Higher Gauge Theories and Superconformal Field Theories in 6d

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Credits for collaborations and discussions/explanations:

- A Deser, B Jurco, S Palmer, P Ritter, L Schmidt, M Wolf ...
- J Baez, U Schreiber

Outline

What are higher structures?

Why should You care?

Examples : L_∞-algebras and L_∞-algebroids

Higher Gauge Theory

Higher Gauge Theory and M5-branes

Our attempt so far...

Reality check

What are higher structures?

"We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance."

G. Moore and N. Seiberg, 1989

"We'll only use as much category theory as is necessary. Famous last words..."

Roman Abramovich

Observation

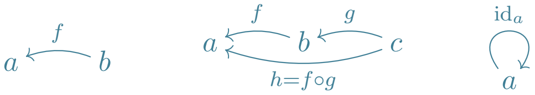
Mathematical objects come with corresponding maps.
Combine them into one entity: Category

Examples:

- Vector spaces and linear maps $\rightarrow$ Vect
- Groups and group homomorphisms $\rightarrow$ Grp
- Topological spaces and homeomorphisms $\rightarrow$ Top
- Smooth manifolds and smooth maps between them $\rightarrow$ Mfd

Category: $\mathscr{C} = \mathscr{C}_1 \rightrightarrows \mathscr{C}_0$

$\mathscr{C}_0$: objects $\mathscr{C}_1$: maps/morphisms



From Categories to Higher Categories

Categories: meta-language, "essence" of mathematical structures

But also: Categories give us more freedom than sets:

- Set theory: objects a, b. Either $a = b$ or $a \neq b$.
- Categories: objects a, b. Relating morphism f: $a \xleftarrow{f} b$

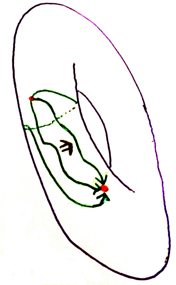
However: What about the morphisms? Relations between them?

Yes, with 2-categories: $a \underset{f_2}{\overset{f_1}{\leftleftarrows}} b$ with $\Downarrow \alpha$, morphisms: set $\rightarrow$ category

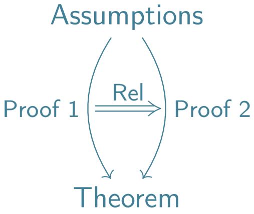
This can be iterated to ∞-categories with general n-morphisms.

Examples

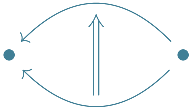
Homotopies:



Relations between proofs:



Parallel Transport of Strings



"Indeed, the subject might better have been called ... archery."

Steve Awodey

Constructing Higher Structures

A mathematical structure ("Bourbaki-style") consists of

- Sets
- Structure Functions
- Structure Equations

"Categorification":

Sets → Categories

Structure Functions → Structure Functors

Structure Equations → Structure Isomorphisms

Example: Group → 2-Group

- Set G → Category $\mathscr{G}$
- product, identity ($\mathbb{1} : * \to \mathsf{G}$), inverse → Functors
- $a(bc) = (ab)c$ → Associator $\mathsf{a} : a \otimes (b \otimes c) \Rightarrow (a \otimes b) \otimes c$
- $\mathbb{1}a = a\mathbb{1} = a$ → Unitors $\mathsf{l}_a : a \otimes \mathbb{1} \Rightarrow a$, $\mathsf{r}_a : \mathbb{1} \otimes a \Rightarrow a$
- $aa^{-1} = a^{-1}a = \mathbb{1}$ → weak inv. $\mathsf{inv}(x) \otimes x \Rightarrow \mathbb{1} \Leftarrow x \otimes \mathsf{inv}(x)$

Note: Process not unique, variants: weak/strict/...

Why should You care?

"Before functoriality, people lived in caves."

Brian Conrad

"It's like déjà vu all over again."

Yogi Berra

In String Theory:

- Point particles $\rightarrow$ Strings : Manifold $M \rightarrow$ Loop Space $\mathcal{L}M$
  Bundles over $\mathcal{L}M$ correspond to higher bundles over M
- Higher form fields: Connections on higher bundles
  - Kalb–Ramond B-field: connective structure on gerbe
  - Higher forms coupling to Dp-branes: higher gerbes
- T-duality/Generalized Geometry:
  - Courant algebroid = symplectic higher Lie algebroid
  - Double/Exceptional Field Theory: gnrlzd Courant algebroids
  - generalized manifolds, e.g. orbifolds = stacks $\in$ bicategory
- String Field Theory:
  - Closed SFT based on higher Lie algebras
  - Open (s)SFT based on higher associative algebras
- (2,0)-theory, (1,0)-theories in F-theory: Higher gauge theories

In Supergravity:

- Higher form fields (as above) belong to higher bundles
- Tensor hierarchies in gauged SUGRA are higher gauge theories

In Field Theory:

- Abstract Definition of TQFTs: higher categories of cobordisms
- AKSZ construction: based on symplectic higher Lie algebroids
- BRST/BV formalism:
  - Any Classical Field Theory $\Rightarrow$ Higher Lie Algebra/L_∞-algebra
  - Any QFT yields loop or quantum L_∞-algebra
- Moduli spaces to gauge field equations: stacks

In Quantum Gravity:

- Quantum spacetimes:
    - Noncommutative geometry only first approximation.
    - Nonassociative spaces from higher geometric quantisation
    - Ultimately: $\mathcal{C}^\infty$(some manifold) $\rightarrow$ some A_∞-algebra
- Visibile in string theory:
    - D1-branes ending on D3-branes: Fuzzy funnel with fuzzy S^2
    - M2-branes ending on M5-branes: Fuzzy funnel with fuzzy S^3

... or should be using them.

