Supergeometry analysis of geometric structure of double field theory

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DFT in supermanifold formulation and group manifold as background geometry,
U. Carow-Watamura, NI, T. Kaneko and S. Watamura,
arXiv:1812.03464. etc.

§1. Introduction

Purposes

• Understanding and clarifying general theory and formulas of geometry of DFT, section conditions, generalized Bianchi identities, etc.

• Analyzing T-duality

• Can we obtain a simple method to compute complicated T-duality equations?

$\rightarrow$ We use super symplectic geometry (topological field theory, BRST-BV formalism)

Plan of Talk

2. Double field theory

3. Supergeometry (graded manifold and pre-QP structure)

4. Generalized fluxes and generalized Bianchi identity

4. Generalized Scherk-Schwarz compactification

5. $GL(2D)$ covariant DFT

§2. Double field theory

Siegel '93, Hull-Zwiebach '09

Let M be an original D-dimensional manifold and $\widetilde{M}$ be a T-dualized manifold.

We first construct a T-duality invariant theory on 2D-dimensional doubled space $\widehat{M}$, and project the theory to physical spacetime to $pr : \widehat{M} \to M$ and $\widetilde{pr} : \widehat{M} \to \widetilde{M}$.

$X^{\hat{M}} = (\tilde{X}_M, X^M)$: coordinates of this doubled space

hat index: 2D dimensional indices,

unhat index: D dimensional indices

$M, N, \cdots$: spacetime indices,

$A, B, \cdots$: tangent flat space indices

We assume $O(D,D)$ an invariant tensor $\eta_{\hat{M}\hat{N}}$.

Generalized Lie derivative and section condition (closure condition)

The generalized Lie derivative a generalized vector $V^{\hat{M}}$ is defined as

$$\mathcal{L}_\Lambda V^{\hat{M}} = \Lambda^{\hat{N}}\partial_{\hat{N}}V^{\hat{M}} + (\eta^{\hat{M}\hat{P}}\eta_{\hat{N}\hat{Q}}\partial_{\hat{P}}\Lambda^{\hat{Q}} - \partial_{\hat{N}}\Lambda^{\hat{M}})V^{\hat{N}},$$

where $\Lambda^{\hat{M}}$ is a gauge parameter.

$\mathcal{L}_\Lambda$ does not satisfy the Leibniz rule,

$$\Delta^M(\Lambda_1,\Lambda_2,V) = \mathcal{L}_{\Lambda_1}(\mathcal{L}_{\Lambda_2}V^M) - \mathcal{L}_{\mathcal{L}_{\Lambda_1}\Lambda_2}V^M - \mathcal{L}_{\Lambda_2}\mathcal{L}_{\Lambda_1}V^M \neq 0$$

Vanishing of $\Delta^M(\Lambda_1, \Lambda_2, V)$ is also called the **closure condition** (the strong section condition), which means that the generalized Lie derivative satisfies

$$[\mathcal{L}_{\Lambda_1}, \mathcal{L}_{\Lambda_2}] = \mathcal{L}_{\mathcal{L}_{\Lambda_1}\Lambda_2}.$$

Closure is always guaranteed, when the section condition is imposed

$$\eta^{\hat{M}\hat{N}}(\partial_{\hat{M}}\Phi)(\partial_{\hat{N}}\Psi) = 0,$$

where Φ and Ψ denote any fields and gauge parameters of DFT.

Generalized metric and generalized vielbein

$\mathcal{H}_{\hat{M}\hat{N}}$: a generalized metric,

$$\mathcal{H}_{\hat{M}\hat{N}} = \begin{pmatrix} g^{MN} & -g^{MP}b_{PN} \\ b_{MP}g^{PN} & g_{MN} - b_{MP}g^{PQ}b_{QN} \end{pmatrix}.$$

$E_{\hat{\hat{A}}}{}^{\hat{M}}$: we introduce the generalized vielbein.

$$E_{\hat{\hat{A}}}{}^{\hat{M}} = \begin{pmatrix} E_A{}^M & E_{BM} \\ E^{AN} & E^B{}_N \end{pmatrix} = \begin{pmatrix} e_A{}^M & e_B{}^L B_{LM} \\ e^A{}_L \beta^{LN} & e^B{}_N + e^B{}_L B_{NK} \beta^{KL} \end{pmatrix}.$$

