magnitude of the denominator in (7.9) is bounded below by $\sqrt{|\lambda|} \geq \sqrt{\epsilon}$ for all $k \in \mathbb{R}^n$ and $|\lambda| \geq \epsilon$. On the other hand, $|k|$ may get arbitrarily close to $\sqrt{\lambda}$ and the denominator in (7.10) is not bounded away from zero. If $\operatorname{Re}\sqrt{\lambda}$ is negative (as is the case for $\operatorname{Im}\lambda \le 0$), we interchange the definitions of the q_j and the m_j. It follows from (7.6), (7.8), and (7.9) that, for $|\lambda| > \epsilon$,

$$|D_\lambda^m D_k^\alpha q_j(k,\lambda)| \le \frac{C(m,\alpha,\epsilon)}{\sqrt{|\lambda|}}$$

with a new constant $C(\alpha, m, \epsilon)$. By Lemmas **??** and (when δ is not an integer) **??**,

$$\|q_i \hat{f}\|_{H^\delta(\mathbb{R}^n)} \le \frac{C(\delta,\epsilon)}{\sqrt{|\lambda|}} \|\hat{f}\|_{H^\delta(\mathbb{R}^n)}$$

for any δ and $0 \le i \le n$. Therefore the operators

$$Q_i : f \mapsto (q_i \hat{f})^\vee$$

satisfy the estimate

$$\|Q_i f\|_{L^2_\delta} \le \frac{C(\delta,\epsilon)}{\sqrt{|\lambda|}} \|f\|_{L^2_\delta}$$

Moreover, the norms of the derivatives of the operators Q_i with respect to λ are also bounded, so that the Q_i are continuous functions of λ in the uniform operator topology. The estimate (7.5), as well as Proposition 7.2, will now follow from the following estimate of the operators

$$M_j : f \mapsto (m_j \hat{f}_j)^\vee$$

combined with Problem 7.7. $\square$

Lemma 7.4. *Let $\delta > \frac{1}{2}$, $|\lambda| > \epsilon > 0$. For all $1 \le i \le n$,*

$$\|M_i f\|_{L^2_{-\delta}} \le C(\epsilon,\delta)\|f\|_{L^2_\delta}$$

Moreover, for $f, g \in L^2_\delta$, the map

$$\lambda \mapsto \langle M_i f, g\rangle$$

has unique limiting values as λ approaches the real axis from above.

Proof. We will prove the lemma for M_1. Let $\eta = \psi(k)$ define the change of coordinates

$$\begin{aligned} \eta_1 &= |k| \\ \eta_j &= k_j \qquad\qquad \text{for } j = 2,\dots,n \end{aligned}$$

By construction the Jacobian, $D\psi = \frac{k_1}{|k|}$, is bounded and bounded away from zero on an open set Ω containing the support of X_1. Let $\tilde{X}_1$ be a C^∞ function, taking values in $[0,1]$, that is supported in Ω and is identically one on the support of X_1. By Lemma **??** and Lemma **??** (when δ is not an integer) $u \mapsto \Phi u = (X_1 u)\circ\psi^{-1}$ and $u \mapsto \tilde{\Phi} u = \left(\frac{\tilde{X}_1 u}{D\psi}\right)\circ\psi^{-1}$ are bounded maps on $H^\delta(\mathbb{R}^n)$ with operator norms depending only on δ and ϵ. Furthermore these operators are norm continuous in λ. (They depend on λ only through X_1 and $\tilde{X}_1$.) Since

$$\begin{aligned}
\langle M_1 f, g\rangle &= \int \frac{X_1(k)}{|k|-\sqrt{\lambda}} \hat{f}(k)\, \overline{\hat{g}(k)}\ \frac{d^n k}{(2\pi)^n} \\
&= \int \frac{1}{|k|-\sqrt{\lambda}} \tilde{X}_1(k)\hat{f}(k)\, \overline{X_1(k)\hat{g}(k)}\ \frac{d^n k}{(2\pi)^n} \\
&= \int \frac{1}{\eta_1-\sqrt{\lambda}} \tilde{X}_1(\psi^{-1}(\eta))\hat{f}(\psi^{-1}(\eta))\, \overline{X_1(\psi^{-1}(\eta))\hat{g}(\psi^{-1}(\eta))}\ \frac{1}{(D\psi)(\psi^{-1}(\eta))}\ \frac{d^n \eta}{(2\pi)^n} \\
&= \int \frac{1}{\eta_1-\sqrt{\lambda}} (\tilde{\Phi}\hat{f})(\eta)\, \overline{(\Phi\hat{g})(\eta)}\ \frac{d^n \eta}{(2\pi)^n} \\
&= \left\langle N_1(\tilde{\Phi}\hat{f})\check{}, (\Phi\hat{g})\check{}\right\rangle \qquad \text{where} \qquad N_1 f = \left(\frac{\hat{f}}{\eta_1-\sqrt{\lambda}}\right)^{\vee}
\end{aligned}$$

it suffices to prove that

$$|\langle N_1 f, g\rangle| \le \widetilde{C}(\delta,\epsilon)\|f\|_{L^2_\delta}\|g\|_{L^2_\delta} \tag{7.11}$$

and that $\langle N_1 f, g\rangle$ converges as λ approaches the real axis from above. Now, by Problem 7.5, N_1 can be also expressed as

(7.12)

$$N_1 f(x_1, x') = i\int_{-\infty}^{x_1} e^{i\sqrt{\lambda}(x_1-y_1)} f(y_1, x')\, dy_1 \qquad \text{with} \qquad x' \in \mathbb{R}^{n-1}$$

Because $\operatorname{Im}\sqrt{\lambda} \ge 0$ and $x_1 - y_1 \ge 0$ on the domain of integration, the exponential in (7.12) is uniformly bounded. Fixing x' for the moment and applying the Cauchy-Schwartz inequality in $\mathbb{R}^1$, we see that, since $\delta > \frac{1}{2}$,

$$\begin{aligned}
|N_1 f(x_1, x')|^2 &\le \int_{-\infty}^{x_1} \left(1+y_1^2\right)^{-\delta} dy_1 \int_{-\infty}^{\infty} |f(y_1,x')|^2 \left(1+y_1^2\right)^{\delta} dy_1 \\
&\le C(\delta)\|f(\cdot,x')\|^2_{L^2_\delta(\mathbb{R}^1)}
\end{aligned}$$

so that

$$\|N_1 f(\cdot,x')\|^2_{L^2_{-\delta}(\mathbb{R}^1)} \le C(\delta)\int_{-\infty}^{\infty} \left(1+x_1^2\right)^{-\delta} dx_1\ \|f(\cdot,x')\|^2_{L^2_\delta(\mathbb{R}^1)}$$

The integral $\tilde{C}(\delta) = C(\delta)\int_{-\infty}^{\infty}\left(1+x_1^2\right)^{-\delta} dx_1$ is finite as long as $\delta > \frac{1}{2}$. Since, for $\delta > 0$,

$$\begin{aligned}
\left(1+x_1^2+|x'|^2\right)^{-\delta} &\le \left(1+x_1^2\right)^{-\delta} \\
\left(1+x_1^2+|x'|^2\right)^{\delta} &\ge \left(1+x_1^2\right)^{\delta}
\end{aligned}$$

we have that

$$\begin{aligned}
\|N_1 f\|^2_{L^2_{-\delta}(\mathbb{R}^n)} &= \int_{-\infty}^{\infty} dx_1 \int_{\mathbb{R}^{n-1}} dx'\ \left(1 + x_1^2 + |x'|^2\right)^{-\delta} |N_1 f(x_1, x')|^2 \\
&\le \int_{\mathbb{R}^{n-1}} dx' \int_{-\infty}^{\infty} dx_1\ \left(1 + x_1^2\right)^{-\delta} |N_1 f(x_1, x')|^2 \\
&= \int_{\mathbb{R}^{n-1}} dx'\ \|N_1 f(\cdot, x')\|^2_{L^2_{-\delta}(\mathbb{R}^1)} \\
&\le \tilde{C}(\delta) \int_{\mathbb{R}^{n-1}} dx' \int_{-\infty}^{\infty} dx_1\ \left(1 + x_1^2\right)^{\delta} |f(x_1, x')|^2 \\
&\le \tilde{C}(\delta) \int_{\mathbb{R}^{n-1}} dx' \int_{-\infty}^{\infty} dx_1\ \left(1 + x_1^2 + |x'|^2\right)^{\delta} |f(x_1, x')|^2
\end{aligned}$$

which implies (7.11). Finally, the existence of limiting values of

$$\langle N_1 f, g\rangle = i \int_{\mathbb{R}^n} \int_{-\infty}^{u_1} e^{i\sqrt{\lambda}(u_1 - y_1)} f(y_1, u')\, \overline{g(u)}\ dy_1 d^n u$$

as $\operatorname{Im}\lambda$ decreases to zero follows from the dominated convergence theorem and the existence of pointwise limits for the function $e^{i\sqrt{\lambda}(x_1 - y_1)}$.

This, together with Problem 7.7, completes the proof of the lemma, Theorem 7.1 and Proposition 7.2. □

Exercise 7.5. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$ and choose $\operatorname{Im}\sqrt{\lambda} > 0$. Define the map

$$\begin{aligned}
N_1 : L^2(\mathbb{R}^n) &\to L^2(\mathbb{R}^n) \\
f \mapsto N_1 f &= \left(\frac{\hat{f}}{\eta_1 - \sqrt{\lambda}}\right)^{\vee}
\end{aligned}$$

Prove that

$$N_1 f(x_1, x') = i \int_{-\infty}^{x_1} e^{i\sqrt{\lambda}(x_1 - y_1)} f(y_1, x')\ dy_1 \qquad \text{with} \qquad x' \in \mathbb{R}^{n-1}$$

Exercise 7.6. Let $\lambda > 0$, $\delta > \frac{1}{2}$ and $f \in L^2_\delta(\mathbb{R}^n)$.

(a) Prove that $G_0(\lambda)f$ is a weak solution of the differential equation $(-\Delta - \lambda)u = f$. In other words, prove that $u = G_0(\lambda)f$ obeys $\langle(-\Delta - \lambda)g, u\rangle = \langle g, f\rangle$ for all $g \in \mathcal{S}(\mathbb{R}^n)$.

(b) Prove that if u is a weak solution of the differential equation $(-\Delta - \lambda)u = f$ and if u is in the range of $G_0(\lambda)$, then $u = G_0(\lambda)f$.

Exercise 7.7. Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be Banach spaces with $\mathcal{B}_3$ the dual space to $\mathcal{B}_2$. Let $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{B}_1$ and $\{g_i\}_{i\in\mathbb{N}} \subset \mathcal{B}_3$ be (strongly) convergent sequences and let $A_i : \mathcal{B}_1 \to \mathcal{B}_2$ be a sequence of operators with uniformly bounded operator norms that converges weakly in the sense that $\lim_{i\to\infty}\langle A_i f, g\rangle$ exists for each $f\in\mathcal{B}_1$ and $g\in\mathcal{B}_3$. Here $\langle A_i f, g\rangle$ means the value of g, viewed as an element of $\mathcal{B}_2^*$, at $A_i f\in\mathcal{B}_2$. Prove that $\lim_{i\to\infty}\langle A_i f_i, g_i\rangle$ converges.

In §7.2 we shall make use of the following lemma. Before stating the lemma we define

$$M_\lambda = \left\{\ k\in\mathbb{R}^n \ \middle|\ k\cdot k-\lambda=0\ \right\}$$

Lemma 7.8. *Let $\lambda>0$ and $f\in L^2_\delta(\mathbb{R}^n)$ with $\delta>\frac{1}{2}$. Then*

$$\operatorname{Im}\int_{\mathbb{R}^n}\big(G_0(\lambda)f\big)(x)\,\bar f(x)\ d^n x = \frac{1}{2^{n+1}\pi^{n-1}\sqrt{\lambda}}\int_{M_\lambda}|\hat f(\omega)|^2 dS_{\sqrt{\lambda}}(\omega) \tag{7.13}$$

where $dS_{\sqrt{\lambda}}(\omega)$ is the surface measure on the sphere M_λ of radius $\sqrt{\lambda}$.

Proof. We apply the Plancherel theorem (**??** b) to

$$\langle G_0(\lambda+i\epsilon)f,f\rangle_{L^2(\mathbb{R}^n,d^n x)} = \Big\langle \frac{\hat f}{k\cdot k-\lambda-i\epsilon},\hat f\Big\rangle_{L^2(\mathbb{R}^n,\frac{d^n k}{(2\pi)^n})}$$

so that

$$\begin{aligned}\operatorname{Im}\langle G_0(\lambda+i\epsilon)f,f\rangle &= \int_{\mathbb{R}^n}\frac{|\hat f(k)|^2\epsilon}{(k\cdot k-\lambda)^2+\epsilon^2}\frac{d^n k}{(2\pi)^n} = \frac{1}{(2\pi)^n}\int_0^\infty\int_{M_{\rho^2}}\frac{|\hat f(\omega)|^2\epsilon}{(\rho^2-\lambda)^2+\epsilon^2}\,dS_\rho(\omega)\,d\rho\\ &= \int_0^\infty\frac{\tilde F(\rho)\epsilon}{(\rho^2-\lambda)^2+\epsilon^2}\,d\rho \qquad \text{where } \tilde F(\rho)=\frac{1}{(2\pi)^n}\int_{M_{\rho^2}}|\hat f(\omega)|^2\ dS_\rho(\omega)\end{aligned}$$

Now make the change of variables $\rho^2-\lambda=\epsilon t$. This gives

$$\operatorname{Im}\langle G_0(\lambda+i\epsilon)f,f\rangle = \int_{-\lambda/\epsilon}^\infty\frac{F_\epsilon(t)}{t^2+1}\ dt \text{ where } F_\epsilon(t)=\frac{1}{2\rho}\tilde F(\rho)\big|_{\rho=\sqrt{\lambda+\epsilon t}}$$

Since $f\in L^2_\delta(\mathbb{R}^n)$, we have that $\hat f\in H^\delta(\mathbb{R}^n)$. Applying Lemma **??** and Problem **??**, in ρ,ω coordinates, yields that $\tilde F(\rho)$ is uniformly bounded and continuous. Hence

$$\lim_{\varepsilon\downarrow 0}\int_{-\lambda/\epsilon}^{-\lambda/(2\epsilon)}\frac{F_\epsilon(t)}{t^2+1}\ dt = \lim_{\varepsilon\downarrow 0}\int_0^{\sqrt{\lambda/2}}\frac{\tilde F(\rho)\epsilon}{(\rho^2-\lambda)^2+\epsilon^2}\ d\rho = 0$$

and, by the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon\downarrow 0}\int_{-\lambda/(2\epsilon)}^\infty\frac{F_\epsilon(t)}{t^2+1}\ dt = \int_{-\infty}^\infty\frac{F_0(t)}{t^2+1}\ dt$$

Hence

$$\operatorname{Im}\langle G_0(\lambda)f,f\rangle = \frac{\tilde F(\sqrt{\lambda})}{2\sqrt{\lambda}}\int_{-\infty}^\infty\frac{1}{t^2+1}\ dt = \frac{\pi}{2\sqrt{\lambda}(2\pi)^n}\int_{M_\lambda}|\hat f(\omega)|^2\ dS_{\sqrt{\lambda}}(\omega)$$

□

The following theorem, originally due to Rellich, will play an important role both in proving the existence of eigenfunctions of H_q and in our treatment of the inverse scattering problem in §7.4.

Theorem 7.9 (Rellich Uniqueness Theorem)**.** *Let $\lambda > 0$, $R > 0$ and $f \in L^2$, with $\operatorname{supp} f \subset \bar{B}_R$, the closed ball of radius R. The following are equivalent.*

(i)$G_0(\lambda)f \in L^2_\mu$ *for some* $\mu > -\frac{1}{2}$

(ii)$\int_{\mathbb{R}^n} f(x)\psi_0(x,\omega)\ d^n x = 0$ *for all* $\omega \in S^{n-1}$

(iii)$\operatorname{supp}(G_0(\lambda)f) \subset \bar{B}_R$

Proof. We shall show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Suppose first that (ii) is false. That is, there exists $k_0 \in M_\lambda$ such that $\hat{f}(k_0) \neq 0$. As $G_0(\lambda)$ commutes with rotations, we may assume that $k_0 = (\sqrt{\lambda}, 0, \cdots, 0)$. Let $\mathcal{U}$ be an open neighbourhood of k_0 in $\mathbb{R}^n$ that is sufficiently small that it does not intersect the support of any $X_j(k)$ with $0 \le j \le n$, $j \neq 1$. The X_j's were defined in the proof of Theorem 7.1. Let $\chi_{\mathcal{U}}$ be a C^∞ function that takes values in $[0,1]$, is one on a neighbourhood of k_0 and is supported in $\mathcal{U}$. In the notation of (7.7)

$$\chi_{\mathcal{U}}(k)(G_0(\lambda)f)\hat{}(k) = \chi_{\mathcal{U}}(k)q_0(k)\hat{f}(k) + \sum_{j=1}^{n}\chi_{\mathcal{U}}(k)q_j(k)m_j(k)\hat{f}(k) = \chi_{\mathcal{U}}(k)q_1(k)\hat{f}(k)m_1(k)$$

Now, if (i) is true, $(G_0(\lambda)f)\hat{}(k) \in H^\mu(\mathbb{R}^n)$ so that,

$$\chi_{\mathcal{U}}(k)q_1(k)\hat{f}(k)m_1(k) \in H^\mu(\mathbb{R}^n)$$

Since we are assuming that f has compact support, $\hat{f}$ and hence $q_1\hat{f}$ is C^∞ in $\mathcal{U}$. Furthermore we are free to choose $\mathcal{U}$ small enough that $q_1\hat{f}$ is bounded away from zero on $\mathcal{U}$. Hence $(\chi_{\mathcal{U}}m_1)\circ\psi^{-1}$, which is the product of $\frac{1}{q_1\hat{f}}$ and $\chi_{\mathcal{U}}q_1\hat{f}m_1$ composed with ψ^{-1}, is still in $H^\mu(\mathbb{R}^n)$. Denote by ζ the inverse Fourier transform of $\chi_{\mathcal{U}}\circ\psi^{-1}$ and observe that the inverse Fourier transform of $m_1\circ\psi^{-1} = \frac{1}{\eta_1 - \sqrt{\lambda}}$ is $-i\delta(x')\, e^{i\sqrt{\lambda}x_1}H(x_1)$ where H is the one dimensional Heaviside function and $\delta(x')$ is the $n-1$ dimensional delta function. We have now shown that, under the assumptions that (ii) is false and (i) is true, the inverse Fourier transform of $(\chi_{\mathcal{U}}m_1)\circ\psi^{-1}$, namely

$$-i\int e^{i\sqrt{\lambda}(x_1-y_1)}H(x_1-y_1)\zeta(y_1,x')\ dy_1 \tag{7.14}$$

is in L^2_μ. But $(1+x_1^2)^s$ is not in $L^1(\mathbb{R})$ for any $s \ge -\frac{1}{2}$. So, by Problem 7.10, the function of (7.14) is not in L^2_s for any $s \ge -\frac{1}{2}$, contradicting our initial hypothesis and proving the implication (i) $\Rightarrow$ (ii). *(ii) $\Rightarrow$ (iii)* For any function f, which is supported is supported inside the ball of radius R, and any $k \in \mathbb{C}^n$,

$$\left|\hat{f}(k)\right| = \left| \int_{\bar{B}_R} e^{-ik\cdot x} f(x)\ d^n x \right| \le e^{R|\operatorname{Im} k|} \int_{\bar{B}_R} |f(x)|\ d^n x$$

If f is C^∞, then the usual integration by parts game gives that

$$\left|\hat{f}(k)\right| \le \frac{C_N}{1+|k|^{2N}} e^{R|\operatorname{Im} k|} \tag{7.15}$$

for all $N \in \mathbb{N}$. In fact, according to the Paley-Wiener theorem, Problem B.13, a function f is C^∞ and supported in $\bar{B}_R$ if and only if $\hat{f}(k)$ extends to a holomorphic function on $\mathbb{C}^n$ which obeys (7.15).

Start by assuming that f is C^∞. We claim first that $\hat{f}$ restricted to the complex manifold $\mathcal{M}_\lambda^{\mathbb{C}} = \{\ \zeta \mid \zeta\cdot\zeta = \lambda\ \}$ is identically zero. To see this consider the power series expansion of $f|_{\mathcal{M}_\lambda^{\mathbb{C}}}$ near $\zeta = (\sqrt{\lambda}, 0, \ldots, 0)$. Let $\widetilde{\zeta} = (\zeta_2, \ldots, \zeta_n)$. Now, $g(\widetilde{\zeta}) = \hat{f}\Big(\sqrt{\lambda - \widetilde{\zeta}\cdot\widetilde{\zeta}}\,, \widetilde{\zeta}\Big)$ is holomorphic in a neighborhood of zero and, by hypothesis (ii), vanishes whenever $\widetilde{\zeta}$ is real. Hence all the coefficients in the power series expansion of g are zero. Thus, g and, since $\mathcal{M}_\lambda^{\mathbb{C}}$ is connected by Problem 7.11, $f|_{\mathcal{M}_\lambda^{\mathbb{C}}}$, are identically zero.

Now consider $\hat{u} = \big(G_0(\lambda) f\big)\hat{}$, which automatically extends to $\zeta \in \mathbb{C}^h$ via

$$\hat{u}(\zeta) = \frac{\hat{f}(\zeta)}{\zeta\cdot\zeta - \lambda} \tag{7.16}$$

By the Cauchy integral formula

$$|\hat{f}(\zeta)| \le \frac{C'_N}{1+|\zeta|^{2N}} e^{R|\operatorname{Im}\zeta|}\ \operatorname{dist}(\zeta, \mathcal{M}_\lambda^{\mathbb{C}}) \tag{7.17}$$

within a distance one, for example, of $\mathcal{M}_\lambda^{\mathbb{C}}$. If $|\zeta\cdot\zeta| \ge \frac{\lambda}{2}$, then, choosing $\sqrt{\zeta\cdot\zeta}$ to have non–negative real part, we have that $\frac{\sqrt{\lambda}}{\sqrt{\zeta\cdot\zeta}}\zeta$ is on $\mathcal{M}_\lambda^{\mathbb{C}}$ so that

$$\operatorname{dist}(\zeta, \mathcal{M}_\lambda^{\mathbb{C}}) \le \Big|\zeta - \frac{\sqrt{\lambda}}{\sqrt{\zeta\cdot\zeta}}\zeta\Big| = \Big|\frac{\sqrt{\zeta\cdot\zeta}-\sqrt{\lambda}}{\sqrt{\zeta\cdot\zeta}}\Big|\,|\zeta| = \Big|\frac{\zeta\cdot\zeta-\lambda}{\sqrt{\zeta\cdot\zeta}(\sqrt{\zeta\cdot\zeta}+\sqrt{\lambda})}\Big|\,|\zeta| \le \frac{\sqrt{2}}{\lambda}\,|\zeta|\,\big|\zeta\cdot\zeta-\lambda\big|$$

On the other hand, if $|\zeta\cdot\zeta| \le \frac{\lambda}{2}$, then

$$\big|\zeta\cdot\zeta - \lambda\big| \ge \frac{\lambda}{2} \ge \frac{\lambda}{2}\frac{\operatorname{dist}(\zeta, \mathcal{M}_\lambda^{\mathbb{C}})}{\sqrt{\lambda}+|\zeta|}$$

since $(\sqrt{\lambda}, 0, \cdots, 0) \in \mathcal{M}_\lambda^{\mathbb{C}}$. In both cases there is a λ–dependent constant D so that

$$\frac{1}{|\zeta\cdot\zeta-\lambda|} \le D\frac{1+|\zeta|^2}{\operatorname{dist}(\zeta, \mathcal{M}_\lambda^{\mathbb{C}})} \tag{7.18}$$

so that (7.16) can have no poles. Hence $\hat{u}(\zeta)$ is holomorphic and satisfies a bound of the form (7.15), so that the Paley-Wiener theorem implies (iii).

Now we extend to a general $f \in L^2$ that is supported in $\bar{B}_R$. Let φ be a nonnegative C_0^∞ function that is supported in the unit ball $\bar{B}_1$ and has total mass one. That is, $\int \varphi(x)\ d^nx = 1$. Set, for each $\varepsilon > 0$, $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\big(\frac{x}{\varepsilon}\big)$ and $f_\varepsilon = \varphi_\varepsilon * f$. Then f_ε is C^∞ and is supported in $\bar{B}_{R+\varepsilon}$. Since $\hat{f}_\varepsilon(k) = \hat{\varphi}_\varepsilon(k)\hat{f}(k)$, f_ε satisfies condition (ii). So by the part of (ii)$\Rightarrow$ (iii) that we have already proven, $u_\varepsilon \equiv G_0(\lambda)f_\varepsilon$ also vanishes outside of $\bar{B}_{R+\varepsilon}$. Since convolution with φ_ε commutes with application of the Laplacian, $u_\varepsilon = \varphi_\varepsilon * u$ where $u = G_0(\lambda)f$. Since u_ε converges to u, locally in L^2, as $\varepsilon \to 0$, u vanishes outside of $\bar{B}_{R+\varepsilon}$ for all $\varepsilon > 0$. This is all we need.

(iii) $\Rightarrow$ (i) Since f vanishes outside $\bar{B}_R$ it is in L^2_δ for any $\delta > \frac{1}{2}$. By Proposition 7.2, $G_0(\lambda)f \in L^2_{-\delta}$ so that the restriction of $G_0(\lambda)f$ to any bounded set is in L^2. Since we are assuming that $G_0(\lambda)f$ vanishes outside of $\bar{B}_R$, it is itself in $L^2(\mathbb{R}^n)$.

□

Exercise 7.10. Let

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

be the Heavyside step function.

(a) Let $\zeta \in \mathcal{S}(\mathbb{R})$ obey $\hat{\zeta}(\alpha) = \int e^{-i\alpha x}\zeta(x)\ dx \neq 0$ for some $\alpha \in \mathbb{R}$. Prove that $z(x) = \int e^{-i\alpha y}H(x-y)\zeta(y)\ dy$ is not in $L^2_s(\mathbb{R})$ for any $s \geq -\frac{1}{2}$.

(b) Let $\zeta \in \mathcal{S}(\mathbb{R}^n)$ obey $\int e^{-i\alpha x_1}\zeta(x_1, 0)\ dx = \int_{\mathbb{R}^{n-1}} \hat{\zeta}(\alpha, k')\ \frac{d^{n-1}k'}{(2\pi)^{n-1}} \neq 0$ for some $\alpha \in \mathbb{R}$. Prove that $z(x_1, x') = \int e^{-i\alpha y_1}H(x_1 - y_1)\zeta(y_1, x')\ dy_1$ is not in $L^2_s(\mathbb{R}^n)$ for any $s \geq -\frac{1}{2}$.

