Modular Invariants for Proper Actions

HAN Fei

National University of Singapore

String and M-Theory : The New Geometry of the 21st Century

The Jacobi theta functions are defined as follows :

$$\theta(v,\tau)=2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\right] ,$$

$$\theta_1(v,\tau)=2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\right] ,$$

$$\theta_2(v,\tau)=\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right] ,$$

$$\theta_3(v,\tau)=\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})\right] .$$

They are all holomorphic functions for $(v,\tau)\in\mathbb{C}\times\mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane. $q=e^{2\pi i\tau}$.

Denote $\theta_i = \theta_i(0,\tau)$. Taking

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4) = \frac{1}{4} + 6\sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d\ odd}} dq^n = \frac{1}{4} + 6q + 6q^2 + \cdots ,$$

$$\varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4 = \frac{1}{16} + \sum_{n=1}^{\infty}\sum_{d|n}(-1)^d d^3 q^n = \frac{1}{16} - q + 7q^2 + \cdots ,$$

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4) = -\frac{1}{8} - 3\sum_{n=1}^{\infty} \sum_{\substack{d|n \\ d\ odd}} dq^{n/2} = -\frac{1}{8} - 3q^{1/2} - 3q + \cdots ,$$

$$\varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4 = \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ n/d\ odd}} d^3 q^{n/2} = q^{1/2} + 8q + \cdots .$$

Geometric Definition

Example

Consider the two-variable series

$$\left[\prod_{n=1}^{\infty}(1-q^n)^{4l}\right]\cdot\left[\prod_{n=1}^{\infty}\left(1+\sum_{i=1}^{\infty}(\lambda^2q^n)^i\right)\left(1+\sum_{i=1}^{\infty}(\lambda^{-2}q^n)^i\right)\right]$$

$$\cdot\left[\sum_{j=0}^{2l-1}\prod_{s\neq j}^{(1)}\frac{1}{\left(\lambda^{\frac{|a_s-a_j|}{2}}-\lambda^{\frac{-|a_s-a_j|}{2}}\right)\prod_{n=1}^{\infty}(1-\lambda^{|a_s-a_j|}q^n)(1-\lambda^{-|a_s-a_j|}q^n)}\right].$$

As $\sum_{i=0}^{2l-1}a_i$ is even, it is not hard to see that in the above series the coefficient of each q^n is a Laurent polynomial of λ with integral coefficients. Denote the above series by

$$P(\cdots,\lambda^{-n},\lambda^{-n+1},\cdots,\lambda^{-1},1,\lambda,\cdots,\lambda^{m-1},\lambda^m,\cdots;q).$$

Geometric Definition

Example

Then the Witten genus of M is

$$\begin{aligned}&\varphi_W^c(M,\tau)\\=&P(\cdots,-|-n-1|,-|-n|,-1,0,-1,\cdots,-|m-2|,-|m-1|,\cdots;q)\\\in&\mathbb{Z}[[q]],\end{aligned}$$

i.e. the q-series obtained by replacing each λ^n with $-|n-1|$ in P.

For the elliptic genera, one has

$$\varphi_1^c(M,\tau)=0,\quad \varphi_2^c(M,\tau)=0.$$

Vanishing and Rigidity

Idea of proof : extend the "*Quantisation Commutes with Induction*" technique by Hochs to "*Quantisation Commutes with Loop Dirac Induction*"

Our proof of the vanishing of Witten genus essentially needs to establish the commutativity of the following diagram :

$$\begin{array}{ccc} K^G_\bullet(M) & \xrightarrow{\ \mathrm{index}_G(\bullet\ \otimes\Theta(T_{\mathbb{C}}M,\tau))\ } & K_\bullet(C^*_r G)[[q]] \\ \uparrow {\scriptstyle K-\mathrm{Ind}^G_K} & & \uparrow {\scriptstyle \text{D-Ind}^{LG}_{LK}} \\ K^K_\bullet(N) & \xrightarrow{\ \mathrm{index}_K(\bullet\ \otimes\Theta(T_{\mathbb{C}}N,\tau))\ } & R(K)[[q]] \end{array} ,$$

where D-Ind^{LG}_{LK} is a loop version of Dirac induction given explicitly by, $\mathrm{index}_G(D_{G/K}\otimes\Theta(\mathfrak{p}_{\mathbb{C}},\tau)\otimes\ \bullet)$.

Vanishing and Rigidity

our proof of the rigidity of the elliptic genera essentially needs to establish the commutativity of the following diagrams :

$$\begin{CD} K^G_\bullet(M) @>{\mathrm{index}_G(\bullet\ \otimes\Delta^+(TM)\oplus\Delta^-(TM))_{\mathbb{C}}\otimes\Theta_1(T_{\mathbb{C}}M,\tau))}>> K_\bullet(C^*_rG)[[q^{\frac{1}{2}}]] \\ @A{K-\mathrm{Ind}^G_K}AA @AA{\mathrm{D}_1\text{-}\mathrm{Ind}^{LG}_{LK}}A \\ K^K_\bullet(N) @>{\mathrm{index}_K(\bullet\ \otimes\Delta^+(TN)\oplus\Delta^-(TN))_{\mathbb{C}}\otimes\Theta_1(T_{\mathbb{C}}N,\tau))}>> R(K)[[q^{\frac{1}{2}}]] \end{CD}$$

and

$$\begin{CD} K^G_\bullet(M) @>{\mathrm{index}_G(\bullet\ \otimes\Theta_2(T_{\mathbb{C}}M,\tau))}>> K_\bullet(C^*_rG)[[q^{\frac{1}{2}}]] \\ @A{K-\mathrm{Ind}^G_K}AA @AA{\mathrm{D}_2\text{-}\mathrm{Ind}^{LG}_{LK}}A \\ K^K_\bullet(N) @>{\mathrm{index}_K(\bullet\ \otimes\Theta_2(T_{\mathbb{C}}N,\tau))}>> R(K)[[q^{\frac{1}{2}}]] \end{CD} \quad ,$$

where $\mathrm{D}_1\text{-}\mathrm{Ind}_{LK}^{LG}$ is a loop version of Dirac induction given explicitly by

$$\mathrm{index}_G(D_{G/K} \otimes (\Delta^+(\mathfrak{p}) \oplus \Delta^-(\mathfrak{p}))_{\mathbb{C}} \otimes \Theta_1(\mathfrak{p}_{\mathbb{C}}, \tau) \otimes \quad \bullet)$$

and $\mathrm{D}_2\text{-}\mathrm{Ind}_{LK}^{LG}$ is a loop version of Dirac induction given explicitly by

$$\mathrm{index}_G(D_{G/K} \otimes \Theta_2(\mathfrak{p}_{\mathbb{C}}, \tau) \otimes \quad \bullet).$$

Thank you very much!