In Generalizing/Deforming mathematical objects:

- Category theory extracts essence of mathematical notions.
- Mathematically consistent generalizations become obvious.

Example: Principal Fiber Bundles (as transition functions)

- Group G as category $\mathsf{G} \rightrightarrows *$, composition: group product
- Cover $\mathcal{U} = \sqcup_a U_a$ of a manifold M yields category $\check{\mathcal{C}}(\mathcal{U})$:

$$(x,U_a) \xleftarrow{(x,U_{ab})} (x,U_b) \qquad (x,U_a) \xleftarrow{(x,U_{ab})} (x,U_b) \xleftarrow{(x,U_{bc})} (x,U_c), \quad (x,U_a) \xleftarrow{(x,U_{ac})} (x,U_c)$$

- $$\begin{array}{ccc} \sqcup_{a,b} U_{ab} & \xrightarrow{g_{ab}} & \mathsf{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \sqcup_a U_a & \xrightarrow{*} & * \end{array}$$

  Transition functions g_{ab},
  cocycle cond. $g_{ab} g_{bc} = g_{ac}$
  cobndries.: $g_{ab}\gamma_b = \gamma_a \tilde{g}_{ab}$

- Generalizations: replace both sides e.g. with higher categories

If we can't escape higher structures, we might as well learn the mathematics behind them.

It's beautiful stuff!

Examples : L_∞-algebras and L_∞-algebroids

N-manifolds, NQ-manifold

- $\mathbb{N}_0$-graded manifold with coordinates of degree $0, 1, 2, \ldots$

$$M^\circ \leftarrow E_1 \oplus E_2 \oplus \ldots$$

manifold (M°) — linear spaces (E_1, E_2)

- NQ-manifold: vector field Q of degree 1, $Q^2 = 0$
- Physicists: think ghost numbers, BRST charge, SFT
- Functions on (M, Q) form differential graded algebra "Chevalley–Eilenberg algebra"

First Example:

- Tangent algebroid $T[1]M$, local coordinate functions x^μ, ξ^μ
- $f(x^\mu, \xi^\mu) \leftrightarrow f(x^\mu, \mathrm{d}x^\mu)$ and $Q = \xi^\mu \frac{\partial}{\partial x^\mu} \quad \leftrightarrow \quad \mathrm{d}x^\mu \frac{\partial}{\partial x^\mu}$
- $\Rightarrow$ Recover de Rham complex: $\mathcal{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$.

NQ-Manifolds

$M^\circ \leftarrow E_1 \oplus E_2 \oplus \ldots$, vector field Q with $|Q| = 1$, $Q^2 = 0$

More Examples:

- Lie algebra $\mathfrak{g}[1]$, coordinate functions ξ^α of degree 1:

$$Q = -\tfrac{1}{2} f^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \frac{\partial}{\partial \xi^\alpha} \quad , \quad Q^2 = 0 \Leftrightarrow \text{Jacobi identity}$$

- $* \leftarrow E_1 \oplus E_2 \oplus \cdots \oplus E_n$: Lie n-algebra
  (indeed equivalent, at least for $n = 2$)
- $M^\circ \leftarrow E_1 \oplus E_2 \oplus \cdots \oplus E_n$: Lie n-algebroid:
- Symplectic Lie n-algebroids: add ω, $\mathcal{L}_Q \omega = 0$:
  - $|\omega|_\mathbb{N} = 0$: Symplectic manifold
  - $|\omega|_\mathbb{N} = 1$: $M \cong T^*[1]M^\circ$, Poisson manifold
  - $|\omega|_\mathbb{N} = 2$: $M \cong T^*[2]E$, $E \to M$ vec bndl: Courant algebroid
  - Symplectic Lie n-algebras: Metric Lie n-algebras

Skipped: Relation higher categories $\Leftrightarrow$ differential graded structs.

Example: Lie 2-Algebras

- Graded vector space: $* \leftarrow W[1] \leftarrow V[2] \leftarrow * \leftarrow \dots$
- Coords: w^a of degree 1 on $W[1]$, v^i of degree 2 on $V[2]$
- Most general vector field Q of degree 1:

$$Q = -m_i^a v^i \frac{\partial}{\partial w^a} - \tfrac{1}{2} m_{ab}^c w^a w^b \frac{\partial}{\partial w^c} - m_{ai}^j w^a v^i \frac{\partial}{\partial v^j} - \tfrac{1}{3!} m_{abc}^i w^a w^b w^c \frac{\partial}{\partial v^i}$$

