Inference for Complex Extreme Events

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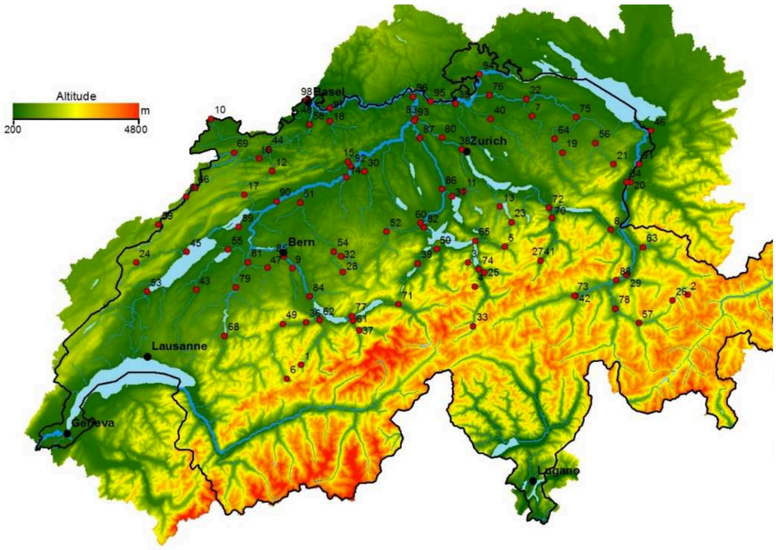
(Not so) Rare events

- Fukushima tsunami just one of many ‘rare events’, e.g.,
  - heavy rain, drought, heatwaves, hurricanes, . . .—likely to have greater impact in future years under climate change
  - stock market ‘corrections’—our pensions, mortgages, savings at risk
- To manage the risk, need estimates of probabilities for events of probability 10^{-4} or 10^{-7} annually, with uncertainties
  - usually based on 25–150 years of data (at very most)
- Basic task is **extrapolation (well) outside the range of any observations**

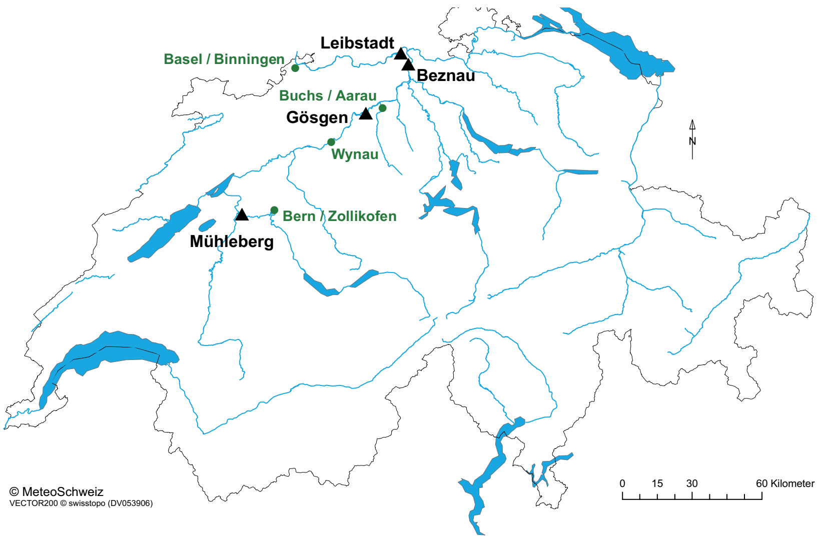
EXAR project



EXAR project

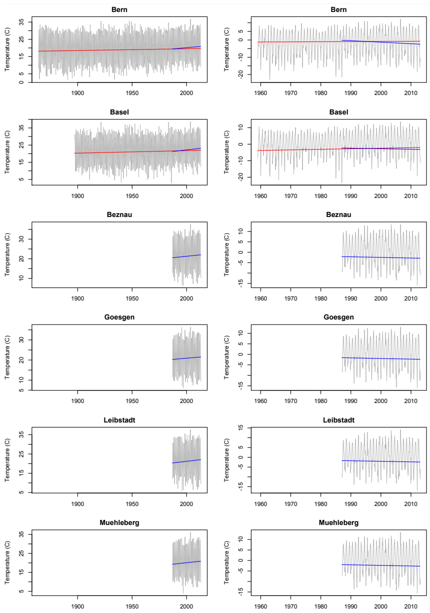
- ☐ Aim to estimate flood risk on Aare river basin up to 2050, taking into account climate change
- ☐ Probabilities needed for events on river network with annual probabilities 10^{-4} (Swiss nuclear power plants are on riverbanks)
- ☐ Must assess combined flooding risk, based on time series of length at most 80 years
- ☐ Several university institutes involved (hydrology, hydraulics, geography, climate science, ...)