$\eta^{\hat{\hat{A}}\hat{\hat{B}}}$: the $O(D,D)$ invariant metric.
$S_{\hat{\hat{A}}\hat{\hat{B}}}$: an $O(1,D-1)\times O(1,D-1)$ invariant double Lorentz metric. The $O(D,D)$

metric $\eta_{\hat{M}\hat{N}}$ and the generalized metric $\mathcal{H}_{\hat{M}\hat{N}}$ are written as

$$\eta_{\hat{M}\hat{N}} = E^{\hat{\bar{A}}}{}_{\hat{M}} \eta_{\hat{\bar{A}}\hat{\bar{B}}} E^{\hat{\bar{B}}}{}_{\hat{N}} \, , \qquad \mathcal{H}_{\hat{M}\hat{N}} = E^{\hat{\bar{A}}}{}_{\hat{M}} S_{\hat{\bar{A}}\hat{\bar{B}}} E^{\hat{\bar{B}}}{}_{\hat{N}} \, .$$

The generalized Lie derivative is

$$\mathcal{L}_\Lambda E^{\hat{M}}_{\bar{A}} = \Lambda^{\hat{N}} \partial_{\hat{N}} E^{\hat{M}}_{\bar{A}} + (\eta^{\hat{M}\hat{P}} \eta_{\hat{N}\hat{Q}} \partial_{\hat{P}} \Lambda^{\hat{Q}} - \partial_{\hat{N}} \Lambda^{\hat{M}}) E^{\hat{N}}_{\bar{A}} .$$

§3. Supergeometry of double field theory

Deser, Stasheff, '14, Deser, Saemann '16, Heller, NI, Watamura, '16

Graded manifold

A **graded manifold** $\mathcal{M} = (M, \mathcal{O}_M)$ on a smooth manifold M is a ringed space which structure sheaf $\mathcal{O}_M$ is $\boldsymbol{Z}$–graded commutative algebras over M, locally isomorphic to $C^\infty(U) \otimes S^\cdot(V)$, where U is a local chart on M, V is a graded vector space and $S^\cdot(V)$ is a free graded commutative ring on V.

Grading is called **degree**.

We denote $C^\infty(\mathcal{M}) = \mathcal{O}_M$.

If degrees are nonnegative, a graded manifold is called an **N-manifold**.

pre-QP-manifold

An N-manifold is called a pre-QP-manifold if it has the following structure.

• ω: a graded symplectic form of degree n on $\mathcal{M}$ and the induced (nondegenerate) Poisson bracket $\{-,-\}$.

• Q: a graded vector field of degree $+1$, satisfying $\mathcal{L}_Q\omega = 0$.

We take a Hamiltonian function $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$ such that $Q(-) = \{\Theta, -\}$.

Note: If $Q^2 = 0$, a pre-QP-manifold is called a **QP-manifold**. $Q^2 = 0$ is equivalent to the classical master equation, $\{\Theta, \Theta\} = 0$.

Note: Θ corresponds to a BRST charge (an AKSZ sigma model).

Example of QP-manifold

Derived bracket construction of Courant algebroid

Roytenberg '99

Let M be a smooth manifold. we consider a graded double cotangent bundle, $\mathcal{M} = T^*[2]T^*[1]M$.

(x^i, p_i): local coordinates of degree $(0, 1)$, on $T^*[1]M$.
(ξ_i, q^i): canonical conjugate coordinates of degree $(2, 1)$ on $T^*[2]$.

This means that the symplectic form is of degree 2,

$$\omega = \delta x^i \wedge \delta \xi_i + \delta q^i \wedge \delta p_i.$$

We consider a Hamiltonian function Θ of degree 3. The simplest Hamiltonian function is

$$\Theta_0 = \xi_i q^i,$$

which trivially satisfies the classical master equation $\{\Theta_0, \Theta_0\} = 0$.

A degree 1 function is $X^i(x)p_i + \alpha_i(x)q^i$, is identified to $X + \alpha = X^i(x)\partial_i + \alpha_i(x)dx^i \in \Gamma(TM \oplus T^*M)$ by the degree shifting map,

$$j : TM \oplus T^*M \to T^*[2]T^*[1]M,$$

defined by $j : (x^i, \partial_i, dx^i) \mapsto (x^i, p_i, q^i)$.

The derived bracket for degree 0 and 1 functions $\{\{-, \Theta_0\}, -\}$ gives operations of a Courant algebroid.

