# 2-Selmer group of even hyperelliptic curves over function fields

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# Outline

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# Odd hyperelliptic curve

Theorem (M. Bhargava and B. Gross (2012))

*When all hyperelliptic curves of fixed genus $n \geq 1$ over $\mathbb{Q}$ having a rational Weierstrass point are ordered by height, the average size of the $2-$Selmer groups of their Jacobians is 3.*

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## Corollary

*The average rank of of the Mordell-Weil groups of the Jacobians of such curves is at most $3/2$.*

# Even hyperelliptic curve

**Theorem (A. Shankar and X. Wang (2014))**

*When all hyperelliptic curves of fixed genus $n \geq 2$ over $\mathbb{Q}$ having a marked rational non-Weierstrass point are ordered by height, the average size of the $2-$Selmer groups of their Jacobians is 6.*

# Even hyperelliptic curve

## Theorem (A. Shankar and X. Wang (2014))

*When all hyperelliptic curves of fixed genus* $n \geq 2$ *over* $\mathbb{Q}$ *having a marked rational non-Weierstrass point are ordered by height, the average size of the* $2-$*Selmer groups of their Jacobians is* $6$.

## Corollary

*The average rank of the Mordell-Weil groups of the Jacobians of the above curves is at most* $5/2$.

# Even hyperelliptic curve

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*When all hyperelliptic curves of fixed genus $n \geq 2$ over $\mathbb{Q}$ having a marked rational non-Weierstrass point are ordered by height, the average size of the 2−Selmer groups of their Jacobians is 6.*

## Corollary

*The average rank of the Mordell-Weil groups of the Jacobians of the above curves is at most $5/2$.*

## Theorem (A. Shankar and X. Wang (2014))

*The proportion of monic even degree hyperelliptic curves having genus $n \geq 4$ that have exactly two rational points is at least* $1 - (48n + 120)2^{-n}$.

# Outline

## Main results

The problem over $\mathbb{Q}$

The problem over $\mathbb{F}_q(C)$

## Proof of the main theorem

Vinberg's representation of type $A_{2n+1}$

Connection to hyperelliptic curves

Canonical reduction theory of G-bundles

Some computations

# Notation

- $k = \mathbb{F}_q$ with $(char(k), 2n+2) = 1$
- $C$ is a smooth, complete, geometrically connected curve over $k$
- $K = k(C)$ the function field of $C$

# Even hyperelliptic curve

An even hyperelliptic curve of genus $n$ is the smooth projective model of the affine curve defined by

$$H : y^2 = x^{2n+2} + c_2 x^{2n} + \cdots + c_{2n+2},$$

where $c_i \in K$, and the tuple $(c_i)_{2 \leq i \leq 2n+2}$ is unique up to the following identification

$$(c_2, c_3, \ldots, c_{2n+2}) \equiv (\lambda^2.c_2, \lambda^3 c_3, \ldots, \lambda^{2n+2}.c_{2n+2}) \qquad \lambda \in K^\times.$$

# Minimal integral model

Fix the data $(c_2, c_3, \ldots, c_{2n+2})$, we define the minimal integral model of $H$ as follows: for each point $v \in |C|$, we can choose an integer $n_v$ which is the smallest integer satisfying that: the tuple

$$(\varpi_v^{2n_v} c_2, \varpi_v^{3n_v} c_3, \cdots, \varpi_v^{(2n+2)n_v} c_{2n+2})$$

has coordinates in $\mathcal{O}_{K_v}$. Given $(n_v)_{v\in|C|}$, we define the invertible sheaf $\mathcal{L}_H \subset K$ whose sections over a Zariski open $U \subset C$ are given by

$$\mathcal{L}_H(U) = K \cap \Big( \prod_{v\in U} \varpi_v^{-n_v} \mathcal{O}_{K_v} \Big).$$

Then $c_i \in H^0(C, \mathcal{L}_H^{\otimes i})$ for all $2 \leq i \leq 2n+2$. Furthermore, the stratum $(\mathcal{L}_H, \underline{c})$ is minimal in the sense that there is no proper subsheaf $\mathcal{M}$ of $\mathcal{L}_H$ such that $c_i \in H^0(C, \mathcal{M}^{\otimes i})$ for all $i$.

