Mixed Hodge structures with modulus

Joint work with Florian Ivorra

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Map of this talk

Our goal. To generalize Deligne's category **MHS** of mixed Hodge structures to that of **MHS** with modulus **MHSM**;

"level $\leq$ **1** parts" (indicated by $_1$) admit geometric description:

$$
\begin{array}{ccccc}
\text{(semi-abel.)} & \subset & \text{(Deligne } \mathbf{1}\text{-motives)} & \cong & \mathbf{MHS}_1 \\
\cap & & \cap & & \cap \\
\text{(comm. alg. gp.)} & \subset & \text{(Laumon } \mathbf{1}\text{-motives)} & \cong & \mathbf{MHSM}_1
\end{array}
$$

2/**3** of this talk is devoted to "level $\leq$ **1** parts", previously known by Barbieri-Viale, Kato, Russell in different languages.

We then explain the whole category **MHSM**, and its application to generalize Kato-Russell's construction of Albanese varieties with modulus to **1**-motives.

Commutative algebraic groups

A commutative algebraic group $\boldsymbol{G}$ over $\mathbb{C}$ is an extension

$$0 \to G_{\mathrm{mul}} \times G_{\mathrm{add}} \to G \to G_{\mathrm{ab}} \to 0,$$

G_{ab} : abel. var., $G_{\mathrm{mul}} \cong \mathbb{G}_m^s$, $G_{\mathrm{add}} \cong \mathbb{G}_a^t$ $(s, t \in \mathbb{Z}_{\geq 0})$.

G is called semi-abelian if $G_{\mathrm{add}} = 0$.

$$\mathbf{AV} = \{\text{ab. var.}\} \subset \mathbf{SA} = \{\text{semi-ab.}\} \subset \mathbf{AG} = \{\text{alg. gp.}\}.$$

X : cpt. Riemann surface, Y : effective divisor on X
$(Y = a_1P_1 + \cdots + a_rP_r;\ P_i \in X$ distinct, $a_i \in \mathbb{Z}_{>0})$.
Generalized Jacobian $\mathrm{Jac}(X, Y) \in \mathbf{AG}$ classifies pairs (L, σ) of a degee zero line b'dl. L on X and $\sigma : L|_Y \cong \mathscr{O}_Y$.

- $Y = \emptyset \Rightarrow \mathrm{Jac}(X, \emptyset) = \mathrm{Jac}(X) \in \mathbf{AV}$;
- Y : reduced (i.e. $a_1 = \cdots = a_r = 1$) $\Rightarrow \mathrm{Jac}(X, Y) \in \mathbf{SA}$.

Duality

$A \in \mathbf{AV} \Rightarrow \exists A^\vee \in \mathbf{AV}$: dual of A; $\quad \mathbf{Jac}(X)^\vee \cong \mathbf{Jac}(X)$.

To extend $^\vee$ to **AG**, we need Laumon **1**-motives:

Def. A Laumon **1**-motive M is a two-term cpx. of the form

$$(*) \quad M = [\mathbb{Z}^s \times \hat{\mathbb{G}}_a^t \to G] \quad (s, t \in \mathbb{Z}_{\geq 0},\ G \in \mathbf{AG}).$$

Regard $G \in \mathbf{AG}$ as a Laumon **1**-motive by $[0 \to G]$.

M in $(*)$ is called a Deligne **1**-motive if $t = 0$ and $G \in \mathbf{SA}$.