- Induces "brackets"/"higher products":
  - $\mu_1(\tau_i) = m_i^a \tau_a$
  - $\mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c \;, \quad \mu_2(\tau_a, \tau_i) = m_{ai}^j \tau_j$
  - $\mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$
- $Q^2 = 0 \Leftrightarrow$ Homotopy Jacobi identities, e.g.
  - $\mu_1(\mu_1(-)) = 0$: μ_1 is a differential
  - $\mu_1(\mu_2(x, y)) = \mu_2(\mu_1(x), y) \pm \mu_2(x, \mu_1(y))$: compatible w. μ_2,
  - $\mu_2(x, \mu_2(y, z)) + \text{cycl.} = \mu_1(\mu_3(x, y, z))$: Jacobiator
- Analogously: Lie 3-, 4-, …-algebras

L_∞-algebras

Lie algebra in bracket picture:

- Vector space $\mathfrak{g}$
- Antisymmetric bilinear product $[-,-] : \wedge^2\mathfrak{g} \to \mathfrak{g}$
- Satisfying Jacobi identity: $\sum_\sigma [x_{\sigma(i)}, [x_{\sigma(j)}, x_{\sigma(k)}]] = 0$

L_∞-algebra in bracket picture:

- Graded vector space $\mathsf{L} = \mathsf{L}_0 \oplus \mathsf{L}_{-1} \oplus \mathsf{L}_{-2} \oplus \dots$
- Graded antisym. multilin. products $\mu_i : \wedge^i \mathsf{L} \to \mathsf{L}$, $|\mu_i| = 2 - i$
- Satisfying higher/homotopy Jacobi identity:

$$\sum_{i+j=n} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \pm \mu_{i+1}(\mu_j(x_{\sigma(1)}, \dots, x_{\sigma(j)}), x_{\sigma(j+1)}, \dots, x_{\sigma(n)}) = 0$$

- Recall: roughly $Q^* = \mu_1 + \mu_2 + \mu_3 + \dots$
- Categorification: e.g. $\mathsf{L} = \mathsf{L}_0 \oplus \mathsf{L}_{-1} \leftrightarrow \mathscr{L} = (\mathsf{L}_{-1} \ltimes \mathsf{L}_0 \rightrightarrows \mathsf{L}_0)$

Morphisms and Quasi-Isomorphisms

- Morphisms of NQ-manifolds clear:

$$M \xrightarrow{\Phi} M'~,\qquad Q \circ \Phi^* = \Phi^* \circ Q$$

- Morphisms of L_∞-algebras $\phi : \mathsf{L} \to \mathsf{L}'$ derived from this:
  - ϕ consists of maps $\phi_i : \wedge^i \mathsf{L} \to \mathsf{L}'$, $\quad |\phi_i| = 1 - i$.
  - ϕ_1: chain map from complex (L, μ_1) to (L', μ_1')
  - ϕ_i, $i > 1$, link higher products between L and L'

New here: Categorical equivalence/quasi-isomorphisms:

$$M \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} M'~,\qquad \begin{array}{c}\Psi \circ \Phi \cong \mathrm{id}_M \\ \Phi \circ \Psi \cong \mathrm{id}_{M'}\end{array} \qquad \Leftrightarrow \qquad \begin{array}{c}\phi \text{ is morphisms} \\ \text{induces } H^\bullet_{\mu_1}(\mathsf{L}) \cong H^\bullet_{\mu_1'}(\mathsf{L}')\end{array}$$

Lie 2-algebra examples:

$$\mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g} \quad \approxeq \quad * \to * \qquad \text{and} \qquad * \to \mathfrak{g} \quad \approxeq \quad \Omega\mathfrak{g} \xrightarrow{e} P_0\mathfrak{g}$$

Consistency: Constructions mostly agnostic to quasi-isomorphisms!

Higher Gauge Theory

"Category theory is the subject where you can leave the definitions as exercises."

John Baez

1st step: Construct Kinematical Data:

- Gauge group $\rightarrow$ Higher gauge group
- Principal Bundle $\rightarrow$ Higher Principal Bundle
- Connection $\rightarrow$?

Local Connections: Lie algebra-valued differential forms.

$\Rightarrow$ Work in unifying category: (functions on) NQ-manifolds:

$$(\Omega^\bullet(M), \mathrm{d}) \rightarrow (T[1]M, Q)\ , \quad (\mathfrak{g}, [-,-]) \rightarrow (\mathfrak{g}, Q)$$

"Mathematics is the art of giving the same name to different things."

Henri Poincaré (1908)

First Attempt

Inspiration from Category Theory: Everything is a morphism.

1st attempt: Consider morphism of dgas: $\mathcal{C}^\infty(\mathfrak{g}[1]) \to \Omega^\bullet(\mathbb{R}^d)$:

- $a : \xi^\alpha \mapsto: A^\alpha \in \Omega^1(\mathbb{R}^d)$, gauge potential $A^\alpha \tau_\alpha \in \Omega^1(\mathbb{R}^d) \otimes \mathfrak{g}$
- $Q\xi^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \mapsto \mathrm{d}A^\alpha = -\frac{1}{2} f^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma$
  equivalently: $\mathrm{d}A + \frac{1}{2}[A, A] = 0$: gauge potential is flat