Swiss nuclear plants



Muhleberg

Air temperature maxima and minima



Guide to extrapolation

- ☐ Advice to students:

  Don't Do It

- ☐ Advice to experts:

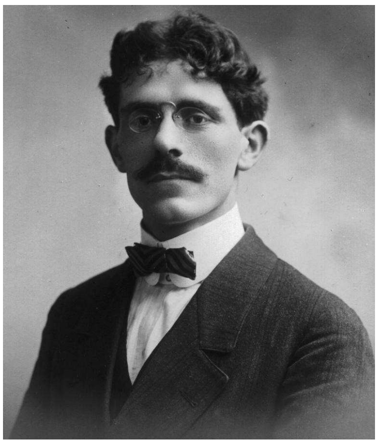
  Don't Do It (Yet)

- ☐ If $X_1, \ldots, X_n \overset{\text{iid}}{\sim} F$, then $M_n = \max(X_1 \ldots, X_n)$ has distribution F^n, but
  - empirical estimate of extreme probability will be 0
  - F^n highly variable if F has to be estimated (always the case)
  - estimation dominated by central observations (relevant to extremes?)
- ☐ Hence base extrapolation on limiting distributions for $(M_n - b_n)/a_n$, for suitable renormalising sequences $a_n > 0$ and b_n, as $n \to \infty$
- ☐ What limits can arise?

Extreme-Value Models

Founders of extreme-value theory

Maurice René Fréchet (1878–1973)
Ronald Alymer Fisher (1890–1962)
Leonard Henry Caleb Tippett (1902–1985)



Modelling maxima

- ☐ A distribution G for maxima must satisfy the **max-stability** relation

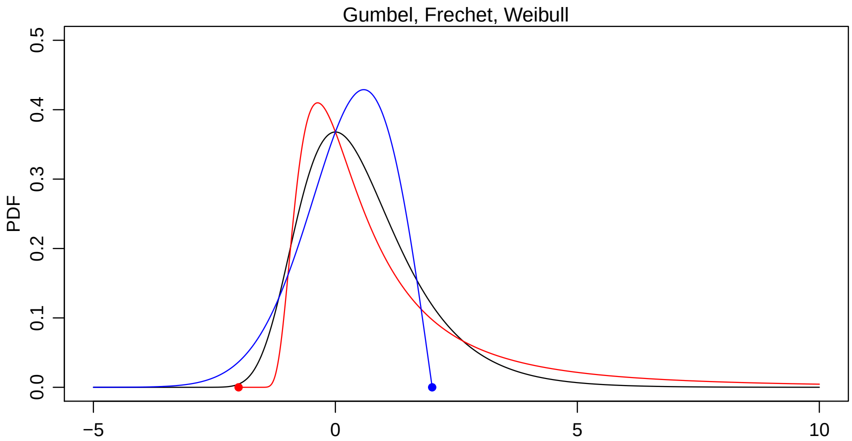
$$G^n(b_n + a_n y) = G(y), \quad m = 1, 2, \dots, \quad \{a_n\} > 0, \{b_n\} \subset \mathbb{R}.$$

- ☐ Only non-trivial solution is the **generalized extreme-value (GEV) distribution**,

$$G(y) = \exp\left\{ - \left[1 + \xi \left(\frac{y - \mu}{\tau} \right) \right]_+^{-1/\xi} \right\},$$

  where $u_+ = \max(u, 0)$, and μ and τ are location and scale parameters.
- ☐ ξ is a shape parameter determining the rate of tail decay, with
  - $\xi > 0$ giving the heavy-tailed (Fréchet) case,
  - $\xi = 0$ giving the light-tailed (Gumbel) case—corresponds to Gaussian data,
  - $\xi < 0$ giving the short-tailed (reverse Weibull) case.
- ☐ ξ is hard to estimate, but crucial because it controls probabilities of large events.

GEV and shape parameter ξ



- ☐ PDFs of the **Gumbel** ($\xi = 0$), the **Fréchet** ($\xi = 0$) and the (reverse) **Weibull** ($\xi < 0$).
- ☐ The Fréchet is bounded below, and the reverse Weibull is bounded above.
- ☐ The standard Weibull is a distribution for minima.

Poisson process

- In terms of the binomial process $N_n(\cdot) = \sum_j I(b_n + a_n X_j \in \cdot)$,

$$(M_n - b_n)/a_n \le y \quad \Leftrightarrow \quad N_n(y, \infty) = 0,$$

and if a limit exists as $n \to \infty$, then

$$\mathrm{P}\left(\frac{M_n - b_n}{a_n} \le y\right) = \left[1 - \frac{n\{1 - F(b_n + a_n y)\}}{n}\right]^n \to \exp\{-\Lambda(y)\},$$

where $\Lambda(y) = -\log G(y) = \{1 + \xi(y - \mu)/\tau\}_+^{-1/\xi}$ is the mean measure of a Poisson process $N(\cdot)$ on $\mathbb{R}$.