# Height of hyperelliptic curves

**Definition**

The height of the hyperelliptic curve $H$ is defined to be the degree of the associated line bundle $\mathcal{L}_H$.

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We are going to consider the following family of hyperelliptic curves:

## Definition

An even hyperelliptic curve $H$ with an associated minimal data $(\mathcal{L}_H, \underline{c})$ is called to be transversal if the discriminant $\Delta(\underline{c}) \in H^0(C, \mathcal{L}_H^{\otimes(2n+1)(2n+2)})$ is square-free.

# Main theorem

Denote $\mathcal{A}^{trans}_{\leq d}$ to be the set of all transversal even hyperelliptic curves of height less than or equal to $d$.

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**Theorem**

*When all transversal even hyperelliptic curves of genus $n \geq 2$ over $K$ are ordered by height, the average size of the $2-$Selmer group of their Jacobians is 6. Equivalently,*

$$\lim_{d\to\infty} \frac{\sum_{H\in\mathcal{A}^{trans}_{\leq d}} \frac{|Sel_2(H)|}{|Aut(H,\infty)|}}{\sum_{H\in\mathcal{A}^{trans}_{\leq d}} \frac{1}{|Aut(H,\infty)|}} = 6.$$

# Notation

- $\mathcal{H}$: the minimal integral model of $H$
- $\mathcal{J}_H$ and $\mathcal{J}_{\mathcal{H}}$ are Jacobian group schemes associated to $H$ and $\mathcal{H}$ respectively.
- Observe that $\mathcal{J}_H$ is the generic fiber of $\mathcal{J}_{\mathcal{H}}$.

# Restatement of the main theorem

## Lemma

*If $H$ is transversal, then $\mathcal{J}_{\mathcal{H}}$ is the Néron model of $\mathcal{J}_H$. Furthermore,*

$$|Sel_2(\mathcal{J}_H)| = |H^1(C, \mathcal{J}_{\mathcal{H}}[2])|.$$

The main theorem is equivalent to:

$$\lim_{d\to\infty} \frac{\sum_{H\in\mathcal{A}^{trans}_{\leq d}} \frac{|H^1(C,\mathcal{J}_{\mathcal{H}}[2])|}{|Aut(H,\infty)|}}{\sum_{H\in\mathcal{A}^{trans}_{\leq d}} \frac{1}{|Aut(H,\infty)|}} = 6.$$

# Outline

Let $(U, Q)$ be the split quadratic space over $k$ of dimension $2n+2$ and discriminant 1. Then for any linear operator $T : U \to U$, we defined its adjoint $T^*$ by the following equation:

$$\langle Tv, w\rangle_Q = \langle v, T^*w\rangle_Q, \qquad \forall v, w \in U.$$

where $\langle v, w\rangle_Q = Q(v+w) - Q(v) - Q(w)$ denotes the bilinear form associated to $Q$. The Vinberg's representation we are going to study is the conjugate action of

$$G := PSO(U) = \{g \in \mathrm{GL}(U) | gg^* = I, \det(g) = 1\}/\mu_2$$

on

$$V = \{T : U \to U | T = T^*, trace(T) = 0\} \cong Sym_0^2(U).$$

# GIT quotient

For each $T \in V$, denote $f_T(x)$ be the characteristic polynomial of $T$:

$$f_T(x) = x^{2n+2} + c_2(T)x^{2n} + \cdots + c_{2n+1}(T)x + c_{2n+2}(T).$$

Then

$$V//G \cong \mathrm{Spec}(k[c_2, c_3, \ldots, c_{2n+2}]) = S.$$

We denote the projection map by $\pi : V \to S$.

# Regular locus

Set

$$V^{reg}(\overline{k}) = \{T \in V(\overline{k}) \mid |\mathrm{Stab}_{G(\overline{k})}(T)| \text{ is finite}\}$$

$$= \{T \in V(\overline{k}) \mid f_T(x) \text{ is its minimal polynomial}\}$$

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$\Longrightarrow$ for any field extension $k \subset F$ and $T \in V^{reg}(F)$,

$$Stab_G(T) \cong (Res_{L/F}\mu_2)_{N=1}/\mu_2,$$

where $L = F[x]/(f_T(x))$.