Improved version

Extend Chevalley–Eilenberg $\mathcal{C}^\infty(\mathfrak{g}[1])$ algebra of $\mathfrak{g}$ to Weil algebra:

$$\mathrm{W}(\mathfrak{g}) := \mathcal{C}^\infty(T[1]\mathfrak{g}[1]) = \mathcal{C}^\infty(\mathfrak{g}[1] \oplus \mathfrak{g}[2])~, \quad \sigma : \mathfrak{g}^*[1] \xrightarrow{\cong} \mathfrak{g}^*[2]$$

$$Q|_{\mathcal{C}^\infty(\mathfrak{g}[1])} = Q_{\mathrm{CE}} + \sigma~, \quad Q_{\mathrm{CE}}\sigma = -\sigma Q_{\mathrm{CE}}$$

Natural morphism of differential graded algebras:

$$\begin{array}{rcl}
(A,F) : \mathrm{W}(\mathfrak{g}) & \longrightarrow & W(\mathbb{R}^d) = \Omega^\bullet(\mathbb{R}^d) \\
\xi^\alpha & \longmapsto & A^\alpha \\
(\sigma\xi^\alpha) = Q\xi^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma & \longmapsto & F^\alpha = (\mathrm{d}A + \tfrac{1}{2}[A,A])^\alpha \\
Q(\sigma\xi^\alpha) = -f^\alpha_{\beta\gamma}(\sigma\xi^\alpha)\xi^\beta & \longmapsto & (\nabla F)^\alpha = 0
\end{array}$$

We obtain gauge potential, curvature and Bianchi identity.
Gauge transformations follow straightforwardly from homotopies

$$(A_t, F_t) : \mathrm{W}(\mathfrak{g}) \quad \longrightarrow \quad W(\mathbb{R}^d) = \Omega^\bullet(\mathbb{R}^d \times I)$$

Complete picture

Let M manifold and $Y \twoheadrightarrow M$ be a cover.

$$\begin{array}{ccc} \Omega^\bullet_{\mathrm{vert}}(Y) & \xleftarrow{\quad A_{\mathrm{vert}} \quad} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & \text{Cartan–Ehresmann 1} & \uparrow \\ \Omega^\bullet(Y) & \xleftarrow{\quad (A,F_A) \quad} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & \text{Cartan–Ehresmann 2} & \uparrow \\ \Omega^\bullet(X) & \xleftarrow[\text{mod } \sim:\ \text{Chern–Weil}]{\quad (c_i) \quad} & \mathrm{inv}(\mathfrak{g}) \end{array}$$

flat connections

connection form is flat canonical form on fibres

connection and curvature

the characteristic form descends to base space

topological invariants

Cartan 1949/1950 Sati, Schreiber, Stasheff 2008

Other, equivalent ways to get essentially same formulas:

- L_∞-algebras and Homotopy Maurer–Cartan equations
- Differentiating Maurer–Cartan forms on L_∞-algebras
- Penrose–Ward transforms

Attention!

$\mathrm{inv}(\mathfrak{g})$ does not respect quasi-isomorphisms!

- Fine for (higher) Chern–Simons theories: $\mathrm{inv}(\mathfrak{g}) = 0$
- Problematic for (1,0)/(2,0)-theories with curvatures $\neq 0$.
- Alternative: simple modification of $W(\mathfrak{g}) \rightarrow$ String structures
- (My) Conclusion:
  - Chern–Simons and Yang–Mills theories: same kinematical data
  - Higher analogues, however, require different kinematical data
  - Source of dismissal of higher non-abelian bundles.

Higher Gauge Theory and M5-branes

"... it can often be profitable to try a technique on a problem even if you know in advance that it cannot possibly solve the problem completely."

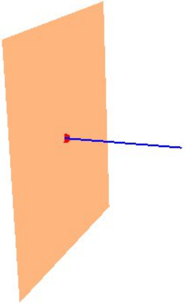
Terence Tao

"Take it with a grin of salt."

Yogi Berra

Motivation: Dynamics of multiple M5-branes

To understand M-theory, an effective description of M5-branes would be very useful.

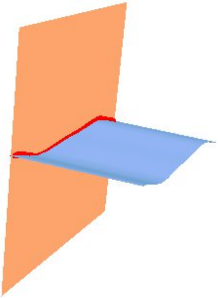


D-branes

- D-branes interact via strings.
- Effective description: theory of endpoints
- Parallel transport of these: Gauge theory
- Study string theory via gauge theory

M5-branes

- M5-branes interact via M2-branes.
- Eff. description: theory of self-dual strings
- Parallel transport: Higher gauge theory
- Long sought $(2,0)$-theory a HGT?



What we know about the (2,0)-theory

Pre-history:

- Conformal QFTs: particularly interesting and important
- Conformal algebra on $\mathbb{R}^{p,q}$: $\mathfrak{so}(p+1, q+1)$
- Supersymmetric extensions only for $p + q \leq 6$ Nahm, 1978
- Examples for $p + q \leq 4$ known for long time
- Belief: $p + q = 4$ maximum for interacting QFTs

String theory: Witten, 1995

- Type IIB superstring theory on $\mathbb{R}^{1,5} \times K_3$
- Moduli space has orbifold singularities of ADE-type
- At singularities: volume of $S^2 \hookrightarrow K_3$ vanishes
- D3-branes wrapping $S^2 \hookrightarrow K_3$ become massless strings
- B-field self-dual: self-dual strings, SUGRA decouples
- $\Rightarrow$ (2,0)-theory, a six-dimensional $\mathcal{N} = (2, 0)$ SCFT