The Dorfman bracket for two generalized vector fields, $X + \alpha$ and $Y + \beta$, is

$$\begin{aligned}[X + \alpha, Y + \beta]_D =& - \{\{X + \alpha, \Theta_0\}, Y + \beta\} \\ =& [X, Y] + \mathcal{L}_X \beta - \iota_Y \alpha,\end{aligned}$$

The anchor map is $\rho(X+\alpha)f = -\{\{X+\alpha, \Theta_0\}, f\} = Xf$.

All the identities of a Courant algebroid are given by the classical master equation $\{\Theta_0, \Theta_0\} = 0$, i.e. $Q^2 = 0$.

Derived bracket construction of generalized Lie derivative

Take 2D dimensional doubled spacetime $\widehat{M}$ with a local coordinate $X^{\hat{M}} = (\tilde{x}_M, x^M)$.

We take a pre-QP-manifold $(\mathcal{M} = T^*[2]T[1]\widehat{M}, \omega, Q)$. Here $Q = \{\Theta, -\}$.

A generalized Lie derivative is defined by a **derived bracket**,

$$\mathcal{L}_V V' = [V, V']_D = [V, V'] \equiv -\{\{V, \Theta\}, V'\},$$

for generalized vector fields V, V', which are functions of degree 1.

Closure condition

In general, $\{\Theta, \Theta\} \neq 0$ on a pre-QP-manifold.

We obtain the following identity of the derived bracket for any $f, g, h \in C^\infty(\mathcal{M})$ using identities of $\{-,-\}$,

$$\begin{aligned}[f,[g,h]] =&\{\{f,\Theta\},\{\{g,\Theta\},h\}\} \\ =&[[f,g],h] + (-1)^{(|f|+n+1)(|g|+n+1)}[g,[f,h]] \\ &+ (-1)^{|g|+n}\frac{1}{2}\{\{\{\{\Theta,\Theta\},f\},g\},h\}.\end{aligned}$$

Case 1, If $\{\Theta, \Theta\} = 0$, the derived bracket $[\cdot, \cdot]$ satisfies the following Leibniz identity of degree $-n+1$,

$$[f, [g, h]] = [[f, g], h] + (-1)^{(|f|-n+1)(|g|-n+1)}[g, [f, h]].$$

$[-, -] = \{\{-, \Theta\}, -\}$: The Dorfman bracket of a Courant algebroid.

Case 2, We can relax the classical master equation as

$$\{\{\{\{\Theta, \Theta\}, f\}, g\}, h\} = 0,$$

which is sufficient for closure of the derived bracket. We call the condition the weak master equation. It is the DFT case!

Deser-Saemann, Bruce-Grabowski

Generalized Lie derivative in DFT in local coordinates

We take a $2D$ dimensional doubled space, $\widehat{M} = \widetilde{M} \times M$,with an $O(D,D)$ invariant metric $\eta_{\hat{M}\hat{N}}$,

Consider an $n = 2$ graded symplectic manifold $\mathcal{M} = T^*[2]T[1](\widetilde{M} \times M)$.

$X^{\hat{M}} = (\tilde{X}_M, X^M)$ is a general coordinate on the base manifold $\widetilde{M} \times M$.

$(X^{\hat{M}}, Q^{\hat{M}}, P_{\hat{M}}, \Xi_{\hat{M}})$: local coordinates on $\mathcal{M}$ of degree $(0,1,1,2)$.

The symplectic structure on $\mathcal{M}$ is

$$\omega = \delta X^{\hat{M}} \wedge \delta \Xi_{\hat{M}} + \delta Q^{\hat{M}} \wedge \delta P_{\hat{M}}.$$

DFT basis

$$Q'^{\hat{M}} := \frac{1}{\sqrt{2}}(Q^{\hat{M}} - \eta^{\hat{M}\hat{N}} P_{\hat{N}}) \quad , \quad P'_{\hat{M}} := \frac{1}{\sqrt{2}}(P_{\hat{M}} + \eta_{\hat{M}\hat{N}} Q^{\hat{N}}),$$

In the DFT basis, Poisson brackets are

$$\{Q'^{\hat{M}}, Q'^{\hat{N}}\} = \eta^{\hat{M}\hat{N}}, \quad \{P'_{\hat{M}}, P'_{\hat{N}}\} = \eta_{\hat{M}\hat{N}}, \quad \{Q'^{\hat{M}}, P'_{\hat{N}}\} = 0.$$

We identify geometric elements and supermanifold elements as follows,

$$j' : \left(X^{\hat{M}}, \partial_{\hat{M}}, \partial_{\hat{M}}, dX^{\hat{M}}\right) \longmapsto (X^{\hat{M}}, \Xi_{\hat{M}}, P'_{\hat{M}}, Q'^{\hat{M}}),$$

with degree shifting. Especially,

$$V^{\hat{M}} \partial_{\hat{M}} \sim V^{\hat{M}} P'_{\hat{M}},$$

Simplest Hamiltonian function

We consider the following $O(D,D)$ invariant degree 3 function,

$$\Theta_0 = \eta^{\hat{M}\hat{N}} \Xi_{\hat{M}} P'_{\hat{N}},$$

which consists only of the coordinate P'_M of the DFT basis.