- Often in practice we assume that the Poisson process applies to observations that exceed some threshold u, and estimate the parameters μ, τ, η, or equivalently fit the **generalised Pareto distribution (GPD)**.
- A parametric model—an 'easy' problem, even if n or u must be chosen and uncertainties are usually much too large for comfort.

General remarks

- Extreme value theory is based on **limiting** models for tails of distributions:
  - Generalised extreme-value distribution (GEV) applies for maxima of an infinite sample,
  - Poisson process model applies for peaks over an 'infinite' threshold,

  both satisfying notions of stability from mathematical considerations.
- Could fit other models, but with weaker mathematical justification.
- In practice models fitted to finite samples, so the models are approximate and extrapolation may be worrisome.
- Relevant data often limited, so helpful if possible to include information from elsewhere.
- **Overwhelming question**: Do we trust extrapolations from mathematical models for real phenomena?
- Now generalize extreme-value paradigm to complex settings . . .

Max-stable processes

- Without loss of generality, we first transform the process so that its marginal distributions are standard Fréchet, GEV(1,1,1), distributions, $e^{-1/z}$, for $z > 0$.
- The GEV distribution is **max-stable**: maxima of independent GEV variables are also GEV—in fact, this is the defining property of the GEV distribution, and allows extrapolation to rare events.
- For the standard Fréchet, GEV(1,1,1), distribution, this means that if $Z, Z_1, \ldots, Z_n \stackrel{\text{iid}}{\sim} \exp(-1/z)$, then for any n,

$$\max\{Z_1, \ldots, Z_n\} \stackrel{D}{=} nZ.$$

- For space/space-time problems we need a process analogue of the GEV, i.e., we seek a process $Z(x)$ such that if $Z_1(x), \ldots, Z_n(x) \stackrel{\text{iid}}{\sim} Z(x)$, then

$$\max\{Z_1(x), \ldots, Z_n(x)\} \stackrel{D}{=} nZ(x), \quad x \in \mathcal{X},$$

where $\mathcal{X}$ represents a space/space-time domain of interest (e.g., a watershed, returns for a stock market over the next 5 years).

Construction of a max-stable process

- Let $W(x)$ be a non-negative random process with $\mathrm{E}\{W(x)\} = 1$ ($x \in \mathcal{X}$), and let (de Haan, 1984)

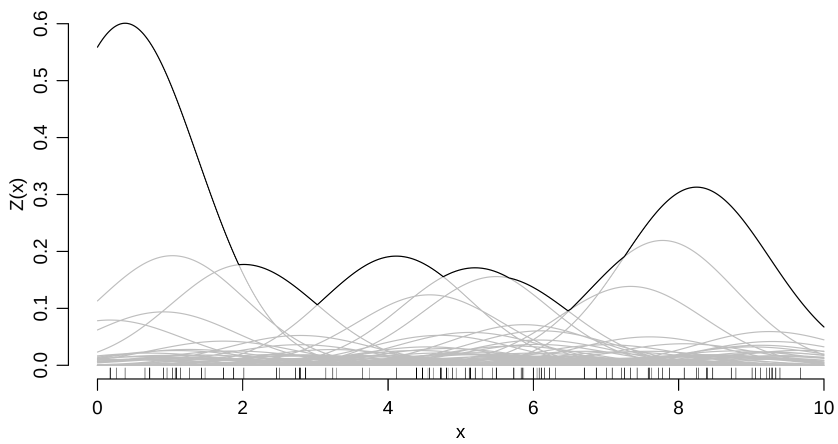
$$Z(x) = \sup_j R_j W_j(x), \quad x \in \mathcal{X}, \tag{1}$$

with $\{R_j\}$ a Poisson process on $\mathbb{R}_+$ of rate $\mathrm{d}r/r^2$ and $\{W_j\}$ replicates of W.
- Then $Q_j(x) = R_j W_j(x)$ is a Poisson process on $\mathbb{R}_+ \times \mathbb{R}_+^{\mathcal{X}}$, and

$$\mathrm{P}\left\{Z(x) \le z(x), x \in \mathcal{X}\right\} = \exp\left(-\mathrm{E}\left[\sup_{x\in\mathcal{X}}\left\{\frac{W(x)}{z(x)}\right\}\right]\right) = \exp\left[-V\{z(x)\}\right],$$

say, where $V\{z(x)\}$ is a void probability for the Q_j, and this gives
  - a **max-stable process** $\{Z(x) : x \in \mathcal{X}\}$, i.e., there exist functions $\{b_n(x)\}$ and $\{a_n(x)\} > 0$ such that
$$Z(x) \stackrel{D}{=} \max_{j=1}^{n}\left\{\frac{Z_j(x) - b_n(x)}{a_n(x)}\right\}, \quad x \in \mathcal{X}.$$
  - $Z(x) \sim$ unit Fréchet at each $x \in \mathcal{X}$.
- Any max-stable process can be expressed using the (non-unique) **spectral representation** (1).