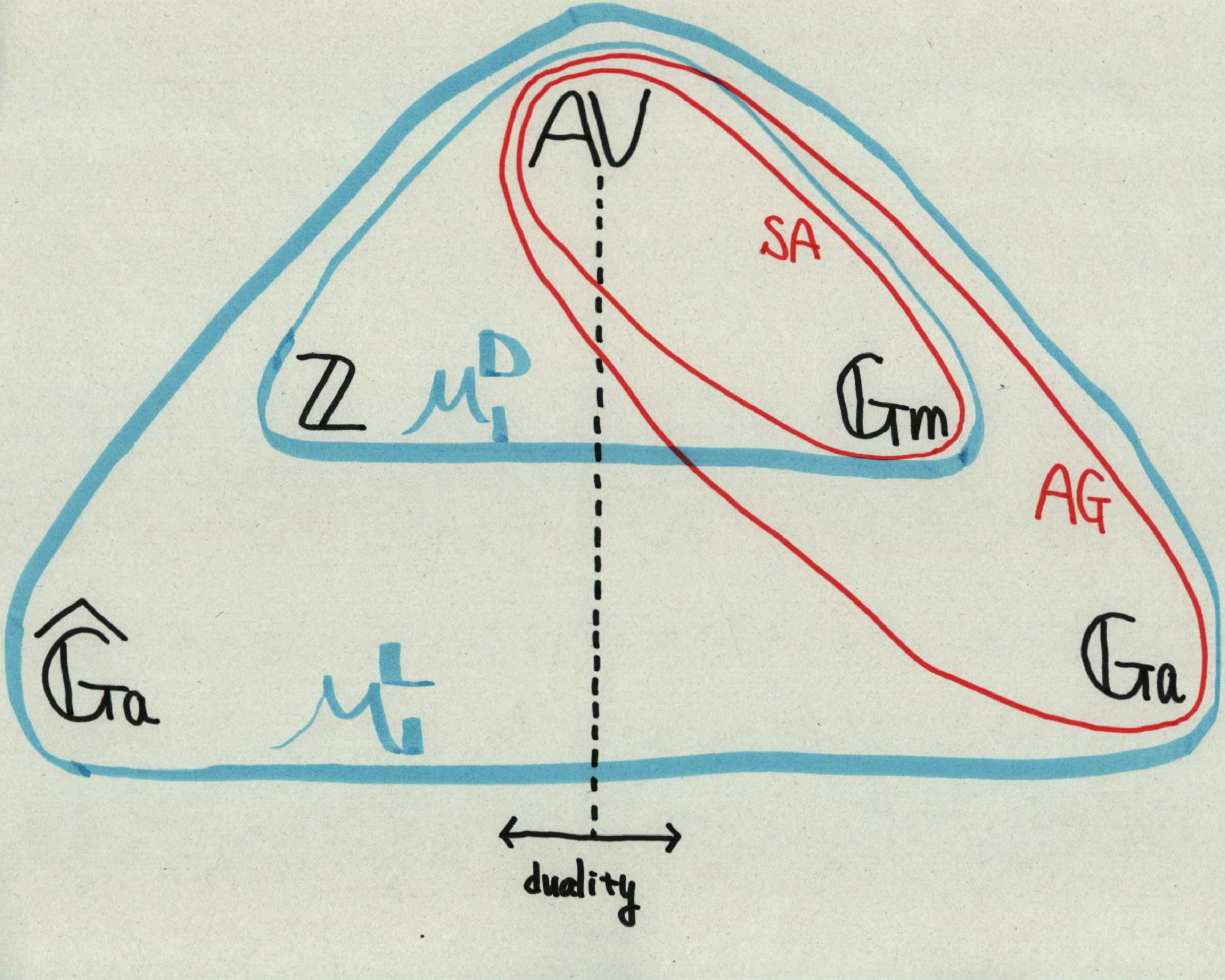
$\mathscr{M}_1^{\mathrm{D}}$ = {Deligne **1**-motives} $\subset \mathscr{M}_1^{\mathrm{L}}$ = {Laumon **1**-motives}

Generalized Jacobian can be generalized to **1**-motives:

Attach $\mathbf{Jac}(X, Y, Z) \in \mathscr{M}_1^{\mathrm{L}}$ to a triple (X, Y, Z) of X : cpt. Riemann surface and Y, Z : eff. divisors s.t. $|Y| \cap |Z| = \emptyset$;

$$\mathbf{Jac}(X, Y, \emptyset) = \mathbf{Jac}(X, Y), \quad \mathbf{Jac}(X, Y, Z)^\vee \cong \mathbf{Jac}(X, Z, Y).$$

This is best explained from the viewpoint of Hodge theory.


