Extended monstrous moonshine

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What is moonshine?

Strange connections between finite groups and modular forms

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Strange connections between finite groups and modular forms

The connections should be "very special"

Infinitely many cases ⇒ not moonshine!

Monstrous Moonshine

Classification of finite simple groups

Any finite simple group is one of the following

- A cyclic group of prime order
- An alternating group A_n $(n \geq 5)$
- A group of Lie type (16 infinite families)
- One of 26 sporadic simple groups

Largest sporadic: Monster $\mathbb{M}$, about $8 \cdot 10^{53}$ elements (Griess 1982).
194 irred. repres. of dim $1, 196883, 21296876, \ldots$

Hauptmodul (or principal modulus)

A Hauptmodul for a discrete subgroup $\Gamma < SL_2(\mathbb{R})$ is a holomorphic function $\mathfrak{H} \to \mathbb{C}$ invariant under Γ, that generates the function field of $\Gamma\backslash\mathfrak{H}$.

J-function as Hauptmodul

The quotient space $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ has genus zero.
Function field generated by J. Fourier expansion:
$q^{-1} + 196884q + 21493760q^2 + \cdots (q = e^{2\pi iz})$

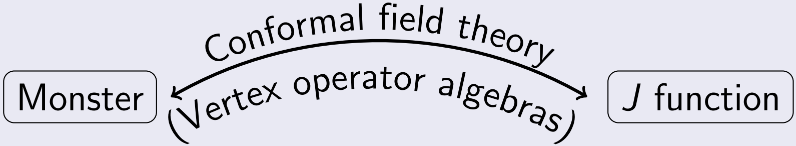
Coefficients of J and Irreducible Monster reps

$$\begin{aligned} 196884 &= 1 + 196883 \text{ (McKay, 1978)} \\ 21493760 &= 1+196883+21296876 \text{ (Thompson, 1979)} \\ 864299970 &= 2\times1+2\times196883+21296876+842609326 \\ \vdots \quad & \quad \vdots \end{aligned}$$

How to continue this sequence?

McKay-Thompson conjecture: Natural graded rep $\bigoplus_{n=0}^{\infty} V_n$ of $\mathbb{M}$ such that $\sum \dim V_n q^{n-1} = J$.

Idea: Physics forms a bridge



Solution: Frenkel, Lepowsky, Meurman 1988

Constructed a vertex operator algebra $V^\natural = \bigoplus_{n\geq 0} V_n^\natural$ (the Moonshine Module), such that $\sum_{n\geq 0}(\dim V_n^\natural)q^{n-1} = J$ and $\mathrm{Aut}\, V^\natural = \mathbb{M}$.

Refined correspondence

Thompson's suggestion: replace graded dimension with graded trace of non-identity elements.

Monstrous Moonshine Conjecture (Conway, Norton 1979)

There is a faithful graded representation $V = \bigoplus_{n \geq 0} V_n$ of the monster $\mathbb{M}$ such that for all $g \in \mathbb{M}$, the series $T_g(\tau) = \sum_{n \geq 0} \mathrm{Tr}(g|V_n) q^{n-1}$ is the q-expansion of a congruence Hauptmodul (= "generates function field of genus 0 $\mathfrak{H}$-quotient").

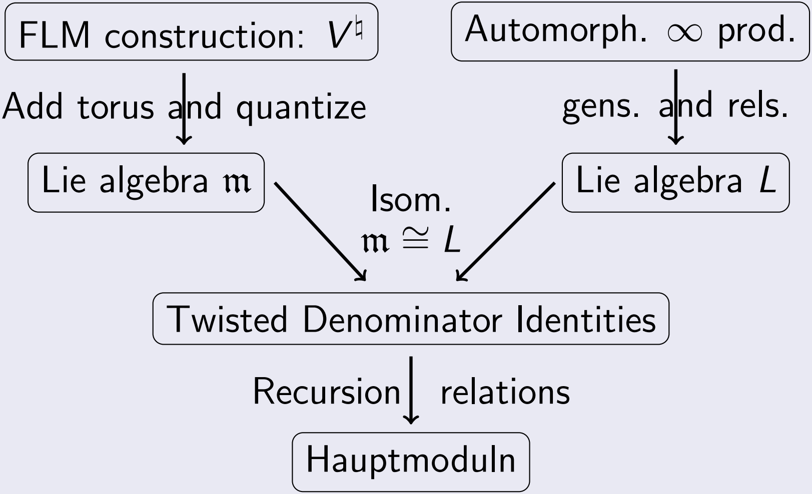
First proof (Atkin, Fong, Smith 1980)

Theorem: A virtual representation of $\mathbb{M}$ exists yielding the desired functions.
No construction.