Exercise 7.11. Prove that if $\lambda \neq 0$, then $\mathcal{M}_\lambda^{\mathbb{C}}$ is a smooth connected manifold.

We shall need one more estimate and its corollary in the next section.

Proposition 7.12 ([Ho, Vol 2, Proposition 14.7.1])**.** *Let $\lambda, \tau > 0$ and let u be a compactly supported L^2 function with $(-\Delta - \lambda)u \in L^2(\mathbb{R}^n)$, then*

$$2\lambda\tau \int_{\mathbb{R}^n} |u|^2|x|^\tau\ d^nx \leq \int_{\mathbb{R}^n} |(-\Delta - \lambda)u|^2|x|^{2+\tau}\ d^nx \tag{7.19}$$

Proof. In polar coordinates

$$-\Delta - \lambda = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_s - \lambda$$

where Δ_s is the Laplacian on the unit sphere. If we set $r = e^t$, then $\frac{\partial}{\partial r} = e^{-t}\frac{\partial}{\partial t}$ and

$$-\Delta - \lambda = e^{-2t}\big(- \frac{\partial^2}{\partial t^2} - (n-2)\frac{\partial}{\partial t} - \Delta_s - \lambda e^{2t}\big)$$

Letting

$$v(t,\omega) = e^{(\frac{\tau+n}{2}-1)t}u(e^t\omega)$$

the integrand of the right hand side of (7.19) becomes

$$\begin{aligned}
|(-\Delta-\lambda)u|^2|x|^{2+\tau} &= e^{(2+\tau)t}\big|e^{-2t}\big(- \frac{\partial^2}{\partial t^2} - (n-2)\frac{\partial}{\partial t} - \Delta_s - \lambda e^{2t}\big)\big(e^{-(\frac{\tau+n}{2}-1)t}v(t,\omega)\big)\big|^2 \\
&= e^{(2+\tau)t}\big|e^{-(\frac{\tau+n}{2}+1)t}\big(- \frac{\partial^2}{\partial t^2} - (n-2-\tau-n+2)\frac{\partial}{\partial t} - \Delta_s + \mu - \lambda e^{2t}\big)v(t,\omega)\big|^2 \\
&= e^{-nt}\big|\big(- \frac{\partial^2}{\partial t^2} + \tau\frac{\partial}{\partial t} - \Delta_s + \mu - \lambda e^{2t}\big)v(t,\omega)\big|^2 \\
&= e^{-nt}\big|(L_1+L_2)v(t,\omega)\big|^2
\end{aligned}$$

where $\mu = (n-2)\big(\frac{\tau+n}{2}-1\big) - \big(\frac{\tau+n}{2}-1\big)^2$ is a real number and

$$\begin{aligned}
L_1 &= -\frac{\partial^2}{\partial t^2} + \mu - \Delta_s - \lambda e^{2t} \\
L_2 &= \tau\frac{\partial}{\partial t}
\end{aligned}$$

Since $d^nx = r^{n-1}\,dr d\omega = e^{nt}\,dt d\omega$, the right hand side of (7.19) becomes

$$M = \iint |L_1v + L_2v|^2\ dtd\omega = \big\|L_1v + L_2v\big\|^2$$

Since

$$L_1^*L_2 + L_2^*L_1 = L_1L_2 - L_2L_1 = 2\lambda\tau e^{2t}$$

we have

$$\begin{aligned}
M &= \big\|L_1v\big\|^2 + \big\|L_2v\big\|^2 + \big(v,(L_1^*L_2 + L_2^*L_1)v\big) \\
&\geq \big(v,(L_1^*L_2 + L_2^*L_1)v\big) \\
&= 2\lambda\tau\int e^{2t}|v|^2 dtdw \\
&= 2\lambda\tau\int |u|^2|x|^\tau d^nx
\end{aligned}$$

□

Corollary 7.13. *Let $\mu > -\frac{1}{2}$, $\lambda > 0$ and $q \in L^\infty(\mathbb{R}^n)$. Suppose that $u \in L^2_\mu(\mathbb{R}^n)$ is outgoing (i.e. $u \in$ Range$(G_0(\lambda))$) and is a weak solution of*

$$(-\Delta - \lambda + q)u = 0$$

If q has compact support, then $u \equiv 0$.

Proof. Pick and $\delta > \frac{1}{2}$. Since q is of compact support, $qu \in L^2_\delta$, and u is a weak solution of $(-\Delta - \lambda)u = -qu$. By Problem 7.6, $u = G_0(\lambda)f$ with $f = -qu$. Applying Theorem 7.9 with $f = -qu$, we conclude that u has compact support and is in L^2. Since $(-\Delta - \lambda)u = -qu$ is also in L^2 (7.19) applies and yields

$$\int_{\mathbb{R}^n} |u|^2|x|^\tau dx \le \frac{1}{2\lambda\tau}\int_{\mathbb{R}^n} |qu|^2|x|^{2+\tau}dx \le \frac{\sup_{x\in\mathbb{R}^n}(|x|^2|q|^2)}{2\lambda\tau}\int_{\mathbb{R}^n}|u|^2|x|^\tau dx$$

Choosing τ large enough that $\frac{\sup_{x\in\mathbb{R}^n}(|x|^2|q|^2)}{2\lambda\tau} < 1$ implies that

$$\int_{\mathbb{R}^n} |u|^2|x|^\tau dx = 0$$

so that $u \equiv 0$. □

7.2. Eigenfunctions for H_q

Theorem 7.14. *Let $\lambda, R > 0$ and $\delta > \frac{1}{2}$. Let $f \in L^2_\delta$ and $q \in L^\infty(\mathbb{R}^n)$ with* $\operatorname{supp} q \subset \bar{B}_R$. *Then there exists a unique $L^2_{-\delta}$ outgoing (i.e. $u \in$ Range$(G_0(\lambda))$) weak solution $u = G_q(\lambda)f$ to*

$$(-\Delta - \lambda + q)u = f \tag{7.20}$$

Moreover, we have the formulas

$$G_q(\lambda) = G_0(\lambda)\big(\mathbb{1} + qG_0(\lambda)\big)^{-1} \tag{7.21}$$

$$G_q(\lambda) = \big(\mathbb{1} + G_0(\lambda)q\big)^{-1}G_0(\lambda) \tag{7.22}$$

Proof. A function $u \in L^2_{-\delta}$ is outgoing if and only if there is a $v \in L^2_\delta$ such that $u = G_0(\lambda)v$. By Problem 7.6, the (weak) equation (7.20) for outgoing u is thus equivalent to the equation

$$\big(\mathbb{1} + qG_0(\lambda)\big)v = f$$

for v. So we must establish the invertability of

$$\big(\mathbb{1} + qG_0(\lambda)\big) : L^2_\delta \to L^2_\delta \tag{7.23}$$

to prove (7.21). Then (7.22) will follow from writing (7.20) as

$$(-\Delta - \lambda)\big(\mathbb{1} + G_0(\lambda)q\big)u = f$$

and observing that $\big(\mathbb{1}+G_0(\lambda)q\big) : L^2_{-\delta} \to L^2_{-\delta}$ has a bounded inverse because it is the Banach space adjoint[1]Let X and Y be Banach spaces and X^* and Y^* be their respective dual spaces. If T is a bounded linear operator from X to Y, then the Banach space adjoint of T, denoted T' is the bounded linear operator from Y^* to X^* defined by $(T'\ell)(x) = \ell(Tx)$ for all $\ell \in Y^*$ and $x \in X$. In the current application, we have $X = Y = L^2_\delta(\mathbb{R}^n)$, $X^* = Y^* =$

[1](1)

$L^2_{-\delta}(\mathbb{R}^n)$ and $\ell(x) = \int_{\mathbb{R}^n} \ell(y)x(y)d^ny$. In this case, the Banach space adjoint is the "L^2–adjoint without the complex conjugate". of $\big(\mathbb{1} + qG_0(\lambda)\big)$.

It remains to establish the invertability of (7.23). We need one additional estimate for $G_0(\lambda)$. If

$$(-\Delta - \lambda)w = g$$

then, by Theorem 7.1,

$$\| - \Delta w\|_{L^2_{-\delta}} \leq \lambda \|w\|_{L^2_{-\delta}} + \|g\|_{L^2_{-\delta}} \leq (C\sqrt{\lambda} + 1)\|g\|_{L^2_{\delta}}$$

Let $\chi(x)$ be a smooth cutoff function which is one on the support of q and is supported inside the ball of radius R. If $\{f_i\}$ is a bounded sequence in $L^2_\delta(\mathbb{R}^n)$, then both $\{\chi G_0(\lambda)f_i\}$ and $\{-\Delta\chi G_0(\lambda)f_i\}$ are bounded sequences in $L^2_{-\delta}(\mathbb{R}^n)$. So $\{\chi G_0(\lambda)f_i\}$ is a bounded sequence in $H^2_0(B_R)$ and, by Rellich's Theorem (Problem **??**), has a subsequence that converges in $L^2(B_R)$ and hence in $L^2_\delta(\mathbb{R}^n)$. Therefore

$$\chi G_0(\lambda) : L^2_\delta(\mathbb{R}^n) \to L^2_\delta(\mathbb{R}^n)$$

is a compact operator (see Definition A.56). Since multiplication by q is a bounded operator, $qG_0(\lambda) = q\chi G_0(\lambda)$ is also compact, by Proposition **??**.

For large λ, it follows from (7.5) that

$$\begin{aligned}\|qG_0(\lambda)\| &\leq \big|(1 + |x|^2)^\delta q(x)\big\|_{L^\infty}\|G_0(\lambda)\| \\ &\leq \|(1 + |x|^2)^\delta q(x)\|_{L^\infty}\frac{C}{\sqrt{\lambda}}\end{aligned}$$

so that $\mathbb{1} + qG_0(\lambda)$ is invertible for λ sufficiently large. By the Fredholm Alternative (Proposition A.70), to the prove bounded invertability of $\mathbb{1} + qG_0(\lambda)$ at any λ we need only prove that the kernel is empty. This can be seen as follows:

Suppose that

$$f + qG_0(\lambda)f = 0$$

then

$$\langle f, G_0(\lambda)f\rangle = -\langle qG_0(\lambda)f, G_0(\lambda)f\rangle \tag{7.24}$$

and, because the right hand side of (7.24) is real, we may conclude via (7.13) that $\hat{f}\big|_{\mathcal{M}_\lambda} = 0$. According to Theorem 7.9, $u = G_0(\lambda)f$ is a compactly supported and hence L^2 solution to $(-\Delta - \lambda)u = f = -qu$. Hence $u \equiv 0$ by Corollary 7.13. By Problem 7.6, $f = 0$ as well and the theorem is proved. ☐

The existence of the generalized eigenfunctions for H_q now follows easily. We have

Corollary 7.15. *Let $\lambda > 0$ and $\omega \in S^{n-1}$. Let $q \in L^\infty$ be of compact support. There exists a unique λ–outgoing eigenfunction for H_q in the direction ω. That is, there exists a unique $\psi_q(\lambda, x, \omega)$ satisfying*

$$(-\Delta - \lambda + q)\psi_q = 0$$

of the form

$$\psi_q = \psi_0(\lambda, x, \omega) + \varphi_q(\lambda, x, \omega)$$

such that $\varphi_q(\lambda, \cdot, \omega) \in L^2_\mu$, for all $\mu < -\frac{1}{2}$, and is λ–outgoing. Moreover,

$$\psi_q(\lambda, \cdot, \omega) = \big(\mathbb{1} - G_q(\lambda)q\big)\psi_0(\lambda, \cdot, \omega) \tag{7.25}$$

and

$$\psi_q(\lambda, \cdot, \omega) = \psi_0(\lambda, \cdot, \omega) - G_0(\lambda)q\psi_q(\lambda, \cdot, \omega)$$

Proof. In terms of φ_q, the equation $(-\Delta - \lambda + q)\psi_q = 0$ is

$$(-\Delta - \lambda + q)\varphi_q = -q\psi_0 \tag{7.26}$$

By Theorem 7.14, this equation has a unique outgoing solution and that solution is

$$\varphi_q = -G_q(\lambda)(q\psi_0(\lambda, \cdot, \omega)) \tag{7.27}$$

Rewriting (7.26) as

$$(-\Delta - \lambda)\varphi_q = -q\psi_0 - q\varphi_q = -q\psi_q$$

yields the last claim. □

Our next step is to extend Theorem 7.9 to H_q. We prove

Theorem 7.16. *Let $\lambda > 0$, $R > 0$ and $f \in L^2$, with $\operatorname{supp} f \subset B_R$ and $\operatorname{supp} q \subset B_R$. The following are equivalent*

(i)$G_q(\lambda)f \in L^2_\mu$ for some $\mu > -\frac{1}{2}$

(ii)$\int_{\mathbb{R}^n} f(x)\psi_q(\lambda, x, \omega)\; d^n x = 0$ for all $\omega \in S^{n-1}$

(iii)$\operatorname{supp}(G_q(\lambda)f) \subset B_R$

Proof. We first note, from (7.22), that

$$\big(\mathbb{1} + G_0(\lambda)q\big)G_q(\lambda)f = G_0(\lambda)f$$

so that

$$G_q(\lambda)f = G_0(\lambda)\big(\mathbb{1} - qG_q(\lambda)\big)f = G_0(\lambda)F \tag{7.28}$$

where $F = \big(\mathbb{1} - qG_q(\lambda)\big)f$. If (i) is satisfied, then Theorem 7.9 yields

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} F(x)\psi_0(x,\omega)\ d^n x \\
&= \int_{\mathbb{R}^n} \big(\big[\mathbb{1} - qG_q(\lambda)\big]f\big)(x)\psi_0(x,\omega)\ d^n x \\
&= \int_{\mathbb{R}^n} f(x)\big(\big[\mathbb{1} - G_q(\lambda)q\big]\psi_0\big)(x,\omega)\ d^n x \\
&= \int_{\mathbb{R}^n} f(x)\psi_q(\lambda,x,\omega)\ d^n x
\end{aligned}$$

which proves that (i) $\Rightarrow$ (ii).

The previous calculation along with Theorem 7.9 implies that

$$\operatorname{supp}(G_0(\lambda)F) \subset B_R$$

which, in view of (7.28), shows that (ii) $\Rightarrow$ (iii). As before, (iii) $\Rightarrow$ (i) is trivial. $\square$

7.3. Asymptotics and the Scattering Amplitude

Let $G_q(\lambda, x, y)$ denote the outgoing Green's kernel defined by

$$\big(G_q(\lambda)f\big)(x) = \int_{\mathbb{R}^n} G_q(\lambda,x,y)f(y)\ d^n y$$

It is, for each fixed y, the solution of

$$(-\Delta_x + q(x) - \lambda)G_q(\lambda,x,y) = \delta(x-y) \tag{7.29}$$

subject to the Sommerfeld radiation condition (see (1.9))

$$\frac{\partial}{\partial r}G_q - i\sqrt{\lambda}G_q = o\Big(\frac{1}{|x|^{\frac{n-1}{2}}}\Big) \qquad \text{as} \qquad |x| \to \infty$$

The "unperturbed" outgoing Green's kernel $G_0(\lambda,x,y)$ can be computed using

$$G_0(\lambda,x,y) = \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^n} \frac{e^{ik\cdot(x-y)}}{|k|^2 - (\lambda + i\epsilon)}\frac{d^n k}{(2\pi)^n}$$

Lemma 7.17. *If $\lambda > 0$, then*

$$G_0(\lambda,x,y) = \frac{i}{4}\Big(\frac{\sqrt{\lambda}}{2\pi|x-y|}\Big)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\sqrt{\lambda}\,|x-y|\,)$$

where $H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z)$ is a Hankel function. In particular, for $n = 3$,

$$G_0(\lambda,x,y) = \frac{e^{i\sqrt{\lambda}\,|x-y|}}{4\pi|x-y|}$$

Asymptotically, for large r,

$$G_0(\lambda, x, y) = \frac{1}{2\sqrt{\lambda}}\Big(\frac{\sqrt{\lambda}}{2\pi|x-y|}\Big)^{\frac{n-1}{2}}\ e^{i(\sqrt{\lambda}\,|x-y|-\frac{n-3}{4}\pi)} + O\Big(\frac{1}{|x-y|^{\frac{n-1}{2}-1}}\Big)$$

Proof. By rotation invariance, we may assume, without loss of generality, that $x - y = (|x-y|, 0, \cdots, 0)$. By scaling $k \to \sqrt{\lambda}\,k$,

$$G_0(\lambda, x, y) = \lim_{\epsilon\downarrow 0} \lambda^{\frac{n}{2}-1}\int_{\mathbb{R}^n} \frac{e^{ik_1\sqrt{\lambda}|x-y|}}{|k|^2 - 1 - i\epsilon/\lambda}\frac{d^n k}{(2\pi)^n} = \lim_{\epsilon\downarrow 0} \lambda^{\frac{n}{2}-1}\int_{\mathbb{R}^n} \frac{e^{ik_1 r}}{|k|^2 - 1 - i\epsilon}\frac{d^n k}{(2\pi)^n}$$

where $r = \sqrt{\lambda}\,|x-y|$. By residues

$$\int_{\mathbb{R}} \frac{e^{ik_1 r}}{k_1^2 + p^2 - 1 - i\epsilon}\frac{dk_1}{2\pi} = 2\pi i \frac{1}{2\pi}\frac{e^{-r\sqrt{p^2-1-i\epsilon}}}{2i\sqrt{p^2-1-i\epsilon}} = \frac{e^{-r\sqrt{p^2-1-i\epsilon}}}{2\sqrt{p^2-1-i\epsilon}}$$

where $p = (k_2, \cdots, k_n)$ and $\sqrt{p^2 - 1 - i\epsilon}$ denotes the square root with positive real part. The corresponding imaginary part is negative. As $\sqrt{p^2-1}$ has an integrable, square root, singularity at $|p| = 1$ and $e^{-r\sqrt{p^2-1}}$ decays exponentially quickly for large $|p|$, the Lebesgue dominated convergence theorem gives

$$G_0(\lambda, x, y) = \tfrac{1}{2}\lambda^{\frac{n}{2}-1}\int_{\mathbb{R}^{n-1}} \frac{e^{-r\sqrt{p^2-1}}}{\sqrt{p^2-1}}\frac{d^{n-1}p}{(2\pi)^{n-1}}$$

Now $\sqrt{p^2-1}$ is positive for $|p| > 1$ and is i times a negative number when $|p| < 1$. Going to spherical coordinates in $\mathbb{R}^{n-1}$

$$G_0(\lambda, x, y) = \frac{\Omega_{n-1}}{2^n\pi^{n-1}}\lambda^{\frac{n}{2}-1}\int_0^\infty \frac{e^{-r\sqrt{\rho^2-1}}}{\sqrt{\rho^2-1}}\ \rho^{n-2}d\rho$$

where $\Omega_m = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ denotes the surface area of a unit sphere in $\mathbb{R}^m$. For $\rho \geq 1$, make the change of variables $\rho = \cosh\tau$, $\frac{1}{\sqrt{\rho^2-1}}d\rho = \frac{1}{\sinh\tau}\sinh\tau\, d\tau = d\tau$

$$\int_1^\infty \frac{e^{-r\sqrt{\rho^2-1}}}{\sqrt{\rho^2-1}}\ \rho^{n-2}d\rho = \int_0^\infty e^{-r\sinh\tau}\ \cosh^{n-2}\tau\ d\tau$$

For $0 \leq \rho \leq 1$, make the change of variables $\rho = \sin\alpha$, $\sqrt{\rho^2-1} = -i\cos\alpha$, $d\rho = \cos\alpha\, d\alpha$

$$\int_0^1 \frac{e^{-r\sqrt{\rho^2-1}}}{\sqrt{\rho^2-1}}\ \rho^{n-2}d\rho = i\int_0^{\frac{\pi}{2}} e^{ir\cos\alpha}\ \sin^{n-2}\alpha\ d\alpha$$

Poisson's integral representation [MO, Third Chapter, §5] for the Bessel functions of the first and second kind are (for $\nu > -\frac{1}{2}$ and $\operatorname{Re} z > 0$)

$$J_\nu(z) = \frac{2(\frac{z}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^{\frac{\pi}{2}} \cos\big(z\cos\alpha\big) \sin^{2\nu}\alpha \; d\alpha$$

$$Y_\nu(z) = \frac{2(\frac{z}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \Big[\int_0^{\frac{\pi}{2}} \sin\big(z\cos\alpha\big) \sin^{2\nu}\alpha \; d\alpha - \int_0^\infty e^{-z\sinh\tau} \cosh^{2\nu}\tau \; d\tau \Big]$$

The Hankel function

$$\begin{aligned} H_\nu^{(1)}(r) &= J_\nu(r) + iY_\nu(r) \\ &= \frac{2(\frac{r}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \Big[\int_0^{\frac{\pi}{2}} e^{ir\cos\alpha} \sin^{2\nu}\alpha \; d\alpha - i\int_0^\infty e^{-r\sinh\tau} \cosh^{2\nu}\tau \; d\tau \Big] \\ &= \frac{2(\frac{r}{2})^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \Big[-i\int_0^\infty \frac{e^{-r\sqrt{\rho^2-1}}}{\sqrt{\rho^2-1}} \; \rho^{n-2} d\rho \Big] \end{aligned}$$

with $\nu = \frac{n-2}{2}$. Thus

$$\begin{aligned} G_0(\lambda,x,y) &= \frac{\Omega_{n-1}}{2^n \pi^{n-1}} \lambda^{\frac{n}{2}-1} i \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{2(\frac{r}{2})^{\frac{n-2}{2}}} H^{(1)}_{\frac{n-2}{2}}(r) \\ &= \frac{i}{4}\Big(\frac{\sqrt{\lambda}}{2\pi|x-y|}\Big)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\sqrt{\lambda}\,|x-y|\,) \end{aligned}$$

Internal remark 6. *Now making the change of variables $s = \rho^2 - 1$,*

$$G_0(\lambda,x,y) = \frac{1}{2^n \pi^{(n-1)/2} \Gamma(\frac{n-1}{2})} \lambda^{\frac{n}{2}-1} \int_{-1}^\infty \frac{e^{-r\sqrt{s}}}{\sqrt{s}} \; (s+1)^{\frac{n-3}{2}} ds$$

For $s > 0$, we make the change of variables $t = \sqrt{s}$

$$\int_0^\infty \frac{e^{-r\sqrt{s}}}{\sqrt{s}} \; (s+1)^{\frac{n-3}{2}} ds = 2\int_0^\infty e^{-rt}(t^2+1)^{\frac{n-3}{2}} \; dt$$

and for $s < 0$, we make the change of variables $\sqrt{s} = -it$

$$\int_{-1}^0 \frac{e^{-r\sqrt{s}}}{\sqrt{s}} \; (s+1)^{\frac{n-3}{2}} ds = 2i\int_0^1 e^{irt}(1-t^2)^{\frac{n-3}{2}} \; dt$$

In particular

$$H^{(1)}_{\frac{1}{2}}(z) = -i\sqrt{\frac{2}{\pi z}} e^{iz}$$

so that, for $n = 3$,

$$G_0(\lambda,x,y) = \frac{e^{i\sqrt{\lambda}\,|x-y|}}{4\pi|x-y|}$$

We already saw the $n = 3$ case in Problem 1.20.

Asymptotically, for large r,

$$H_\nu^{(1)}(\sqrt{\lambda}\,r) = \sqrt{\frac{2}{\pi\sqrt{\lambda}\,r}}\,e^{i(\sqrt{\lambda}\,r - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} + O\big(\frac{1}{r^{3/2}}\big)$$

This is formula (9.2.3) in [AS]. So

$$\begin{aligned} G_0(\lambda,x,y) &= \frac{i}{4}\big(\frac{\sqrt{\lambda}}{2\pi|x-y|}\big)^{\frac{n-2}{2}}\sqrt{\frac{2}{\pi\sqrt{\lambda}\,|x-y|}}\,e^{i(\sqrt{\lambda}\,|x-y| - \frac{n}{4}\pi + \frac{1}{4}\pi)} + O\Big(\frac{1}{|x-y|^{\frac{n+1}{2}}}\Big) \\ &= \frac{1}{2\sqrt{\lambda}}\big(\frac{\sqrt{\lambda}}{2\pi|x-y|}\big)^{\frac{n-1}{2}}\ e^{i(\sqrt{\lambda}\,|x-y| - \frac{n-3}{4}\pi)} + O\Big(\frac{1}{|x-y|^{\frac{n+1}{2}}}\Big) \end{aligned}$$

□

Internal remark 7. *Alternatively, we can observe that, by rotation invariance G_0 depends on x and y only through $r = |x-y|$ and obeys, for $r > 0$,*

$$0 = (-\Delta - \lambda)G_0 = -\big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \lambda\big)G_0$$

This, up to a change of variables, Bessel's equation. It's general solution may be found choosing $a = n-1$, $b = \lambda$, $d = 2$ and $c = 0$, and consequently, $\alpha = \nu = -\frac{n-2}{2}$, $\beta = \sqrt{\lambda}$ and $\gamma = 1$ in

Exercise 7.18. Let $\beta, \gamma > 0$. Prove that $Z_\nu(x)$ obeys Bessel's equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

if and only if $y(x) = x^\alpha Z_\nu(\beta x^\gamma)$ obeys

$$x^2 y'' + axy' + (bx^d + c)y = 0$$

with

$$\alpha = \frac{1-a}{2} \qquad \beta = \frac{2\sqrt{b}}{d} \qquad \gamma = \frac{d}{2} \qquad \nu = \frac{2}{d}\sqrt{\big(\frac{1-a}{2}\big)^2 - c}$$

The general solution of the Bessel equation of order ν is $cJ_\nu(x) + dY_\nu(x)$ where J_ν and Y_ν are Bessel functions of the first and second kind, respectively. For our purposes, it is more convenient to use the Hankel functions

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \qquad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$$

as a basis for the space of solutions. So far, we know that

$$G_0(\lambda,x,y) = c_{n,\lambda}\frac{1}{|x-y|^{\frac{n-2}{2}}}H^{(1)}_{\frac{n-2}{2}}(\sqrt{\lambda}\,|x-y|) + d_{n,\lambda}\frac{1}{|x-y|^{\frac{n-2}{2}}}H^{(2)}_{\frac{n-2}{2}}(\sqrt{\lambda}\,|x-y|)$$

for some constants $c_{n,\lambda}$ and $d_{n,\lambda}$. Asymptotically, for large r,

$$H^{(1)}_\nu(\sqrt{\lambda}\,r) = \sqrt{\frac{2}{\pi\sqrt{\lambda}\,r}}\, e^{i(\sqrt{\lambda}\,r-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} + O\big(\frac{1}{r^{3/2}}\big)$$

$$H^{(2)}_\nu(\sqrt{\lambda}\,r) = \sqrt{\frac{2}{\pi\sqrt{\lambda}\,r}}\, e^{-i(\sqrt{\lambda}\,r-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} + O\big(\frac{1}{r^{3/2}}\big)$$

These are formulae (9.2.3) and (9.2.4) in [AS]. To satisfy the Sommerfeld radiation condition we need to choose $d_{n,\lambda} = 0$. The coefficient $c_{n,\lambda}$ is chosen to give a unit delta function at the origin, or, equivalently, to give the correct coefficient in the asymptotic behaviour for large $|x-y|$.