Making a Smith (1990) process

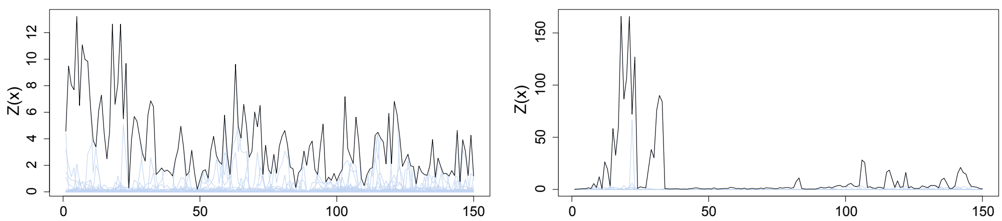


Comments

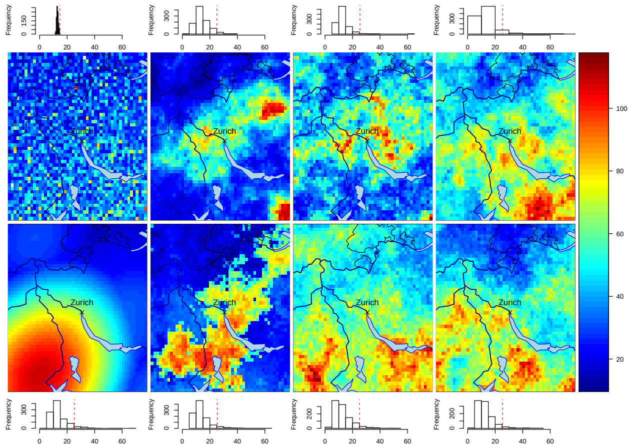
- ☐ Numerous max-stable models now exist, some more ‘realistic’ than others
- ☐ Particularly flexible example is the Brown–Resnick process (), which takes

$$W(x) = \exp\{\varepsilon(x) - \gamma(x)\},$$

where $\varepsilon(x)$ is a stationary or intrinsically stationary Gaussian process with semi-variance or semivariogram $\gamma(x)$—can use panoply of functions γ from spatial statistics, or can invent your own.



Realisations from spatial models



Top: results from the latent variable, Student t copula, Hüsler–Reiss copula and extremal-t copula models. Bottom: results from the Smith, Schlather, geometric Gaussian and Brown–Resnick models. The histograms are of 1000 realisations of a summary of rainfall centred on Zürich, and the vertical lines correspond to the realizations shown.

Extremal coefficient

- For any set $\mathcal{D} \subset \mathcal{X}$, homogeneity of V means that a max-stable model satisfies

$$\mathrm{P}\{Z(x) \le z, x \in \mathcal{D}\} = \exp\{-V_{\mathcal{D}}(z)\} = \exp\{-V_{\mathcal{D}}(1)/z\} = \left(e^{-1/z}\right)^{V_{\mathcal{D}}(1)}, \quad z > 0,$$

  and the **extremal coefficient**

$$\theta_{\mathcal{D}} = V_{\mathcal{D}}(1)$$

  summarises the degree of dependence of the extremes in $\mathcal{D}$.
- In particular, the pairwise version,

$$\theta(x, x') = \mathrm{E}\left[\max\left\{W(x), W(x')\right\}\right], \quad x, x' \in \mathcal{X},$$

  can be regarded as an analogue of the correlation coefficient, with

$$\text{(total dependence)} \quad 1 \le \theta(x, x') \le 2 \quad \text{(independence)},$$

  and the interpretation

$$\mathrm{P}\left\{Z(x') > z \mid Z(x) > z\right\} \sim 2 - \theta(x, x'), \quad z \to \infty.$$

- θ can be estimated nonparametrically, either as a basis for model checking, or for subsequent semiparametric estimation of V.

Likelihood inference

- Suppose we have independent (annual) maxima observed at $\mathcal{D} = \{x_1, \ldots, x_D\} \subset \mathcal{X}$ for n years, so the data for each year have joint distribution

$$\mathrm{P}\{Z(x_1) \le z_1, \ldots, Z(x_D) \le z_D\} = \exp\{-V(z_1, \ldots, z_D)\}, \quad z_1, \ldots, z_D > 0.$$

- The formulation of the model using its CDF means that to compute the likelihood function we must differentiate e^{-V} with respect to $z_1, \ldots, z_D$, leading to combinatorial explosion:

$$-V_1 e^{-V}, \quad (V_1V_2 - V_{12})e^{-V}, \quad (-V_1V_2V_3 + V_{12}V_3[3] - V_{123})e^{-V}, \quad \ldots,$$

  with about 10^5 terms for $D = 10$. Clearly this is infeasible for realistic applications, so we need to avoid this, by
  - using a composite (usually a pairwise) likelihood; or
  - using the timing of events to chose the term of the partition in the likelihood;
  - using threshold exceedances.
- In any case we must compute (many) derivatives of V, and sometimes integrate them ... can be painful.