# Stabilizer group scheme over S

**Theorem**

*There exists a unique group scheme $I_S$ over $S$ equipped with an isomorphism $\pi^* I_S \to Stab_G$ over $V^{reg}$. This isomorphism is $G-$equivariant, thus, as a corollary, there is a $\mathbb{G}_m-$equivariant isomorphism of stacks $[BI_S] \cong [V^{reg}/G]$, where $BI_S$ is the relative classifying stack of $I_S$ over $S$.*

# Outline

# The generalized Jacobian group scheme

For each $\underline{c} = (c_2, c_3, \ldots, c_{2n+2}) \in S$, the associated polynomial

$$f_{\underline{c}}(x) = x^{2n+2} + c_2 x^{2n} + \cdots + c_{2n+1}x + c_{2n+2}$$

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Set

$$\mathcal{J}_{V^{reg}} := \mathcal{J}_S \times_S V^{reg}$$

# Stabilizer group scheme and Jacobian

**Theorem**

*There exists a canonical $G-$equivariant isomorphism over $V^{reg}$ between the stabilizer scheme $Stab_G$ and $\mathcal{J}_{V^{reg}}[2]$.*

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**Corollary**

*The above isomorphism induces an isomorphism over $S$ from $I_S$ to $\mathcal{J}_S[2]$.*

$\Longrightarrow$ an isomorphism between stacks

$$B\mathcal{J}_S[2] \cong [V^{reg}/G]$$

# An interpretation of $H^1(C, \mathcal{J}_{\mathcal{H}}[2])$

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- Set $\mathcal{M} = Hom(C, [B\mathcal{J}_S[2]/\mathbb{G}_m])$

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$$\Longrightarrow \text{a base map } \ b : \mathcal{M} \to \mathcal{A}$$

$$\Longrightarrow H^1(C, \mathcal{J}_{\mathcal{H}}[2]) = b^{-1}(\alpha_H)$$

# Counting points on stacks

We also have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\quad b \quad} & \mathcal{A} \\
& {\scriptstyle \pi_{\mathcal{M}}} \searrow \quad \swarrow {\scriptstyle \pi_{\mathcal{A}}} & \\
& Hom(C, B\mathbb{G}_m) &
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$\Longrightarrow$ for any line bundle $\mathcal{F}$ over $C$,

$$|\mathcal{M}_{\mathcal{F}}(k)| = \sum_{H \in \mathcal{A}_{\mathcal{F}}(k)} |H^1(C, \mathcal{J}_{\mathcal{H}}[2])|.$$

Now it is enough to prove that

$$\lim_{deg(\mathcal{F})\to\infty} \frac{|\mathcal{M}_{\mathcal{F}}^{trans}(k)|}{|\mathcal{A}_{\mathcal{F}}^{trans}(k)|} = 6,$$

where $\mathcal{M}_{\mathcal{F}}^{trans} = b^{-1}(\mathcal{A}_{\mathcal{F}}^{trans})$.

# Another interpretation of $\mathcal{M}_{\mathcal{F}}(k)$

From the isomorphism:

$$[B\mathcal{J}_S[2]/\mathbb{G}_m] \cong [V^{reg}/(G \times \mathbb{G}_m)]$$

$$\Longrightarrow \mathcal{M} \cong Hom(C, [V^{reg}/(G \times \mathbb{G}_m)]).$$

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$\Longrightarrow$ a $k-$point of $\mathcal{M}_{\mathcal{F}}$ is a pair $(\mathcal{E}, s)$, where $\mathcal{E}$ is a principal $G-$bundle, and $s$ is a section of

$$V^{reg}(\mathcal{E}, \mathcal{F}) = (V^{reg} \times^G \mathcal{E}) \otimes \mathcal{F}$$

# Outline

As algebraic groups over $k$:

$$G = \mathrm{PSO}(U) \cong \mathrm{GSO}(U)/\mathbb{G}_m,$$

where $\mathbb{G}_m$ denotes the center of $\mathrm{GSO}(U)$.
$\Longrightarrow$ $G-$bundles $\leftrightarrow$ $\mathrm{GSO}(U)/\mathbb{G}_m-$bundles.
Moreover, any $\mathrm{GSO}(U)/\mathbb{G}_m-$bundle can be lifted to a $\mathrm{GSO}(U)-$bundle uniquely up to tensor twist by a line bundle.