More on the (2,0)-theory

- Also appears in M-theory — Witten, Strominger 1995/1996
  - self-dual strings: boundaries of M2- between M5-branes
  - become massless, if M5-branes approach each other
  - description of stacks of parallel M5-branes
- Field content: $\mathcal{N} = (2,0)$ tensor multiplet — Nahm 1978
  - a self-dual 3-form field strength
  - five (Goldstone) scalars
  - fermionic partners
- Observables: Wilson surfaces, i.e. parallel transport of strings
- Belief: No Lagrangian description
- As important as $\mathcal{N} = 4$ super Yang-Mills for string theory
- Huge interest in string theory: AGT, AdS_7-CFT_6, S-duality, ...
- Mathematics: Geom. Langlands, Khovanov Homology, ...

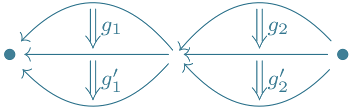
A successful M5-brane model should have the following properties:

- Contain an interacting, self-dual 2-form gauge potential
- Based on a sound mathematical foundation: higher bundles
- Field content of the $(2,0)$-theory, $\mathcal{N} = (1,0)$ supersymmetric
- Gauge structure natural, match some expectations (ADE, ...)
- Non-trivial coupling, interacting field theory
- Possible restriction to free $\mathcal{N} = (2,0)$ tensor multiplet
- contains the non-abelian self-dual string soliton as BPS state
- Reduction to 4d SYM theory with ADE gauge algebras
- and to 3d Chern–Simons-matter models with discrete coupling
- Explain S-duality after reduction to 4d
- Match expected moduli space of $(2,0)$-theory
- ...

BTW: help expanding this list appreciated!

Arguments against existence of classical M5-brane model

Non-abelian parallel transport of strings problematic:



Consistency of parallel transport requires:

$$(g_1'g_2')(g_1g_2) = (g_1'g_1)(g_2'g_2)$$

This renders group G abelian.

Eckmann and Hilton, 1962
Physicists 80'ies and 90'ies

Way out: 2-categories, Higher Gauge Theory.

Two operations $\circ$ and $\otimes$ satisfying Interchange Law:

$$(g_1' \otimes g_2') \circ (g_1 \otimes g_2) = (g_1' \circ g_1) \otimes (g_2' \circ g_2) \ .$$

Objection 2: No Coupling Constant

Standard objection beyond the previous no-go theorem:

- theory at conformal fixed points $\Rightarrow$ no dimensionful parameter
- fixed points are isolated $\Rightarrow$ no dimensionless parameter
- "No parameters $\Rightarrow$ no classical limit $\Rightarrow$ no Lagrangian."
  string theory folklore
- Furthermore: no continuous deformations of free theory
  Bekaert, Henneaux, Sevrin (1999)

Answers:

- Same arguments for M2-brane Schwarz, 2004
- There, integer parameters arose from orbifold $\mathbb{R}^8/\mathbb{Z}_k$
- Same should happen for M5-branes

Objection 3: Dimensional Reduction Unclear

Final common objection: Dimensional reduction is unclear.

- (2,0)-theory should reduce to $\mathcal{N} = 2$ SYM theory in 5d
- Reduction on $\mathbb{R}^{1,4} \times S^1$, radius R yields volume form $2\pi R \, \mathrm{d}^5 x$
- Conformal invariance of $F \wedge *F$ requires volume form $\frac{1}{R} \mathrm{d}^5 x$

Our solution:

- Reduction to $\mathcal{N} = 2$ SYM in 4d works fine
- Can dimensionally oxidize to 5d SYM afterwards (?)

Our attempt so far...

"Problems worthy of attack prove their worth by hitting back."

Piet Hein

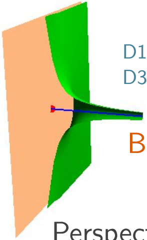
"If you ask me anything I don't know, I'm not going to answer."

Yogi Berra

First Questions: Which higher Lie algebra to take?

Guidance from BPS self-dual strings

The non-abelian self-dual string



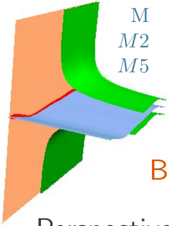
	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration

Perspective of D3:

Bogomolny monopole eqn.

$$F = \nabla^2 = *\nabla\Phi \text{ on } \mathbb{R}^3$$



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration

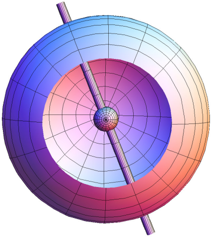
Perspective of M5:

Abelian Self-dual string eqn.

$$H := \mathrm{d}B = *\mathrm{d}\Phi \text{ on } \mathbb{R}^4$$

Monopoles

- Solution to Bogomolny equation $F = *\nabla\phi$
- Abelian: singular on $\mathbb{R}^3$, Dirac strings
- Principal bundle over S^2
- Non-Abelian: non-singular on $\mathbb{R}^3$



$$\mathsf{U}(1) \hookrightarrow \mathsf{SU}(2) \cong S^3 \xrightarrow{\pi} S^2 \qquad \mathsf{SU}(2) \hookrightarrow S^2 \times \mathsf{SU}(2) \xrightarrow{\mathrm{pr}} S^2$$

$$\rho: \mathsf{U}(1) \dashrightarrow \mathsf{SU}(2), \quad \pi\times\mathrm{id}: \mathsf{SU}(2) \cong S^3 \dashrightarrow S^2 \times \mathsf{SU}(2), \quad \mathrm{id}: S^2 \dashrightarrow S^2$$