Second proof (Borcherds 1992)

Theorem: The Conway-Norton conjecture holds for $V^\natural$.

Outline of Borcherds's proof



Additional Moonshine Phenomena

Theorem (Ogg 1974)

The primes p such that $X_0(p)^+ = X_0(p)/\langle w_p \rangle$ has genus zero are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$$

These are the primes p such that all supersingular elliptic curves over $\overline{\mathbb{F}_p}$ have j invariant in $\mathbb{F}_p$.

Ogg's Jack Daniels problem

Explain why these are precisely the primes that divide the order of the Monster.

Borcherds's half-solution

For each $p | \#\mathbb{M}$, there is a conjugacy class pA, such that $T_g(\tau)$ is a Hauptmodul of $X_0(p)^+$ for g in pA.

The other half (still open)

Explain why $V^\natural$ has so many automorphisms.

Positivity phenomena

For g in pA, the coefficients of $T_g(\tau)$ are non-negative integers.
E.g. for 2A, $T_g(\tau) = q^{-1} + 4372q + 96256q^2 + \cdots$ is a Hauptmodul for $\Gamma_0(2)^+$.

Extra phenomena (Conway-Norton, Queen)

The coefficients of $T_g(\tau)$ appear to be dimensions of representations of centralizers.
E.g., For g in 2A, $C_{\mathbb{M}}(g) \cong 2.\mathbb{B}$, with irreducible representations: $1, 4371, 96255, 96256, \ldots$

Two explanations conjectured!

1. Generalized Moonshine (Norton 1987): Graded representations $V(g)$ of $C_{\mathbb{M}}(g)$ in characteristic zero, traces of centralizing elements are Hauptmoduln.
2. Modular moonshine (Ryba 1994): Graded representations V^p of $C_{\mathbb{M}}(g)$ in characteristic p, Brauer characters of p-regular centralizing elements are Hauptmoduln.

Complementary history

	Generalized	Modular
1990s	Objects $V(g)$ exist (Dong-Li-Mason 1997)	Hauptmoduln assuming existence (Borcherds-Ryba 1996)
2010s	Hauptmoduln	Objects V^p exist

Key advance:

Theory of cyclic orbifolds of vertex operator algebras.

What is a vertex operator algebra?

1. $V = \bigoplus_{n\in\mathbb{Z}} V_n$, a graded vector space
2. $\mathbf{1} \in V_0$, an "identity element"
3. $\omega \in V_2$ a "Virasoro element"
4. $Y : V \otimes V \to V((z))$, "multiplication"

satisfying

1. $Y(\mathbf{1}, z)x = x,\ Y(x, z)\mathbf{1} \in x + zV[[z]]$
2. coefficients of $Y(\omega, z)$ give Virasoro action
3. "commutativity and associativity".
4. $\forall n, \dim V_n < \infty$. For $n \ll 0$, $V_n = 0$.

Lattice vertex operator algebras

L an even positive definite lattice.

$V_L = \mathbb{C}[L] \otimes \mathrm{Sym}_{\mathbb{C}}\, x(L \otimes \mathbb{C})[x]$.

Graded dimension is $\frac{\theta_L(\tau)}{\eta^{\mathrm{rank}\, L}}$.

E.g., For Leech lattice, get $J + 24$.

V-Modules

For V - vertex operator algebra, a V-module is a vector space M with an action map $Y^M : V \otimes M \to M((z))$ satisfying some compatibility.
V is **holomorphic** if all V-modules are direct sums of V.

Theorem (Dong 1994)

All V_L-modules are direct sums of $V_{L+\alpha}$ for $\alpha \in L^\vee/L$.
In particular, V_L holomorphic $\Leftrightarrow$ L unimodular.