A derived bracket using this Θ_0 gives the generalized Lie derivative on a generalized vector field V,

$$\begin{aligned} \mathcal{L}_\Lambda V = [\Lambda, V]_D = & - \{\{\Lambda, \Theta_0\}, V\} \\ = & \Lambda^{\hat{N}} \partial_{\hat{N}} V^{\hat{M}} + (\eta^{\hat{M}\hat{P}} \eta_{\hat{N}\hat{Q}} \partial_{\hat{P}} \Lambda^{\hat{Q}} - \partial_{\hat{N}} \Lambda^{\hat{M}}) V^{\hat{N}}. \end{aligned}$$

Closure condition

The classical master equation is not satisfied,

$$\{\Theta_0, \Theta_0\} = \eta^{\hat{M}\hat{N}} \Xi_{\hat{M}} \Xi_{\hat{N}} \neq 0.$$

We impose the closure condition, $\{\{\{\{\Theta_0, \Theta_0\}, f\}, g\}, h\} = 0$, which is

$$2(\partial^{\hat{M}} V_1^{\hat{N}} V_{2\hat{N}} \partial_{\hat{M}} V_3^{\hat{Q}} - 2\partial^{\hat{M}} V_1^{[\hat{P}} \partial_{\hat{M}} V_2^{\hat{Q}]} V_{3\hat{P}}) P'_{\hat{Q}} = 0.$$

This condition is rewritten as the section condition,

$$\partial^{\hat{M}} V_1^{\hat{P}} \partial_{\hat{M}} V_2^{\hat{Q}} = 0.$$

A similar condition is obtained for functions on the doubled spacetime.

§4. Twist and generalized fluxes

We introduce **fluxes** in DFT by a canonical transformation called twist.

Twist

$$e^{\delta_\alpha} f = f + \{f, \alpha\} + \frac{1}{2}\{\{f, \alpha\}, \alpha\} + \cdots ,$$

for $f \in C^\infty(\mathcal{M})$. Here α is a local function of **degree** 2, corresponding to a gerbe connection (a stack of groupoids). It is degree-preserving and obeys

$$\{e^{\delta_\alpha} f, e^{\delta_\alpha} g\} = e^{\delta_\alpha}\{f, g\},$$

for all $f, g \in C^\infty(\mathcal{M})$,

Note

If a Hamiltonian function Θ is twisted by α, $\Theta \to \Theta' = e^{\delta_\alpha}\Theta$, then a twist changes the closure condition.

$$\{\{\{\{\Theta', \Theta'\}, f\}, g\}, h\} = 0,$$

which is equivalent to

$$e^{\delta_\alpha}\{\{\{\{\Theta, \Theta\}, e^{-\delta_\alpha} f\}, e^{-\delta_\alpha} g\}, e^{-\delta_\alpha} h\} = 0.$$

• A twist does not change a D-dimensional physical spacetime $M \subset \widetilde{M}$.

• A twist introduces 'connection' terms to the section condition for a generalized vector field.

Local Lorentz frame

$\bar{Q}^{\hat{\bar{A}}}, \bar{P}_{\hat{\bar{A}}}$: flat tangent and cotangent coordinates of degree 1 corresponding to the local Lorentz frame. The DFT basis is

$$\bar{Q}'^{\hat{\bar{A}}} := \frac{1}{\sqrt{2}}(\bar{Q}^{\hat{\bar{A}}} - \eta^{\hat{\bar{A}}\hat{\bar{B}}}\bar{P}_{\hat{\bar{B}}}) \quad , \quad \bar{P}'_{\hat{\bar{A}}} := \frac{1}{\sqrt{2}}(\bar{P}_{\hat{\bar{A}}} + \eta_{\hat{\bar{A}}\hat{\bar{B}}}\bar{Q}'^{\hat{\bar{B}}})$$