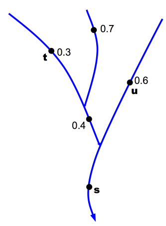
Extremal dependence on river network

- Sources of dependence between data at locations x_1 and x_2 on the network $\mathcal{X}$:
  - **flow-dependence**; x_2 is downstream of x_1, or vice versa
  - **'geo'-dependence**: the same events may impact nearby watersheds
- Overall semi-variogram

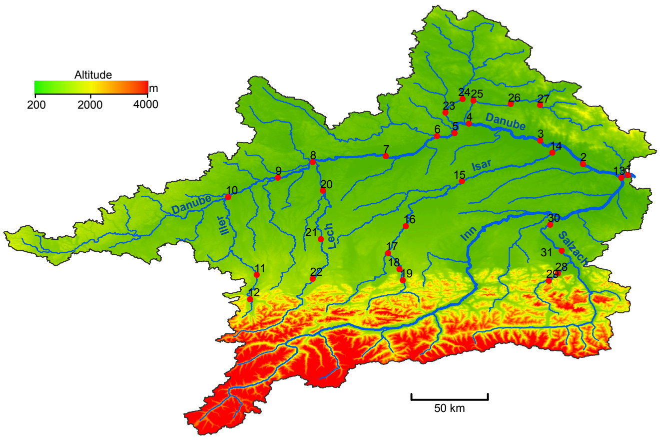
$$\gamma(x_1, x_2) = \lambda_{\mathrm{RIV}}\{1 - C_{\mathrm{RIV}}(x_1, x_2)\} + \lambda_{\mathrm{GEO}}\,\gamma_{\mathrm{GEO}}(x_1, x_2), \quad x_1, x_2 \in \mathcal{X},$$

  where $\lambda_{\mathrm{RIV}}, \lambda_{\mathrm{EUC}} > 0$.
- Flow-dependence in terms of shortest river distance $d(\cdot, \cdot)$:

$$\begin{aligned} C_{\mathrm{RIV}}(s,u) &= C_1\{d(s,u)\} \times \sqrt{0.6}, \\ C_{\mathrm{RIV}}(s,t) &= C_1\{d(s,t)\} \times \sqrt{0.4 \times 0.3}, \\ C_{\mathrm{RIV}}(u,t) &= 0, \\ C_1(h) &= \exp(-h/\theta), \quad \theta > 0. \end{aligned}$$



Upper Danube Basin

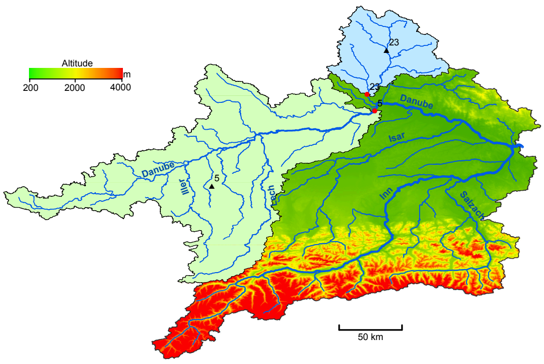


Asadi, Davison, Engelke (2015) *Annals of Applied Statistics*

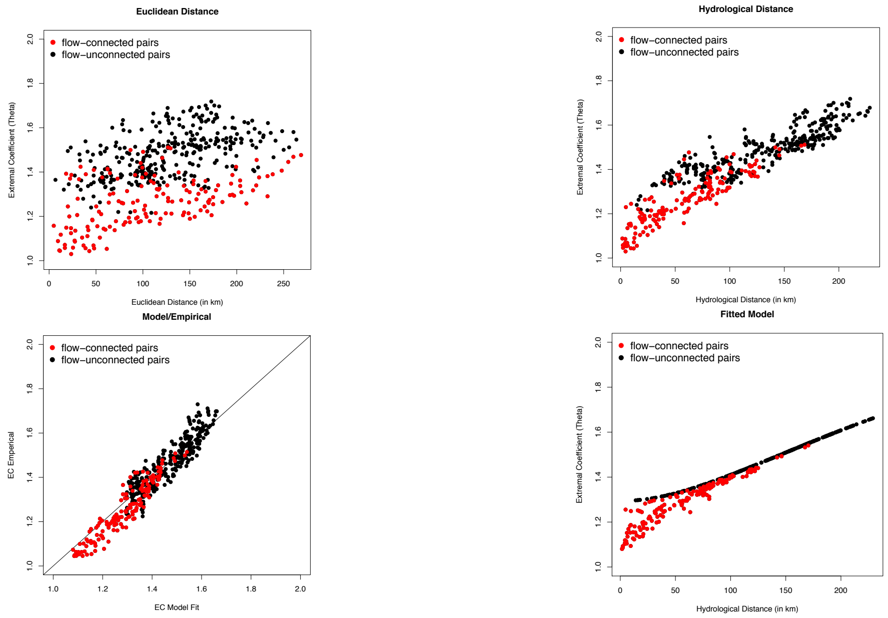
Extremal dependence on river network

- Introduce **hydrological location** of each station, as $h(x) \in \mathbb{R}^2$ as centroid of its sub-catchment, and define dependence measure

$$\gamma_{\text{EUC}}(x_1, x_2) = \|h(x_1) - h(x_2)\|^{\alpha}, \quad \alpha \in (0, 2].$$