Corollary 7.19. *Let $\lambda > 0$ and $\omega \in S^{n-1}$. Let $q \in L^\infty$ be of compact support.*

(a) There is a C^∞ function $a_q(\sqrt{\lambda},\ \cdot\ ,\omega) : S^{n-1} \to \mathbb{C}$ such that, asymptotically for large $|x|$,

$$\psi_q(\lambda,x,\omega) = e^{i\sqrt{\lambda}\,x\cdot\omega} + \frac{a_q(\sqrt{\lambda},\theta,\omega)}{|x|^{\frac{n-1}{2}}} e^{i\sqrt{\lambda}\,|x|} + O\big(|x|^{-\frac{n-1}{2}-1}\big)$$

where $\theta = \frac{x}{|x|}$. **??? Regularity in λ, ω? ???**

(b) There is a constant $C_{\sqrt{\lambda},n}$ such that

$$G_q(\lambda,x,y) = C_{\sqrt{\lambda},n} \frac{e^{i\sqrt{\lambda}\,|x|}}{|x|^{\frac{n-1}{2}}} \psi_q(\lambda,y,-\frac{x}{|x|}) + O\Big(|x|^{-\frac{n-1}{2}-1}\Big)$$

Proof. (a) Rewriting the last conclusion of Corollary 7.15 in terms of the Green's kernel gives the Lipman–Schwinger equation (see (1.10))

$$\psi_q(\lambda,x,\omega) = e^{i\sqrt{\lambda}\,x\cdot\omega} - \int G_0(\lambda,x,y)\, q(y)\psi_q(\lambda,y,\omega)\, d^n y \tag{7.30}$$

For y in the support of q and large $|x|$,

$$(x-y)^2 = x^2-2x\cdot y+y^2 = x^2\big(1-2\frac{\theta\cdot y}{|x|}+\frac{y^2}{x^2}\big) \implies |x-y| = |x|-\theta\cdot y+O\big(\frac{1}{|x|}\big)$$

so that

$$G_0(\lambda,x,y) = C_{\sqrt{\lambda},n}\frac{1}{|x|^{\frac{n-1}{2}}}\ e^{i\sqrt{\lambda}\,(|x|-\theta\cdot y)} + O\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big) \tag{7.31}$$

Substituting this into (7.30), defining

$$a_q(\sqrt{\lambda},\theta,\omega) = -C_{\sqrt{\lambda},n}\int e^{-i\sqrt{\lambda}\,\theta\cdot y}\, q(y)\psi_q(\lambda,y,\omega)\, d^n y$$

and noting that ψ_q is locally L^2 and hence locally L^1 gives part (a).

(b) By (7.21),

$$G_q = (\mathbb{1} - G_q q)G_0$$

From (7.31)

$$G_0(\lambda, y, x) = G_0(\lambda, x, y) = C_{\sqrt{\lambda},n} \frac{e^{i\sqrt{\lambda}\,|x|}}{|x|^{\frac{n-1}{2}}} \psi_0(y, -\theta) \ + O\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big)$$

so that, by (7.25)

$$\begin{aligned} G_q(\lambda, x, y) = G_q(\lambda, y, x) &= C_{\sqrt{\lambda},n} \frac{e^{i\sqrt{\lambda}\,|x|}}{|x|^{\frac{n-1}{2}}} (\mathbb{1} - G_q q)\psi_0(\ \cdot\ , -\theta) + O\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big) \\ &= C_{\sqrt{\lambda},n} \frac{e^{i\sqrt{\lambda}\,|x|}}{|x|^{\frac{n-1}{2}}} \psi_q(y, -\theta) + O\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big) \end{aligned}$$

Internal remark 8. *In the y variable, $qO\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big)$ is of compact support and L^∞ with L^∞ norm $O\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big)$. Consequently $G_q qO\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big)$ is $L^2_{-\delta}$ and hence locally L^2 in y.*

□

7.4. Inverse Scattering at Fixed Energy

Let $n \geq 3$ and $q \in L^\infty(\mathbb{R}^n)$ be supported in $B_R = \big\{\ x \in \mathbb{R}^n\ \big|\ |x| < R\ \big\}$. Let $\lambda \in \mathbb{R}\backslash\{0\}$ and $\omega \in S^{n-1} = \big\{\ x \in \mathbb{R}^n\ \big|\ |x| = 1\ \big\}$. In the stationary approach to scattering theory by a potential [RS3, §XI.6] one shows that there exists a unique outgoing solution $\psi_q(\lambda, x, \omega)$ of

$$L_q\psi = \big(-\Delta + q - \lambda^2\big)\psi(\lambda, x, \omega) = 0 \tag{7.32}$$

of the form

$$\psi_q(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} + \frac{a_q(\lambda, \theta, \omega)}{|x|^{\frac{n-1}{2}}} e^{i\lambda|x|} + O\big(|x|^{-\frac{n-1}{2}-1}\big) \tag{7.33}$$

where $\theta = \frac{x}{|x|}$. We have done this, but with λ replaced by $\sqrt{\lambda}$, in Corollaries 7.15 and 7.19.

Definition 7.20. The function $a_q(\lambda, \theta, \omega)$ in (7.33) is called the *scattering amplitude*. The function $\psi_q(\lambda, x, \omega)$ is called an outgoing eigenfunction.

*The scattering amplitude measures, roughly speaking, the amplitude, measured in the direction $\theta = \frac{x}{|x|}$, of a spherical wave produced by the potential q when it interacts with plane waves of energy λ^2 moving in the direction ω. We discussed a classical analog of this picture in §**??**.3. In particular (7.33) is the analog for the Schrödinger equation of (1.11).*

We now fix $\lambda_0 \in \mathbb{R}\backslash\{0\}$. The subject of inverse scattering at fixed energy is the study of the invertibility of the map

$$q \mapsto \mathcal{A}_{\lambda_0}(q) = a_q(\lambda_0, \cdot, \cdot) \tag{7.34}$$

Novikov [No] proved that $\mathcal{A}_{\lambda_0}$ is injective. Several approachs [No,Ra,W] have been used to prove this result and various extensions. In this section we outline a proof that reduces the problem to proving the injectivity of the Dirichlet–to–Neumann map $\Lambda_{q-\lambda_0^2}$ acting on a large ball.

Theorem 7.21. *Let $n \geq 3$ and $q_1, q_2 \in L^\infty(\mathbb{R}^n)$ both be supported in B_R. Assume that*

$$\mathcal{A}_{\lambda_0}(q_1) = \mathcal{A}_{\lambda_0}(q_2)$$

Then

$$E(u_1, u_2) = \int_{B_R} (q_1 - q_2) u_1 \overline{u_2}\ d^n x = 0 \tag{7.35}$$

for all $u_i \in H^2(B_R)$ obeying

$$\big(-\Delta + q_1 - \lambda_0^2\big)u_1 = 0 \qquad \big(-\Delta + q_2 - \lambda_0^2\big)u_2 = 0 \tag{7.36}$$

Consequently $q_1 = q_2$.

By the divergence theorem, for all u_i obeying (7.36),

(7.37)

$$\begin{aligned} E(u_1,u_2) &= \int_{B_R} \big((\Delta u_1)\overline{u_2} - u_1(\overline{\Delta u_2})\big)\ d^n x = \int_{B_R} \nabla\cdot\big((\nabla u_1)\overline{u_2} - u_1(\overline{\nabla u_2})\big)\ d^n x \\ &= \int_{\partial B_R} \big(\frac{\partial u_1}{\partial r}\overline{u_2} - u_1 \overline{\frac{\partial u_2}{\partial r}}\big)\ dS \end{aligned}$$

where dS denotes the surface measure on ∂B_R. Note, in particular, that the outgoing eigenfunctions ψ_{q_i} satisfy (7.36). We first show

Lemma 7.22. *Under the hypotheses of Theorem 7.21,*

$$E\big(\psi_{q_1}(\lambda_0, \cdot, \omega), \psi_{q_2}(\lambda_0, \cdot, \omega)\big) = 0 \qquad \text{for all } \omega \in S^{n-1}$$

Lemma 7.23. *Let $q \in L^\infty(\mathbb{R}^n)$ be supported in the closed ball $\bar{B}_{R'} \subset B_R$ and define*

$$\mathcal{E} = \operatorname{span}\big\{\ \psi_q(\lambda_0, \cdot, \omega)\ \big|\ \omega \in S^{n-1}\ \big\}$$

Then $\mathcal{E}$ is dense in

$$N(L_q) = \big\{\ u \in H^2(B_R)\ \big|\ (-\Delta + q - \lambda_0^2)u = 0\ \big\}$$

with respect to the $H^2(B_{R'})$ topology.

These two lemmata imply (7.35). That $q_1 = q_2$ then follows by applying the portion of the proof of Theorem 4.12 that starts with (4.14).

Internal remark 9. *(4.14) applies to all H^1 solutions. But the special solutions used later in the proof were all H^2.*

Proof of Lemma 7.22. Since $a_{q_1}(\lambda_0,\theta,\omega) = a_{q_2}(\lambda_0,\theta,\omega)$ for all θ and ω, it follows from (7.33) that

$$\psi_{q_1}(\lambda_0,x,\omega) - \psi_{q_2}(\lambda_0,x,\omega) = O\big(|x|^{-\frac{n-1}{2}-1}\big)$$

is in $L^2(\mathbb{R}^n)$. We also have that

$$\big(-\Delta-\lambda_0^2\big)\big(\psi_{q_1}(\lambda_0,\,\cdot\,,\omega)-\psi_{q_2}(\lambda_0,\,\cdot\,,\omega)\big) = -q_1\psi_{q_1}(\lambda_0,\,\cdot\,,\omega)+q_2\psi_{q_2}(\lambda_0,\,\cdot\,,\omega)$$

is supported in B_R. By (7.27) and (7.21), both $\psi_{q_j}(\lambda_0,\,\cdot\,,\omega)-e^{i\lambda x\cdot\omega}$, $j=1,2$ are $G_0(\lambda_0^2)$–outgoing so that $\psi_{q_1}(\lambda_0,\,\cdot\,,\omega) - \psi_{q_2}(\lambda_0,\,\cdot\,,\omega)$ is as well. Hence, by the Rellich uniqueness theorem (Theorem 7.9) with $f = -q_1\psi_{q_1} + q_2\psi_{q_2}$,

$$\psi_{q_1}(\lambda_0,x,\omega) - \psi_{q_2}(\lambda_0,x,\omega) = 0 \qquad \text{for all } |x| > R$$

Let $\chi(x)$ be a smooth cutoff function which is one on a neighbourhood of ∂B_R and is supported inside $B_{2R}\setminus B_{\frac12 R}$. Then, as we saw in the proof of Theorem 7.14, $\chi\psi_{q_j}(\lambda_0,\,\cdot\,,\omega)\in H_0^2(B_{2R}\setminus B_{\frac12 R})$ and hence $\frac{\partial}{\partial r}\chi\psi_{q_j}(\lambda_0,\,\cdot\,,\omega)\in H_0^1(B_{2R}\setminus B_{\frac12 R})$ for $j=1,2$. So by the restriction to the boundary theorem (Theorem **??**)

$$\begin{aligned}\psi_{q_1}(\lambda_0,\,\cdot\,,\omega)\big|_{\partial B_R} &= \psi_{q_2}(\lambda_0,\,\cdot\,,\omega)\big|_{\partial B_R}\\ \frac{\partial}{\partial r}\psi_{q_1}(\lambda_0,\,\cdot\,,\omega)\big|_{\partial B_R} &= \frac{\partial}{\partial r}\psi_{q_2}(\lambda_0,\,\cdot\,,\omega)\big|_{\partial B_R}\end{aligned}$$

as elements of $H^{\frac12}(\partial B_R)\subset L^2(\partial B_R)$. So the Lemma follows from (7.37). $\square$

Proof of Lemma 7.23. First we prove that $\mathcal{E}$ is dense in $N(L_q)$ with respect to the $L^2(B_R)$ topology. To do so it suffices to prove that if $f\in N(L_q)$ obeys

$$\int_{B_R} f(x)\,\psi_q(\lambda_0,x,\omega)\;d^nx = 0 \qquad \text{for all } \omega\in S^{n-1} \tag{7.38}$$

then f must vanish. Define

$$w(x) = \int_{B_R} G_q(\lambda_0,x,y)f(y)\;d^ny \tag{7.39}$$

Then, by (7.29),

$$-\Delta w - \lambda_0^2 w = \chi_{B_R} f - qw$$

vanishes outside B_R and, by Corollary **??** and (7.38),

$$w(x) = O\Big(|x|^{-\frac{n-1}{2}-1}\Big)$$

is in $L^2(\mathbb{R}^n)$. As $\chi_{B_R} f - qw$ is also in $L^2(\mathbb{R}^n)$, we have that, as in the proof of Theorem 7.14, $w \in H^2(B_{2R})$. By the Rellich uniqueness theorem, we obtain that w vanishes outside B_R and therefore

$$w\big|_{\partial B_R} = \frac{\partial w}{\partial r}\big|_{B_R} = 0 \tag{7.40}$$

Now using the divergence theorem and $(-\Delta + q - \lambda_0^2)w = \chi_{B_R} f$ we have

$$\begin{aligned}\int_{B_R} |f|^2 \, d^n x &= \int_{B_R} \bar{f}(x)\,(-\Delta + q - \lambda_0^2) w \, d^n x \\ &= \int_{B_R} w(x)\,(-\Delta + q - \lambda_0^2)\bar{f}(x)\, d^n x - \int_{\partial B_R} \Big(\frac{\partial w}{\partial r}\bar{f} - w\frac{\partial \bar{f}}{\partial r}\Big)\, dS\end{aligned}$$

Since q is real–valued and $f \in N(L_q)$, we have that $(-\Delta + q - \lambda_0^2)\bar{f} = 0$ in B_R and conclude that $f = 0$.

Now we finish the proof by showing density in $H^2(B_{R'})$. By the interior regularity Proposition **??**, with $\gamma = \mathbb{1}$ and $\ell = 2$, there is a constant $C = C(R, R')$ such that

(7.41)

$$\begin{aligned}\|u\|_{H^2(B_{R'})} &\le C\Big(\| - \Delta u\|_{L^2(B_R)} + \|u\|_{L^2(B_R)}\Big) \\ &\le C\Big(\|(-\Delta + q - \lambda_0^2)u\|_{L^2(B_R)} + \|(q - \lambda_0^2)u\|_{L^2(B_R)} + \|u\|_{L^2(B_R)}\Big) \\ &\le C'\Big(\|(-\Delta + q - \lambda_0^2)u\|_{L^2(B_R)} + \|u\|_{L^2(B_R)}\Big)\end{aligned}$$

for all $u \in H^2(B_R)$. Now let $f \in N(L_q)$. We already know that there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{E}$ such that

$$\lim_{j\to\infty} \|f - f_j\|_{L^2(B_R)} = 0$$

Then from (7.41) we conclude that

$$\lim_{j\to\infty} \|f - f_j\|_{H^2(B_{R'})} = 0$$

concluding the proof of Lemma 7.23. □

Appendix A

Functional Analysis

In this appendix we provide a summary (mostly without proofs) of the most basic definitions and results concerning Banach and Hilbert spaces (§A.1) and bounded operators (§A.2). We also develop (with proofs) the most basic results concerning compact operators (§A.3).

A.1. Banach and Hilbert Spaces

Definition A.1 (Vector Space)**.** A vector space over $\mathbb{C}$ is a set $\mathcal{V}$ equipped with two operations,

$$(\mathbf{v}, \mathbf{w}) \in \mathcal{V} \times \mathcal{V} \mapsto \mathbf{v} + \mathbf{w} \in \mathcal{V} \qquad (\alpha, \mathbf{v}) \in \mathbb{C} \times \mathcal{V} \mapsto \alpha\mathbf{v} \in \mathcal{V}$$

called addition and scalar multiplication, respectively, that obey the following axioms.

Additive Axioms:

There is an element $\mathbf{0} \in \mathcal{V}$ and, for each $\mathbf{x} \in \mathcal{V}$ there is an element $-\mathbf{x} \in \mathcal{V}$ such that, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$,

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
(2) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
(3) $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$
(4) $(-\mathbf{x}) + \mathbf{x} = \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

Multiplicative Axioms:

For every $\mathbf{x} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$,

(5) $0\mathbf{x} = \mathbf{0}$
(6) $1\mathbf{x} = \mathbf{x}$
(7) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

Distributive Axioms:

For every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$,

(8) $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

(9) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

Definition A.2 (Subspace)**.** A subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is called a *linear subspace* of $\mathcal{V}$ if it is closed under addition and scalar multiplication. That is, if $\mathbf{x}+\mathbf{y} \in \mathcal{W}$ and $\alpha\mathbf{x} \in \mathcal{W}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{W}$ and all $\alpha \in \mathbb{C}$. Then $\mathcal{W}$ is itself a vector space over $\mathbb{C}$.

Definition A.3 (Inner Product)**.**
(a) An inner product on a vector space $\mathcal{V}$ is a function

$$(\mathbf{x}, \mathbf{y}) \in \mathcal{V} \times \mathcal{V} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{C}$$

that obeys

(1) $\langle \alpha\mathbf{x}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle$, $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (linearity in the first argument)

(2) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (conjugate symmetry)

(3) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ if $\mathbf{x} \neq \mathbf{0}$ (positive–definiteness)

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha \in \mathbb{C}$.

(b) Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal* with respect to the inner product $\langle \cdot\, , \cdot \rangle$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

(c) We'll use the terms "inner product space" or "pre–Hilbert space" to mean a vector space over $\mathbb{C}$ equipped with an inner product.

Definition A.4 (Norm)**.**
(a) A norm on a vector space $\mathcal{V}$ is a function $\mathbf{x} \in \mathcal{V} \mapsto \|\mathbf{x}\| \in [0, \infty)$ that obeys

(1) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(2) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$

(3) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha \in \mathbb{C}$.

(b) A sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ is said to be *Cauchy* with respect to the norm $\|\cdot\|$ if

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\text{ s.t. }\; m, n > N \implies \|\mathbf{v}_n - \mathbf{v}_m\| < \varepsilon$$

(c) A sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{V}$ is said to *converge* to $\mathbf{v}$ in the norm $\|\cdot\|$ if

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\text{ s.t. }\; n > N \implies \|\mathbf{v}_n - \mathbf{v}\| < \varepsilon$$

(d) A normed vector space is said to be *complete* if every Cauchy sequence converges.

(e) A subset $\mathcal{D}$ of a normed vector space $\mathcal{V}$ is said to be *dense* in $\mathcal{V}$ if $\overline{\mathcal{D}} = \mathcal{V}$, where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$. That is, if every element of $\mathcal{V}$ is a limit of a sequence of elements of $\mathcal{D}$.

Theorem A.5. *Let $\langle\ \cdot\ ,\ \cdot\rangle$ be an inner product on a vector space $\mathcal{V}$ and set $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$ for all $\mathbf{x} \in \mathcal{V}$.*

(a) The inner product is sesquilinear. That is,

$$\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}\rangle = \alpha\,\langle \mathbf{x}, \mathbf{z}\rangle + \beta\,\langle \mathbf{y}, \mathbf{z}\rangle$$
$$\langle \mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z}\rangle = \overline{\alpha}\,\langle \mathbf{x}, \mathbf{y}\rangle + \overline{\beta}\,\langle \mathbf{x}, \mathbf{z}\rangle$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$.[1]

(b) $\|\mathbf{x}\|$ is a norm.

(c) The inner product and associated norm obeys

(1) (Cauchy–Schwarz inequality) $\big|\,\langle \mathbf{x}, \mathbf{y}\rangle\,\big| \le \|\mathbf{x}\|\ \|\mathbf{y}\|$

(2) (Parallelogram law) $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$

(3) (Polarization identities)

$$\begin{aligned}
&\langle \mathbf{x}, \mathbf{y}\rangle \\
&= \tfrac{1}{2}\big\{\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\big\} + \tfrac{1}{2i}\big\{\|\mathbf{x}+i\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\big\} \\
&= \tfrac{1}{4}\big\{\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\big\} + \tfrac{1}{4i}\big\{\|\mathbf{x}+i\mathbf{y}\|^2 - \|\mathbf{x}-i\mathbf{y}\|^2\big\}
\end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$

Proof. (a) is obvious.

(b) See [**RS**, Theorem II.2] or [**Co**, Corollary 1.5].

(c) (1) See [**RS**, Corollary to Theorem II.1] or [**Co**, paragraph 1.4].

(c) (2) and (3) are obvious.

□

Lemma A.6. *Let $\|\ \cdot\ \|$ be a norm on a vector space $\mathcal{V}$. There exists an inner product $\langle\ \cdot\ ,\ \cdot\ \rangle$ on $\mathcal{V}$ such that*

$$\langle \mathbf{x}, \mathbf{x}\rangle = \|\mathbf{x}\|^2 \qquad \text{for all } \mathbf{x} \in \mathcal{V}$$

if and only if $\|\ \cdot\ \|$ obeys the parallelogram law.

[1] Physicists and mathematical physicists tend to use the convention that inner products are linear in the second argument and conjugate linear in the first.

Proof. This is Excercise A.7, below. □

Exercise A.7. Let $\| \cdot \|$ be a norm on a vector space $\mathcal{V}$. Prove that there exists an inner product $\langle \cdot , \cdot \rangle$ on $\mathcal{V}$ such that

$$\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 \qquad \text{for all } \mathbf{x} \in \mathcal{V}$$

if and only if $\| \cdot \|$ obeys the parallelogram law.

Definition A.8 (Banach Space)**.**
(a) A Banach space is a complete normed vector space.

(b) Two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be isometric if there exists a map $U : \mathcal{B}_1 \to \mathcal{B}_2$ that is

(1) linear (meaning that $U(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha U(\mathbf{x})+\beta U(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{B}_1$ and $\alpha, \beta \in \mathbb{C}$)
(2) surjective (also called onto)
(3) isometric (meaning that $\|U\mathbf{x}\|_{\mathcal{B}_2} = \|\mathbf{x}\|_{\mathcal{B}_1}$ for all $\mathbf{x} \in \mathcal{B}_1$). This implies that U is injective (also called 1–1).

Definition A.9 (Hilbert Space)**.**
(a) A Hilbert space $\mathcal{H}$ is a complex inner product space that is complete under the associated norm.

(b) Two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ are said to be isomorphic (denoted $\mathcal{H}_1 \cong \mathcal{H}_2$) if there exists a map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that is

(1) linear
(2) onto
(3) inner product preserving (meaning that $\langle U\mathbf{x}, U\mathbf{y}\rangle_{\mathcal{H}_2} = \langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{H}_1}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}_1$)

Such a map, U, is called *unitary.*

Theorem A.10 (Completion)**.** *If $\big(\mathcal{V}, \langle \cdot , \cdot \rangle_{\mathcal{V}}\big)$ is any inner product space, then there exists a Hilbert space $\big(\mathcal{H}, \langle \cdot, , \cdot \rangle_{\mathcal{H}}\big)$ and a map $U : \mathcal{V} \to \mathcal{H}$ such that*

(1) U is 1–1,
(2) U is linear,
(3) $\langle U\mathbf{x}, U\mathbf{y}\rangle_{\mathcal{H}} = \langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{V}}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and
(4) $U(\mathcal{V}) = \big\{ \ U\mathbf{x} \ \big| \ \mathbf{x} \in \mathcal{V} \ \big\}$ is dense in $\mathcal{H}$. If $\mathcal{V}$ is complete, then $U(\mathcal{V}) = \mathcal{H}$.

$\mathcal{H}$ is called the completion of $\mathcal{V}$.

Proof. See [**RS**, Theorem I.3 and Problem 1 of Chapter II]. □

Example A.11.
(a) $\mathbb{C}^n = \left\{ \mathbf{x} = (x_1, \cdots x_n) \,\middle|\, x_1, \cdots x_n \in \mathbb{C} \right\}$ together with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum\limits_{\ell=1}^{n} x_\ell \bar{y}_\ell$ is a Hilbert space.

(b) If $1 \le p < \infty$, then $\ell^p = \left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, \sum_{n=1}^\infty |x_n|^p < \infty \right\}$ together with the norm $\big\|(x_n)_{n\in\mathbb{N}}\big\|_p = \left[\sum_{n=1}^\infty |x_n|^p\right]^{1/p}$ is a Banach space. Here, and in the next two examples, each entry x_n in the sequence $(x_n)_{n\in\mathbb{N}}$ is to be a complex number.

(c) $\ell^2 = \left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, \sum_{n=1}^\infty |x_n|^2 < \infty \right\}$ is a Hilbert space with the inner product $\big\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \big\rangle = \sum_{n=1}^\infty x_n\, \overline{y_n}$.

(d) $\ell^\infty = \left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, \sup\limits_n |x_n| < \infty \right\}$ and $c_0 = \left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, \lim\limits_{n\to\infty} x_n = 0 \right\}$ are both Banach spaces with the norm $\big\|(x_n)_{n\in\mathbb{N}}\big\|_\infty = \sup\limits_n |x_n|$.

(e) Let $\mathcal{X}$ be a metric space (or more generally a topological space) and

$$\begin{aligned} C(\mathcal{X}) &= \left\{ f : \mathcal{X} \to \mathbb{C} \,\middle|\, f \text{ continuous, bounded} \right\} \\ C_0(\mathcal{X}) &= \left\{ f : \mathcal{X} \to \mathbb{C} \,\middle|\, f \text{ continuous, compact support} \right\} \end{aligned}$$

If $\mathcal{X}$ is a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ for some $n \in \mathbb{N}$, let

$$C_\infty(\mathcal{X}) = \left\{ f : \mathcal{X} \to \mathbb{C} \,\middle|\, f \text{ continuous, } \lim_{|x|\to\infty} f(x) = 0 \right\}$$

Then $C(\mathcal{X})$ and $C_\infty(\mathcal{X})$ are Banach spaces with the norm $\|f\| = \sup\limits_{x\in\mathcal{X}} |f(x)|$. $C_0(\mathcal{X})$ is a normed vector space, but need not be complete.