# Canonical reduction of $\mathrm{GSO}(2n+2)-$bundles

Let $\mathcal{E}$ be a $\mathrm{GSO}(2n+2)-$bundle. Then there exists uniquely a parabolic subgroup $P \subset \mathrm{GSO}(U)$ with Levi quotient $L$ and the associated $P-$bundle $\mathcal{E}_P$ such that

1. We have an isomorphism $\mathcal{E} \cong \mathcal{E}_P(\mathrm{GSO}(U))$, where $\mathcal{E}_P(\mathrm{GSO}(U))$ is the quotient $(\mathcal{E}_P \times \mathrm{GSO}(U))/P$ with the following action of $P$ on $\mathcal{E}_P \times \mathrm{GSO}(U)$ : for any $h \in P, e \in \mathcal{E}_P$, and $g \in \mathrm{GSO(U)}$ then $h.(e,g) = (h.e, h^{-1}g)$.

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2. The Levi bundle $\mathcal{E}_L$ associated, by extension of structure group, to $\mathcal{E}_P$ for the projection $P \to L$ is semi-stable.

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2. The Levi bundle $\mathcal{E}_L$ associated, by extension of structure group, to $\mathcal{E}_P$ for the projection $P \to L$ is semi-stable.
3. For every non-trivial character $\chi$ of $P$ which is a non-negative linear combination of simple roots with respect to some Borel subgroup contained in $P$, the line bundle $\chi_* \mathcal{E}_P$ on $C$ has positive degree.

Assume that the Levi subgroup

$$L = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_t} \times \mathrm{GSO}(2h).$$

$\Longrightarrow$ a flag of isotropic subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_t \subset V_t^* \subset \cdots \subset V_1^* \subset U,$$

where $\dim(V_i/V_{i-1}) = n_i$ for $1 \le i \le t$, and $\dim(V_t^*/V_t) = 2h$.
$\Longrightarrow$ a filtration of the vector bundle $\mathcal{E} \times^{\mathrm{GSO}(U)} U$:

$$0 \subset \mathcal{E}_P \times^P V_1 \subset \cdots \subset \mathcal{E}_P \times^P V_t \subset \mathcal{E}_P \times^P V_t^* \subset \cdots \subset \mathcal{E}_P \times^P V_1^*$$

satisfying that the quotient bundles

$$X_i = \mathcal{E}_P \times^P V_i/(\mathcal{E}_P \times^P V_{i-1}),\ \ 1 \le i \le t$$

and

$$X_{t+1} = (\mathcal{E}_P \times^P V_t^*)/(\mathcal{E}_P \times^P V_t)$$

are semistable.

Moreover,

$$(\mathcal{E}_P \times^P V_{i-1}^*)/(\mathcal{E}_P \times^P V_i^*) \cong X_i^\vee \otimes \mathcal{L}$$

and

$$X_{t+1} \cong X_{t+1}^\vee \otimes \mathcal{L}.$$

Denote the slope of $X_i$ by $\mu_i$, then the "canonical conditions" imply that:

$$\mu_1 > \mu_2 > \cdots > \mu_t > \mu_{t+1} = d/2 \text{ if } h > 0,$$
$$\mu_1 > \mu_2 > \cdots > \mu_t \quad \text{and } \mu_{t-1} + \mu_t > d \text{ if } h = 0.$$