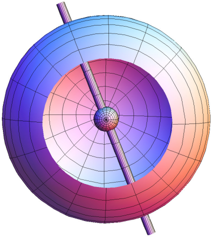
abelian, Dirac

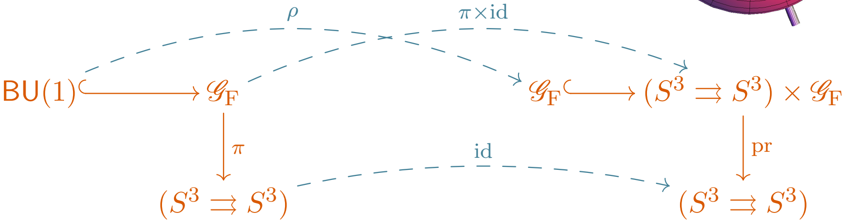
non-Abelian, ’t Hooft-Polyakov

$\Rightarrow$ Choose $\mathsf{SU}(2)$, as trivialization possible.

Self-Dual Strings

- Abelian: singular on $\mathbb{R}^4$, Dirac strings
- Solution to $H = *\nabla\phi$
- Gerbe over S^3
- Non-Abelian: ?





abelian

non-Abelian ?

$\Rightarrow$ Choose $\mathscr{G}_F$, with 2-group structure: String 2-group

(many other reasons for this)

- String 2-group $\mathscr{G}_F$ and M-theory: many reasons long story...
- $\mathscr{G}_F$ is analogue of $\mathsf{Spin}(3) \cong \mathsf{SU}(2)$ from many perspectives
- Lie differentiate (e.g. Demessie, CS (2016))
- Result:
  String Lie 2-algebra $\mathfrak{string}(3) = \left(\mathfrak{su}(2) \xleftarrow{\mu_1=0} \mathbb{R}[1]\right)$ with
  $$Q\xi^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma~, \qquad Qb = -\tfrac{1}{3!} f_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma$$
  or
  $$\mu_2(x_1,x_2) = [x_1,x_2]~, \qquad \mu_3(x_1,x_2,x_3) = (x_1,[x_2,x_3])$$
- Equivalently: (quasi-isomorphic):
  $$P_0\mathfrak{su}(2) \hookleftarrow \hat{\Omega}\mathfrak{su}(2)$$

Remarks:

- Can be defined for any ADE Lie algebra $\mathfrak{g} \to \mathfrak{string}(\mathfrak{g})$
- Can twist the Weil algebra to $\tilde{W}(\mathfrak{string}(\mathfrak{g}))$ by inv. polynomial

Higher Chern–Simons with String Lie 2-algebra

- Recall: Chevalley-Eilenberg algebra of String Lie 2-algebra $\mathfrak{g}$:

$$\mathrm{CE}(\mathfrak{g}) = \mathcal{C}^\infty(\mathbb{R}[2] \to \mathfrak{su}(2)[1]) \;,$$
$$Q\xi^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \quad \text{and} \quad Qb = \tfrac{1}{3!} f_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma \;.$$

- Double to Weil algebra:

$$\mathrm{W}(\mathfrak{g}) := \mathcal{C}^\infty(T[1]\mathfrak{g}[1]) = \mathcal{C}^\infty(\mathfrak{g}[1] \oplus \mathfrak{g}[2]) \;, \quad \sigma : \mathfrak{g}^*[1] \xrightarrow{\cong} \mathfrak{g}^*[2]$$
$$Q|_{\mathcal{C}^\infty(\mathfrak{g}[1])} = Q_{\mathrm{CE}} + \sigma \;, \quad Q_{\mathrm{CE}}\sigma = -\sigma Q_{\mathrm{CE}}$$

- Potentials/curvatures/Bianchi identities from dga-morphisms

$$(A, B, F, H) : \mathrm{W}(\mathfrak{g}) \longrightarrow \Omega^\bullet(M) = W(M)$$
$$\xi^\alpha \longmapsto A^\alpha \in \Omega^1(M) \quad \text{and} \quad b \longmapsto B \in \Omega^2(M)$$
$$(\sigma\xi^\alpha) = Q\xi^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \longmapsto F^\alpha = (\mathrm{d}A + \tfrac{1}{2}[A, A])^\alpha$$
$$(\sigma b) = Qb - \tfrac{1}{3!} f_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma \longmapsto H = \mathrm{d}B - \tfrac{1}{3!}(A, [A, A])$$

- Bianchi identities: $\nabla F = 0$ and $\mathrm{d}H = -\tfrac{1}{2}(\mathrm{d}A, [A, A])$
- Gauge trafos and Top. invariants derived as above