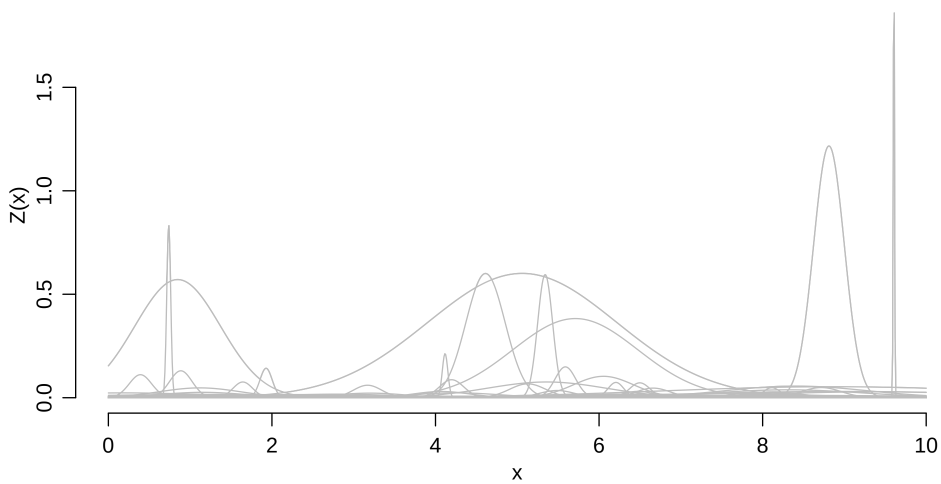


Extremal coefficients



Threshold exceedances slide 29

Individual events



Exceedances and risk functions

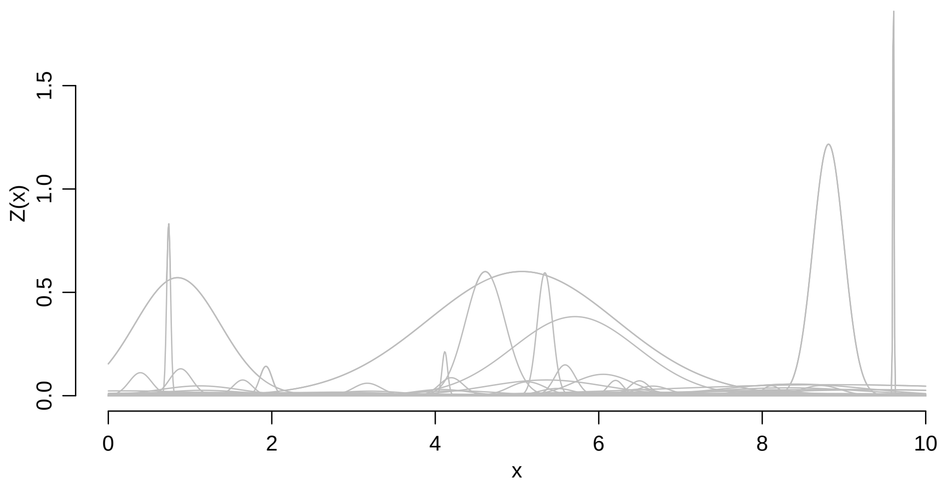
- Modelling threshold exceedances is widely used in (scalar) practice:
  - more flexible than using maxima
  - statistically more efficient, makes better use of data
- For scalar data, choosing rare events is easy: either they're big or they're small.
- For multivariate data, we need to say what 'direction' is extreme
- Do this via a scalar **risk function** f applied to the individual events $Q_j(x) = R_j W_j(x)$ of the max-stable process
  - Choose those events Q_j for which $f(Q_j)$ exceeds a threshold u
  - **Red**: extremes on $[0, 2]$, selected using risk function