(f) Let $1 \le p \le \infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space, with X a set, $\mathcal{M}$ a σ–algebra and μ a measure. For $p < \infty$, set

$$\begin{aligned} \mathcal{L}^p(X, \mathcal{M}, \mu) &= \left\{ \varphi : X \to \mathbb{C} \,\middle|\, \varphi \ \mathcal{M}\text{–measurable, } \textstyle\int |\varphi(x)|^p \, d\mu(x) < \infty \right\} \\ \|\varphi\|_p &= \left[\int |\varphi(x)|^p \, d\mu(x) \right]^{1/p} \end{aligned}$$

For $p = \infty$, set[2]

$$\begin{aligned} \mathcal{L}^\infty(X, \mathcal{M}, \mu) &= \left\{ \varphi : X \to \mathbb{C} \,\middle|\, \varphi \ \mathcal{M}\text{–measurable, } \operatorname{ess\,sup} |\varphi(x)| < \infty \right\} \\ \|\varphi\|_\infty &= \operatorname{ess\,sup} |\varphi(x)| \end{aligned}$$

This is not quite a Banach space because any function φ that is zero almost everywhere has "norm" zero. So we define an equivalence relation on

[2]The essential supremum of $|\varphi|$, with respect to the measure μ, is denoted $\operatorname{ess\,sup}_{x\in X} |\varphi(x)|$ and is defined to be $\inf\{\, a \ge 0 \mid |\varphi(x)| \le a$ almost everywhere with respect to $\mu \,\}$.

$\mathcal{L}^p(X, \mathcal{M}, \mu)$ by

$$\varphi \sim \psi \iff \varphi = \psi \text{ a.e.}$$

As usual, the equivalence class of $\varphi \in \mathcal{L}^p(X, \mathcal{M}, \mu)$ is

$$[\varphi] = \left\{ \ \psi \in \mathcal{L}^p(X, \mathcal{M}, \mu) \ \middle| \ \psi \sim \varphi \ \right\}$$

Then

$$L^p(X, \mathcal{M}, \mu) = \left\{ \ [\varphi] \ \middle| \ \varphi \in \mathcal{L}^p(X, \mathcal{M}, \mu) \ \right\}$$

is a Banach space with

$$[\varphi] + [\psi] = [\varphi + \psi] \qquad a[\varphi] = [a\varphi] \qquad \big\|[\varphi]\big\|_p = \|\varphi\|_p$$

for all $\varphi, \psi \in \mathcal{L}^p(X, \mathcal{M}, \mu)$ and $a \in \mathbb{C}$, and $L^2(X, \mathcal{M}, \mu)$ is a Hilbert space with inner product

$$\langle [\varphi], [\psi] \rangle = \int \varphi(x)\, \overline{\psi(x)}\ d\mu(x)$$

for all $\varphi, \psi \in \mathcal{L}^2(X, \mathcal{M}, \mu)$. It is standard to write φ in place of $[\varphi]$.

(g) Let D be an open subset of $\mathbb{C}$. Then

$$A^2(D) = \left\{ \ \varphi : D \to \mathbb{C} \ \middle| \ \varphi \text{ analytic}, \quad \textstyle\int_D |\varphi(x+iy)|^2\ dx\, dy < \infty \ \right\}$$

is a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle = \int_D \varphi(x+iy)\, \overline{\psi(x+iy)}\ dxdy$$

(h) Let $\ell \geq 0$ be an integer and Ω be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we shall use $\partial^\alpha \varphi(x)$ to denote the partial derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \varphi(x)$. The order of this partial derivative is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define

$$\|\varphi\|_{H^\ell(\Omega)} = \left\{ \sum_{|\alpha| \leq \ell} \int_\Omega \left|\partial^\alpha \varphi(x)\right|^2\ d^n x \right\}^{1/2}$$

for each $\varphi \in C^\ell(\Omega)$ for which the right hand side is finite. The Sobolev space $H^\ell(\Omega)$ is the completion of the vector space $\left\{ \ \varphi \in C^\ell(\Omega) \ \middle| \ \|\varphi\|_{H^\ell(\Omega)} < \infty \ \right\}$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{H^\ell(\Omega)} = \sum_{|\alpha| \leq \ell} \int_\Omega \partial^\alpha \varphi(x)\, \overline{\partial^\alpha \psi(x)}\ d^n x$$

Similarly, $H_0^\ell(\Omega)$ is the completion of $C_0^\infty(\Omega)$.

Theorem A.12. *Let $-\infty < a < b < \infty$ and $1 \le p < \infty$. The following sets of functions are dense in $L^p\big([a,b]\big)$.*

(a) simple functions (functions of the form $\sum_{j=1}^n a_j \chi_{E_j}(x)$ with $n \in \mathbb{N}$ and the sets E_j measurable)

(b) step functions (functions of the form $\sum_{j=1}^n a_j \chi_{E_j}(x)$ with $n \in \mathbb{N}$ and the sets E_j intervals)

(c) continuous functions that vanish at a and b

(d) periodic C^∞ functions of period $b-a$

(e) C^∞ functions that are supported in (a,b)

Here $\chi_E(x)$ denotes the characteristic function of the set E.

Proof. See Exercises A.13 and A.14, below. □

Exercise A.13. Let $\epsilon > 0$ and $-\infty < a < b < \infty$. Let m be Lebesgue measure and $f : [a,b] \to \mathbb{R}$ be a Lebesgue–measurable function.

(a) Prove that there exists an $M \in [0,\infty)$ such that

$$m\big\{\ x \in [a,b]\ \big|\ |f(x)| \ge M\ \big\} \le \epsilon$$

(b) Assume that $f : [a,b] \to [c,C]$. Prove that there exists a simple function s such that $c \le s(x) \le C$ and $|f(x) - s(x)| \le \epsilon$ for all $x \in [a,b]$. A simple function is, by definition, of the form $\sum_{j=1}^n a_j \chi_{E_j}(x)$ with $n \in \mathbb{N}$ and the sets E_j measurable.

(c) Let $s : [a,b] \to [c,C]$ be a simple function. Prove that there exists a step function $g : [a,b] \to [c,C]$ such that the measure

$$m\big\{\ x \in [a,b]\ \big|\ s(x) \ne g(x)\ \big\} < \epsilon$$

A step function is, by definition, of the form $\sum_{i=1}^n a_i \chi_{E_i}(x)$ with $n \in \mathbb{N}$ and the sets E_i intervals.

(d) Let $g : [a,b] \to [c,C]$ be a step function. Prove that there exists a continuous function $h : [a,b] \to [c,C]$ such that $h(a) = h(b) = 0$ and

$$m\big\{\ x \in [a,b]\ \big|\ g(x) \ne h(x)\ \big\} < \epsilon$$

Exercise A.14. Let $-\infty < a < b < \infty$ and $1 \le p < \infty$. Prove that the following sets of functions are dense in $L^p\big([a,b]\big)$.

(a) simple functions

(b) step functions

(c) continuous functions that vanish at a and b

(d) periodic C^∞ functions of period $b-a$

(e) C^∞ functions that are supported in (a,b)

Definition A.15 (Basis)**.** Let $\mathcal{B}$ be a Banach space and $\mathcal{H}$ a Hilbert space.

(a) A subset S of $\mathcal{H}$ is an *orthonormal subset* if each vector in S is of length one and each pair of distinct vectors in S is orthogonal.

(b) An *orthonormal basis* (or complete orthonormal system) for $\mathcal{H}$ is an orthonormal subset of $\mathcal{H}$, which is maximal in the sense that it is not properly contained in any other orthonormal subset of $\mathcal{H}$.

(c) A *Schauder basis* for $\mathcal{B}$ is a sequence $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{B}$ such that for each $v \in \mathcal{B}$ there is a unique sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $v = \sum_{n=1}^\infty \alpha_n \mathbf{e}_n$.

(d) An *algebraic basis* (or Hamel basis) for $\mathcal{B}$ is a subset $\mathcal{S} \subset \mathcal{B}$ such that each $\mathbf{x} \in \mathcal{B}$ has a unique representation as a finite linear combination of elements of $\mathcal{S}$. This is the case if and only if every finite subset of $\mathcal{S}$ is linearly independent and each $\mathbf{x} \in \mathcal{B}$ has some representation as a finite linear combination of elements of $\mathcal{S}$.

Theorem A.16. *Every Hilbert space has an orthonormal basis.*

Proof. See [**RS**, Theorem II.5] or [**Co**, Proposition 4.2]. □

Theorem A.17. *Every vector space has an algebraic basis.*

Proof. This is Exercise A.19, below. □

Theorem A.18. *Let $\{\mathbf{e}_i\}_{i\in\mathcal{I}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Then, for each $\mathbf{x} \in \mathcal{H}$, $\{\ i \in \mathcal{I}\ \big|\ \langle \mathbf{e}_i, \mathbf{x}\rangle \neq 0\ \}$ is countable[3] and*

$$\mathbf{x} = \sum_{i\in\mathcal{I}} \langle \mathbf{x}, \mathbf{e}_i\rangle\ \mathbf{e}_i \qquad \|\mathbf{x}\|^2 = \sum_{i\in\mathcal{I}} |\ \langle \mathbf{x}, \mathbf{e}_i\rangle\ |^2$$

(The right hand sides converge independent of order.)

Conversely, if $\{c_i\}_{i\in\mathcal{I}} \subset \mathbb{C}$ and $\sum_{i\in\mathcal{I}} |c_i|^2 < \infty$, then $\sum_{i\in\mathcal{I}} c_i\,\mathbf{e}_i$ converges to an element of $\mathcal{H}$.

Proof. See [**RS**, Theorem II.6] or [**Co**, Theorem 4.13]. □

Exercise A.19. Prove that every vector space has an algebraic basis.

Hint: Use Zorn's Lemma (which is equivalent to the axiom of choice). It says that if a nonempty set $\mathfrak{S}$

(1) is partially ordered and

[3] We'll include finite in countable.

(2) has the property that every linearly ordered subset has an upper bound

then $\mathfrak{S}$ has a maximal element.

Exercise A.20. Prove that $B = \big\{\mathbf{e}_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}\big\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2\big([0, 2\pi]\big)$.

Definition A.21 (Separable)**.** A metric space is said to be separable if it has a countable dense subset.

Lemma A.22. *A metric space $\big(\mathcal{M}, d\big)$ fails to be separable if and only if there is an $\varepsilon > 0$ and an uncountable subset $\big\{m_i\big\}_{i\in\mathcal{I}} \subset$ with $d(m_i, m_j) \geq \varepsilon$ for all $i, j \in \mathcal{I}$ with $i \neq j$.*

Proof. This is Exercise A.23, below. $\square$

Exercise A.23. Prove that a metric space $\big(\mathcal{M}, d\big)$ fails to be separable if and only if there is an $\varepsilon > 0$ and an uncountable subset $\big\{m_i\big\}_{i\in\mathcal{I}} \subset \mathcal{M}$ with $d(m_i, m_j) \geq \varepsilon$ for all $i, j \in \mathcal{I}$ with $i \neq j$.

Theorem A.24. *Let $\mathcal{H}$ be a Hilbert space.*

(a) $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis.

(b) If $\dim \mathcal{H} = n \in \mathbb{N}$, then $\mathcal{H} \cong \mathbb{C}^n$.

(c) If $\mathcal{H}$ is separable but is not of finite dimension, then $\mathcal{H} \cong \ell^2$.

Proof. See [**RS**, Theorem II.7] or [**Co**, Theorem 5.4 and Corollary 5.5] $\square$

Example A.25.
(a) As $L^2([0, 2\pi])$ has a countable, orthonormal basis, it is separable and isomorphic to ℓ^2.

(b) ℓ^∞ is not separable. To see this define, for each subset $S \subset \mathbb{N}$,

$$x^{(S)} = \big(x_n^{(S)}\big)_{n\in\mathbb{N}} \in \ell^\infty$$

by

$$x_n^{(S)} = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases}$$

This is an uncountable family of elements of ℓ^∞ with $\|x^{(S)} - x^{(T)}\|_\infty = 1$ for all distinct subsets S, T of $\mathbb{N}$.

Definition A.26 (Orthogonal Complement)**.** The orthogonal complement, $\mathcal{M}^\perp$, of any subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, is defined to be

$$\mathcal{M}^\perp = \big\{\ \mathbf{y} \in \mathcal{H} \ \big|\ \langle \mathbf{x}, \mathbf{y}\rangle = 0 \text{ for all } \mathbf{x} \in \mathcal{M}\ \big\}$$

Theorem A.27. *Let $\mathcal{M}$ be a linear subspace of a Hilbert space $\mathcal{H}$. Then*

(a) $\mathcal{M}^\perp$ is a closed linear subspace of $\mathcal{H}$.

(b) $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$

(c) $\left(\mathcal{M}^\perp\right)^\perp = \overline{\mathcal{M}}$ (the closure of $\mathcal{M}$)

Proof. See [**RS**, Problem 6 of Chapter 2] and [**Co**, Corollary 2.9]. □

Theorem A.28 (Projection)**.** *Let $\mathcal{M}$ be a closed linear subspace of a Hilbert space $\mathcal{H}$. Then each $\mathbf{x} \in \mathcal{H}$ has a unique representation $\mathbf{x} = \mathbf{x}^\parallel + \mathbf{x}^\perp$ with $\mathbf{x}^\parallel \in \mathcal{M}$ and $\mathbf{x}^\perp \in \mathcal{M}^\perp$.*

Proof. See [**RS**, Theorem II.3] or [**Co**, Theorem 2.6]. □

A.2. Bounded Linear Operators

Definition A.29 (Linear Operator)**.** Let $\mathcal{B}$, $\tilde{\mathcal{B}}$ be Banach spaces and $\mathcal{H}$, $\tilde{\mathcal{H}}$ be Hilbert spaces.

(a) Let $\mathcal{D}$ be a linear subspace of $\mathcal{B}$. A map $A : \mathcal{D} \to \tilde{\mathcal{B}}$ is called a *linear operator* if it obeys

$$A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A(\mathbf{x}) + \beta A(\mathbf{y}) \qquad \text{for all } \alpha, \beta \in \mathbb{C} \text{ and } \mathbf{x}, \mathbf{y} \in \mathcal{D}$$

One usually denotes the image of $\mathbf{x}$ under A as $A\mathbf{x}$, rather than $A(\mathbf{x})$. The set $\mathcal{D}$ is called the domain of A and is generally denoted $\mathcal{D}(A)$. One often calls A a "linear operator on $\mathcal{B}$" even when its domain is a proper subset of $\mathcal{B}$.

(b) A linear operator $A : \mathcal{D} \to \tilde{\mathcal{B}}$ is said to be *bounded* if

$$\|A\| = \sup_{\mathbf{0} \neq \mathbf{x} \in \mathcal{D}} \frac{\|A\mathbf{x}\|_{\tilde{\mathcal{B}}}}{\|\mathbf{x}\|_{\mathcal{B}}} = \sup_{\substack{\mathbf{x} \in \mathcal{D} \\ \|\mathbf{x}\|_{\mathcal{B}} = 1}} \|A\mathbf{x}\|_{\tilde{\mathcal{B}}} \tag{A.1}$$

is finite. The set of all bounded, linear operators defined on $\mathcal{B}$ and taking values in $\tilde{\mathcal{B}}$ is denoted $\mathcal{L}(\mathcal{B}, \tilde{\mathcal{B}})$. With the norm (A.1), it is itself a Banach space. The set of all bounded, linear operators defined on $\mathcal{B}$ and taking values in $\mathcal{B}$ is denoted $\mathcal{L}(\mathcal{B})$.

(c) A *linear functional* on $\mathcal{B}$ is a linear operator $f : \mathcal{B} \to \mathbb{C}$. A *bounded linear functional* on $\mathcal{B}$ is a linear operator $f : \mathcal{B} \to \mathbb{C}$ for which

$$\sup_{\mathbf{0} \neq \mathbf{x} \in \mathcal{B}} \frac{|f(\mathbf{x})|}{\|\mathbf{x}\|_{\mathcal{B}}}$$

is finite.

(d) The *dual space* of a Banach space $\mathcal{B}$ is the space $\mathcal{B}'$ of all bounded linear functionals on $\mathcal{B}$. The dual space is itself a Banach space.

(e) Let $T : \mathcal{D}(T) \subset \mathcal{H} \to \tilde{\mathcal{H}}$ be a linear operator. Denote

$$\mathcal{D}(T^*) = \left\{ \varphi \in \tilde{\mathcal{H}} \;\middle|\; \exists!\ \eta \in \mathcal{H} \ \text{ s.t. } \ \langle \varphi, T\psi\rangle_{\tilde{\mathcal{H}}} = \langle \eta, \psi\rangle_{\mathcal{H}} \quad \forall \ \ \psi \in \mathcal{D}(T) \right\}$$

If $\varphi \in \mathcal{D}(T^*)$ the corresponding η is denoted $T^*\varphi$. Thus $T^*\varphi$ is the unique vector in $\mathcal{H}$ such that

$$\langle \varphi, T\psi\rangle_{\tilde{\mathcal{H}}} = \langle T^*\varphi, \psi\rangle_{\mathcal{H}} \qquad \text{for all } \psi \in \mathcal{D}(T)$$

The operator T^* is called the adjoint of T.

Proposition A.30. *The normed vector space $\mathcal{L}(\mathcal{B}, \tilde{\mathcal{B}})$, with the norm* (A.1)*, is a Banach space.*

Proof. See [**RS**, Theorem III.2]. □

Lemma A.31. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then there is a linear operator $W : \mathcal{H} \to \mathcal{H}$ which is defined on all of $\mathcal{H}$, but is* **not** *bounded.*

Proof. This is Exercise A.32, below. □

Exercise A.32. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Construct a linear operator $W : \mathcal{H} \to \mathcal{H}$ which is defined on all of $\mathcal{H}$, but is **not** bounded. (Hint: use an algebraic basis.)

Example A.33. (a) *Matrices*: Let $n \in \mathbb{N}$. An $n \times n$ matrix $\big[M_{i,j}\big]_{1\le i,j\le n}$ is naturally associated to the operator $M : \mathbb{C}^n \to \mathbb{C}^n$ determined by

$$(M\mathbf{x})_i = \sum_{j=1}^{n} M_{i,j} x_j$$

The adjoint operator is associated to the matrix $\big[M^*_{i,j} = \overline{M_{j,i}}\,\big]_{1\le i,j\le n}$.

(b) *Multiplication Operators*: Let $1 \le p < \infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$ be measurable. If the essential supremum of f is finite, then

$$\begin{aligned} M_f : L^p(X, \mathcal{M}, \mu) &\to L^p(X, \mathcal{M}, \mu) \\ \varphi(x) &\mapsto (f\varphi)(x) = f(x)\,\varphi(x) \end{aligned}$$

is a bounded linear operator with $\|M_f\| = \operatorname*{ess\,sup}_{x\in X} |f(x)|$. On the other hand, if the essential supremum of f is infinite, then M_f will not be defined on all of $L^p(X,\mathcal{M},\mu)$ (as a map into $L^p(X,\mathcal{M},\mu)$) and will not be bounded (as a map into $L^p(X,\mathcal{M},\mu)$). In the case $p=2$, $M_f^* = M_{\overline{f}}$.

(c) *Projection Operators*: Let $\mathcal{H}$ be Hilbert space and let $\mathcal{M}$ be a nonempty, closed, linear subspace of $\mathcal{H}$. Define the map $P:\mathcal{H}\to\mathcal{H}$ by

$$P\mathbf{x} = \mathbf{x}^{\parallel}$$

where $\mathbf{x} = \mathbf{x}^{\perp} + \mathbf{x}^{\parallel}$ is the decomposition of Theorem A.28.a It is a bounded linear operator with $\|P\| = 1$, called the orthogonal projection on $\mathcal{M}$. It obeys

$$P^2 = P \qquad P^* = P$$

where P^* is the adjoint of P. We'll see, in Lemma A.35 below, that conversely, if $P:\mathcal{H}\to\mathcal{H}$ is a bounded linear operator that obeys $P^2 = P$ and $P^* = P$, then P is orthogonal projection on $\mathcal{M} = \mathrm{range}(P)$.

(d) *Integral Operators*: Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces and $T : X\times Y\to\mathbb{C}$ be a function that is measurable with respect to $\mathcal{M}\otimes\mathcal{N}$. Let $1\le p\le\infty$ and $\varphi\in L^p(Y,\mathcal{N},\nu)$. Define, for each $x\in X$ for which the function $y\mapsto T(x,y)\varphi(y)$ is in $L^1(Y,\mathcal{N},\nu)$,

$$(T\varphi)(x) = \int_Y T(x,y)\varphi(y)\ d\nu(y) \tag{A.2}$$

(1) If

$$M_1 = \operatorname*{ess\ sup}_{x\in X}\int_Y |T(x,y)|\ d\nu(y) < \infty$$
$$M_2 = \operatorname*{ess\ sup}_{y\in Y}\int_X |T(x,y)|\ d\mu(x) < \infty$$

then (A.2) is a bounded operator $T : L^p(Y,\mathcal{N},\nu)\to L^p(X,\mathcal{M},\mu)$ with norm $\|T\|\le M_1^{1-\frac{1}{p}} M_2^{\frac{1}{p}}$.

(2) If the Hilbert–Schmidt norm

$$\|T\|_{\mathrm{H.S.}} = \Big[\int_{X\times Y} |T(x,y)|^2\ d\mu\times\nu(x,y)\Big]^{1/2} < \infty$$

then (A.2) is a bounded operator $T : L^2(Y,\mathcal{N},\nu)\to L^2(X,\mathcal{M},\mu)$ with norm $\|T\|\le\|T\|_{\mathrm{H.S.}}$.

In the case $p=2$,

$$(T^*\psi)(y) = \int_x \overline{T(x,y)}\psi(x)\ d\mu(x)$$

(e) *Differential Operators*: Let Ω be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Recall that if $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we use $\partial^\alpha u(x)$ to denote the partial derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ to denote the order of this partial derivative. For any finite subset $\mathcal{I} \subset \mathbb{N}_0^n$ and any family $\big\{a_\alpha(x)\big\}_{\alpha\in\mathcal{I}}$ of bounded, measurable functions on Ω the map

$$\varphi(x) \mapsto \sum_{\alpha\in\mathcal{I}} a_\alpha(x)\, \partial^\alpha \varphi(x)$$

is a linear map on $C^\infty(\Omega) \subset L^2(\Omega)$. But it is not bounded as a map from $L^2(\Omega)$ to $L^2(\Omega)$.

Exercise A.34. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ–finite measure spaces and $T : X \times Y \to \mathbb{C}$ be a function that is measurable with respect to $\mathcal{M} \otimes \mathcal{N}$.

(a) Assume that

$$M_1 = \operatorname*{ess\,sup}_{x\in X} \int_Y |T(x, y)|\ d\nu(y) < \infty$$
$$M_2 = \operatorname*{ess\,sup}_{y\in Y} \int_X |T(x, y)|\ d\mu(x) < \infty$$

Let $1 \le p \le \infty$. Prove that

$$(T\varphi)(x) = \int_Y T(x, y)\varphi(y)\ d\nu(y)$$

defines a bounded operator $T : L^p(Y, \mathcal{N}, \nu) \to L^p(X, \mathcal{M}, \mu)$ with norm $\|T\| \le M_1^{1-\frac{1}{p}} M_2^{\frac{1}{p}}$.

(b) Assume that

$$\|T\|_{\mathrm{H.S.}} = \Big[\int_{X\times Y} |T(x, y)|^2\ d\,\mu \times \nu\,(x, y)\Big]^{1/2} < \infty$$

Prove that $(T\varphi)(x) = \int_Y T(x, y)\varphi(y)\ d\nu(y)$ defines a bounded operator from $L^2(Y, \mathcal{N}, \nu)$ to $L^2(X, \mathcal{M}, \mu)$ with norm $\|T\| \le \|T\|_{\mathrm{H.S.}}$.

Lemma A.35. *Let $\mathcal{H}$ be a Hilbert space. Let $P : \mathcal{H} \to \mathcal{H}$ be a bounded operator that obeys*

$$P^2 = P \qquad P^* = P$$

Then P is orthogonal projection on the range of P.

Proof. This is Exercise A.36, below. □

Exercise A.36. Let $\mathcal{H}$ be a Hilbert space. Let $P : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator that obeys

$$P^2 = P \qquad P^* = P$$

Prove that P is orthogonal projection on the range of P.

Lemma A.37. *Let E and F be orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Then $E + F$ is again an orthogonal projection if and only if $EF = FE = 0$.*

Proof. This is Exercise A.38, below. □

Exercise A.38. Let E and F be orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Prove that $E + F$ is again an orthogonal projection if and only if $EF = FE = 0$. What is the geometric significance of the condition $EF = FE = 0$?

Theorem A.39. *Let $\mathcal{V}$ and $\tilde{\mathcal{V}}$ be normed vector spaces and let $T : \mathcal{V} \to \tilde{\mathcal{V}}$ be a linear operator. The following are equivalent.*

(1) T is continuous at every $\mathbf{x} \in \mathcal{V}$.

(2) T is continuous at one $\mathbf{x}_0 \in \mathcal{V}$.

(3) T is bounded.

Proof. The proof is trivial. □

Theorem A.40. *Let $\mathcal{B}$ be a Banach space.*

(a) Let $\mathcal{S}$ be a subspace of $\mathcal{B}$ and $\lambda \in \mathcal{S}'$. Then there is a $\Lambda \in \mathcal{B}'$ such that $\|\Lambda\|_{\mathcal{B}'} = \|\lambda\|_{\mathcal{S}'}$ and $\Lambda(\mathbf{x}) = \lambda(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{S}$.

(b) Let $\mathbf{x} \in \mathcal{B}$. There is a nonzero $\Lambda \in \mathcal{B}'$ such that $\big|\Lambda(\mathbf{x})\big| = \|\Lambda\|_{\mathcal{B}'} \|\mathbf{x}\|_{\mathcal{B}}$.

(c) Let $\mathcal{Y}$ be a subspace of $\mathcal{B}$ and $\mathbf{x} \in \mathcal{B}$ with the distance from $\mathbf{x}$ to $\mathcal{Y}$ being d. There is a $\Lambda \in \mathcal{B}'$ such that $\|\Lambda\|_{\mathcal{B}'} \le 1$, $\Lambda(\mathbf{x}) = d$ and $\Lambda(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{Y}$.