AV
SA
AG
$\mathbb{Z}$
μ_1^D
$\mathbb{G}_m$
$\hat{\mathbb{G}}_a$
μ_1^L
$\mathbb{G}_a$
duality

AV and HS_1

Recall. $\exists$ equiv. of cat. $\mathbf{AV} \cong \mathbf{HS_1}$: Hodge str. of level ≤ 1.

Def. A Hodge structure of level ≤ 1 is a pair $H = (H_{\mathbb{Z}}, F^0)$ of

a) $H_{\mathbb{Z}}$: free $\mathbb{Z}$-module of finite rank,

b) $F^0 \subset H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$: $\mathbb{C}$-subspace;

subject to conditions ($H_{\mathbb{C}} = F^0 \oplus \overline{F^0}$ and "polarizable").

- $\mathbf{Jac}(X) \leftrightarrow H_1(X, \mathbb{Z})$ for X : cpt. Riemann surf.
- $^\vee : \mathbf{AV} \to \mathbf{AV}$ corresponds to an easy linear algebra operation $\mathbf{Hom}(-, \mathbb{Z}(1))$ on $\mathbf{HS_1}$.
- $J(X)^\vee \cong J(X)$ explained by Poincaré duality.

The equivalence $\mathbf{AV} \cong \mathbf{HS_1}$ is extended to $\mathscr{M}_1^{\mathrm{D}}$ by Deligne.

$\mathcal{M}_1^{\mathrm{D}}$ and $\mathbf{MHS_1}$

Thm. (Deligne). $\exists$ equiv. of cat. $\mathcal{M}_1^{\mathrm{D}} \cong \mathbf{MHS_1}$.

Def. A mixed HS of level ≤ 1 $H = (H_{\mathbb{Z}}, F^0, W_{-1}, W_{-2})$ is:

a) $H_{\mathbb{Z}}$: free $\mathbb{Z}$-module of finite rank,

b) $F^0 \subset H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$: $\mathbb{C}$-subspace,

c) $W_{-2} \subset W_{-1} \subset H_{\mathbb{Z}}$: $\mathbb{Z}$-submodules,

s.t. $H_{\mathbb{Z}}/W_{-1} \cong \mathbb{Z}^s, W_{-1}/W_{-2} \in \mathbf{HS_1}, W_{-2} \cong \mathbb{Z}(1)^t$ $(s, t \in \mathbb{Z}_{\ge 0})$

- (X, Y, Z) as above, with Y, Z reduced $\Rightarrow$
  $\mathrm{Jac}(X, Y, Z) \leftrightarrow H_1(X \setminus Y, Z, \mathbb{Z})$: relative homology.
- $^\vee : \mathcal{M}_1^D \to \mathcal{M}_1^D \leftrightarrow$ an easy lin. alg. operation on $\mathbf{MHS_1}$.
- Poincaré duality explains $\mathrm{Jac}(X, Y, Z)^\vee \cong \mathrm{Jac}(X, Z, Y)$.

The equivalence $\mathcal{M}_1^{\mathrm{D}} \cong \mathbf{MHS_1}$ is extended to $\mathcal{M}_1^{\mathrm{L}}$.

$\mathscr{M}_1^{\mathrm{L}}$ and $\mathbf{MHSM_1}$

At least **3** known categories: $\mathscr{M}_1^{\mathrm{L}} \cong \mathbf{FHS_1} \cong \mathcal{H}_1 \cong \mathbf{MHSM_1}$.

- $\mathbf{FHS_1}$: formal HS (Barbieri-Viale, 2007)
- $\mathcal{H}_1$: **MHS** with additive part (Kato and Russell, 2012)
- $\mathbf{MHSM_1}$: **MHS** with modulus (Ivorra and Y.).

Def. A **MHSM of level $\leq$ 1** $\mathscr{H} = (H, U, V, \mathscr{F})$ consists of:

a) $H = (H_{\mathbb{Z}}, F^0, W_{-1}, W_{-2}) \in \mathbf{MHS_1}$.

b) U, V : finite dim. $\mathbb{C}$-vector sp.

c) $\mathscr{F} \subset H_{\mathbb{C}} \oplus U \oplus V$: $\mathbb{C}$-subspace.

satisfying certain conditions (details come later).