Theorem (van Ekeren, Möller, Scheithauer 2015)

V holomorphic VOA, g "anomaly-free" automorphism of order n. Then there exist:

- a "generalized VOA" ${}^gV = \bigoplus_{i,j \in \mathbb{Z}/n\mathbb{Z}} {}^gV^{i,j}$.
- a pair of commuting automorphisms (g, g^*) giving decomposition into $V^{i,j}$.
- An isomorphism $V = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} {}^gV^{i,0}$
- a holomorphic VOA $V/g = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} {}^gV^{0,j}$

Cyclic orbifold duality: $(V, g) \leftrightarrow (V/g, g^*)$

Special case: First construction of $V^\natural$ (1988)

V_Λ - Leech lattice VOA
σ lifted from the -1-automorphism of Λ.
Then $V^\natural = (V_\Lambda)/\sigma$.

Now we have 51 constructions

We can take any σ that is fixed-point free, with "no massless states". (51 algebraic conjugacy classes).

Generalized Monstrous Moonshine

Generalized Moonshine Conjecture (Norton 1987):

- $g \in \mathbb{M} \Rightarrow V(g)$ graded proj. rep. of $C_{\mathbb{M}}(g)$
- $(g, h), gh = hg \Rightarrow Z(g, h; \tau)$ holomorphic on $\mathfrak{H}$

1. q-expansion of $Z(g, h; \tau)$ is graded trace of (a lift of) h on $V(g)$.
2. Z is invariant under simultaneous conjugation of the pair (g, h) up to scalars.
3. $Z(g, h; \tau)$ constant or a Hauptmodul.
4. $Z(g, h; \frac{a\tau+b}{c\tau+d})$ proportional to $Z(g^a h^c, g^b h^d; \tau)$.
5. $Z(g, h; \tau) = J(\tau)$ if and only if $g = h = 1$.

Brute force solution (like Atkin-Fong-Smith)?

This is a finite problem:

- Finitely many conjugacy classes of commuting pairs, and possible levels are bounded.
- Central extensions of centralizers “can be computed”.

Not finite enough for 2018

- We still haven't classified the commuting pairs.
- We still don't know character tables of all centralizers, let alone central extensions.

Physics Language (Dixon, Ginsparg, Harvey 1988)

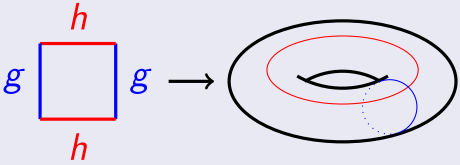
$V(g)$ - twisted sectors of a conformal field theory with $\mathbb{M}$-symmetry.
$Z(g, h; \tau)$ - genus 1 partition functions (with (g, h)-twisted boundary conditions).
All except Hauptmodul claim (3) “follow” from conformal field theory considerations.

Algebraic Interpretation

$V(g) =$ irreducible g-twisted $V^\natural$-module $V^\natural(g)$
$Z(g, h; \tau) = \mathrm{Tr}(\tilde{h} q^{L(0)-1} | V(g))$ for a lift $\tilde{h}$.

Geometric interpretation of Z

Physicists draw boundary conditions as colorings.



Commuting pair (g, h) gives hom $\pi_1(E, e) \to \mathbb{M}$.
$SL_2(\mathbb{Z})$ action changes generating pair.
Ignoring scalar ambiguities, claims (2) and (4) say that Z is a function on the moduli space of elliptic curves with principal $\mathbb{M}$-bundles.

First Breakthrough (Dong, Li, Mason 1997)

- Existence and uniqueness (up to isom.) of $V^\natural(g)$.
- Convergence of power series defining Z.
- Settles claims (1), (2), (5).
- Reduces $SL_2(\mathbb{Z})$ claim (4) to "g-rationality".

Theorem (C, Miyamoto 2016)

Category of g-twisted $V^\natural$-modules is semisimple.
This resolves the $SL_2(\mathbb{Z})$-compatibility claim (4).