Adjustment for non-zero curvatures

1. Kinematical data
   - Readily from dga-morphisms $W(\mathfrak{string}(3)) \to \Omega^\bullet(\mathbb{R}^4)$
   - twist Weil algebra Sati, Schreiber, Stasheff (2009)
   - Get: string structures

$$A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}~, \quad B \in \Omega^2(\mathbb{R}^4) \otimes \mathfrak{u}(1)~,$$
$$F = \mathrm{d}A + \tfrac{1}{2}[A, A]~, \quad H = \mathrm{d}B + \tfrac{1}{2}(A, \mathrm{d}A) + \tfrac{1}{3!}(A, [A, A])~,$$
$$\nabla F = 0~, \quad \mathrm{d}H = -(F, F)$$

   - Add by hand: Higgs field $\phi \in \Omega^0(\mathbb{R}^4) \otimes \mathfrak{u}(1)$

2. Dynamical principle Schmidt, CS (2017)
   - Obvious: $H = *\mathrm{d}\phi$, implying $\mathrm{d}H = (F, F) = *\Box\phi$
   - Motivates: $F = \pm * F$
   - Full picture:
     $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, instanton + anti-instanton $c_2(F) = 0$

Conclusions so far

EOM matches story known from (1,0)-theories, so what?

- Higher analogue of $\mathsf{SU}(2) \cong \mathsf{Spin}(3)$ is $\mathsf{String}(3)$
- String structures allow for gauge invariant field equations
- Examples of truly non-abelian and non-trivial higher bundles
- Agnostic about quasi-isomorphs.: also for $P_0\mathfrak{su}(2) \hookleftarrow \hat{\Omega}\mathfrak{su}(2)$

The 6d superconformal field theory

Extension from BPS to (1,0)-theory

Look for candidate theory in the literature and find:

> 6d (1,0)-model derived from tensor hierarchies
> Samtleben, Sezgin, Wimmer (2011)

Open problems with this model:

- Issue 1: Choice of gauge structure unclear
- Issue 2: cubic interactions
- Issue 3: scalar fields with wrong sign kinetic term
- Issue 4: Self-duality of 3-form imposed by hand
- Issue 5: Unclear, how to fulfill "wishlist"

Previous observation:

- Gauge structure is Lie 3-algebra with "extra structure."
  Palmer, CS (2013), Samtleben et al. (2014)

New: Schmidt, CS (2017)

- Idea: use $\mathfrak{string}(\mathfrak{g})$ as gauge structure in this model
- Issue: need suitable notion of inner product for action
- Inner product/cyclic L_∞-algebras $\Leftrightarrow$ symplectic NQ-manifold
- Consequence: Extend $\mathfrak{string}(\mathfrak{g})$ from

$$(\mathfrak{g} \longleftarrow \mathbb{R} \xleftarrow{\text{id}} \mathbb{R}) \;\cong \mathfrak{g}$$

to symplectic graded vector space $T^*[2]\mathfrak{string}(\mathfrak{g})$:

$$\begin{array}{ccccc} \mathbb{R}^* \xleftarrow{\mu_1=\text{id}} & \mathbb{R}^*[1] & & \mathfrak{g}^*[2] & \xleftarrow{\mu_1=\text{id}} \mathfrak{g}^*[3] \\ \oplus & \oplus & & \oplus & \\ \mathfrak{g} & \mathbb{R}[1] & \xleftarrow{\mu_1=\text{id}} & \mathbb{R}[2] & \end{array}$$

- This carries natural inner product
- Has necessary extra structure for (1,0)-model

Properties of resulting (1,0)-theory

Field content:

- (1,0) tensor multiplet (ϕ, χ^i, B), values in $\mathbb{R}^2$, $\phi = \phi_s + \phi_r$, ...
- (1,0) vector multiplet (A, λ^i, Y^{ij}), values in $\mathfrak{g} \oplus \mathbb{R}$
- C-field, values in $\mathbb{R} \oplus \mathfrak{g}^*$

Action (schematically):

$$S = \int_{\mathbb{R}^{1,5}} \Big(\mathcal{H}_r \wedge *\mathcal{H}_s + \mathrm{d}\phi_r \wedge *\mathrm{d}\phi_s - *\bar{\chi}_r \not{\partial} \chi_s + \mathcal{H}_s \wedge *(\bar{\lambda}, \gamma_{(3)} \lambda) + *(Y, \bar{\lambda})\chi_s$$
$$+ \phi_s \big((\mathcal{F}, *\mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \not{\nabla} \lambda) \big) + (\bar{\lambda}, \mathcal{F}) \wedge *\gamma_{(2)} \chi_s$$
$$+ \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \Big)$$

This solves problems 1 and 2:

- Choice of gauge structure for ADE-(2,0)-theories clear.
- No cubic interaction term for scalar fields

Completing the theory

Adding Pasti-Sorokin-Tonin-type action:

- Recall: PST action has self-duality of H as equation of motion
- Bosonic part of (1,0)-theory was PST completed
  Bandos, Sorokin, Samtleben (2013)
- Full PST action announced, never appeared (not possible?)
- With string structure, construction possible and simplifies

Adding matter fields:

- Add hypermultiplet to get fields of (2,0)-tensor multiplet
- General construction and couplings discussed
  Samtleben, Sezgin, Wimmer (2012)
- Can make concrete choices with twisted string structures

$\Rightarrow$ A (1,0)-theory in 6d satisfying many of the "wishlist" items.