Twists in DFT

DFT has the following three twists,

$$E := E_{\hat{\bar{A}}}{}^{\hat{M}}(X)\eta^{\hat{\bar{A}}\hat{\bar{B}}}P'_{\hat{M}}\bar{P}'_{\hat{\bar{B}}},$$
$$u := u_{\hat{P}}{}^{\hat{M}}(X)\eta^{\hat{N}\hat{P}}P'_{\hat{M}}P'_{\hat{N}}, \quad \bar{u} := \bar{u}_{\hat{\bar{A}}}{}^{\hat{\bar{B}}}(X)\eta^{\hat{\bar{C}}\hat{\bar{A}}}\bar{P}'_{\hat{\bar{B}}}\bar{P}'_{\hat{\bar{C}}}.$$

We have the following formulas of twists,

$$e^{\frac{\pi}{2}\delta_E} P'_{\hat{M}} = E^{\hat{\bar{A}}}{}_{\hat{M}} \bar{P}'_{\hat{\bar{A}}}, \quad e^{\frac{\pi}{2}\delta_E} \bar{P}'_{\hat{\bar{A}}} = -E_{\hat{\bar{A}}}{}^{\hat{M}} P'_{\hat{M}},$$

$$e^{\frac{\pi}{2}\delta_E} \Xi_{\hat{M}} = \Xi_{\hat{M}} - \frac{1}{2}\Omega_{\hat{M}\hat{N}\hat{P}} P'^{\hat{N}} P'^{\hat{P}} + \frac{1}{2}\Omega_{\hat{M}\hat{N}\hat{P}} E_{\hat{\bar{A}}}{}^{\hat{N}} E_{\hat{\bar{C}}}{}^{\hat{P}} \bar{P}'^{\hat{\bar{A}}} \bar{P}'^{\hat{\bar{C}}}.$$

where $\Omega_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} := E_{\hat{\bar{A}}}{}^{\hat{M}} \partial_{\hat{M}} E_{\hat{\bar{B}}}{}^{\hat{N}} E_{\hat{\bar{C}}\hat{N}}$ is a generalized Weitzenböck connection, and $\Omega_{\hat{M}\hat{N}\hat{P}} = E^{\hat{\bar{A}}}{}_{\hat{M}} E^{\hat{\bar{B}}}{}_{\hat{N}} E^{\hat{\bar{C}}}{}_{\hat{P}} \Omega_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}$.

Then, the twisted Hamiltonian function becomes,

$$\Theta_F = e^{\frac{\pi}{2}\delta_E}\Theta_0 = E_{\hat{\bar{A}}}{}^{\hat{M}} \Xi_{\hat{M}} \bar{P}'^{\hat{\bar{A}}} + \frac{1}{3!}\mathcal{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} \bar{P}'^{\hat{\bar{A}}} \bar{P}'^{\hat{\bar{B}}} \bar{P}'^{\hat{\bar{C}}} + \frac{1}{2}\Phi_{\hat{\bar{C}}\hat{M}\hat{N}} P'^{\hat{M}} P'^{\hat{N}} \bar{P}'^{\hat{\bar{C}}}$$

where

$$\mathcal{F}_{\hat{\hat{A}}\hat{\hat{B}}\hat{\hat{C}}} = 3\Omega_{[\hat{\hat{A}}\hat{\hat{B}}\hat{\hat{C}}]}, \qquad \Phi_{\hat{\hat{C}}\hat{M}\hat{N}} = -\Omega_{\hat{\hat{C}}\hat{\hat{A}}\hat{\hat{B}}} E^{\hat{\hat{A}}}{}_{\hat{M}} E^{\hat{\hat{B}}}{}_{\hat{N}}$$

We obtained the correct forms of a generalized flux and a generalized Weitzenböck connection. Aldazabal, Baron, Marques, Nunez, '11

§5. Generalized Bianchi identity via pre-QP-manifold

Bianchi identity of fluxes in SUGRA

In a QP-manifold (SUGRA), the Bianchi identity of fluxes is equivalent to the classical master equation $\{\Theta, \Theta\} = 0$ for a Hamiltonian twisted by fluxes Θ.

A general form of Θ on a D-dimensional manifold M is of degree 3 and identifications of fluxes is,

$$\Theta = \rho^M{}_N(x)\xi_M q^N + \pi^{MN}(x)\xi_M p_N + \frac{1}{3!}H_{MNP}(x)q^L q^M q^N + \frac{1}{2}F^N_{LM}(x)q^L q^M p_N + \frac{1}{2}Q^{MN}_L(x)q^L p_M p_N + \frac{1}{3!}R^{LMN}(x)p_L p_M p_N.$$

1. Original Neveu-Schwarz H-flux

$H = dB$, $F = 0$, $Q = 0$, $R = 0$.

$$\Theta_1 = e^{\delta_B}\Theta_0 = \xi_M q^M + \frac{1}{3!}H_{LMN}(x)q^L q^M q^N,$$

where $B = \frac{1}{2}B_{MN}(x)q^M q^N$.