# Semistable filtration of $(\mathcal{E} \times^{\mathrm{GSO}(U)} V)$

we obtain the following "matrix filtration" of $\mathrm{Sym}^2_0(\mathcal{E}) \otimes \mathcal{L}^\vee$:

$$\begin{array}{cccccccc}
\mathrm{Sym}^2(X_1)\otimes\mathcal{L}^\vee & X_1\otimes X_2\otimes\mathcal{L}^\vee & \cdots & X_1\otimes X_t\otimes\mathcal{L}^\vee & X_1\otimes X_{t+1}^\vee & X_1\otimes X_t^\vee & \cdots & X_1\otimes X_1^\vee \\
X_2\otimes X_1\otimes\mathcal{L}^\vee & \mathrm{Sym}^2(X_2)\otimes\mathcal{L}^\vee & \cdots & X_2\otimes X_t\otimes\mathcal{L}^\vee & X_2\otimes X_{t+1}^\vee & X_2\otimes X_t^\vee & \cdots & X_2\otimes X_1^\vee \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
X_t\otimes X_1\otimes\mathcal{L}^\vee & X_t\otimes X_2\otimes\mathcal{L}^\vee & \cdots & \mathrm{Sym}^2(X_t)\otimes\mathcal{L}^\vee & X_t\otimes X_{t+1}^\vee & X_t\otimes X_t^\vee & \cdots & X_t\otimes X_1^\vee \\
X_{t+1}^\vee\otimes X_1 & X_{t+1}^\vee\otimes X_2 & \cdots & X_{t+1}^\vee\otimes X_t & \mathrm{Sym}^2_0(X_{t+1})\otimes\mathcal{L}^\vee & X_{t+1}\otimes X_t^\vee & \cdots & X_{t+1}\otimes X_1^\vee \\
X_t^\vee\otimes X_1 & X_t^\vee\otimes X_2 & \cdots & X_t^\vee\otimes X_t & X_t^\vee\otimes X_{t+1} & \mathrm{Sym}^2(X_t^\vee)\otimes\mathcal{L} & \cdots & X_t^\vee\otimes X_1^\vee\otimes\mathcal{L} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
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\end{array}$$

# Outline

# The case P=B the Borel subgroup

The main contributors to the average

$$\lim_{\deg(\mathcal{F})\to\infty} \frac{|\mathcal{M}^{trans}_{\mathcal{F},B}(k)|}{|\mathcal{A}^{trans}_{\mathcal{F}}|}$$

are

$$\mathcal{E} = X_1 \oplus \cdots \oplus X_{n+1} \oplus (X_{n+1}^{\vee} \otimes \mathcal{L}) \oplus \cdots \oplus (X_1^{\vee} \otimes \mathcal{L})$$

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$$\lim_{\deg(\mathcal{F})\to\infty} \frac{|\mathcal{M}^{trans}_{\mathcal{F},B}(k)|}{|\mathcal{A}^{trans}_{\mathcal{F}}|}$$

are

$$\mathcal{E} = X_1 \oplus \cdots \oplus X_{n+1} \oplus (X^\vee_{n+1} \otimes \mathcal{L}) \oplus \cdots \oplus (X^\vee_1 \otimes \mathcal{L})$$

satisfying that

$$\mu_i = \mu_{i+1} + f \ \forall\ 1 \leq i \leq n,$$

where $f = \deg(\mathcal{F})$, $\mu_i = \deg(X_i)$.

# Case 1: $2\mu_{n+1} - d = f$

For any $(\mathcal{E}, s) \in \mathcal{M}_{\mathcal{F}}^{trans}$, where $s$ is a section of

$$(V \times^{\mathrm{GSO}(U)} \mathcal{E}) \otimes \mathcal{F} = \mathrm{Sym}^2_0(\mathcal{E}) \otimes \mathcal{L}^\vee \otimes \mathcal{F},$$

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then $s$ is of the following form:

$$\begin{pmatrix} * & * & \cdots & * & * & * \\ x_1 & * & \cdots & * & * & * \\ 0 & x_2 & \cdots & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_2 & * & * \\ 0 & 0 & \cdots & 0 & x_1 & * \end{pmatrix}$$

where $x_i \in k^*$.

$$\xrightarrow{g.s.g^*} \begin{pmatrix} 0 & 0 & \cdots & 0 & * & * \\ 1 & 0 & \cdots & * & * & * \\ 0 & 1 & \cdots & * & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \text{ for some } g \in \mathrm{GSO}(U)(K)$$

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The Kostant section $\kappa_1$

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The Kostant section $\kappa_1$

$\implies$ This case contributes 1 to the average.