(d) Let $\mathbf{x} \in \mathcal{B}$. Then

$$\|\mathbf{x}\|_{\mathcal{B}} = \sup_{\substack{\Lambda \in \mathcal{B}' \\ \|\Lambda\|_{\mathcal{B}'} = 1}} \big|\Lambda(\mathbf{x})\big|$$

Proof. Part (a) is [**RS**, Corollary 1 of Theorem III.6 (the Hahn–Banach theorem)]. The other parts follow easily from part (a) and, in the case of part (d), the definition of $\|\Lambda\|_{\mathcal{B}'}$ and part (b). □

Theorem A.41 (The B.L.T. Theorem)**.** *Let $\mathcal{V}$ be a dense linear subspace of a Banach space $\mathcal{B}$. Let $\tilde{\mathcal{B}}$ be a second Banach space and $T : \mathcal{V} \to \tilde{\mathcal{B}}$ be a bounded linear operator. Then there is a unique bounded linear operator $\tilde{T} : \mathcal{B} \to \tilde{\mathcal{B}}$ such that $T\mathbf{x} = \tilde{T}\mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$. Furthermore $\|T\| = \|\tilde{T}\|$.*

Proof. See [**RS**, Theorem I.7]. □

Example A.42. We define the Fourier transform as a unitary operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. To start we define Schwartz space to be

$$\mathcal{S}(\mathbb{R}) = \left\{ \varphi : \mathbb{R} \to \mathbb{C} \;\middle|\; \varphi \text{ is } C^\infty,\ \|\varphi\|_{n,m} < \infty \text{ for all integers } n, m \geq 0 \right\}$$

where $\|\varphi\|_{n,m} = \sup_{x\in\mathbb{R}} \left| x^n \frac{d^m\varphi}{dx^m}(x) \right|$. Next we define the Fourier transform and inverse Fourier transform on $\mathcal{S}(\mathbb{R})$ by

$$\hat{\varphi}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} \varphi(x)\, dx$$
$$\check{\psi}(x) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \psi(\xi)\, d\xi$$

and verify that the linear operators $\varphi \mapsto \hat{\varphi}$ and $\psi \mapsto \check{\psi}$ each map $\mathcal{S}(\mathbb{R})$ into (in fact onto) $\mathcal{S}(\mathbb{R})$ and are inverses of each other and obey

$$\int_{-\infty}^{\infty} \varphi(x)\, \overline{\psi(x)}\, dx = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(\xi)\, \overline{\hat{\psi}(\xi)}\, d\xi$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$. Then the B.L.T. theorem provides us with the unique bounded extension of the map $\varphi \mapsto \hat{\varphi}$ to $L^2(\mathbb{R})$, which we call $\mathcal{F}$. For the details, see Appendix B.

Theorem A.43 (Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and $\lambda \in \mathcal{H}^*$ be a bounded linear functional on $\mathcal{H}$. Then there is a unique $\mathbf{y}_\lambda \in \mathcal{H}$ such that*

$$\lambda(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y}_\lambda \rangle$$

for all $\mathbf{x} \in \mathcal{H}$. Furthermore $\|\lambda\|_{\mathcal{H}^*} = \|\mathbf{y}_\lambda\|_{\mathcal{H}}$.

Proof. See [**RS**, Theorem II.4] or [**Co**, Theorem 3.4]. □

Corollary A.44. *Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ and $C \geq 0$ obey*

(1) $B(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha B(\mathbf{x}, \mathbf{z}) + \beta B(\mathbf{y}, \mathbf{z})$

(2) $B(\mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z}) = \bar{\alpha} B(\mathbf{x}, \mathbf{y}) + \bar{\beta} B(\mathbf{x}, \mathbf{z})$

(3) $\left| B(\mathbf{x}, \mathbf{y}) \right| \leq C \|\mathbf{x}\|\, \|\mathbf{y}\|$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. Then there is a unique $A \in \mathcal{L}(\mathcal{H})$ such that $B(\mathbf{x}, \mathbf{y}) = \langle A\mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Furthermore $\|A\| \leq C$.

Corollary A.45. *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be Hilbert spaces and $T : \mathcal{H} \to \tilde{\mathcal{H}}$ be a bounded linear operator. Then the adjoint T^* of T is a bounded linear operator defined on all of $\tilde{\mathcal{H}}$.*

Proof. This is Exercise A.46, below. □

Exercise A.46. Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be Hilbert spaces and $T : \mathcal{H} \to \tilde{\mathcal{H}}$ be a bounded linear operator. Prove that the adjoint, T^*, of T is a bounded linear operator defined on all of $\tilde{\mathcal{H}}$.

Definition A.47 (Operator Topologies)**.** Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be Banach spaces. Let $T : \mathcal{B} \to \tilde{\mathcal{B}}$ and, for each $n \in \mathbb{N}$, $T_n : \mathcal{B} \to \tilde{\mathcal{B}}$ be bounded linear operators.

(a) The sequence of $\{T_n\}_{n\in\mathbb{N}}$ of operators is said to converge *uniformly* or *in norm* to T if

$$\lim_{n\to\infty} \|T - T_n\| = 0$$

(b) The sequence of $\{T_n\}_{n\in\mathbb{N}}$ of operators is said to converge *strongly* to T if

$$\lim_{n\to\infty} \|T\mathbf{x} - T_n\mathbf{x}\|_{\tilde{\mathcal{B}}} = 0 \qquad \text{for each } \mathbf{x} \in \mathcal{B}$$

(c) The sequence of $\{T_n\}_{n\in\mathbb{N}}$ of operators is said to converge *weakly* to T if

$$\lim_{n\to\infty} \ell\big(T_n\mathbf{x}\big) = \ell\big(T\mathbf{x}\big) \qquad \text{for each } \mathbf{x} \in \mathcal{B} \text{ and each } \ell \in \tilde{\mathcal{B}}'$$

In the event that $\tilde{\mathcal{B}}$ is a Hilbert space, this is equivalent to

$$\lim_{n\to\infty} \langle T_n\mathbf{x}, \mathbf{y}\rangle_{\tilde{\mathcal{B}}} = \langle T\mathbf{x}, \mathbf{y}\rangle_{\tilde{\mathcal{B}}} \qquad \text{for each } \mathbf{x} \in \mathcal{B} \text{ and each } \mathbf{y} \in \tilde{\mathcal{B}}$$

Remark A.48 (Operator Topologies)**.** Since

$$\big|\ell\big((T_n - T)\mathbf{x}\big)\big| \le \|\ell\|_{\tilde{\mathcal{B}}'} \big\|(T_n - T)\mathbf{x}\big\|_{\tilde{\mathcal{B}}}$$

and

$$\big\|(T_n - T)\mathbf{x}\big\|_{\tilde{\mathcal{B}}} \le \|T_n - T\|\, \|\mathbf{x}\|_{\mathcal{B}}$$

we have

$$\text{norm convergence} \implies \text{strong convergence} \implies \text{weak convergence}$$

In general the other implications are false, unless $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are finite dimensional. This is illustrated by the following

Example A.49 (Operator Topologies)**.** Let $\mathcal{B} = \tilde{\mathcal{B}} = \ell^2$.

(a) Let

$$P_n\big(x_1, x_2, x_3 \cdots\big) = \big(\overbrace{0, \cdots, 0}^{n \text{ places}}, x_{n+1}, x_{n+2}, x_{n+3}, \cdots\big)$$

be projection on the orthogonal complement of the first n components. Then for each fixed $\mathbf{x} \in \ell^2$, $\lim_{n\to\infty} P_n\mathbf{x} = \mathbf{0}$ so that P_n converges strongly to 0 as $n \to \infty$. But, for any $n > m$,

$$(P_n - P_m)\big(x_1, x_2, x_3 \cdots\big) = \big(\overbrace{0, \cdots, 0}^{m \text{ places}}, x_{m+1}, x_{m+2}, \cdots, x_n, 0, \cdots\big)$$

so that there is a vector $\mathbf{x} \in \ell^2$ with $(P_n - P_m)\mathbf{x} = \mathbf{x}$. Consequently $\|P_n - P_m\| = 1$ and the sequence $\{P_n\}_{n\in\mathbb{N}}$ is not Cauchy and does not converge in norm.

(b) Let

$$R_n\big(x_1, x_2, x_3 \cdots\big) = \big(\overbrace{0, \cdots, 0}^{n \text{ places}}, x_1, x_2, x_3, \cdots\big)$$

be right shift by n places. For any $\mathbf{x}, \mathbf{y} \in \ell^2$

$$\big|\langle R_n\mathbf{x}, \mathbf{y}\rangle\big| = \big|\langle R_n\mathbf{x}, P_n\mathbf{y}\rangle\big| \le \|R_n\mathbf{x}\|\,\|P_n\mathbf{y}\| = \|\mathbf{x}\|\,\|P_n\mathbf{y}\| \xrightarrow{n\to\infty} 0$$

So R_n converges weakly to zero as $n \to \infty$. On the other hand, $\|R_n\mathbf{x}\| = \|\mathbf{x}\|$ for all $n \in \mathbb{N}$ and $\mathbf{x} \in \ell^2$. So the R_n does not converge strongly or in norm. (If R_n did converge either strongly or in norm to some R, the fact that $R_n \xrightarrow{\text{weakly}} 0$ would force $R = 0$.)

Theorem A.50 (Neumann Expansion)**.** *Let T be a bounded linear operator on the Banach space $\mathcal{B}$ whose operator norm $\|T\| < 1$. Then $\mathbb{1}-T$ is bijective and has a bounded inverse and furthermore*

$$(\mathbb{1} - T)^{-1} = \sum_{n=0}^{\infty} T^n \qquad \big\|(\mathbb{1} - T)^{-1}\big\| \le \tfrac{1}{1-\|T\|} \qquad \big\|(\mathbb{1} - T)^{-1} - \mathbb{1}\big\| \le \tfrac{\|T\|}{1-\|T\|}$$

The series $\sum_{n=0}^{\infty} T^n$ converges in norm.

Theorem A.51 (Adjoints)**.** *Let $\mathcal{H}$ be a Hilbert space and $S, T \in \mathcal{L}(\mathcal{H})$.*

(a) The map $A \mapsto A^*$ is a conjugate linear isometric isomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})$. In particular

$$(\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^* \qquad \|A^*\| = \|A\|$$

for all $A, B \in \mathcal{L}(\mathcal{H})$ and all $\alpha, \beta \in \mathbb{C}$.

(b) $(TS)^* = S^*T^*$

(c) $(T^*)^* = T$

(d) If T has a bounded inverse, then T^* has a bounded inverse and $(T^*)^{-1} = (T^{-1})^*$.

(e) The map $A \mapsto A^*$ is continuous in the weak and uniform topologies. That is, if $\{A_n\}_{n\in\mathbb{N}}$ converges to A weakly (in norm), then $\{A_n^*\}_{n\in\mathbb{N}}$ converges to A^* weakly (in norm). The map $A \mapsto A^*$ is continuous in the strong topology if and only if $\mathcal{H}$ is finite dimensional.

(f) $\|T^*T\| = \|T\|^2$

(g) If $T = T^*$, then $\|T\| = \sup\big\{\ |\langle T\mathbf{x}, \mathbf{x}\rangle|\ \big|\ \mathbf{x} \in \mathcal{H},\ \|\mathbf{x}\| = 1\ \big\}$.

Proof. See [**RS**, Theorem VI.3] and Example A.54 and Proposition A.52, below, or [**Co**, Propositions 2,6, 2.7, 2.13]. □

Proposition A.52. *Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator.*

(a) We have

$$\|T\| = \sup_{\substack{\mathbf{x},vy\in\mathcal{H} \\ \mathbf{x},\mathbf{y}\neq\mathbf{0}}} \tfrac{|\langle T\mathbf{x},\mathbf{y}\rangle|}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$$

(b) Assume in addition that $T = T^*$. Then

$$\|T\| = \sup_{\substack{\mathbf{x}\in\mathcal{H} \\ \mathbf{x}\neq\mathbf{0}}} \tfrac{|\langle T\mathbf{x},\mathbf{x}\rangle|}{\|\mathbf{x}\|^2}$$

Proof. This is Exercise A.53, below. □

Exercise A.53. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator.

(a) Prove that

$$\|T\| = \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathcal{H} \\ \mathbf{x},\mathbf{y}\neq\mathbf{0}}} \tfrac{|\langle T\mathbf{x},\mathbf{y}\rangle|}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$$

(b) Assume in addition that $T = T^*$. Prove that

$$\|T\| = \sup_{\substack{\mathbf{x}\in\mathcal{H} \\ \mathbf{x}\neq\mathbf{0}}} \tfrac{|\langle T\mathbf{x},\mathbf{x}\rangle|}{\|\mathbf{x}\|^2}$$

(c) Find an example which shows that the equation of part (b) can fail if $T \neq T^*$.

Example A.54. Let $\mathcal{H} = \ell^2$ and define the right and left shift operators by

$$\begin{aligned} L(x_1, x_2, x_3, \cdots) &= (x_2, x_3, \cdots) \\ R(x_1, x_2, x_3, \cdots) &= (0, x_1, x_2, x_3, \cdots) \end{aligned}$$

First observe that $\|L\| = \|R\| = 1$ and that

$$\langle L\mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{\infty} (L\mathbf{x})_j\,\overline{y_j} = \sum_{j=1}^{\infty} x_{j+1}\,\overline{y_j} = \sum_{i=2}^{\infty} x_i\,\overline{y_{i-1}} = \sum_{i=1}^{\infty} x_i\,\overline{(R\mathbf{y})_i} = \langle \mathbf{x}, R\mathbf{y}\rangle$$

so that $L^* = R$ and $R^* = L$. Next observe that, for each $n \in \mathbb{N}$ and $\mathbf{x} \in \ell^2$,

$$\|L^n\mathbf{x}\|^2 = \sum_{m=n+1}^{\infty} |x_m|^2 \xrightarrow{n\to\infty} 0$$
$$\|R^n\mathbf{x}\|^2 = \sum_{m=1}^{\infty} |x_m|^2 = \|\mathbf{x}\|^2$$

Thus, as $n \to \infty$, L^n converges strongly to zero, but $L^{n*} = R^n$ does not converge strongly to anything. On the other hand, L^{n*} does converge weakly to zero since, for all $\mathbf{x}, \mathbf{y} \in \ell^2$,

$$\big|\left<R^n\mathbf{x}, \mathbf{y}\right>\big| = \big|\left<L^{n*}\mathbf{x}, \mathbf{y}\right>\big| = \big|\left<\mathbf{x}, L^n\mathbf{y}\right>\big| \le \|\mathbf{x}\|\,\|L^n\mathbf{y}\| \xrightarrow{n\to\infty} 0$$

Theorem A.55 (Principle of Uniform Boundedness etc.)**.** *Unless otherwise stated, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is linear and has domain $\mathcal{X}$.*

(a) T is bounded if and only if

$$T^{-1}\big\{\ \mathbf{y} \in \mathcal{Y}\ \big|\ \|\mathbf{y}\|_{\mathcal{Y}} \le 1\ \big\} = \big\{\ \mathbf{x} \in \mathcal{X}\ \big|\ \|T\mathbf{x}\|_{\mathcal{Y}} \le 1\ \big\}$$

has nonempty interior. ($\mathcal{X}, \mathcal{Y}$ need not be complete.)

(b) Principle of Uniform Boundedness: *Let $\mathcal{F} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$.*

If, for each $\mathbf{x} \in X$,	$\big\{\ \|T\mathbf{x}\|\ \big|\ T \in \mathcal{F}\ \big\}$ *is bounded,*
then	$\big\{\ \|T\|\ \big|\ T \in \mathcal{F}\ \big\}$ *is bounded,*

($\mathcal{Y}$ need not be complete.)

(c) If $B : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ is bilinear and continuous in each variable separately (i.e. $B(\mathbf{x}, \mathbf{y})$ is continuous in $\mathbf{x}$ for each fixed $\mathbf{y}$ and vice versa), then $B(\mathbf{x}, \mathbf{y})$ is jointly continuous (i.e. if $\lim_{n\to\infty} \mathbf{x}_n = 0$ and $\lim_{n\to\infty} \mathbf{y}_n = 0$, then $\lim_{n\to\infty} B(\mathbf{x}_n, \mathbf{y}_n) = 0$).

(d) Open Mapping Theorem: *If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is surjective (i.e. onto) and if $\mathcal{O}$ is an open subset of $\mathcal{X}$, then $T\mathcal{O} = \big\{\ T\mathbf{x}\ \big|\ \mathbf{x} \in \mathcal{O}\ \big\}$ is an open subset of $\mathcal{Y}$.*

(e) Inverse Mapping Theorem: *If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is bijective (i.e. 1–1 and onto), then T^{-1} is bounded.*

(f) Closed Graph Theorem: *The graph of T is defined to be*

$$\Gamma(T) = \big\{\ (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}\ \big|\ \mathbf{y} = T\mathbf{x}\ \big\}$$

Then

$$T \text{ is bounded} \iff \Gamma(T) \text{ is closed}$$

In other words, T is bounded if and only if

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}, \ \lim_{n\to\infty} T\mathbf{x}_n = \mathbf{y} \implies \mathbf{y} = T\mathbf{x}$$

(g) Hellinger–Toeplitz Theorem: *Let T be an everywhere defined linear operator on $\mathcal{H}$ that obeys $\langle \mathbf{x}, T\mathbf{y}\rangle = \langle T\mathbf{x}, \mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Then T is bounded.*

Proof. (a) See Proposition at the beginning of [**RS**, §III.5].

(b) See [**RS**, Theorem III.9] or [**Co**, Theorem 14.1].

(c) See the Corollary to [**RS**, Theorem III.9].

(d) See [**RS**, Theorem III.10] or [**Co**, Theorem 12.1].

(e) See [**RS**, Theorem III.11] or [**Co**, Theorem 12.5].

(f) See [**RS**, Theorem III.12] or [**Co**, Theorem 12.6].

(g) See the Corollary to [**RS**, Theorem III.12]. □

A.3. Compact Operators

In this section we provide an introduction to compact linear operators on Banach and Hilbert spaces. These operators behave very much like familiar finite dimensional matrices, without necessarily having finite rank. For more thorough treatments, see [**RS**, §VI.5, VI.6] *or*[**Y**].

Definition A.56. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A linear operator $C: \mathcal{X} \to \mathcal{Y}$ is said to be compact if for each bounded sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathcal{X}$, there is a subsequence of $\{Cx_i\}_{i\in\mathbb{N}}$ that is convergent.

Example A.57. Let $a < b$ and $c < d$. If $C : [c,d]\times[a,b] \to \mathbb{C}$ is continuous, then the integral operator

$$(Cf)(y) = \int_a^b C(y,x)f(x)\ dx$$

is compact as an operator from $\mathcal{X} = C[a,b]$, the space of continuous functions on $[a,b]$ with supremum norm, to $\mathcal{Y} = C[c,d]$.

Exercise A.58. Use the Arzelà–Ascoli theorem ([**RS**, Theorem 1.28] or [**Co**, Theorem 3.8]) to prove that the operator C of Example A.57 is compact.

Example A.59 (Hilbert–Schmidt Operators)**.** Let $\langle X, \mu\rangle$ and $\langle Y, \nu\rangle$ be measure spaces and let $k(x, y)$ be a measurable function on $X \times Y$ with

$$\int_{X\times Y} |k(x,y)|^2\ d\mu(x)d\nu(y) < \infty$$

Then

$$(Kf)(x) = \int_Y k(x,y)f(y)\ d\nu(y)$$

is a compact map from $L^2(Y, d\nu)$ to $L^2(X, d\mu)$. Such an operator is called Hilbert–Schmidt.

Proof. Let $\{f_i\}_{i\in\mathbb{N}}$ be a bounded sequence in $L^2(Y, d\nu)$. By part (c) of Exercise A.60, below, $\{f_i\}_{i\in\mathbb{N}}$ has a weakly convergent subsequence. By throwing away all but this subsequence, we may assume that $\{f_i\}_{i\in\mathbb{N}}$ converges weakly to $f \in L^2(Y, d\nu)$.

We now show that $\{Kf_i\}_{i\in\mathbb{N}}$ converges strongly to $Kf \in L^2(X, d\mu)$. Since $\int_{X\times Y} |k(x,y)|^2\ d\mu(x)d\nu(y) < \infty$ we have that $\int_Y |k(x,y)|^2\ d\nu(y) < \infty$ for almost every $x \in X$. For any such $x \in X$,

$$\begin{aligned}\lim_{i\to\infty}\int_Y k(x,y)f_i(y)\ d\nu(y) &= \lim_{i\to\infty}\left\langle f_i\, ,\, \overline{k(x,\ \cdot\ )}\right\rangle_{L^2(Y,d\nu)} \\ &= \left\langle f\, ,\, \overline{k(x,\ \cdot\ )}\right\rangle_{L^2(Y,d\nu)} \\ &= \int_Y k(x,y)f(y)\ d\nu(y)\end{aligned}$$

Furthermore, by Cauchy–Schwarz,

$$\begin{aligned}\big|(Kf_i)(x)\big| &\le \int_Y \big|k(x,y)f_i(y)\big|\ d\nu(y) \\ &\le \|f_i\|_{L^2(Y,d\nu)}\sqrt{\int_Y \big|k(x,y)\big|^2\ d\nu(y)} \\ &\le \sup_i \|f_i\|_{L^2(Y,d\nu)}\sqrt{\int_Y \big|k(x,y)\big|^2\ d\nu(y)} \equiv H(x)\end{aligned}$$

Thus we have shown that $(Kf_i)(x)$ converges pointwise to $(Kf)(x)$ for almost every x and is bounded, for all i by the function $H(x)$ which is square integrable with respect to $d\mu(x)$. Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{i\to\infty}\|Kf - Kf_i\|^2_{L^2(X,d\mu)} = \lim_{i\to\infty}\int_X \big|(Kf)(x) - (Kf_i)(x)\big|^2\ d\mu(x) = 0$$

□

Exercise A.60. Let $\mathcal{H}$ be a Hilbert Space. A sequence $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ is said to converge weakly to $f \in \mathcal{H}$ if

$$\lim_{i\to\infty} \langle f_i, g\rangle = \langle f, g\rangle$$

for all $g \in \mathcal{H}$.

(a) Give an example of a sequence that converges weakly but not strongly.

(b) Prove that if $\{f_i\}_{i\in\mathbb{N}}$ converges weakly to f, then $\|f\| \le \liminf_{i\to\infty} \|f_i\|$. Prove that if $\{f_i\}_{i\in\mathbb{N}}$ converges weakly to f and $\|f\| = \lim_{i\to\infty} \|f_i\|$, then $\{f_i\}_{i\in\mathbb{N}}$ converges strongly to f.

(c) Prove that $\mathcal{H}$ is weakly sequentially compact. That is, every bounded sequence in $\mathcal{H}$ has a weakly convergent subsequence.

Example A.61 (Nuclear Operators)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and denote by $\mathcal{X}'$ the dual space of $\mathcal{X}$. That is, the space of bounded linear functionals on $\mathcal{X}$. If $\{x'_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $\mathcal{X}'$, $\{y_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $\mathcal{Y}$ and $\{c_i\}_{i\in\mathbb{N}}$ is a set of complex numbers obeying $\sum_i |c_i| < \infty$, then

$$Kx = \sum_{i=1}^{\infty} c_i \; x'_i(x) \; y_i$$

is called a nuclear operator from $\mathcal{X}$ to $\mathcal{Y}$. Since

$$\sum_{i=1}^{\infty} |c_i| \; \big|x'_i(x)\big| \; \|y_i\|_{\mathcal{Y}} \le \|x\|_{\mathcal{X}} \sup_i \|y_i\|_{\mathcal{Y}} \sup_i \|x'_i\|_{\mathcal{X}'} \sum_{i=1}^{\infty} |c_i|$$

the series defining Kx converges strongly and K is a bounded operator of norm at most $\sup_i \|y_i\|_{\mathcal{Y}} \sup_i \|x'_i\|_{\mathcal{X}'} \sum_{i=1}^{\infty} |c_i|$.

Exercise A.62. Prove that any nuclear operator is compact.

Proposition A.63. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces.*

(a) If $C : \mathcal{X} \to \mathcal{Y}$ is a compact operator, then C is a bounded operator.

(b) If $C_1, C_2 : \mathcal{X} \to \mathcal{Y}$ are compact operators and $\alpha_1, \alpha_2 \in \mathbb{C}$, then the operator $\alpha_1 C_1 + \alpha_2 C_2$ is compact.

(c) If $C : \mathcal{X} \to \mathcal{Y}$ is a compact operator and $B_{\mathcal{X}} : \mathcal{Z} \to \mathcal{X}$ and $B_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Z}$ are bounded operators, then $CB_{\mathcal{X}}$ and $B_{\mathcal{Y}}C$ are compact.

(d) Let, for each $i \in \mathbb{N}$, $C_i : \mathcal{X} \to \mathcal{Y}$ be a compact operator. If the C_i's converge in operator norm to an operator $C : \mathcal{X} \to \mathcal{Y}$, then C is compact.

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ be a bounded sequence in $\mathcal{X}$.

(a) This is Exercise A.64, below.

(b) Since C_1 is compact, there is a subsequence $\{x_{i_\ell}\}_{\ell\in\mathbb{N}}$ such that $C_1x_{i_\ell}$ converges in $\mathcal{Y}$. Since C_2 is compact, there is a subsequence $\{x_{i_{\ell_m}}\}_{m\in\mathbb{N}}$ of the bounded sequence $\{x_{i_\ell}\}_{\ell\in\mathbb{N}}$ such that $C_2x_{i_{\ell_m}}$ converges in $\mathcal{Y}$. Then $\alpha_1C_1x_{i_{\ell_m}} + \alpha_2C_2x_{i_{\ell_m}}$ also converges in $\mathcal{Y}$.

(c) Let $\{z_i\}_{i\in\mathbb{N}}$ be a bounded sequence in $\mathcal{Z}$. Since $B_\mathcal{X}$ is bounded, $\{B_\mathcal{X}z_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $\mathcal{X}$. Since C is compact, there is a subsequence $\{B_\mathcal{X}z_{i_\ell}\}_{\ell\in\mathbb{N}}$ such that $CB_\mathcal{X}z_{i_\ell}$ converges in $\mathcal{Y}$.

Since C is compact, there is a subsequence $\{x_{i_\ell}\}_{\ell\in\mathbb{N}}$ such that Cx_{i_ℓ} converges in $\mathcal{Y}$. Since $C_\mathcal{Y}$ is bounded, $B_\mathcal{Y}Cx_{i_\ell}$ converges in $\mathcal{Y}$.