$\{\Theta_1, \Theta_1\} = 0$ is equivalent to $dH = 0$.

2. Fluxes with metric

Blumenhagen-Deser-Plauschinn-Rennecke '12

$$
\begin{aligned}
H &= \nabla B \\
F &= T + \beta^{\sharp} H \\
Q &= \nabla \beta + \wedge^2 \beta^{\sharp} H, \\
R &= [\beta, \beta]_S^{\nabla} + \wedge^3 \beta^{\sharp} H,
\end{aligned}
$$

where ∇ is a covariant derivative with respect to the Riemannian connection and T is a torsion tensor. Four fluxes satisfy complicated Bianchi identity.

Corresponding Hamiltonian function

Let

$$B = \frac{1}{2} B_{MN}(x) q^M q^N, \quad \beta = \frac{1}{2} \beta^{MN}(x) p_M p_N,$$
$$e = e_A{}^M(x) q^A p_M, \quad e^{-1} = e^A{}_M(x) q^M p_A.$$

and consider twist $\Theta_2 = e^{-\delta_e} e^{\delta_{e^{-1}}} e^{-\delta_e} e^{-\delta_\beta} \Theta_1$.

From Θ_2, we obtain forms H, F, Q, R in the previous page, and

$$\{\Theta_2, \Theta_2\} = 0,$$

gives the correct Bianchi identity of H, F, Q, R.

Heller, NI, Watamura '16

Generalized Bianchi identity of generalized fluxes in DFT

The Hamiltonian function with generalized fluxes is

$$\Theta_F = E_{\hat{\hat{A}}}{}^{\hat{M}} \Xi_{\hat{M}} \bar{P}'^{\hat{\hat{A}}} + \frac{1}{3!} \mathcal{F}_{\hat{\hat{A}}\hat{\hat{B}}\hat{\hat{C}}} \bar{P}'^{\hat{\hat{A}}} \bar{P}'^{\hat{\hat{B}}} \bar{P}'^{\hat{\hat{C}}} + \frac{1}{2} \Phi_{\hat{\hat{C}}\hat{M}\hat{N}} P'^{\hat{M}} P'^{\hat{N}} \bar{P}'^{\hat{\hat{C}}}$$

pre-Bianchi identity

Carow-Watamura, NI, Kaneko and Watamura, '18

In a pre-QP-manifold, $\{\Theta, \Theta\} \neq 0$. Then, we propose a weak version of the classical master equation

$$\mathcal{B}(\Theta_F, \Theta_0, \alpha) = \{\Theta_F, \Theta_F\} - e^{\delta_\alpha} \{\Theta_0, \Theta_0\} = 0.$$

where α is a canonical transformation function of degree 2, and Θ_0 is a Hamiltonian function without fluxes. A generalized Bianchi identity is derived from this equation.

We choose a twist by $\alpha = E = E_{\hat{\bar{A}}}{}^{\hat{M}}\eta^{\hat{\bar{A}}\hat{\bar{B}}}P'_{\hat{M}}\bar{P}'_{\hat{\bar{B}}}$, we obtain