Dimensional reductions

Recall from wishlist:

- ...
- → Reduction to 4d SYM theory with ADE gauge algebras
- → and to 3d Chern–Simons-matter models with discrete coupling
- ...

Crucial consistency check: Reduction to D-branes/SYM theory

$$S = \int_{\mathbb{R}^{1,5}} \Big(\langle \mathcal{H}, *\mathcal{H} \rangle + \langle \mathrm{d}\phi, *\mathrm{d}\phi \rangle - *\langle \bar{\chi}, \partial\!\!\!/ \chi \rangle + \mathcal{H}_s \wedge *(\bar{\lambda}, \gamma_{(3)}\lambda) + *(Y, \bar{\lambda})\chi_s$$
$$+ \phi_s \big((\mathcal{F}, *\mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \nabla\!\!\!\!/ \lambda) \big) + (\bar{\lambda}, \mathcal{F}) \wedge *\gamma_{(2)}\chi_s$$
$$+ \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \Big)$$

- Start from ADE-String Lie 3-algebra
- Anticipate 4d gauge couplings:

$$\tau = \tau_1 + \mathrm{i}\tau_2 = \frac{\theta}{2\pi} + \frac{\mathrm{i}}{g^2_{\mathrm{YM}}} ,$$

- VEVs from compactification on T^2 along x^9 and x^{10}

$$\langle \phi_s \rangle = -\frac{1}{32\pi^2} \frac{\tau_2}{R_9 R_{10}} \quad \text{and} \quad \langle B_s \rangle = \frac{1}{16\pi^2} \frac{\tau_1}{R_9 R_{10}}$$

- Strong coupling expansion around VEVs (cf. M2 $\rightarrow$ D2)
- $\Rightarrow$ 4d $\mathcal{N} = 4$ SYM with ADE-gauge group and θ-term

Consistency check: Reduction to M2-brane models

Additional consistency check: Reduction to M2-brane models

- Replace $\mathbb{R}^{1,5}$ by $\mathbb{R}^{1,2} \times S^3$.
- Assumptions:
  - String Lie 3-algebra of $\mathfrak{su}(n) \times \mathfrak{su}(n)$
  - A trivial on S^3, non-trivial on $\mathbb{R}^{1,2}$
  - B trivial on $\mathbb{R}^{1,2}$
  - B encodes abelian gerbe with DD class k on S^3.
- Recall: $\mathcal{H} = \mathrm{d}B + \mathrm{cs}(A)$
- Then we get the integer Chern–Simons coupling:

$$\mathcal{H} \wedge *\mathcal{H} \to k \mathrm{vol}_{S^3}\ \mathrm{cs}(A)$$

$$\int_{\mathbb{R}^{1,5}} \mathcal{H} \wedge *\mathcal{H} \to k \int_{\mathbb{R}^{1,2}} \mathrm{cs}(A)$$

- Altogether: Chern–Simons matter theory of ABJM type.
- Note: This theory has $\mathcal{N} = 4$, different potential from ABJM.

Reality check

"The first law of physical mathematics: Every cloud has a silver lining."

Yuri Manin

"It ain't over till it's over."

Yogi Berra

Open Problems

Our model is not the desired (2,0)-theory!

Problems:

- Free Yang–Mills multiplet contradicts $\mathcal{N} = (2, 0)$ SUSY
- Moduli space of vacua is not that of multiple M5-branes
- S-duality unclear
- PST mechanism relies on $\phi_s > 0$
- Scalar field with wrong sign kinetic term
- Model not compatible with categorical equivalence

Turn problems into hints of solution:

- Scalar field with wrong sign kinetic term
  (rigid feature of Samtleben et al. model)
- Model not compatible with categorical equivalence
  (rigid feature of Samtleben et al. model)

Last point: the model of Samtleben et al. is too rigid:

$$(X_r)_s{}^t = f^t_{rs} + d^t_{rs} = f^t_{[rs]} + d^t_{(rs)}$$

Next steps/work in progress:

- String 2-algebra $\rightarrow$ Lie 2-algebras with right branching
- Metric twisted Weil algebras and categorical equivalence
  L Schmidt & CS, arXiv 1901.?????
- Rederive SUSY action in bigger picture

Summary:

- Higher gauge theory classically underlies M-theory
- Higher analogue of $\mathsf{SU}(2)$ is $\mathsf{String}(3)$
- There is non-abelian self-dual string
- There is classical action with many of desired features
- However: Clear differences to (2,0)-theory

Soon to come:

- ▷ Understand generalization of String Structure (WIP)
- ▷ Understand Categorical Equivalence, Higher Twists (WIP)
- ▷ Study $\mathcal{N} = (1,0)$-models (next on our list)
- ▷ Link to categorified integrability, fuzzy S^3, etc. (future)
- ▷ Better understanding of M-theory (far future)

Announcement

1 Postdoc (3 years) + 1 PhD position (3.5 years)

Mathematics of M5-branes

starting Sep/Oct 2019

More: Contact me if interested.

Higher Gauge Theories and Superconformal Field Theories in 6d

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Heriot-Watt University, Edinburgh

Workshop “String and M-Theory ...,” Singapore, 10.12.2018