(d) Let $\{x_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $\mathcal{X}$ and set

$$X = \sup_j \|x_j\|_\mathcal{X}$$

For each fixed $i \in \mathbb{N}$, $\{C_ix_j\}_{j\in\mathbb{N}}$ has a convergent subsequence, since C_i is compact by hypothesis. By taking subsequences of subsequences and using the diagonal trick, we can find a subsequence $\{x_{j_\ell}\}_{\ell\in\mathbb{N}}$ such that $\lim_{\ell\to\infty} C_ix_{j_\ell}$ exists for each $i \in \mathbb{N}$. It suffices for us to prove that $\{Cx_{j_\ell}\}_{\ell\in\mathbb{N}}$ is Cauchy. Let $\varepsilon > 0$. Since the C_i's converge in operator norm to C, there is an $I \in \mathbb{N}$ such that $\|C - C_i\| < \frac{\varepsilon}{3X}$ for all $i \geq I$. Since $\{C_Ix_{j_\ell}\}_{\ell\in\mathbb{N}}$ is Cauchy, there is an $L \in \mathbb{N}$ such that $\left\|C_Ix_{j_\ell} - C_Ix_{j_m}\right\|_\mathcal{Y} < \frac{\varepsilon}{3}$ for all $\ell, m > L$. Hence if $\ell, m > L$, then

$$\begin{aligned}\left\|Cx_{j_\ell} - Cx_{j_m}\right\|_\mathcal{Y} &\leq \left\|Cx_{j_\ell} - C_Ix_{j_\ell}\right\|_\mathcal{Y} + \left\|C_Ix_{j_\ell} - C_Ix_{j_m}\right\|_\mathcal{Y} \\ &\qquad\qquad + \left\|C_Ix_{j_m} - Cx_{j_m}\right\|_\mathcal{Y} \\ &\leq X\|C - C_I\| + \left\|C_Ix_{j_\ell} - C_Ix_{j_m}\right\|_\mathcal{Y} + X\|C_I - C\| \\ &< X\tfrac{\varepsilon}{3X} + \tfrac{\varepsilon}{3} + X\tfrac{\varepsilon}{3X} \\ &= \varepsilon\end{aligned}$$

□

Exercise A.64. Prove that compact operators are necessarily bounded.

Proposition A.65. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Denote by $\mathcal{X}'$ and $\mathcal{Y}'$ their dual spaces. That is, $\mathcal{X}'$ (resp. $\mathcal{Y}'$) is the Banach space of bounded linear functionals on $\mathcal{X}$ (resp. $\mathcal{Y}$). The adjoint, $C^* : \mathcal{Y}' \to \mathcal{X}'$, of a bounded operator $C : \mathcal{X} \to \mathcal{Y}$ is determined by*

$$(C^*\eta)(x) = \eta(Cx) \qquad \text{for all } \eta \in \mathcal{Y}' \text{ and } x \in \mathcal{X}$$

A bounded operator $C : \mathcal{X} \to \mathcal{Y}$ is compact if and only if C^* is compact.

Proof. First assume that C is compact. Let $\{\eta_i\}_{i\in\mathbb{N}}$ be a bounded subset of $\mathcal{Y}'$ and set

$$Y' = \sup_i \|\eta_i\|_{\mathcal{Y}'}$$

Let $B = \left\{ x \in \mathcal{X} \;\middle|\; \|x\|_{\mathcal{X}} \le 1 \right\}$ be the unit ball in $\mathcal{X}$. Since C is compact, $\overline{CB}$, which is the closure of $\left\{ Cx \in \mathcal{X} \;\middle|\; \|x\|_{\mathcal{X}} \le 1 \right\}$, is a compact subset of $\mathcal{Y}$. We shall apply Arzelà–Ascoli ([**RS**, Theorem 1.28] or [**Co**, Theorem 3.8]) to the sequence of functions

$$f_i : y \in \overline{CB} \mapsto \eta_i(y) \in \mathbb{C}$$

Since

$$\big|f_i(y)\big| \le Y'\|y\|_{\mathcal{Y}} \le Y'\|C\|$$

the sequence is uniformly bounded. Since

$$\big|f_i(y) - f_i(\tilde y)\big| \le Y'\|y - \tilde y\|_{\mathcal{Y}}$$

it is equicontinuous. So, by Arzelà–Ascoli, there is a subsequence f_{i_ℓ} that converges uniformly on $\overline{CB}$. Since

$$\begin{aligned}\|C^*\eta_i - C^*\eta_j\|_{\mathcal{X}'} &= \sup_{x\in B}\big|(C^*\eta_i)(x) - (C^*\eta_j)(x)\big| = \sup_{x\in B}\big|\eta_i(Cx) - \eta_j(Cx)\big| \\ &= \sup_{x\in B}\big|f_i(Cx) - f_j(Cx)\big| \qquad = \sup_{y\in CB}\big|f_i(y) - f_j(y)\big|\end{aligned}$$

the sequence $\{C^*\eta_{i_\ell}\}_{\ell\in\mathbb{N}}$ is Cauchy in $\mathcal{X}'$.

Conversely, assume that C^* is compact. Let $\{x_i\}_{i\in\mathbb{N}}$ be a bounded sequence in $\mathcal{X}$. By the implication that we have already proven, the adjoint, $C^{**} : \mathcal{X}'' \to \mathcal{Y}''$, of C^* is compact. We may naturally view $\mathcal{X}$ as a closed subspace of $\mathcal{X}''$ and $\mathcal{Y}$ as a closed subspace of $\mathcal{Y}''$. So we may view $\{x_i\}_{i\in\mathbb{N}}$ as a bounded sequence in $\mathcal{X}''$. Then $\{C^{**}x_i\}_{i\in\mathbb{N}}$ has a subsequence $\{C^{**}x_{i_\ell}\}_{\ell\in\mathbb{N}}$ that converges in $\mathcal{Y}''$. For any $\eta \in \mathcal{Y}'$ and $x \in \mathcal{X}$ (we'll write X for x, when we want to think of it as an element of $\mathcal{X}''$),

$$\begin{aligned}(C^{**}X)(\eta) &= X(C^*\eta) && \text{by the definition of “adjoint”} \\ &= (C^*\eta)(x) && \text{by the identification of } \mathcal{X} \text{ with a subset of } \mathcal{X}'' \\ &= \eta(Cx) && \text{by the definition of “adjoint”}\end{aligned}$$

Thus $C^{**}x \in \mathcal{Y}''$ is $Cx \in \mathcal{Y}$, viewed as an element of $\mathcal{Y}''$ and $\{Cx_{i_\ell}\}_{\ell\in\mathbb{N}}$ converges in $\mathcal{Y}$. □

It is the spectral properties of compact operators that make them act very much like matrices. Perhaps it is more appropriate to say that the spectral properties of noncompact operators are often very different from those of matrices. A simple, yet typical, example of this is given in Exercise A.68, below. We start with careful definitions of “eigenvalue” like terms. For a *thorough, but still readable, treatment of the spectral theory of self–adjoint operators on Hilbert spaces, see* [**RS**].

Definition A.66. Let $\mathcal{X}$ be a Banach space and $B : \mathcal{X} \to \mathcal{X}$ be a linear operator defined on $\mathcal{X}$.

(a) The number $\lambda \in \mathbb{C}$ is said to be in the resolvent set of B if the operator $B - \lambda\mathbb{1}$ is bijective (one–to–one and onto) with bounded inverse. We shall use $\rho(B)$ to denote the resolvent set of B.

(b) The number $\lambda \in \mathbb{C}$ is said to be in the spectrum of B if it is not in the resolvent set of B. We write $\sigma(B) = \mathbb{C} \setminus \rho(B)$.

(c) The number $\lambda \in \mathbb{C}$ is said to be an eigenvalue of B if there is a nonzero vector $x \in \mathcal{X}$, called an eigenvector corresponding to λ, such that $Bx = \lambda x$. The set of all eigenvalues of B is called the point spectrum of B.

Proposition A.67. *Let $\mathcal{X}$ be a Banach space and $B : \mathcal{X} \to \mathcal{X}$ be a linear operator defined on $\mathcal{X}$.*

(a) If $|\lambda| > \|B\|$, then $\lambda \in \rho(B)$.

(b) $\rho(B)$ is an open subset of $\mathbb{C}$.

(c) If λ is an eigenvalue of B, then $\lambda \in \sigma(B)$.

Proof. (a) Since $\frac{\|B\|}{|\lambda|} < 1$, the series $-\frac{1}{\lambda}\sum_{m=0}^{\infty}\left(\frac{B}{\lambda}\right)^m$ converges in operator norm to a bounded operator R on $\mathcal{X}$. As

$$(B - \lambda\mathbb{1})R = R(B - \lambda\mathbb{1}) = -\sum_{m=0}^{\infty}\left(\tfrac{B}{\lambda}\right)^{m+1} + \sum_{m=0}^{\infty}\left(\tfrac{B}{\lambda}\right)^m = \mathbb{1}$$

$R = \big(B - \lambda\mathbb{1}\big)^{-1}$ and $\lambda \in \rho(B)$.

(b) Let $\mu \in \rho(B)$ and denote by $\big(B - \mu\mathbb{1}\big)^{-1}$ the inverse of $B - \mu\mathbb{1}$. By hypothesis, this inverse is a bounded operator on $\mathcal{X}$. If

$$|\lambda - \mu| < \big\|\big(B - \mu\mathbb{1}\big)^{-1}\big\|$$

then the series $\big(B-\mu\mathbb{1}\big)^{-1}\sum_{m=0}^{\infty}(\mu-\lambda)^m\big(B-\mu\mathbb{1}\big)^{-m}$ converges in operator norm to a bounded operator $\tilde{R}$ on $\mathcal{X}$. As

$$\begin{aligned}(B - \lambda\mathbb{1})\tilde{R} = \tilde{R}(B - \lambda\mathbb{1}) &= \tilde{R}(B - \mu\mathbb{1}) + (\mu - \lambda)\tilde{R}\\ &= \sum_{m=0}^{\infty}(\mu - \lambda)^m\big(B - \mu\mathbb{1}\big)^{-m} + \sum_{m=0}^{\infty}(\mu - \lambda)^{m+1}\big(B - \mu\mathbb{1}\big)^{-m-1}\\ &= \mathbb{1}\end{aligned}$$

$\tilde{R}$ is the operator inverse of $\big(B - \lambda\mathbb{1}\big)$ and $\lambda \in \rho(B)$. This shows that

$$\left\{ \lambda \in \mathbb{C} \;\middle|\; |\lambda - \mu| < \big\|\big(B - \mu\mathbb{1}\big)^{-1}\big\| \right\} \subset \rho(B)$$

and that $\rho(B)$ is open.

(c) If λ is an eigenvalue of B, then $B - \lambda\mathbb{1}$ has a nontrivial kernel, namely all of the eigenvectors corresponding to λ. Thus $\lambda \notin \rho(B)$. $\square$

The next example shows that, for operators acting on infinite dimensional spaces, even nice operators, the bulk of the spectrum need not consist of eigenvalues.

Exercise A.68. Let $\mathcal{H} = L^2(X, \mu)$ for some measure space $\langle X, \mu \rangle$. Let $f : X \to \mathbb{C}$ be a bounded measurable function on X. Let A be the bounded linear operator on $\mathcal{H}$ given by multiplication by $f(x)$.

(a) Prove that $\lambda \in \sigma(A)$ if and only if

$$\forall \epsilon > 0 \quad \mu\big\{\, x \in X \;\big|\; |f(x) - \lambda| < \epsilon \,\big\} > 0$$

(b) Prove that λ is an eigenvalue of A if and only if

$$\mu\big\{\, x \in X \;\big|\; f(x) = \lambda \,\big\} > 0$$

(c) Let X be the open interval $(0, 1)$, μ be Lebesgue measure on $(0, 1)$ and $f(x) = x$. Find the spectrum of A, the operator on $\mathcal{H}$ given by multiplication by x. Also find all of the eigenvalues of A.

We next prove that if C is a compact operator, then $\sigma(C) \setminus \{0\}$ consists only eigenvalues of finite multiplicity. If there are infinitely many different eigenvalues, they must converge to zero. We first need the following technical lemma.

Lemma A.69. *Let $\mathcal{X}$ be a Banach space and $B : \mathcal{X} \to \mathcal{X}$ be a compact operator. If λ is a nonzero complex number, then the range of $C - \lambda\mathbb{1}$ is a closed linear subspace of $\mathcal{X}$.*

Proof. Denote by $\mathcal{R}$ and $\mathcal{K}$ the range and kernel, respectively, of $C - \lambda\mathbb{1}$. Let $y \in \overline{\mathcal{R}}$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$ such that $(C - \lambda\mathbb{1})x_n$ converges to y. Denote by ρ_n the distance from x_n to $\mathcal{K}$. For each $n \in \mathbb{N}$, there is a $z_n \in \mathcal{K}$ such that $\rho_n \le \|x_n - z_n\| < \rho_n + \frac{1}{n}$. Then $\tilde{x}_n = x_n - z_n$ obeys

$$\lim_{n\to\infty} (C - \lambda\mathbb{1})\tilde{x}_n = \lim_{n\to\infty} (C - \lambda\mathbb{1})x_n = y$$

We first consider the case that $\{\rho_n\}_{n\in\mathbb{N}}$ is bounded. Then the sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is bounded, and, since C is compact, there is a subsequence

$\{\tilde{x}_{n_\ell}\}_{\ell\in\mathbb{N}}$ such that $C\tilde{x}_{n_\ell}$ converges in $\mathcal{X}$, say to $\tilde{y}$. Then

$$\tilde{x}_{n_\ell} = \tfrac{1}{\lambda}\big[C\tilde{x}_{n_\ell} - \big(C - \lambda\mathbb{1}\big)\tilde{x}_{n_\ell}\big]$$

converges in $\mathcal{X}$ to $x = \frac{1}{\lambda}(\tilde{y} - y)$. Since C is bounded, $\tilde{y} = Cx$ and $y = (C - \lambda\mathbb{1})x \in \mathcal{R}$.

Finally, we consider the case that $\{\rho_n\}_{n\in\mathbb{N}}$ is not bounded. Then, possibly restricting to a subsequence, we may assume that $\lim_{n\to\infty}\rho_n = \infty$. As the sequence $\big\{\frac{\tilde{x}_n}{\|\tilde{x}_n\|}\big\}_{n\in\mathbb{N}}$ is bounded and C is still compact, there is a subsequence $\big\{\frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|}\big\}_{\ell\in\mathbb{N}}$ such that $C\frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|}$ converges in $\mathcal{X}$, say to $\tilde{z}$. As

$$\lim_{n\to\infty}(C - \lambda\mathbb{1})\tfrac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} = \tfrac{y}{\lim_{n\to\infty}\|\tilde{x}_{n_\ell}\|} = 0$$

we have

$$\lim_{n\to\infty}\tfrac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} = \tfrac{1}{\lambda}\lim_{n\to\infty}\big[C\tfrac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} - \big(C - \lambda\mathbb{1}\big)\tfrac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|}\big] = \tfrac{\tilde{z}}{\lambda}$$

and hence

$$\big(C - \lambda\mathbb{1}\big)\tilde{z} = \lambda\lim_{n\to\infty}(C - \lambda\mathbb{1})\tfrac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} = 0$$

In other words, $\tilde{z} \in \mathcal{K}$. This provides a contradiction, since $\tilde{x}_n$ is a distance ρ_n from $\mathcal{K}$ so that $\frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ is a distance $\frac{\rho_n}{\|\tilde{x}_n\|} \ge \frac{\rho_n}{\rho_n + 1/n}$ from $\mathcal{K}$. As $\lim_{n\to\infty}\frac{\rho_n}{\rho_n+1/n} = 1$, $\frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|}$ cannot converge to a point of $\mathcal{K}$. □

Proposition A.70 (The Fredholm Alternative)**.** *Let $C : \mathcal{X} \to \mathcal{X}$ be a compact operator on the Banach space $\mathcal{X}$. If λ is a nonzero complex number, then either λ is an eigenvalue of C or $\lambda \in \rho(C)$.*

Proof. Suppose that λ is not an eigenvalue of C. Then, by definition, $C - \lambda\mathbb{1}$ is one–to–one. By lemma A.69, the range of $C - \lambda\mathbb{1}$ is closed. We now claim that the range of $C - \lambda\mathbb{1}$ is all of $\mathcal{X}$. If not, $\mathcal{X}_1 = (C - \lambda\mathbb{1})\mathcal{X}$ is a proper closed subspace of $\mathcal{X}$. Since the restriction of C to $\mathcal{X}_1$ is still compact, $\mathcal{X}_2 = (C-\lambda\mathbb{1})\mathcal{X}_1$ is a closed subspace of $\mathcal{X}_1$. If $\mathcal{X}_2$ were not a proper subspace of $\mathcal{X}_1$, then for each $x \in \mathcal{X}\setminus\mathcal{X}_1$, there would be a vector $x' \in \mathcal{X}_1$ with $(C-\lambda\mathbb{1})x' = (C-\lambda\mathbb{1})x$ and this would contradict the assumption that $C - \lambda\mathbb{1}$ is one–to–one. Thus $\mathcal{X}_2 = (C - \lambda\mathbb{1})\mathcal{X}_1$ is a proper closed subspace of $\mathcal{X}_1$. Continuing in this way, we can generate a sequence $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ of subspaces of $\mathcal{X}$ with $\mathcal{X}_{n+1} = (C-\lambda\mathbb{1})\mathcal{X}_n$ and $\mathcal{X}_{n+1}$ a proper closed subspace of $\mathcal{X}_n$. By Exercise A.71, below, there is, for each $n \in \mathbb{N}$, a unit vector $x_n \in \mathcal{X}_n\setminus\mathcal{X}_{n+1}$ whose distance from $\mathcal{X}_{n+1}$ is at least $\frac{1}{2}$. If $n > m$,

$$\tfrac{1}{\lambda}\big(Cx_m - Cx_n\big) = x_m - \tilde{x}_m$$

with

$$\tilde{x}_m = -\tfrac{1}{\lambda}\big(C-\lambda\mathbb{1}\big)x_m + \tfrac{1}{\lambda}Cx_n = -\tfrac{1}{\lambda}\big(C-\lambda\mathbb{1}\big)x_m + \tfrac{1}{\lambda}(C-\lambda\mathbb{1})x_n + x_n \in \mathcal{X}_{m+1}$$

Hence $\|Cx_m - Cx_n\| \ge \frac{|\lambda|}{2}$ for all $n > m$ and $\{Cx_n\}_{n\in\mathbb{N}}$ may not contain any convergent subsequence, contradicting the compactness of C.

So $C - \lambda\mathbb{1}$ is both one–to–one and onto. The boundedness of the inverse map is an immediate consequence of the inverse mapping theorem (part (e) of Theorem A.55), But it is also easy to prove boundedness directly and we do that now. If $\big(C - \lambda\mathbb{1}\big)^{-1}$ is not bounded, there is a sequence of unit vectors $x_n \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \big\|(C - \lambda\mathbb{1})x_n\big\| = 0 \implies \lim_{n\to\infty} (C - \lambda\mathbb{1})x_n = 0$$

Since C is compact, there is a subsequence $\big\{x_{n_m}\big\}_{m\in\mathbb{N}}$ such that Cx_{n_m} converges, say to y. But then

$$\lim_{m\to\infty} x_{n_m} = \lim_{m\to\infty} \tfrac{1}{\lambda}Cx_{n_m} - \lim_{m\to\infty} \tfrac{1}{\lambda}(C - \lambda\mathbb{1})x_{n_m} = \tfrac{y}{\lambda}$$

and

$$Cy = \lambda C \lim_{m\to\infty} x_{n_m} = \lambda y$$

As $\|y\| = |\lambda| \ne 0$, this contradicts the assumption that λ is not an eigenvalue of C. □

Exercise A.71. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a proper closed subspace of $\mathcal{X}$. Let $0 < \rho < 1$. Prove that there is a unit vector $x \in \mathcal{X} \setminus \mathcal{Y}$ whose distance from $\mathcal{Y}$ is at least ρ.

Exercise A.72. Let $\mathcal{X}$ be an infinite dimensional Banach space. Prove that the identity operator on $\mathcal{X}$ is not compact.

Proposition A.73 (The Spectrum of Compact Operators)**.** *Let $C : \mathcal{X} \to \mathcal{X}$ be a compact operator on the Banach space $\mathcal{X}$. The spectrum of C consists of at most countably many points. For any $\varepsilon > 0$, $\big\{\ \lambda \in \sigma(C)\ \big|\ |\lambda| > \varepsilon\ \big\}$ is finite. If $0 \ne \lambda \in \sigma(C)$, then λ is an eigenvalue of C of finite multiplicity.*

Proof. We have already proven, in Proposition A.70, that any nonzero number in the spectrum of C is an eigenvalue and we have also already proven, in Proposition A.67, that $\sigma(C) \subset \big\{\ \lambda \in \mathbb{C}\ \big|\ |\lambda| \le \|C\|\ \big\}$. Since eigenvectors corresponding to different eigenvalues are necessarily independent, it suffices to prove that there cannot exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of independent eigenvectors of C whose corresponding eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ converge to $\lambda \ne 0$.

Denote by $\mathcal{X}_n$ the span of $\{x_1,\ x_2,\ \cdots\ ,\ x_n\}$. By Exercise A.71, there is, for each $n \ge 2$, a unit vector $y_n \in \mathcal{X}_n$ whose distance from $\mathcal{X}_{n-1}$ is at least $\frac{1}{2}$. If $n > m$,

$$\tfrac{1}{\lambda_n}Cy_n - \tfrac{1}{\lambda_m}Cy_m = y_n - \tilde{y}_n$$

with

$$\tilde{y}_n = -\tfrac{1}{\lambda_n}\big(C - \lambda_n\mathbb{1}\big)y_n + \tfrac{1}{\lambda_m}Cy_m \in \mathcal{X}_{n-1}$$

since $(C - \lambda_n \mathbb{1})\mathcal{X}_n \subset \mathcal{X}_{n-1}$ and $C\mathcal{X}_m \subset \mathcal{X}_m \subset \mathcal{X}_{n-1}$. Hence

$$\left\|\tfrac{1}{\lambda_n} C y_n - \tfrac{1}{\lambda_m} C y_m\right\| \geq \tfrac{1}{2} \qquad \text{for all } n > m$$

By assumption $\lim_{n\to\infty} \lambda_n = \lambda \neq 0$, so that $\left\|C y_n - C y_m\right\| \geq \frac{|\lambda|}{4}$ for all $n > m$ sufficiently large. Thus $\{C y_n\}_{n\in\mathbb{N}}$ may not contain any convergent subsequence, contradicting the compactness of C. □

Exercise A.74. Let $\mathcal{X}$ be an infinite dimensional Banach space and let $C : \mathcal{X} \to \mathcal{X}$ a compact operator. Prove that $0 \in \sigma(C)$.

Exercise A.75. Let $\mathcal{H}$ be a separable Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be any sequence of complex numbers that converges to 0. Prove that the operator defined by

$$C\Big(\sum_{n=1}^{\infty} \alpha_n e_n\Big) = \sum_{n=1}^{\infty} \mu_n \alpha_n e_{n+1}$$

is compact and has $\sigma(C) = \{0\}$.

The Fourier Transform and Tempered Distributions

In this appendix, we provide a summary of the most basic definitions and results concerning the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\ d^n x$$

and tempered distributions. For a more extensive treatment of Fourier transforms, see, for example, [**RS2**, §IX.1]. *For a more extensive treatment of tempered distributions see, for example,* [**RS**, §V.3]. *We shall use the standard multi–index notation that if* $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, *where* $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$, *then* x^α *denotes* $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ *and* $\partial^\alpha u(x)$ *denotes the partial derivative* $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}u(x)$. *The order of this partial derivative is* $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

B.1. Schwartz Space

Definition B.1. (a) Schwartz space is the vector space

$$\mathcal{S}(\mathbb{R}^n) = \left\{\ u \in C^\infty(\mathbb{R}^n)\ \Big|\ \sup_{x\in\mathbb{R}^n} \big|(1+|x|^m)\partial^\alpha u(x)\big| < \infty\ \forall\ m \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n\ \right\}$$

of all C^∞ functions on $\mathbb{R}^n$ all of whose derivatives (including the function itself) decay faster than any polynomial at infinity.

(b) Define, for each $\alpha, \beta \in \mathbb{N}_0^n$ and each $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta \varphi(x)\right|$$

Then

(1) $\|\varphi\|_{\alpha,\beta} \ge 0$

(2) $\|a\varphi\|_{\alpha,\beta} = |a|\,\|\varphi\|_{\alpha,\beta}$

(3) $\|\varphi + \psi\|_{\alpha,\beta} \le \|\varphi\|_{\alpha,\beta} + \|\psi\|_{\alpha,\beta}$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $a \in \mathbb{C}$. These are precisely the defining conditions for $\|\ \cdot\ \|_{\alpha,\beta}$ to be a semi–norm. In order for $\|\ \cdot\ \|_{\alpha,\beta}$ to be a norm it must also obey $\|\varphi\|_{\alpha,\beta} = 0 \iff \varphi = 0$. This is the case if and only if $|\beta| = 0$. If $|\beta| \ne 0$ the constant function $\varphi(x) = 1$ has $\|\varphi\|_{\alpha,\beta} = 0$.

Example B.2.
(a) For any polynomial $P(x)$, the function $\varphi(x) = P(x)e^{-|x|^2}$ is in Schwartz space. This is because, firstly, for any $\alpha, \beta \in \mathbb{N}_0$, $x^\alpha \partial^\beta \varphi$ is again a polynomial times $e^{-|x|^2}$ and, secondly,

$$e^{-|x|^2} = \frac{1}{e^{|x|^2}} \le \frac{1}{1 + |x|^2 + \frac{1}{2!}|x|^4 + \cdots + \frac{1}{p!}|x|^{2p}} \tag{B.1}$$

for every $p \in \mathbb{N}$. Consequently, $x^\alpha \partial^\beta \varphi$ is bounded.

(b) If φ is $C^\infty(\mathbb{R}^n)$ and of compact support then $\varphi \in \mathcal{S}(\mathbb{R}^n)$. One such function, with $n = 1$, is

$$\varphi(x) = \begin{cases} 0 & \text{if } |x| \ge 1 \\ e^{-\frac{1}{(x-1)^2}} e^{-\frac{1}{(x+1)^2}} & \text{if } -1 < x < 1 \end{cases}$$

The heart of the proof that this function really is C^∞ at $x = \pm 1$ is the observation that, for any $p \ge 0$, $\lim\limits_{y\to 0} \frac{1}{|y|^p} e^{-\frac{1}{y^2}} = 0$, which follows immediately from (B.1) with $x = \frac{1}{y}$.