$$
\begin{aligned}
&\mathcal{B}(\Theta_F, \Theta_0, E) \\
=&(2\partial_{\hat{N}}E_{\hat{\bar{C}}}{}^{\hat{M}}E_{\hat{\bar{D}}}{}^{\hat{N}} + \eta^{\hat{\bar{A}}\hat{\bar{B}}}E_{\hat{\bar{A}}}{}^{\hat{M}}\mathcal{F}_{\hat{\bar{B}}\hat{\bar{C}}\hat{\bar{D}}} - \eta^{\hat{M}\hat{N}}\Omega_{\hat{N}\hat{Q}\hat{U}}E_{\hat{\bar{C}}}{}^{\hat{Q}}E_{\hat{\bar{D}}}{}^{\hat{U}})\Xi_{\hat{M}}\bar{P}'^{\hat{\bar{C}}}\bar{P}'^{\hat{\bar{D}}} \\
&+ (\eta^{\hat{\bar{A}}\hat{\bar{B}}}E_{\hat{\bar{A}}}{}^{\hat{M}}\Phi_{\hat{\bar{B}}\hat{N}\hat{P}} + \eta^{\hat{M}\hat{Q}}\Omega_{\hat{Q}\hat{N}\hat{P}})\Xi_{\hat{M}}P'^{\hat{N}}P'^{\hat{P}} \\
&+ \Big(-\frac{2}{3!}E_{\hat{\bar{A}}}{}^{\hat{M}}\partial_{\hat{M}}\mathcal{F}_{\hat{\bar{B}}\hat{\bar{C}}\hat{\bar{D}}} + \frac{3}{4}\eta^{\hat{\bar{E}}\hat{\bar{F}}}\mathcal{F}_{\hat{\bar{E}}\hat{\bar{A}}\hat{\bar{B}}}\mathcal{F}_{\hat{\bar{F}}\hat{\bar{C}}\hat{\bar{D}}} - \frac{1}{4}\eta^{\hat{\bar{E}}\hat{\bar{F}}}\Omega_{\hat{\bar{E}}\hat{\bar{A}}\hat{\bar{B}}}\Omega_{\hat{\bar{F}}\hat{\bar{C}}\hat{\bar{D}}}\Big)\bar{P}'^{\hat{\bar{A}}}\bar{P}'^{\hat{\bar{B}}}\bar{P}'^{\hat{\bar{C}}}\bar{P}'^{\hat{\bar{D}}} \\
&+ \Big(-E_{\hat{\bar{A}}}{}^{\hat{P}}\partial_{\hat{P}}\Phi_{\hat{\bar{B}}\hat{M}\hat{N}} + \frac{1}{2}\eta^{\hat{\bar{C}}\hat{\bar{D}}}\mathcal{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}\Phi_{\hat{\bar{D}}\hat{M}\hat{N}} \\
&- \eta^{\hat{Q}\hat{R}}\Phi_{\hat{\bar{A}}\hat{Q}\hat{M}}\Phi_{\hat{\bar{B}}\hat{R}\hat{N}} + \frac{1}{2}\eta^{\hat{P}\hat{R}}\Omega_{\hat{P}\hat{M}\hat{N}}\Omega_{\hat{R}\hat{Q}\hat{U}}E_{\hat{\bar{A}}}{}^{\hat{Q}}E_{\hat{\bar{B}}}{}^{\hat{U}}\Big)P'^{\hat{M}}P'^{\hat{N}}\bar{P}'^{\hat{\bar{A}}}\bar{P}'^{\hat{\bar{B}}} \\
&+ \frac{1}{4}(\eta^{\hat{R}\hat{S}}\Phi_{\hat{R}\hat{M}\hat{N}}\Phi_{\hat{S}\hat{P}\hat{Q}} - \eta^{\hat{R}\hat{S}}\Omega_{\hat{R}\hat{M}\hat{N}}\Omega_{\hat{S}\hat{P}\hat{Q}})P'^{\hat{M}}P'^{\hat{N}}P'^{\hat{P}}P'^{\hat{Q}}.
\end{aligned}
$$

The pre-Bianchi identity is

$$2\partial_{\hat{N}} E_{[\hat{\hat{C}}}{}^{\hat{M}} E_{\hat{\hat{D}}]}{}^{\hat{N}} + E^{\hat{\hat{A}}\hat{M}} \mathcal{F}_{\hat{\hat{B}}\hat{\hat{C}}\hat{\hat{D}}} - \Omega^{\hat{M}}{}_{\hat{Q}\hat{U}} E_{\hat{\hat{C}}}{}^{\hat{Q}} E_{\hat{\hat{D}}}{}^{\hat{U}} = 0,$$

$$E^{\hat{\hat{A}}\hat{M}} \Phi_{\hat{\hat{A}}\hat{N}\hat{P}} + \Omega^{\hat{M}}{}_{\hat{N}\hat{P}} = 0,$$

$$-\frac{2}{3!} E_{[\hat{\hat{A}}}{}^{\hat{M}} \partial_{\hat{M}} \mathcal{F}_{\hat{\hat{B}}\hat{\hat{C}}\hat{\hat{D}}]} + \frac{3}{4} \mathcal{F}_{\hat{\hat{E}}[\hat{\hat{A}}\hat{\hat{B}}} \mathcal{F}^{\hat{\hat{F}}}{}_{\hat{\hat{C}}\hat{\hat{D}}]} - \frac{1}{4} \Omega_{\hat{\hat{E}}[\hat{\hat{A}}\hat{\hat{B}}} \Omega^{\hat{\hat{E}}}{}_{\hat{\hat{C}}\hat{\hat{D}}]} = 0,$$