# Case 2: $-2\mu_{n+1} + d = f$

Any section $s$ is of the form:

$$\begin{pmatrix} * & \cdots & * & * & * & \cdots & * & * \\ x_1 & \cdots & * & * & * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & x_n & * & x_{n+1} & \cdots & * & * \\ 0 & \cdots & 0 & * & * & \cdots & * & * \\ 0 & \cdots & 0 & 0 & x_n & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & x_1 & * \end{pmatrix}$$

where $x_i \in k^\times$.

$$\xrightarrow{g.s.g^*} \begin{pmatrix} 0 & & & & & & a & b \\ 1 & \ddots & & & & \iddots & \iddots & a \\ & \ddots & \ddots & & \iddots & \iddots & \iddots & \\ & & & d & c & \iddots & & \\ & 1 & 0 & 1 & d & & & \\ & & e & 0 & & & & \\ & & & 1 & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{pmatrix}$$

$$
\xrightarrow{g.s.g^*}
\begin{pmatrix}
0 & \ddots & & & & & a & b \\
1 & \ddots & \ddots & & & & & a \\
 & & & d & c & & & \\
 & 1 & 0 & 1 & d & & & \\
 & & e & 0 & & & & \\
 & & & 1 & \ddots & & & \\
 & & & & \ddots & \ddots & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & 1 & 0
\end{pmatrix}
$$

The Kostant section $\kappa_2$

$$\xrightarrow{g.s.g^*} \begin{pmatrix} 0 & \ddots & & & & & & a & b \\ 1 & \ddots & & & & & \iddots & \iddots & a \\ & \ddots & \ddots & & & \iddots & \iddots & \iddots & \\ & & & & d & c & \iddots & & \\ & & 1 & 0 & 1 & d & & & \\ & & & e & 0 & & & & \\ & & & & 1 & \ddots & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 1 & 0 \end{pmatrix}$$

**The Kostant section $\kappa_2$**

$\Longrightarrow$ This case contributes 1 to the average.

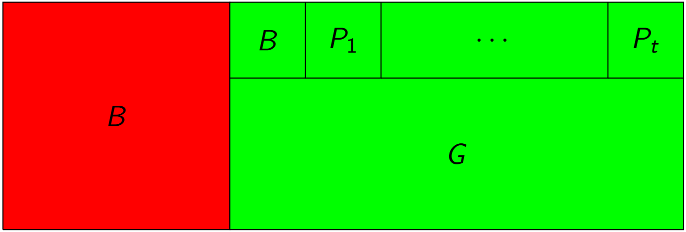
# The case P=B the Borel subgroup

$$\begin{pmatrix}
0 & & & & & & & a & b\\
1 & \ddots & & & & & \iddots & \iddots & a\\
 & \ddots & \ddots & & & \iddots & \iddots & & \\
 & & & & d & c & & & \\
 & & 1 & 0 & e & d & & & \\
 & & & 1 & 0 & & & & \\
 & & & & 1 & \ddots & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & & & & \ddots & \ddots & \\
 & & & & & & & 1 & 0
\end{pmatrix}
\qquad
\begin{pmatrix}
0 & & & & & & & a & b\\
1 & \ddots & & & & & \iddots & \iddots & a\\
 & \ddots & \ddots & & & \iddots & \iddots & & \\
 & & & & d & c & & & \\
 & & 1 & 0 & 1 & d & & & \\
 & & & e & 0 & & & & \\
 & & & & 1 & \ddots & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & & & & \ddots & \ddots & \\
 & & & & & & & 1 & 0
\end{pmatrix}$$

The Kostant section $\kappa_1$

The Kostant section $\kappa_2$

# The whole picture




$$Vol(red) = 2$$

$$Vol(green) = 4 = \tau(G)$$

# Summary

- $6 =$ the number of Kostant sections $+ \tau(G)$.

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# Summary

- $6 =$ the number of Kostant sections $+ \tau(G)$.
- By a similar method, we also can give an upperbound for the average in general case (remove the transversal condition) if we assume $\mathrm{char}(k)$ is big enough.
- The method that was used here, is partially similar to the method in the paper "Average size of 2-Selmer groups of elliptic curves over function fields" of Q.P. Ho, V.B. Le Hung, and B.C. Ngo.

Thank you!