Next, we introduce a metric on $\mathcal{S}(\mathbb{R}^n)$ which is chosen so that φ and ψ are close together if and only if $\|\varphi - \psi\|_{\alpha,\beta}$ is small for every α, β. The details are given in the following

Theorem B.3. *Define $d : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ by*

$$d(\varphi, \psi) = \sum_{\alpha,\beta\in\mathbb{N}_0^n} 2^{-|\alpha|-|\beta|} \frac{\|\varphi - \psi\|_{\alpha,\beta}}{1 + \|\varphi - \psi\|_{\alpha,\beta}}$$

Then

(a) $d(\varphi, \psi)$ is well–defined for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and is a metric.

(b) With this metric, $\mathcal{S}(\mathbb{R}^n)$ is a complete metric space.

(c) In this metric $\varphi = \lim_{k\to\infty} \varphi_k$ *if and only if* $\lim_{k\to\infty} \|\varphi_k - \varphi\|_{\alpha,\beta} = 0$ *for every* $\alpha, \beta \in \mathbb{N}_0^n$.

Proof. (a) To prove that $\sum_{\alpha,\beta\in\mathbb{N}_0^n} 2^{-|\alpha|-|\beta|} \frac{\|\varphi-\psi\|_{\alpha,\beta}}{1+\|\varphi-\psi\|_{\alpha,\beta}}$ is well–defined it suffices to observe, firstly, that $\frac{A}{1+A} \le 1$ for every $A \ge 0$ and, secondly, that

$$\sum_{\alpha,\beta\in\mathbb{N}_0^n}^{\infty} 2^{-|\alpha|-|\beta|} = \Big[\sum_{\ell=0}^{\infty} 2^{-\ell}\Big]^{2n}$$

converges because the geometric series converges.

- The metric axiom $d(\varphi, \psi) \ge 0$ is obvious.
- The metric axiom that $d(\varphi, \psi) = 0 \implies \varphi = \psi$ is obvious because $d(\varphi, \psi) = 0$ forces the $\alpha = \beta = 0$ term in its definition, namely $\frac{\|\varphi-\psi\|_{0,0}}{1+\|\varphi-\psi\|_{0,0}}$, to vanish. And that first term is zero if and only if its numerator $\|\varphi - \psi\|_{0,0} = \sup_{x\in\mathbb{R}^n} |\varphi(x) - \psi(x)|$ is zero.
- The metric axiom $d(\varphi, \psi) = d(\psi, \varphi)$ is obvious.
- The triangle inequality follows from
$$\frac{\|\varphi - \psi\|_{\alpha,\beta}}{1 + \|\varphi - \psi\|_{\alpha,\beta}} \le \frac{\|\varphi - \zeta\|_{\alpha,\beta}}{1 + \|\varphi - \zeta\|_{\alpha,\beta}} + \frac{\|\zeta - \psi\|_{\alpha,\beta}}{1 + \|\zeta - \psi\|_{\alpha,\beta}}$$
which is proven as follows. We supress the subscripts α, β. Because $\frac{x}{1+x} = 1 - \frac{1}{1+x}$ is an increasing function of x
$$\begin{aligned} \tfrac{\|\varphi-\psi\|}{1+\|\varphi-\psi\|} &\le \tfrac{\|\varphi-\zeta\|+\|\zeta-\psi\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} \\ &= \tfrac{\|\varphi-\zeta\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} + \tfrac{\|\zeta-\psi\|}{1+\|\varphi-\zeta\|+\|\zeta-\psi\|} \\ &\le \tfrac{\|\varphi-\zeta\|}{1+\|\varphi-\zeta\|} + \tfrac{\|\zeta-\psi\|}{1+\|\zeta-\psi\|} \end{aligned}$$

(c) For the "only if" part, assume that $\varphi = \lim_{k\to\infty} \varphi_k$ and let $\alpha, \beta \in \mathbb{N}_0$. Then

$$d(\varphi, \varphi_k) \ge 2^{-|\alpha|-|\beta|} \frac{\|\varphi - \varphi_k\|_{\alpha,\beta}}{1 + \|\varphi - \varphi_k\|_{\alpha,\beta}} \implies \lim_{k\to\infty} \frac{\|\varphi - \varphi_k\|_{\alpha,\beta}}{1 + \|\varphi - \varphi_k\|_{\alpha,\beta}} = 0$$

For any $0 < \varepsilon < \frac{1}{2}$ and $x > 0$,

$$\tfrac{x}{1+x} < \varepsilon \implies x < \varepsilon(1 + x) \implies x - \varepsilon x < \varepsilon \implies x < \tfrac{\varepsilon}{1-\varepsilon} < 2\varepsilon$$

Hence $\lim_{k\to 0} \|\varphi - \varphi_k\|_{\alpha,\beta} = 0$ too.

For the "if" part assume that $\lim_{k\to\infty} \|\varphi_k - \varphi\|_{\alpha,\beta} = 0$ for every $\alpha, \beta \in \mathbb{N}_0$. We must prove that, as a consequence, $\varphi = \lim_{k\to\infty} \varphi_k$. The idea is that, in the definition of $d(\varphi, \psi)$, the sum of all terms with $|\alpha|$ or $|\beta|$ large is small, regardless of what φ and ψ are. Precisely, write $\psi_k = \varphi - \varphi_k$ and note that, for every $M \in \mathbb{N}$

$$\begin{aligned}
d(\varphi_k,\varphi) &= \sum_{\alpha,\beta\in\mathbb{N}_0^n} 2^{-|\alpha|-|\beta|}\frac{\|\psi_k\|_{\alpha,\beta}}{1+\|\psi_k\|_{\alpha,\beta}} \\
&= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|,|\beta|\le Mn}} 2^{-|\alpha|-|\beta|}\frac{\|\psi_k\|_{\alpha,\beta}}{1+\|\psi_k\|_{\alpha,\beta}} + \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha| \text{ or } |\beta|>Mn}}^{\infty} 2^{-|\alpha|-|\beta|}\frac{\|\psi_k\|_{\alpha,\beta}}{1+\|\psi_k\|_{\alpha,\beta}} \\
&\le \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|,|\beta|\le Mn}} 2^{-|\alpha|-|\beta|}\frac{\|\psi_k\|_{\alpha,\beta}}{1+\|\psi_k\|_{\alpha,\beta}} + \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha| \text{ or } |\beta|>Mn}}^{\infty} 2^{-|\alpha|-|\beta|} \\
&\le \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|,|\beta|\le Mn}} 2^{-|\alpha|-|\beta|}\frac{\|\psi_k\|_{\alpha,\beta}}{1+\|\psi_k\|_{\alpha,\beta}} + 2n\Big\{\sum_{m=M+1}^{\infty} 2^{-m}\Big\}\Big\{\sum_{m=0}^{\infty} 2^{-m}\Big\}^{2n-1} \\
&= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|,|\beta|\le Mn}} 2^{-|\alpha|-|\beta|}\frac{\|\psi_k\|_{\alpha,\beta}}{1+\|\psi_k\|_{\alpha,\beta}} + 2n\Big\{\frac{1}{2^M}\Big\}\{2\}^{2n-1}
\end{aligned}$$

Let $\varepsilon > 0$ and choose M so that $\frac{1}{2^M} \le \frac{\varepsilon}{4n2^{2n-1}}$ and hence

$$2n\Big\{\frac{1}{2^M}\Big\}\{2\}^{2n-1} \le \frac{\varepsilon}{2}$$

For each $\alpha,\beta \in \mathbb{N}_0$, $\lim\limits_{k\to\infty}\|\psi_k\|_{\alpha,\beta} = 0$ so that there is a $K_{\alpha,\beta}$ for which $k \ge K_{\alpha,\beta}$ implies $\|\psi_k\|_{\alpha,\beta} < \frac{\varepsilon}{2^{2n+1}}$. Set

$$K = \max\Big\{\ K_{\alpha,\beta}\ \Big|\ \alpha,\beta\in\mathbb{N}_0,\ |\alpha|,|\beta|\le M\ \Big\}$$

If $k \ge K$, then

$$\begin{aligned}
d(\varphi_k,\varphi) &\le \sum_{\substack{\alpha,\beta\in\mathbb{N}_0\\ |\alpha|,|\beta|\le Mn}} 2^{-|\alpha|-|\beta|}\frac{\|\psi_k\|_{\alpha,\beta}}{1+\|\psi_k\|_{\alpha,\beta}} + 2n\Big\{\frac{1}{2^M}\Big\}\{2\}^{2n-1} \\
&< \tfrac{\varepsilon}{2} + \sum_{\alpha,\beta\in\mathbb{N}_0} 2^{-|\alpha|-|\beta|}\tfrac{\varepsilon}{2^{2n+1}} \\
&= \varepsilon
\end{aligned}$$

(b) Let $\{\varphi_k\}$ be a Cauchy sequence with respect to the metric d. Then, as in part (c), for each $\alpha,\beta \in \mathbb{N}_0$, $\lim\limits_{k,k'\to\infty}\big\|\varphi_k - \varphi_{k'}\big\|_{\alpha,\beta} = 0$. In particular, $\lim\limits_{k,k'\to\infty}\big\|\varphi_k - \varphi_{k'}\big\|_{0,0} = 0$, so that the sequence $\{\varphi_k\}$ is Cauchy in the set, $\mathcal{C}(\mathbb{R}^n)$, of all bounded, continuous functions on $\mathbb{R}^n$ equipped with the uniform metric. Since $\mathcal{C}(\mathbb{R}^n)$ is complete, there exists a continuous function φ such that $\{\varphi_k\}$ converges uniformly to φ. As well, for each $\beta \in \mathbb{N}_0$,

$\lim\limits_{k,k'\to\infty} \left\|\varphi_k - \varphi_{k'}\right\|_{0,\beta} = 0$ so that the sequence $\left\{\partial^\beta\varphi_k\right\}$ of β^{th} derivatives is Cauchy in $\mathcal{C}(\mathbb{R}^n)$ and there exists a continuous function φ_β such that $\left\{\partial^\beta\varphi_k\right\}$ converges uniformly to φ_β. This ensures that φ is C^∞ with $\partial^\beta\varphi = \varphi_\beta$ for each $\beta \in \mathbb{N}_0$. Finally, we have that, for each $\alpha, \beta \in \mathbb{N}_0$, there is a $K_{\alpha,\beta}$ such that $\left|x^\alpha\partial^\beta\varphi_k(x) - x^\alpha\partial^\beta\varphi_{k'}(x)\right| < \varepsilon$ for all $k, k' \geq K_{\alpha,\beta}$ and all $x \in \mathbb{R}^n$. Consequently, if $k \geq K_{\alpha,\beta}$,

$$\begin{aligned}
\left\|\varphi_k - \varphi\right\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n} \left|x^\alpha\left(\partial^\beta\varphi_k(x) - \partial^\beta\varphi(x)\right)\right| \\
&= \sup_{x\in\mathbb{R}^n} \lim_{k'\to\infty} |x^\alpha|\left|\partial^\beta\varphi_k(x) - \partial^\beta\varphi_{k'}(x)\right| \\
&\leq \sup_{x\in\mathbb{R}^n} \varepsilon = \varepsilon
\end{aligned}$$

So, by part (c), $\left\{\varphi_k\right\}$ converges to φ with respect to the metric d. $\square$

Lemma B.4. *Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Then*

(a) $a\varphi + b\psi \in \mathcal{S}(\mathbb{R}^n)$ for all $a, b \in \mathbb{C}$ and

(b) $\partial^\gamma\varphi \in \mathcal{S}(\mathbb{R}^n)$ for all $\gamma \in \mathbb{N}_0^n$ and

(c) $\varphi\zeta \in \mathcal{S}(\mathbb{R}^n)$ for all C^∞ functions ζ that are polynomially bounded and have polynomially bounded derivatives and

(d) the convolution $(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(y)\psi(x-y)\ d^ny \in \mathcal{S}(\mathbb{R}^n)$.

Proof. These are all pretty obvious. Parts (a) and (b) are immediate consequences of the bounds

$$\begin{aligned}
\|a\varphi + b\psi\|_{\alpha,\beta} &\leq |a|\ \|\varphi\|_{\alpha,\beta} + |\beta|\ \|\psi\|_{\alpha,\beta} \\
\|\partial^\gamma\varphi\|_{\alpha,\beta} &= \|\varphi\|_{\alpha,\beta+\gamma}
\end{aligned}$$

which are true for all $\alpha, \beta, \gamma \in \mathbb{N}_0$ and $a, b \in \mathbb{C}$.

For part (c), let $\alpha, \beta \in \mathbb{N}_0^n$. By hypothesis, there is an $L \in \mathbb{N}$ such that $\Big[\sum_{\substack{\alpha'\in\mathbb{N}_0^n\\ |\alpha'|\leq L}} \left|x^{\alpha'}\right|\Big]^{-1}\partial^{\beta'}\zeta$ is uniformly bounded for all $\beta' \leq \beta$. (Here, $\beta' \leq \beta$ means that $\beta'_j \leq \beta_j$ for each $1 \leq j \leq n$. Also recall that, when α' is the zero vector, $x^{\alpha'} = 1$.) By the product rule

$$\partial^\beta(\varphi\zeta) = \sum_{\substack{\beta'\in\mathbb{N}_0^n\\ \beta'\leq\beta}} \tbinom{\beta}{\beta'}\ \partial^{\beta-\beta'}\varphi\ \partial^{\beta'}\zeta$$

where $\binom{\beta}{\beta'} = \prod_{j=1}^n \frac{\beta_j!}{\beta'_j!(\beta-\beta')_j!}$, and part (c) follows from

$$\|\varphi\zeta\|_{\alpha,\beta} \leq \sum_{\substack{\alpha',\beta'\in\mathbb{N}_0^n\\ |\alpha'|\leq L,\ \beta'\leq\beta}} \tbinom{\beta}{\beta'}\Big\|\Big[\sum_{\substack{\alpha''\in\mathbb{N}_0^n\\ |\alpha''|\leq L}} \left|x^{\alpha''}\right|\Big]^{-1}\partial^{\beta'}\zeta\Big\|_{L^\infty(\mathbb{R}^n)}\ \|\varphi\|_{\alpha+\alpha',\beta-\beta'}$$

The proof of part (d) is similar to that of part (c) but uses that

∘ the function $\Big[\sum_{\substack{\alpha'\in\mathbb{N}_0^n\\ |\alpha'|\le n+1}} \big|y^{\alpha'}\big|\Big]^{-1} \in L^1(\mathbb{R}^n)$ and

∘ $|x^\alpha| \le \sum\limits_{\substack{\alpha'\in\mathbb{N}_0^n\\ \alpha'\le\alpha}} \binom{\alpha}{\alpha'}\, |y^{\alpha'}||(x-y)^{\alpha-\alpha'}|$

∘ All derivatives of $\varphi(y)\psi(x-y)$ with respect to x are absolutely integrable with respect to y, so that we are allowed to move derivatives with respect to x inside the integral $\int_{\mathbb{R}^n} \varphi(y)\psi(x-y)\ d^n y$. □

B.2. The Fourier Transform

Definition B.5. The Fourier transform $\hat{f}(\xi)$ of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\ d^n x \tag{B.2a}$$

Since $f(x)$, and hence $e^{-i\xi\cdot x}f(x)$, is a continuous function of x which is bounded by a constant times $\frac{1}{1+|x|^{2n}}$, the integral exists and $\hat{f}(\xi)$ is a well–defined complex number for each $\xi \in \mathbb{R}^n$. We shall show in Theorem B.9, below that the map $f \mapsto \hat{f}$ is a continuous, linear map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and furthermore that this map is one–to–one and onto with the inverse map being the inverse Fourier transform given by

$$\check{g}(x) = \int_{\mathbb{R}^n} e^{i\xi\cdot x} g(\xi)\ \tfrac{d^n\xi}{(2\pi)^n} \tag{B.2b}$$

The computational properties of the Fourier transform are given in

Theorem B.6. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$. Then*

(a) The Fourier transform of $af(x)+bg(x)$ is $a\hat{f}(\xi) + b\hat{g}(\xi)$.

(b) If $\beta \in \mathbb{N}_0$, then the Fourier transform of $\partial^\beta f(x)$ is $i^{|\beta|}\xi^\beta \hat{f}(\xi)$.

(c) The Fourier transform, $\hat{f}(\xi)$, of $f(x)$ is infinitely differentiable and, for each $\beta \in \mathbb{N}_0$, $\frac{\partial^\beta}{\partial\xi^\beta}\hat{f}(\xi)$ is the Fourier transform of $(-i)^{|\beta|}x^\beta f(x)$.

(d) Let $a \in \mathbb{R}^n$. The Fourier transform of the translated function $(T_a f)(x) = f(x-a)$ is $e^{-ia\cdot\xi}\hat{f}(\xi)$.

(e) The Fourier transform of $f(x) = e^{-|x|^2/2}$ is $\hat{f}(\xi) = \big(2\pi\big)^{n/2} e^{-|\xi|^2/2}$.

(f) $\int_{\mathbb{R}^n} f(x)\,\overline{g(x)}\ d^n x = \int_{\mathbb{R}^n} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\ \frac{d^n\xi}{(2\pi)^n}$

(g) The Fourier transform of the convolution $h = f * g$ is $\hat{h}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

Proof. (a) is obvious.

(b) By induction, it suffices to prove the case $|\beta| = 1$. We do so for $n = 1$. By integration by parts, the Fourier transform of the first derivative $f'(x)$ is

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\ dx &= -\int_{-\infty}^{\infty} f(x)\big(\tfrac{d}{dx} e^{-i\xi x}\big)\ dx = i\xi \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\ dx \\ &= i\xi \hat{f}(\xi)\end{aligned}$$

The boundary terms vanished because $\lim\limits_{x\to\infty} e^{-i\xi x} f(x) = \lim\limits_{x\to-\infty} e^{-i\xi x} f(x) = 0.$

(c) Again, by induction, it suffices to prove the case $|\beta| = 1$. Again, we do so for $n = 1$.

$$\begin{aligned}\tfrac{d}{d\xi}\hat{f}(\xi) = \tfrac{d}{d\xi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\ dx &= \int_{-\infty}^{\infty} \tfrac{\partial}{\partial\xi}\big(e^{-i\xi x} f(x)\big)\ dx \\ &= \int_{-\infty}^{\infty} (-ix) e^{-i\xi x} f(x)\ dx\end{aligned}$$

is indeed $-i$ times the Fourier transform of $xf(x)$. The second equality, in which the derivative with respect to ξ was moved past the integral sign is justified by Problem B.7, below.

(d) is obvious — just make the change of variables $x' = x - a$ in the integral defining the Fourier transform of $T_a f$.

(e) Since the integral defining $\hat{f}(\xi)$ factorizes, it suffices to condsider $n = 1$. By part (c) of this Theorem, $\frac{d}{d\xi}\hat{f}(\xi)$ is the Fourier transform of $-ixf(x) = -ixe^{-x^2/2} = i\frac{d}{dx}e^{-x^2/2} = if'(x)$. Thus by parts (a) and (b) of this Theorem, $\frac{d}{d\xi}\hat{f}(\xi) = -\xi\hat{f}(\xi)$ and

$$\tfrac{d}{d\xi}\big\{\hat{f}(\xi)e^{\xi^2/2}\big\} = e^{\xi^2/2}\big\{\tfrac{d}{d\xi}\hat{f}(\xi) + \xi\hat{f}(\xi)\big\} = 0$$

for all $\xi \in \mathbb{R}$. Consequently $\hat{f}(\xi)e^{\xi^2/2}$ must be some constant, independent of ξ. Hence to determine $\hat{f}(\xi)$ we need only to determine the value of that constant, which we may do by computing $\hat{f}(\xi)e^{\xi^2/2}\big|_{\xi=0} = \hat{f}(0)$. Since $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-x^2/2}\ dx > 0$, it is determined by

$$\begin{aligned}\hat{f}(0)^2 &= \Big[\int_{-\infty}^{\infty} e^{-x^2/2}\ dx\Big]^2 = \Big[\int_{-\infty}^{\infty} e^{-x^2/2}\ dx\Big]\Big[\int_{-\infty}^{\infty} e^{-y^2/2}\ dx\Big] \\ &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dxdy\end{aligned}$$

Changing to polar coordinates,

$$\hat{f}(0)^2 = \int_0^{\infty} dr\ r \int_0^{2\pi} d\theta\ e^{-r^2/2} = 2\pi \int_0^{\infty} dr\ re^{-r^2/2} = 2\pi\Big[-e^{-r^2/2}\Big]_0^{\infty} = 2\pi$$

Thus $\hat f(0) = \sqrt{2\pi}$ which tells us that $\hat f(\xi)e^{\xi^2/2} = \sqrt{2\pi}$ and hence that $\hat f(\xi) = \sqrt{2\pi}e^{-\xi^2/2}$ for all ξ.

(f) By the definition in (B.2a),

$$\begin{aligned}
\int \hat f(\xi)\,\overline{\hat g(\xi)}\ \tfrac{d^n\xi}{(2\pi)^n} &= \int \tfrac{d^n\xi}{(2\pi)^n} \int d^nx\ e^{-i\xi\cdot x} f(x)\,\overline{\hat g(\xi)}\\
&= \int d^nx \int \tfrac{d^n\xi}{(2\pi)^n}\ e^{-i\xi\cdot x} f(x)\,\overline{\hat g(\xi)}\\
&= \int d^nx\ f(x)\Big[\ \overline{\int \tfrac{d^n\xi}{(2\pi)^n}\ e^{i\xi\cdot x}\hat g(\xi)}\ \Big] = \int d^nx\ f(x)\,\overline{g(x)}
\end{aligned}$$

The last equality uses Theorem B.9, below.

(g) By the definition in (B.2a) and Lemma B.4.d,

$$\begin{aligned}
\hat h(\xi) &= \int_{\mathbb{R}^n} d^nx\ e^{-i\xi\cdot x} h(x) = \int_{\mathbb{R}^n} d^nx \int_{\mathbb{R}^n} d^ny\ e^{-i\xi\cdot y} f(y)\ e^{-i\xi\cdot(x-y)} g(x-y)\\
&= \int_{\mathbb{R}^n} d^ny\ e^{-i\xi\cdot y} f(y) \int_{\mathbb{R}^n} d^nx\ e^{-i\xi\cdot(x-y)} g(x-y) \qquad \text{by Fubini}\\
&= \int_{\mathbb{R}^n} d^ny\ e^{-i\xi\cdot y} f(y) \int_{\mathbb{R}^n} d^nx'\ e^{-i\xi\cdot x'} g(x') \qquad \text{with } x' = x-y\\
&= \hat f(\xi)\,\hat g(\xi)
\end{aligned}$$

□

Exercise B.7. Let $f : (-\infty,\infty)\times[c,d] \to \mathbb{C}$ be continuous. Assume that $\frac{\partial f}{\partial y}$ exists and is continuous and that there is a constant C such that

$$|f(x,y)|, \left|\tfrac{\partial f}{\partial y}(x,y)\right| \le \tfrac{C}{1+x^2} \qquad \text{and} \qquad \left|\tfrac{\partial f}{\partial y}(x,y) - \tfrac{\partial f}{\partial y}(x,y')\right| \le C\tfrac{|y-y'|}{1+x^2}$$

for all $-\infty < x < \infty$ and $c \le y, y' \le d$. Prove that $g(y) = \int_{-\infty}^{\infty} f(x,y)\ dx$ is differentiable with $g'(y) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial y}(x,y)\ dx$.

Exercise B.8. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Set

$$r(x) = f(-x) \qquad c(x) = \overline{f(x)}$$

Prove that

$$\hat r(\xi) = \overline{\hat c(\xi)} \qquad \hat c(\xi) = \overline{\hat f(-\xi)}$$

Theorem B.9. *The maps*

$$\begin{aligned}
f(x) \in \mathcal{S}(\mathbb{R}^n) &\mapsto \hat f(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\ d^nx\\
g(\xi) \in \mathcal{S}(\mathbb{R}^n) &\mapsto \check g(x) = \int_{\mathbb{R}^n} e^{i\xi\cdot x} g(\xi)\ \tfrac{d^n\xi}{(2\pi)^n}
\end{aligned}$$

are one–to–one, continuous, linear maps from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$ and are inverses of each other.

Proof. That $\hat f$ is linear in f was Theorem B.6.a.

We now assume that $f \in \mathcal{S}(\mathbb{R}^n)$ and prove that $\hat f(\xi) \in \mathcal{S}(\mathbb{R}^n)$. For simplicity of notation, we assume that $n = 1$. Let α, β be nonnegative integers. By parts (b) and (c) of Theorem B.6 followed by the product rule, $\xi^\alpha \frac{d^\beta}{d\xi^\beta}\hat f(\xi)$ is the Fourier transform of

$$\begin{aligned}(-i)^\alpha \tfrac{d^\alpha}{dx^\alpha}\big[(-ix)^\beta f(x)\big] &= (-i)^{\alpha+\beta}\sum_{\ell=0}^{\min\{\alpha,\beta\}} \tbinom{\alpha}{\ell}\big(\tfrac{d^\ell}{dx^\ell}x^\beta\big)\big(\tfrac{d^{\alpha-\ell}}{dx^{\alpha-\ell}}f(x)\big)\\ &= (-i)^{\alpha+\beta}\sum_{\ell=0}^{\min\{\alpha,\beta\}} \tbinom{\alpha}{\ell}\tfrac{\beta!}{(\beta-\ell)!}x^{\beta-\ell}f^{(\alpha-\ell)}(x)\end{aligned}$$

Hence

$$\begin{aligned}\|\hat f(\xi)\|_{\alpha,\beta} &= \sup_{\xi\in\mathbb{R}}\big|\xi^\alpha \tfrac{d^\beta}{d\xi^\beta}\hat f(\xi)\big|\\ &= \sup_{\xi\in\mathbb{R}}\bigg|\int_{-\infty}^{\infty} e^{-i\xi x}\Big[\sum_{\ell=0}^{\min\{\alpha,\beta\}}\tbinom{\alpha}{\ell}\tfrac{\beta!}{(\beta-\ell)!}x^{\beta-\ell}f^{(\alpha-\ell)}(x)\Big]dx\bigg|\\ &\le \sum_{\ell=0}^{\min\{\alpha,\beta\}}\tbinom{\alpha}{\ell}\tfrac{\beta!}{(\beta-\ell)!}\int_{-\infty}^{\infty}\big|x^{\beta-\ell}f^{(\alpha-\ell)}(x)\big|dx\\ &= \sum_{\ell=0}^{\min\{\alpha,\beta\}}\tbinom{\alpha}{\ell}\tfrac{\beta!}{(\beta-\ell)!}\int_{-\infty}^{\infty}\tfrac{1}{1+x^2}\big\{|x|^{\beta-\ell}+|x|^{\beta-\ell+2}\big\}\big|f^{(\alpha-\ell)}(x)\big|dx\\ &\le \sum_{\ell=0}^{\min\{\alpha,\beta\}}\tbinom{\alpha}{\ell}\tfrac{\beta!}{(\beta-\ell)!}\big\{\|f\|_{\beta-\ell,\alpha-\ell}+\|f\|_{\beta-\ell+2,\alpha-\ell}\big\}\int_{-\infty}^{\infty}\tfrac{1}{1+x^2}\,dx\\ &= \sum_{\ell=0}^{\min\{\alpha,\beta\}}\pi\tbinom{\alpha}{\ell}\tfrac{\beta!}{(\beta-\ell)!}\big\{\|f\|_{\beta-\ell,\alpha-\ell}+\|f\|_{\beta-\ell+2,\alpha-\ell}\big\}\end{aligned} \tag{B.3}$$

Since $f \in \mathcal{S}(\mathbb{R})$, the right hand side is finite. The corresponding argument for general n proves that, when $f \in \mathcal{S}(\mathbb{R}^n)$, $\|\hat f\|_{\alpha,\beta}$ is finite for all $\alpha, \beta \in \mathbb{N}_0^n$, so that $\hat f \in \mathcal{S}(\mathbb{R}^n)$.