$$f(Q) = \int_0^2 Q(x)\,dx$$

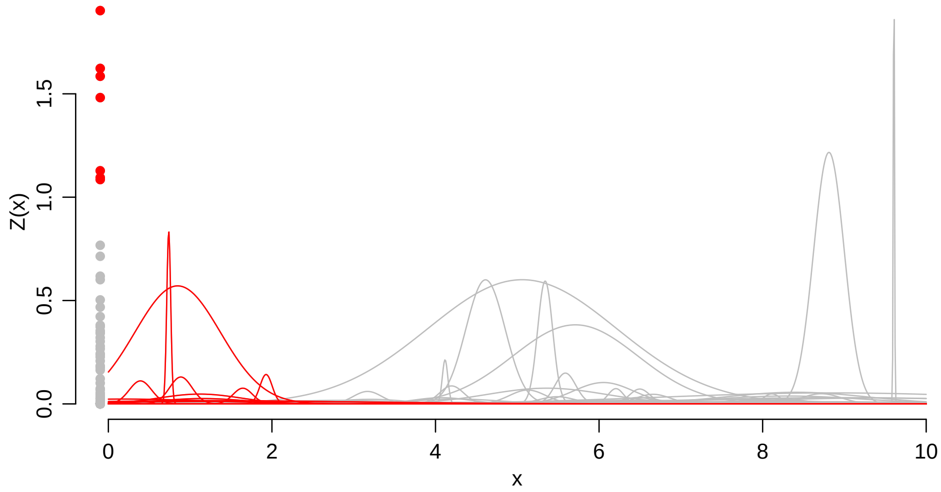
  - **Blue**: most intense events, selected using risk function

$$f(Q) = \max Q(x)$$

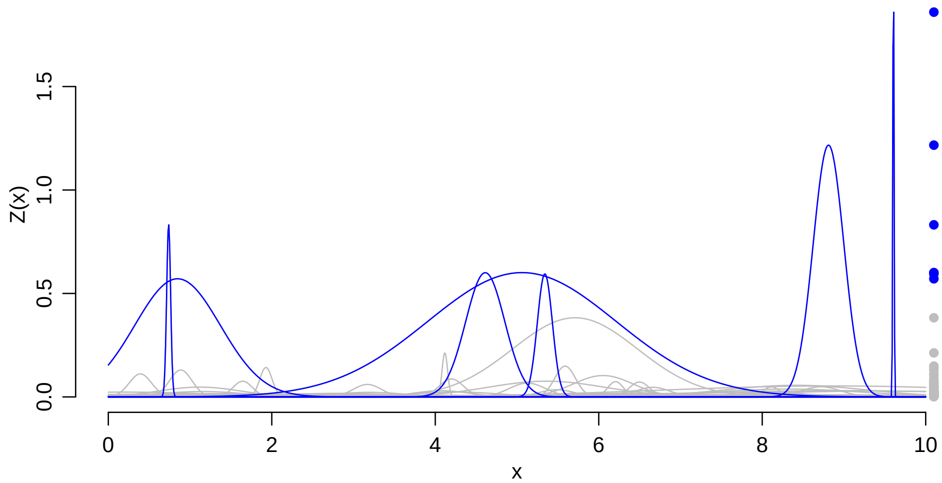
Individual events



Extremes in $[0, 2]$



Highest peaks anywhere

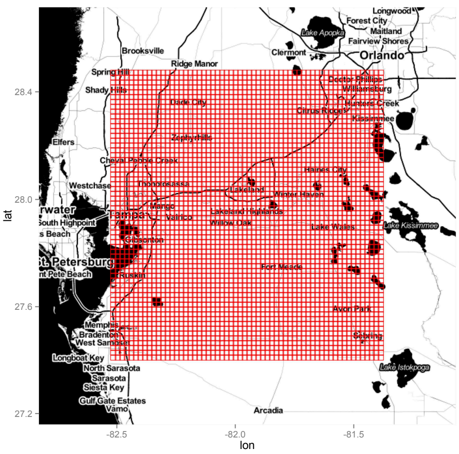


Inference

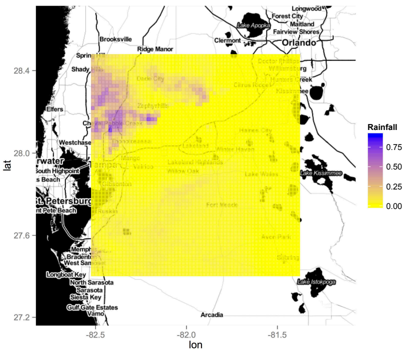
- ☐ Fitting for the 'exceedances' Q_j is (in principle) much easier than for the max-stable process $Z(x)$:
  - likelihoods can be constructed, at least for Gaussian-based $Q(x)$ but
  - they entail lots of burdensome integrals to compute norming constants.
- ☐ Fixes
  - estimate the integrals using quasi-Monte Carlo or other methods,
  - avoid likelihood inference, using the gradient score to dodge computing the norming constants.
- ☐ Big problems ($D \approx 1000$s) feasible with the gradient score, smaller ones ($D \approx 100$s) with quasi-Monte Carlo approximation.