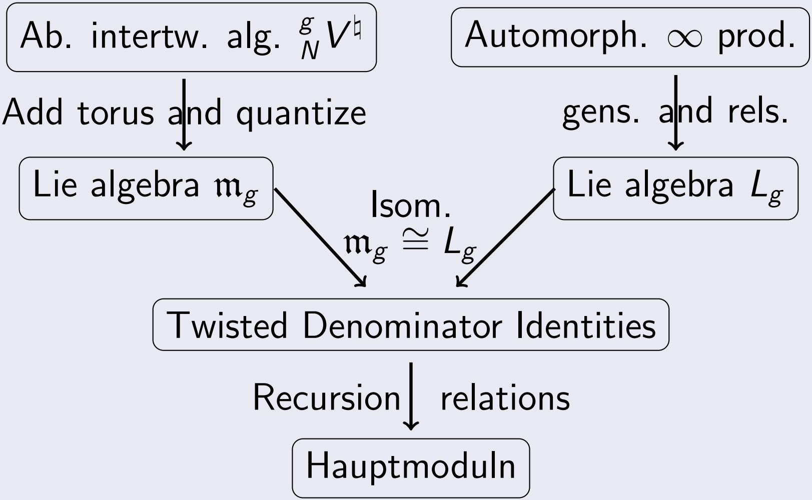
On to claim (3)

We now need to show that all $Z(g, h; \tau)$ are Hauptmoduln or constant.

Second Breakthrough (Höhn 2003)

Generalized Moonshine for 2A (Baby monster case).
- Gives outline for proving Hauptmodul claim (3).

Borcherds-Höhn program for Hauptmoduln



Right side (C 2009)

Borcherds products of the form:

$$T_g(\sigma) - T_g(-1/\tau) = p^{-1} \prod_{m>0, n \in \frac{1}{N}\mathbb{Z}} (1 - p^m q^n)^{c^g_{m,n}(mn)}$$

- Exponent $c^g_{m,n}(mn)$ is q^{mn}-coefficient of a v.v. modular function formed from $\{T_{g^i}(\tau)\}_{i=0}^{N-1}$.
- L_g is a $\mathbb{Z} \oplus \frac{1}{N}\mathbb{Z}$-graded BKM Lie algebra.
- Simple roots of multiplicity $c^g_{1,n}(n)$ in degree $(1, n)$.

Add a torus and quantize

- Take a graded tensor product with a lattice abelian intertwining algebra $V_{II_{1,1}(-1/N)}$
- Get conformal VA, $c = 26$, graded by 2d lattice, has invariant form.
- Apply a bosonic string quantization functor.
- For Fricke g (i.e., $T_g(\tau) = T_g(-1/N\tau)$), get a BKM Lie algebra $\mathfrak{m}_g$ with real simple root.
- graded by $II_{1,1}(-1/N) \cong \mathbb{Z} \oplus \frac{1}{N}\mathbb{Z}$.

Comparison

Borcherds-Kac-Moody Lie algebras:
- $\mathfrak{m}_g$ has canonical projective action of $C_{\mathbb{M}}(g)$.
- L_g has "nice shape": known simple roots, good homology.

Isomorphism from matching root multiplicities: $\dim(L_g)_{m,n} = (\mathfrak{m}_g)_{m,n} = c^g_{m,n}(mn)$.

Transport de structure $\Rightarrow L_g$ gets $\widetilde{C_{\mathbb{M}}(g)}$ action.

End of proof (C 2016)

Virtual $\widetilde{C_{\mathbb{M}}(g)}$-module isom $H_*(E_g, \mathbb{C}) \cong \bigwedge^* E_g$ implies equivariant Hecke operators $n\hat{T}_n$ given by

$$n\hat{T}_n Z(g,h,\tau) = \sum_{ad=n, 0\leq b<d} Z(g^d, g^{-b}h^a, \tfrac{a\tau+b}{d})$$

act by monic polynomials on $Z(g,h,\tau)$.

- Hauptmodul condition follows (C 2008).
- Constants come from (g,h) such that all $g^a h^c$ are non-Fricke when $(a,c)=1$, using claim (4).

This resolves the final claim (3).

Modular Moonshine

Ryba's conjecture

For each g in conjugacy class pA ($p | \#\mathbb{M}$), there is a vertex algebra $V^p = \bigoplus_{n \geq 0} V^p_n$ over $\mathbb{F}_p$ with an action of $C_{\mathbb{M}}(g)$, such that for all p-regular elements h, the Brauer character

$$\sum_{n \geq 0} \hat{\mathrm{Tr}}(h | V^p_n) q^{n-1}$$

is the q-expansion of the Hauptmodul $T_{gh}(\tau)$.

