Siegel series and intersection numbers

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Definition (Gross-Keating invariant)

$T = (b_{ij})$: a non-degenerate (half-integral) symmetric matrix over A of size $n \times n$, where A : a finite extension of $\mathbb{Z}_p$.
Let $S(T)$ be the set of all $(a_1, \cdots, a_n) \in \mathbb{Z}^n_{\geq 0}$ satisfying:

$$\begin{cases} (a_1, \cdots, a_n) \textit{ is non-decreasing}; \\ \mathrm{ord}(b_{ii}) \geq a_i; \\ \mathrm{ord}(2b_{ij}) \geq (a_i + a_j)/2. \end{cases}$$

Put

$$\mathbf{S}(\{T\}) = \bigcup_{U \in \mathrm{GL}_n(A)} S(T[U]).$$

The Gross-Keating invariant $\mathrm{GK}(T)$ of T is the greatest element of $\mathbf{S}(\{T\})$ with respect to the lexicographic order $\succeq$ on $\mathbb{Z}^n_{\geq 0}$.

Notation

A: the ring of integers of a local field F, π: a uniformizer, f: the cardinality of the residue field κ

(L, q_L): a quadratic A-lattice, $H_k = \oplus_k \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

Definition (Siegel series)

The Siegel series is the polynomial $\mathcal{F}_L(X)$ of X such that

$$\mathcal{F}_L(f^{-k}) = \lim_{N \to \infty} f^{-N \dim \mathrm{O}_F(L \otimes F, H_k \otimes F)} \# \mathrm{O}_A(L, H_k)(A/\pi^N A),$$

where $k/2 \geq$ *the rank of* L.

Here, $\mathrm{O}_A(L, H_k)(A/\pi^N A)$ is the set of linear maps from $L \otimes A/\pi^N A$ to $H_k \otimes A/\pi^N A$ preserving the associated quadratic forms.

Main theorem

Theorem (C-, Yamauchi)

$$\mathcal{F}_L(X) = \sum_{m=1}^{d} \left(c_m \cdot f^{(n+1)m} \cdot X^{2m} \cdot \sum_{L' \in \mathcal{G}_{L,d,m}} \mathcal{F}_{L'}(X) \right) +$$

$$(1 - X)(1 - f^d X)^{-1} \cdot \left(\prod_{i=1}^{d} (1 - f^{2i} X^2) \right) \cdot \mathcal{F}_{L_0^{(d,n)}}(f^d X),$$

where $c_m = -\left(\binom{m}{1}_f \cdot c_1 + \binom{m}{2}_f \cdot c_2 + \cdots + \binom{m}{m-1}_f \cdot c_{m-1} \right) + 1.$

Ingredients in the proof of the main theorem

Ingredients of the proof of the theorem

1. Stratification of a p-adic manifold
2. geometric description of each stratum and invariants
3. Grassmannian
4. Lattice counting argument
5. Ikeda-Kataurada's theorem

Modular polynomial

Definition (Modular polynomial)

A modular polynomial ϕ_m for $m \in \mathbb{Z}_{>0}$ is

$$\phi_m : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}, (j, j') \mapsto \prod_{\tilde{E}' \to E'} (j - j(\tilde{E}')).$$

Here, the product is over isogenies $\tilde{E}' \to E'$ of degree m (up to isomorphism).

Remark

1. $\phi_m \in \mathbb{Z}[x, y]$
2. $\phi_m(x, y) = \pm\phi_m(y, x)$ ("$-$" precisely if m is a square).