If $[\mathbb{Z}^s \times \hat{\mathbb{G}}_a^t \to G] \leftrightarrow (H, U, V, \mathscr{F})$ by $\mathscr{M}_1^{\mathrm{L}} \cong \mathbf{MHSM_1}$,

$$[\mathbb{Z}^s \to G/G_{\mathrm{add}}] \leftrightarrow H \text{ by } \mathscr{M}_1^{\mathrm{D}} \cong \mathbf{MHS_1},$$

$$U \cong \mathrm{Lie}(G_{\mathrm{add}}), \quad V \cong \mathrm{Lie}(\hat{\mathbb{G}}_a^t).$$

Jacobian 1-motives

For (X, Y, Z) as above, Y, Z not. nec. reduced,
$\mathbf{Jac}(X, Y, Z) \leftrightarrow (H, U, V, \mathscr{F})$ under $\mathscr{M}_1^{\mathrm{L}} \cong \mathbf{MHSM_1}$ with

- $H = H_1(X \setminus Y_{\mathrm{red}}, Z_{\mathrm{red}}, \mathbb{Z}) \in \mathbf{MHS_1}$ (Deligne),
- $U = H^0(X, \mathscr{O}_X(-Y_{\mathrm{red}})/\mathscr{O}_X(-Y))$.
- $V = H^0(X, \mathscr{O}_X(Z - Z_{\mathrm{red}})/\mathscr{O}_X)$.
- $\mathscr{F} := \mathrm{Im}(H^0(X, \Omega_X^1(Z)) \to H^1(X, [\mathscr{O}_X(-Y) \to \Omega_X^1(Z)]))$ $\cong H_{\mathbb{C}} \oplus U \oplus V$. (This map tuns out to be injective.)

N.B. Y, Z: reduced $\Rightarrow U = V = 0$, i.e. $\mathbf{Jac}(X, Y, Z) \in \mathscr{M}_1^{\mathrm{D}}$.

$^\vee : \mathscr{M}_1^L \to \mathscr{M}_1^L \leftrightarrow$ an easy lin. alg. operation on $\mathbf{MHSM_1}$.

Poincaré and Serre dualities imply

$$\mathbf{Jac}(X, Y, Z)^\vee \cong \mathbf{Jac}(X, Z, Y).$$

MHS of arbitrary level

MHS : Deligne's cat. of mixed Hodge structures.

Formal property. **MHS** is abelian; it contains $\mathbf{MHS_1}$.

Geometry. X : smooth proper variety of dimension d,
$Y, Z \subset X$: effective reduced divisors, $|Y| \cap |Z| = \emptyset$.
$\exists H^n(X, Y, Z) \in \mathbf{MHS}$ s.t. $H_{\mathbb{Z}} = H^n(X \setminus Z, Y, \mathbb{Z})$ $(n \in \mathbb{Z})$.

Duality. $H^n(X, Y, Z)^\vee \cong H^{2d-n}(X, Z, Y)(d)/(\text{tor})$.

Albanese. One has $H^{2d-1}(X, Y, Z)(d)/(\text{tor}) \in \mathbf{MHS_1}$, and it corresponds to the Albanese 1-motive $\mathrm{Alb}(X, Y, Z) \in \mathscr{M}_1^{\mathrm{D}}$ via $\mathbf{MHS_1} \cong \mathscr{M}_1^{\mathrm{D}}$. [$d = 1 \Rightarrow \mathrm{Alb}(X, Y, Z) = \mathrm{Jac}(X, Y, Z)$.]

NB. Deligne constructed $H^n(S) \in \mathbf{MHS}$ for any variety S.
One has $H^{2d-1}(S)(d)/(\text{torsion}) \in \mathbf{MHS_1}$ if $d = \dim S$, and it corresponds to the Albanese 1-motive $\mathrm{Alb}(S) \in \mathscr{M}_1^{\mathrm{D}}$.