Borcherds-Ryba interpretation 1996

$V^p = H^0(g, V^\natural_{\mathbb{Z}})$, where $V^\natural_{\mathbb{Z}}$ is a self-dual integral form of $V^\natural$ (i.e., a VOA over $\mathbb{Z}$ with $\mathbb{M}$-symmetry).

Theorem (Borcherds-Ryba 1996, Borcherds 1998)

If $V^\natural_{\mathbb{Z}}$ exists, then $V^p := H^0(g, V^\natural_{\mathbb{Z}})$ works.

Theorem (C 2017)

$V^\natural_{\mathbb{Z}}$ exists.

How to construct a self-dual integral form?

Existing constructions (e.g., by cyclic orbifold) have denominators.

For example, order n orbifold construction requires $\frac{1}{n}$ and $e^{\pi i/n}$.

Solution

Do orbifold constructions of many different orders, and glue using faithfully flat descent.

Unification?

Recall

Conjugacy classes pA yield reps. of $C_{\mathbb{M}}(g)$:
$V(g)$ (char. 0) and V^g (char. p)
Same p-regular characters!
In fact, for any $g \in \mathbb{M}$, we get reps of $C_{\mathbb{M}}(g)$:
$V(g)$ (char. 0) and $\hat{H}^*(g, V^\natural_{\mathbb{Z}})$ (char. $|g|$).

Question

Is there an integral structure that produces both?

Main obstruction

Sometimes $\hat{H}^1(g, V_{\mathbb{Z}}^\natural) \neq 0$.

Conjecture (Borcherds-Ryba 1996)

$\hat{H}^1(g, V_{\mathbb{Z}}^\natural) = 0$ if and only if
$T_g(\tau) = \sum \mathrm{Tr}(g|V_n^\natural)q^{n-1}$ has a pole at 0.

Definition

We say g is Fricke if $T_g(\tau)$ has a pole at 0.
Equivalently, $T_g(\tau)$ is invariant under the Fricke involution $w_N : \tau \mapsto -\frac{1}{N\tau}$.
g is non-Fricke if $T_g(\tau)$ is regular at 0.

Fricke versus non-Fricke

- $\mathbb{M}$ has 141 Fricke classes, and 53 non-Fricke classes
- $T_g(\tau)$ non-negative coeffs. $\Leftrightarrow$ g Fricke.
- $Z(g, h, \tau)$ has a pole at ∞ if and only if g is Fricke.
- $V^\natural/g \cong \begin{cases} V^\natural & g \text{ is Fricke} \\ V_\Lambda & g \text{ non-Fricke} \end{cases}$

Conjecture (Borcherds 1998)

There is a rule that assigns to any $g \in \mathbb{M}$ of order n, a $\frac{1}{n}\mathbb{Z}$-graded $\mathbb{Z}[e^{2\pi i/n}]$-module V_g with an action of a central extension $\mathbb{Z}/n\mathbb{Z}.C_{\mathbb{M}}(g)$, such that

- $V_g \otimes_{\mathbb{Z}[e^{2\pi i/n}]} \mathbb{C} \cong V(g)$ as $\mathbb{Z}/n\mathbb{Z}.C_{\mathbb{M}}(g)$-reps.
- g Fricke $\Rightarrow$ $V_g \otimes_{\mathbb{Z}[e^{2\pi i/n}]} \mathbb{Z}/n\mathbb{Z} \cong \hat{H}^0(g, V^\natural_{\mathbb{Z}})$ as $\mathbb{Z}/n\mathbb{Z}.C_{\mathbb{M}}(g)$-reps.
- $\hat{H}^*(h, V_g) = V_{gh} \otimes \mathbb{Z}/|h|\mathbb{Z}$ when g, h commute and have coprime order.
- (additional compatibilities)

Current progress

Twisted modules $V(g)$ can be defined over $\mathbb{Z}[\frac{1}{n}, e^{2\pi i/n}]$, but removing $\frac{1}{n}$ is tricky.
Looks like $\hat{H}^1(g, V^\natural_\mathbb{Z}) = 0$ for Fricke g (in progress)

What we really need

- Canonical lifts of $\hat{H}^*(g, V^\natural_\mathbb{Z})$ to characteristic zero.
- Meaningful interpretation of these objects.