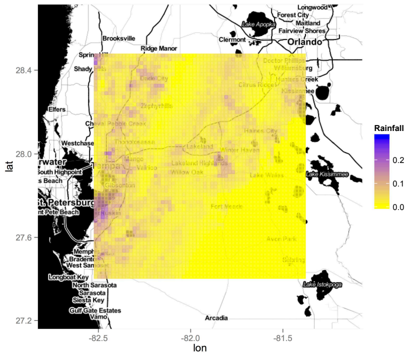
Extreme rainfall over Florida

- ☐ 15-minute radar rainfall measurements over Florida from 1994–2010
- ☐ We focus on a 120 km× 120 km square south-west of Orlando and on the wet season, i.e., June to September.



Florida rainfall





Local rainfall Spatially dispersed rainfall

Risk functionals

- We define two risk functionals

$$f_{\max}(X^*) = \left[\sum_{i=1}^{\ell} \{X^*(s_i)\}^{20}\right]^{1/20}, \quad f_{\text{sum}}(X^*) = \left[\sum_{i=1}^{\ell} \{X^*(s_i)\}^{\xi_0}\right]^{1/\xi_0},$$

  where $\ell = 3600$ is the number of grid cells.
- Here
  - $f_{\max}$ is a continuous and differentiable approximation of $\max_{i=1,\ldots,\ell} X^*(s_i)$ which satisfies the requirements for the gradient score,
  - f_{sum} selects events with large spatial cover. The power ξ_0 approximately transforms the data X^* back to a scale where summing observations has a physical meaning.

Spatial model and parameter estimates

- Non-separable semi-variogram model

$$\gamma(x_i, x_j) = \left\| \frac{\Omega(x_i - x_j)}{\tau} \right\|^{\kappa}, \quad x_i, x_j \in [0, 120]^2, \quad i, j \in \{1, \ldots 3600\},$$

with $0 < \kappa \leqslant 2, \tau > 0$ and anisotropy matrix

$$\Omega = \begin{bmatrix} \cos\eta & -\sin\eta \\ a\sin\eta & a\cos\eta \end{bmatrix}, \quad \eta \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right], \quad a > 1.$$

- Fitted parameters obtained for both risk functionals with exceedances of $f_{\max}(X^*)$ and $f_{\text{sum}}(X^*)$ over the 99 quantile:

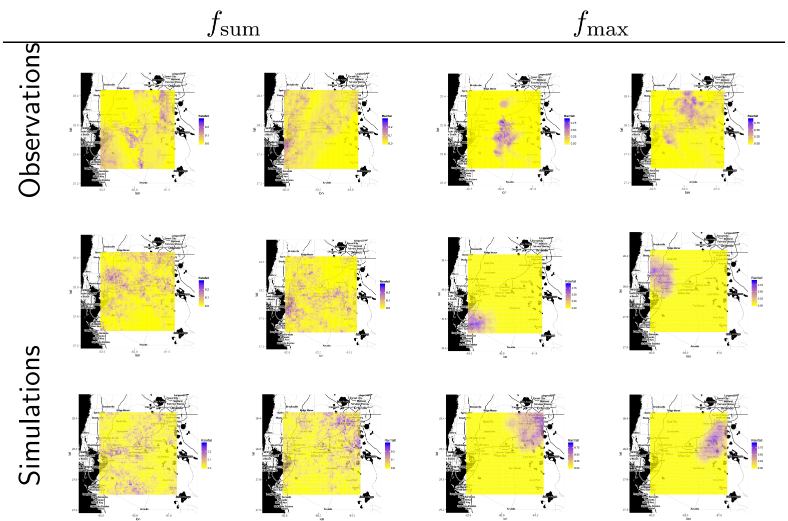
	κ	τ	η	a
$f_{\max}$	$1.192_{0.02}$	$9.06_{0.19}$	$0.08_{0.61}$	$1.008_{0.005}$
f_{sum}	$0.326_{0.007}$	$46.67_{0.018}$	$-0.30_{0.10}$	$1.064_{0.017}$

  – $f_{\max}$ estimates are quite smooth with a small scale, they capture high quantiles and induce a model similar to that in earlier work.

  – For f_{sum}, the semi-variogram is rougher but with a much larger scale, which is consistent with large-scale events.

  – Anisotropy does not seem significant.

Simulated extreme rainfall



15-minute cumulated rainfall (inches): observed (first row) and simulated (second and third rows) for the risk functionals f_{sum} (left) and $f_{\max}$ (right) with intensity equivalent to the 0.99 quantile.

Closing

- ☐ Basic ideas on maxima and point processes extend to spatial and space-time settings.
- ☐ Max-stable processes give asymptotic dependence models—asymptotic independence can be bothersome in practice, but can account for it (up to a point).
- ☐ Can fit such models using
  - pairwise likelihood (can be inefficient),
  - full likelihood (needs additional information, difficult with large D),
  - Bayesian methods, or
  - gradient score methods.
- ☐ Model-checking possible, using simulation from fitted models and other techniques—but difficult to validate far into tails, because of lack of data.
- ☐ Currently much research in area (e.g., threshold models, non-stationarity, gridded data, non-Euclidean spaces, ...).

Some reading

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