It also proves that the map $f \mapsto \hat f$ is continuous, since if the sequence $\{f_j\}_{j\in\mathbb{N}}$ converges to f in $\mathcal{S}(\mathbb{R}^n)$, then replacing f by $f - f_j$ in (B.3), or its analog for general n, shows that $\|\hat f - \hat f_j\|_{\alpha,\beta}$ converges to zero as $j \to \infty$, for all $\alpha, \beta \in \mathbb{N}_0^n$. So $\{\hat f_j\}_{j\in\mathbb{N}}$ converges to $\hat f$ in $\mathcal{S}(\mathbb{R})$ too.

The proof that the map $g(\xi) \mapsto \check{g}(x)$ is a continuous, linear map from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$ is similar.

We now assume that $f(x) \in \mathcal{S}(\mathbb{R}^n)$ and prove that the inverse Fourier transform of $\hat{f}(\xi)$ is $f(x)$. In symbols, we prove that

$$f(x) = \int_{\mathbb{R}^n} e^{i\xi\cdot x}\hat{f}(\xi)\ \tfrac{d^n\xi}{(2\pi)^n} \tag{B.4}$$

We first prove the ($x = 0$, $n = 1$) special case that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)\ \tfrac{d\xi}{2\pi} \tag{B.5}$$

Write

$$f(x) = f(0)e^{-x^2/2} + xh(x) \quad \text{where } h(x) = \begin{cases} \frac{1}{x}\big(f(x) - f(0)e^{-x^2/2}\big) & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

By Problem B.10, below, the function $h \in \mathcal{S}(\mathbb{R})$. So, by parts (e) and (c) of Theorem B.6,

$$\hat{f}(\xi) = \sqrt{2\pi}f(0)e^{-\xi^2/2} + i\tfrac{d}{d\xi}\hat{h}(\xi)$$

and

$$\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\ d\xi = \tfrac{f(0)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\xi^2/2}\ d\xi + \tfrac{i}{2\pi}\int_{-\infty}^{\infty} \tfrac{d}{d\xi}\hat{h}(\xi)\ d\xi$$

The first term

$$\tfrac{f(0)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\xi^2/2}\ d\xi = f(0)$$

by the computation at the end of the proof of Theorem B.6.e. The second term is $\frac{i}{2\pi}$ times

$$\int_{-\infty}^{\infty} \tfrac{d}{d\xi}\hat{h}(\xi)\ d\xi = \lim_{A,B\to\infty}\int_{-A}^{B} \tfrac{d}{d\xi}\hat{h}(\xi)\ d\xi = \lim_{A,B\to\infty}\big[\hat{h}(B) - \hat{h}(-A)\big] = 0$$

Here we have used the fundamental theorem of calculus and the decay at $\pm\infty$ which follows from the fact that $\hat{h} \in \mathcal{S}(\mathbb{R})$, which, in turn, follows from $h \in \mathcal{S}(\mathbb{R})$. This completes the proof of (B.5). The proof of the analog of (B.5) for general n is similar. Replacing f by $T_{-x}f$ and using $f(x) = \big(T_{-x}f\big)(0)$ and $\widehat{T_{-x}f}(\xi) = e^{i\xi\cdot x}\hat{f}(\xi)$ gives (B.4).

The proof that

$$g(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\check{g}(x)\ d^n x \tag{B.6}$$

is similar. The formulae (B.4) and (B.6) show that the maps $f(x) \mapsto \hat{f}(\xi)$ and $g(\xi) \mapsto \check{g}(x)$ are onto $\mathcal{S}(\mathbb{R})$ and are inverses of each other. $\square$

Exercise B.10. Let $f \in \mathcal{S}(\mathbb{R})$ and define

$$h(x) = \begin{cases} \frac{1}{x}\big(f(x) - f(0)e^{-x^2/2}\big) & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

Prove that $h \in \mathcal{S}(\mathbb{R})$.

Theorem B.11. *The Fourier transform (B.2a) has a unique continuous extension to $L^2(\mathbb{R}^n)$. The inverse Fourier transform (B.2b) has unique continuous extension to $L^2(\mathbb{R}^n)$. The two extensions are inverses of each other.*

Proof. This is an immediate consequence of the B.L.T. Theorem A.41, Theorem B.9, Theorem B.6.f (which implies that the Fourier and inverse Fourier tranforms are bounded operators with respect to the $L^2(\mathbb{R}^n)$ norm) and a simple extension of Problem A.14 (which implies that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$). □

Lemma B.12 (the Riemann–Lebesgue lemma)**.** *The Fourier transform (B.2a) extends uniquely to a bounded map from $L^1(\mathbb{R}^n)$ to $C_\infty(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity.*

Proof. By Theorem B.9, the Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$, which is dense in $L^1(\mathbb{R}^n)$ (by a simple extension of Problem A.14), into $\mathcal{S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$. It now suffices to observe that

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$$

and apply the B.L.T. Theorem A.41. □

Exercise B.13. The goal of this problem is to prove the Paley–Wiener theorem, which says that a function f is C^∞ and supported in the closed ball $\bar{B}_R = \{\, x \in \mathbb{R}^n \bigm| |x| \le R \,\}$ if and only if $\hat{f}(\xi)$ extends to a holomorphic function on $\mathbb{C}^n$ which obeys

$$\big|\hat{f}(\xi)\big| \le \tfrac{C_N}{1+|\xi|^{2N}} e^{R|\operatorname{Im}\xi|} \qquad \text{for all } N \in \mathbb{N} \tag{B.7}$$

(a) Let $f \in C_0^\infty(\mathbb{R}^n)$ be supported in $\bar{B}_R$. Prove that $\hat{f}(\xi)$ extends to a holomorphic function on $\mathbb{C}^n$ and that, for each $N \in \mathbb{N}$, there is a constant C_N such that (B.7) holds.

(b) Assume that the Fourier transform $\hat{f}(\xi)$ of a function $f(x)$ extends to a holomorphic function on $\mathbb{C}^n$ and that, for each $N \in \mathbb{N}$, there is a constant C_N such that (B.7) holds. Let $\eta \in \mathbb{R}^n$. Prove that

$$f(x) = e^{-\eta\cdot x} \int e^{i\xi\cdot x} \hat{f}(\xi + i\eta)\, \tfrac{d^n\xi}{(2\pi)^n}$$

(c) Prove that, under the hypotheses of part (b), $f(x)$ is supported in $\bar{B}_R$.

B.3. Tempered Distributions

The theory of tempered distributions allows us to give a rigorous meaning to the Dirac delta function. It is "defined", on a handwaving level, by the properties that

(1) $\delta(x) = 0$ *except when* $x = 0$

(2) $\delta(0)$ *is "so infinite" that*

(3) the area under its graph is one.

Still on a handwaving level, if f *is any continuous function, then the functions* $f(x)\delta(x)$ *and* $f(0)\delta(x)$ *are the same since they are both zero for every* $x \neq 0$. *Consequently*

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \int_{-\infty}^{\infty} f(0)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0) \tag{B.8}$$

That $\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$ *is by far the most important property of the Dirac delta function. But there is no Riemann integrable function* $\delta(x)$ *that satisfies* (B.8).

Exercise B.14. Prove that there is no Riemann integrable function $\delta(x)$ that satisfies (B.8).

The basic idea which allows us to make make rigorous sense of (B.8) *is to generalize the meaning of "a function on* $\mathbb{R}$*". We shall call the generalization a "tempered distribution on* $\mathbb{R}$*". Of course a function on* $\mathbb{R}$*, in the conventional sense, is a rule which assigns a number to each* $x \in \mathbb{R}$. *A tempered distribution will be a rule which assigns a number to each nice function on* $\mathbb{R}$. *We will associate to the conventional function* $f : \mathbb{R} \to \mathbb{C}$ *the tempered distribution which assigns to the nice function* $\varphi(x)$ *the number* $\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$. *The tempered distribution which corresponds to the Dirac delta function will assign to* $\varphi(x)$ *the number* $\varphi(0)$. *The space of "nice functions" used by tempered distributions is the Schwartz space of Definition B.1.*

Definition B.15 (Tempered Distributions)**.** The space of all tempered distributions on $\mathbb{R}^n$, denoted $\mathcal{S}'(\mathbb{R}^n)$, is the dual space of $\mathcal{S}(\mathbb{R}^n)$. That is, it is the set of all functions

$$f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$$

that are linear and continuous. One usually denotes by $\langle f, \varphi \rangle$ the value in $\mathbb{C}$ that the distribution $f \in \mathcal{S}'(\mathbb{R})$ assigns to $\varphi \in \mathcal{S}(\mathbb{R}^n)$. In this notation,

∘ that f is linear means that $\langle f, a\varphi + b\psi \rangle = a\,\langle f, \varphi \rangle + b\,\langle f, \psi \rangle$ for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and all $a, b \in \mathbb{C}$.

∘ that f is continuous means that if $\varphi = \lim\limits_{n\to\infty} \varphi_n$ in $\mathcal{S}(\mathbb{R}^n)$, then $\langle f, \varphi\rangle = \lim\limits_{n\to\infty} \langle f, \varphi_n\rangle$.

Example B.16. (a) Here is the motivating example for the whole subject. Let $f : \mathbb{R}^n \to \mathbb{C}$ be any function that is polynomially bounded (that is, there is a polynomial $P(x)$ such that $|f(x)| \le P(x)$ for all $x \in \mathbb{R}^n$) and that is Riemann integrable on $[-M, M]^n$ for each $M > 0$. Then

$$f : \varphi \in \mathcal{S}(\mathbb{R}^n) \mapsto \langle f, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\ d^n x$$

is a tempered distribution. The integral converges because every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ decays faster at infinity than one over any polynomial. See Problem B.17, below. The linearity in φ of $\langle f, \varphi\rangle$ is obvious. The continuity in φ of $\langle f, \varphi\rangle$ follows easily from Problem B.17 (generalized to $\mathbb{R}^n$) and Theorem B.18, below.

(b) The Dirac delta function, on $\mathbb{R}$, and more generally the Dirac delta function translated to $b \in \mathbb{R}$, are defined as tempered distributions by

$$\langle \delta, \varphi\rangle = \varphi(0) \qquad \langle \delta_b, \varphi\rangle = \varphi(b)$$

Once again, the linearity in φ is obvious and the continuity in φ is easily verified if one applies Theorem B.18.

(c) The derivative of the Dirac delta function δ_b is defined by

$$\left\langle \delta_b', \varphi\right\rangle = -\varphi'(b)$$

The reason for the name "derivative of the Dirac delta function" will be given in the section on differentiation, later. See Definition B.21.

(d) The principal value of $\frac{1}{x}$, with x running over $\mathbb{R}$, is defined by

$$\left\langle P\tfrac{1}{x}, \varphi\right\rangle = \lim_{\varepsilon\to 0+} \int_{|x|>\varepsilon} \tfrac{\varphi(x)}{x}\ dx$$

The first thing that we have to do is verify that the limit above actually exists. This is not a trivial statement, because not only is $\frac{\varphi(x)}{x}$ not integrable on $[-1, 1]$ if $\varphi(0) \neq 0$ (because, for x near zero, $\frac{\varphi(x)}{x} \approx \frac{\varphi(0)}{x}$), but $\int_0^1 \frac{1}{x} dx$ and $\int_{-1}^0 \frac{1}{x}\ dx$ do not even exist as improper integrals:

$$\begin{aligned}
\int_0^1 \tfrac{1}{x}\ dx &= \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \tfrac{1}{x}\ dx\ = \lim_{\varepsilon\to 0+} \ln\tfrac{1}{\varepsilon} = \infty \\
\int_{-1}^0 \tfrac{1}{x}\ dx &= \lim_{\varepsilon\to 0+} \int_{-1}^{-\varepsilon} \tfrac{1}{x}\ dx = \lim_{\varepsilon\to 0+} \ln\varepsilon = -\infty
\end{aligned}$$

Here is the verification that the limit defining $\left\langle P\frac{1}{x},\varphi\right\rangle$ exists

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{|x|>\varepsilon}\tfrac{\varphi(x)}{x}\,dx &= \lim_{\substack{\varepsilon\to 0+\\ M,M'\to\infty}}\left\{\int_\varepsilon^M \tfrac{\varphi(x)}{x}dx+\int_{-M'}^{-\varepsilon}\tfrac{\varphi(x)}{x}\,dx\right\}\\ &= \lim_{\substack{\varepsilon\to 0+\\ M,M'\to\infty}}\left\{\int_\varepsilon^1 \tfrac{\varphi(x)}{x}dx+\int_1^M\tfrac{\varphi(x)}{x}dx+\int_{-M'}^{-1}\tfrac{\varphi(x)}{x}\,dx+\int_{-1}^{-\varepsilon}\tfrac{\varphi(x)}{x}\,dx\right\}\\ &= \lim_{\substack{\varepsilon\to 0+\\ M,M'\to\infty}}\left\{\int_\varepsilon^1 \tfrac{\varphi(x)-\varphi(-x)}{x}\,dx+\int_1^M\tfrac{\varphi(x)}{x}\,dx+\int_{-M'}^{-1}\tfrac{\varphi(x)}{x}\,dx\right\}\end{aligned}$$

The first integral converges because, by the mean value theorem, we have, for some c between x and $-x$,

$$\left|\tfrac{\varphi(x)-\varphi(-x)}{x}\right| = \left|\tfrac{\varphi'(c)\,2x}{x}\right| \le 2\|\varphi\|_{0,1}$$

The second and third integrals converge because, for $|x|\ge 1$

$$\left|\tfrac{\varphi(x)}{x}\right| \le \tfrac{1}{x^2}|x\varphi(x)| \le \tfrac{1}{x^2}\|\varphi\|_{1,0}$$

These bounds give both that $\left\langle P\frac{1}{x},\varphi\right\rangle$ is well–defined and

$$\begin{aligned}\left|\left\langle P\tfrac{1}{x},\varphi\right\rangle\right| &\le 2\|\varphi\|_{0,1}\int_0^1 dx+\|\varphi\|_{1,0}\int_1^\infty \tfrac{1}{x^2}\,dx+\|\varphi\|_{1,0}\int_{-\infty}^{-1}\tfrac{1}{x^2}\,dx\\ &= 2\|\varphi\|_{0,1}+2\|\varphi\|_{1,0}\end{aligned}$$

Linearity is again obvious. Continuity again follows by Theorem B.18, below.

Exercise B.17. Let $f:\mathbb{R}\to\mathbb{C}$ be Riemann integrable on $[-M,M]$ for all $M>0$ and obey the bound $|f(x)|\le P(x)$ for all $x\in\mathbb{R}$, where $P(x)$ is the polynomial $P(x)=\sum_{m=N_-}^{N_+}a_mx^m$ and $N_\pm$ are nonnegative integers.

(a) Prove that there is a constant $C>0$ such that

$$|f(x)|(1+x^2)\le C\big(|x|^{N_-}+|x|^{N_++2}\big)$$

for all $x\in\mathbb{R}$.

(b) Prove that

$$\int_{-\infty}^{\infty}|f(x)\varphi(x)|\,dx\le \pi C\big(\|\varphi\|_{N_-,0}+\|\varphi\|_{N_++2,0}\big)$$

for all $\varphi\in\mathcal{S}(\mathbb{R})$.

Theorem B.18 (Continuity Test). *A linear map*

$$f : \varphi \in \mathcal{S}(\mathbb{R}^n) \mapsto \langle f, \varphi \rangle \in \mathbb{C}$$

is continuous if and only if there are constants $C > 0$ and $N \in \mathbb{N}$ such that

$$\big|\, \langle f, \varphi \rangle \,\big| \le C \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|,|\beta| \le N}} \|\varphi\|_{\alpha,\beta}$$

Proof. $\Leftarrow$: Assume that $\big|\, \langle f, \varphi \rangle \,\big| \le C \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|,|\beta| \le N}} \|\varphi\|_{\alpha,\beta}$ and that the sequence $\big\{\varphi_k\big\}_{k\in\mathbb{N}}$ converges to φ in $\mathcal{S}(\mathbb{R}^n)$. Then

$$\big|\, \langle f, \varphi \rangle - \langle f, \varphi_k \rangle \,\big| = \big|\, \langle f, \varphi - \varphi_k \rangle \,\big| \le C \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|,|\beta| \le N}} \|\varphi - \varphi_k\|_{\alpha,\beta}$$

converges to zero as $k \to \infty$. So f is continuous.

$\Rightarrow$: Assume that $f \in \mathcal{S}'(\mathbb{R}^n)$. In particular f is continuous at $\varphi = 0$. Then there is a $\delta > 0$ such that

$$d(\psi, 0) < \delta \implies \big|\, \langle f, \psi \rangle \,\big| < 1$$

Choose N so that $\sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha| \text{ or } |\beta| > N}} 2^{-|\alpha|-|\beta|} < \frac{\delta}{2}$ and consider any $\psi \in \mathcal{S}(\mathbb{R}^n)$ that obeys

$$\sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|,|\beta| \le N}} \|\psi\|_{\alpha,\beta} \le \tfrac{\delta}{2}$$

For any such ψ we have

$$\begin{aligned} d(\psi, 0) &= \sum_{\alpha,\beta \in \mathbb{N}_0^n} 2^{-|\alpha|-|\beta|} \tfrac{\|\psi\|_{\alpha,\beta}}{1+\|\psi\|_{\alpha,\beta}} \le \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|,|\beta| \le N}} \|\psi\|_{\alpha,\beta} + \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha| \text{ or } |\beta| > N}} 2^{-|\alpha|-|\beta|} \\ &\implies d(\psi, 0) < \delta \\ &\implies \big|\, \langle f, \psi \rangle \,\big| < 1 \end{aligned}$$

Consequently, for any $0 \ne \varphi \in \mathcal{S}(\mathbb{R}^n)$, setting

$$\psi = \tfrac{\delta}{2} \Bigg[\sum_{\substack{\alpha',\beta' \in \mathbb{N}_0^n \\ |\alpha'|,|\beta'| \le N}} \|\varphi\|_{\alpha',\beta'} \Bigg]^{-1} \varphi$$

we have

$$\sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|,|\beta| \le N}} \|\psi\|_{\alpha,\beta} = \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|,|\beta| \le N}} \tfrac{\delta}{2} \Bigg[\sum_{\substack{\alpha',\beta' \in \mathbb{N}_0^n \\ |\alpha'|,|\beta'| \le N}} \|\varphi\|_{\alpha',\beta'} \Bigg]^{-1} \|\varphi\|_{\alpha,\beta} = \tfrac{\delta}{2}$$

and hence

$$\big|\left\langle f,\varphi\right\rangle\big| = \tfrac{2}{\delta}\Bigg[\sum_{\substack{\alpha',\beta'\in\mathbb{N}_0^n\\|\alpha'|,|\beta'|\le N}}\|\varphi\|_{\alpha',\beta'}\Bigg]\,\big|\left\langle f,\psi\right\rangle\big| < \tfrac{2}{\delta}\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|,|\beta|\le N}}\|\varphi\|_{\alpha,\beta}$$

as desired. □

B.4. Operations on Tempered Distributions

We now define a number of operations like, for example, addition and differentiation, on tempered distributions. The motivation for all of these definitions comes from Example B.16.a with $f \in \mathcal{S}(\mathbb{R}^n)$. Then we can view f both as a conventional function and as a tempered distribution. We will define each operation in such a way that when it is applied to $f \in \mathcal{S}(\mathbb{R}^n)$, viewed as a distribution, it yields the same answer as when the operation is applied to f viewed as an ordinary function, with the result viewed as a distribution. As a trivial example, suppose that we wish to define multiplication by 7. If $f \in \mathcal{S}(\mathbb{R}^n)$ is viewed as an ordinary function, applying the operation of multiplication by 7 to it gives the ordinary function $7f$. But $7f$ can again be viewed as the distribution $\left\langle 7f,\varphi\right\rangle = \int 7f(x)\,\varphi(x)\ d^nx = 7\left\langle f,\varphi\right\rangle$. So we would define the operation of multiplication by 7 applied to any distribution f as the distribution $7f$ defined by $\left\langle 7f,\varphi\right\rangle = 7\left\langle f,\varphi\right\rangle$.

Addition and Scalar Multiplication.

Motivation. If $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $a, b \in \mathbb{C}$, then

$$\begin{aligned}\int_{\mathbb{R}^n}\big[af(x)+bg(x)\big]\,\varphi(x)\ d^nx &= a\int_{\mathbb{R}^n} f(x)\,\varphi(x)\ d^nx + b\int_{\mathbb{R}^n} g(x)\,\varphi(x)\ d^nx\\ &= a\left\langle f,\varphi\right\rangle + b\left\langle g,\varphi\right\rangle\end{aligned}$$

Definition B.19. If $f, g \in \mathcal{S}'(\mathbb{R}^n)$ and $a, b \in \mathbb{C}$, then define $af+bg \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\left\langle af+bg,\varphi\right\rangle = a\left\langle f,\varphi\right\rangle + b\left\langle g,\varphi\right\rangle$$

Theorem B.20. *If $f, g \in \mathcal{S}'(\mathbb{R}^n)$ and $a, b \in \mathbb{C}$, then $af + bg$, defined above, is a well–defined element of $\mathcal{S}'(\mathbb{R}^n)$. The operations of addition and scalar multiplication so defined obey the usual vector space axioms of Definition A.1.*

Proof. Trivial. □

Differentiation.

Motivation. If $f \in \mathcal{S}(\mathbb{R})$, then, by integration by parts,

$$\int_{-\infty}^{\infty} f'(x)\,\varphi(x)\ dx = -\int_{-\infty}^{\infty} f(x)\,\varphi'(x)\ dx \qquad \text{(the boundary terms vanish)}$$

More generally, if $f \in \mathcal{S}(\mathbb{R}^n)$, and $\gamma \in \mathbb{N}_0^n$,

$$\int_{\mathbb{R}^n} (\partial^\gamma f)(x)\,\varphi(x)\ d^n x = (-1)^{|\gamma|}\int_{\mathbb{R}^n} f(x)\,(\partial^\gamma \varphi)(x)\ d^n x$$

Definition B.21. If $\gamma \in \mathbb{N}_0^n$, we define the γ^{th} derivative of $f \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \partial^\gamma f, \varphi\rangle = (-1)^{|\gamma|}\langle f, \partial^\gamma \varphi\rangle$$

Since $\|\partial^\gamma \varphi\|_{\alpha,\beta} = \|\varphi\|_{\alpha,\beta+\gamma}$ the right hand side gives a well–defined element of $\mathcal{S}'(\mathbb{R}^n)$.

Remark B.22. Note that *every* derivative of *every* distribution *always* exists.

Example B.23. The Heavyside unit function

$$H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(on $\mathbb{R}$) may also be viewed as the tempered distribution

$$\langle H, \varphi\rangle = \int_0^\infty \varphi(x)\,dx$$

(in $\mathcal{S}'(\mathbb{R})$) via Example B.16.a. The derivative of this distribution is

$$\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^\infty \varphi'(x)\,dx = -\big[\varphi(x)\big]_0^\infty = \varphi(0) = \langle \delta, \varphi\rangle$$

Thus H' is the Dirac delta function.

Fourier Transform.

Motivation. If f and φ are both in $\mathcal{S}(\mathbb{R}^n)$, then, writing $\varphi(\xi) = \overline{\psi(\xi)}$

$$\begin{aligned}\langle \hat{f}, \varphi\rangle &= \int_{\mathbb{R}^n} \hat{f}(\xi)\varphi(\xi)\ d^n\xi = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\psi(\xi)}\ d^n\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} f(x)\overline{\check{\psi}(x)}\ d^n x \qquad \text{(by Theorem B.6.f and Theorem B.9)} \\ &= \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\ d^n x\end{aligned}$$

since

$$\begin{aligned}\overline{\check{\psi}(x)} &= \overline{\int_{\mathbb{R}^n} e^{i\xi\cdot x}\psi(\xi)\ \tfrac{d^n\xi}{(2\pi)^n}} = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\overline{\psi(\xi)}\ \tfrac{d^n\xi}{(2\pi)^n} = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(\xi)\ \tfrac{d^n\xi}{(2\pi)^n} \\ &= \tfrac{1}{(2\pi)^n}\hat{\varphi}(x)\end{aligned}$$

Definition B.24. The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ is the tempered distribution defined by

$$\left\langle \hat{f}, \varphi \right\rangle = \left\langle f, \hat{\varphi} \right\rangle$$

It is well–defined by Theorem B.9.

Example B.25. The Fourier transform of the Dirac delta function is given by

$$\left\langle \hat{\delta}, \varphi \right\rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(x)\ dx = \langle 1, \varphi \rangle$$

That is, $\hat{\delta}$ is the constant function 1.

Example B.26. The Fourier transform of the constant function 1, viewed as a tempered distribution, is

$$\left\langle \hat{1}, \varphi \right\rangle = \langle 1, \hat{\varphi} \rangle = \int_{-\infty}^{\infty} \hat{\varphi}(\xi)\ d\xi = 2\pi\varphi(0)$$

by (B.5). That is, the Fourier transform of the constant function 1 is $2\pi\delta(\xi)$.

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