MHSM of arbitrary level

Def. A **MHS with modulus** is $\mathscr{H} = (H, U^\bullet, V^\bullet, \{\mathscr{F}^p\}_{p\in\mathbb{Z}})$,

a) $H \in \mathbf{MHS}$,

b) $\cdots \to U^p \xrightarrow{u^p} U^{p-1} \xrightarrow{u^{p-1}} \cdots$: chain of $\mathbb{C}$-linear maps,

c) $\cdots \to V^p \xrightarrow{v^p} V^{p-1} \xrightarrow{v^{p-1}} \cdots$: chain of $\mathbb{C}$-linear maps,

d) $\mathscr{F}^p \subset H_\mathbb{C} \oplus U^p \oplus V^p$: $\mathbb{C}$-subspaces ($\forall p \in \mathbb{Z}$),

subject to the following conditions:

- $\dim_\mathbb{C}[\oplus_p(U^p \oplus V^p)] < \infty$.
- $(\mathrm{id}_{H_\mathbb{C}} \oplus u^p \oplus v^p)(\mathscr{F}^p) \subset \mathscr{F}^{p-1}$.
- For $x \in H_\mathbb{C}$: $x \in F^pH_\mathbb{C} \Leftrightarrow \exists u \in U^p$ s.t. $x + u \in \mathscr{F}^p$.
- $\mathscr{F}^p \hookrightarrow H_\mathbb{C} \oplus U^p \oplus V^p \twoheadrightarrow V^p$: surj.
- $U^p \hookrightarrow H_\mathbb{C} \oplus U^p \oplus V^p \twoheadrightarrow H_\mathbb{C} \oplus U^p \oplus V^p/\mathscr{F}^p$: inj.

Main results

Formal property. **MHSM** is abelian; it contains $\mathbf{MHSM_1}$.

Geometry. X : smooth proper variety of dimension d.
$Y, Z \subset X$: eff. divisors, not nec. reduced, $|Y| \cap |Z| = \emptyset$,
$(Y + Z)_{\mathrm{red}}$: strict normal crossing. $n \in \mathbb{Z}$.
$\exists \mathscr{H}^n(X, Y, Z) \in \mathbf{MHSM}$ s.t. $H = H^n(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}) \in \mathbf{MHS}$.

Duality. $\mathscr{H}^n(X, Y, Z)^\vee \cong \mathscr{H}^{2d-n}(X, Z, Y)(d)/(\mathrm{tor})$.

Albanese. $\mathscr{H}^{2d-1}(X, Y, Z)(d)/(\mathrm{tor}) \in \mathbf{MHSM_1}$, corresponding to Albanese **1**-motive $\mathbf{Alb}(X, Y, Z) \in \mathscr{M}_1^{\mathrm{L}}$ via $\mathbf{MHSM_1} \cong \mathscr{M}_1^{\mathrm{L}}$.

Jacobian. When $d = 1$, we have $\mathbf{Alb}(X, Y, Z) = \mathbf{Jac}(X, Y, Z)$, and $\mathbf{Jac}(X, Y, Z)^\vee \cong \mathbf{Jac}(X, Z, Y)$ by Duality (above).

Relation with previous works

Kato and Russell (2012) constructed $\mathbf{Alb}(X, Y) \in \mathbf{AG}$ for smooth proper X, eff. divisor Y (generalizing $\mathbf{Jac}(X, Y)$ for curves). It agrees with our $\mathbf{Alb}(X, Y, \emptyset)$.

Bloch and Srinivas (2000) constructed enriched **HS** and $H^n(S) \in \mathbf{EHS}$ for proper (but possibly singular) var. S. Barbieri-Viale (2007) and Mazzari (2011) generalized **EHS** to formal **HS**. This is (almost) same as **MHSM** with $V^\bullet = 0$.

Deligne constructed $H^n(S) \in \mathbf{MHS}$ for arbitrary (possibly singular/non-proper) S, which is basis of all constructions.

Problem. Can one define $H^n(S) \in \mathbf{MHSM}$ for such S?