$$-E_{[\hat{\hat{A}}}{}^{\hat{P}} \partial_{\hat{P}} \Phi_{\hat{\hat{B}}]\hat{M}\hat{N}} + \frac{1}{2} \mathcal{F}^{\hat{\hat{C}}}{}_{\hat{\hat{A}}\hat{\hat{B}}} \Phi_{\hat{\hat{C}}\hat{M}\hat{N}} - \Phi_{[\hat{\hat{A}}[\hat{M}}{}^{\hat{Q}} \Phi_{\hat{\hat{B}}]\hat{N}]\hat{Q}} + \frac{1}{2} \Omega^{\hat{P}}{}_{\hat{M}\hat{N}} \Omega_{\hat{P}\hat{Q}\hat{U}} E_{\hat{\hat{A}}}{}^{\hat{Q}} E_{\hat{\hat{B}}}{}^{\hat{U}} = 0,$$

$$\Phi_{\hat{R}[\hat{M}\hat{N}} \Phi^{\hat{R}}{}_{\hat{P}\hat{Q}]} - \Omega_{\hat{R}[\hat{M}\hat{N}} \Omega^{\hat{R}}{}_{\hat{P}\hat{Q}]} = 0.$$

1st and 2nd: local expressions of $\mathcal{F}_{\hat{\hat{A}}\hat{\hat{B}}\hat{\hat{C}}}$ and $\Phi_{\hat{\hat{A}}\hat{N}\hat{P}}$.
3rd: the generalized Bianchi identity in DFT in Aldazabal, Marques, Nunez, '13, Geissbühler, Marques, Nunez, Penas, '13.
4th: another generalized Bianchi identity for $\Phi_{\hat{\hat{A}}\hat{M}\hat{N}}$.
5th: trivially satisfied.

General form

The most general degree 3 Hamiltonian which consist of $(X^{\hat{M}}, \Xi_{\hat{M}}, P'^{\hat{M}}, \bar{P}'^{\hat{\bar{C}}})$.

$$
\begin{aligned}
\Theta_F =& \bar{\rho}_{\hat{\bar{A}}}{}^{\hat{M}}(X)\Xi_{\hat{M}}\bar{P}'^{\hat{\bar{A}}} + \rho_{\hat{N}}{}^{\hat{M}}(X)\Xi_{\hat{M}}P'^{\hat{N}} + \frac{1}{3!}\mathcal{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}(X)\bar{P}'^{\hat{\bar{A}}}\bar{P}'^{\hat{\bar{B}}}\bar{P}'^{\hat{\bar{C}}} \\
&+ \frac{1}{2}\Phi_{\hat{\bar{C}}\hat{M}\hat{N}}(X)P'^{\hat{M}}P'^{\hat{N}}\bar{P}'^{\hat{\bar{C}}} \\
&+ \frac{1}{2}\Delta_{\hat{\bar{A}}\hat{\bar{B}}\hat{M}}(X)P'^{\hat{M}}\bar{P}'^{\hat{\bar{A}}}\bar{P}'^{\hat{\bar{B}}} + \frac{1}{3!}\Psi_{\hat{M}\hat{N}\hat{P}}(X)P'^{\hat{M}}P'^{\hat{N}}P'^{\hat{P}}, \\
\Theta_0 =& \eta^{\hat{M}\hat{N}}\Xi_{\hat{M}}P'_{\hat{N}}.
\end{aligned}
$$

We obtain more general generalized Bianchi identity.

§6. Generalized Scherk-Schwarz twist as supergeometry twist

We apply our method to a concrete application, which is a generalized Scherk-Schwarz (GSS) compactification.

Generalized Scherk-Schwarz (GSS) compactification Aldazabal, Baron, Marques, Nunez, '11, Grana, Marques, '12, Berman, Lee, '13

The $2D$-dimensional target space splits into $2d$-dimensional external space and $2(D-d)$-dimensional internal space, $X = (\mathbb{X}, \mathbb{Y})$.

GSS ansatz of splits for each field are

$$\mathcal{E}^{\hat{\hat{A}}}{}_{\hat{M}}(X) = \widehat{E}^{\hat{\hat{A}}}{}_{\hat{I}}(\mathbb{X}) U^{\hat{I}}{}_{\hat{M}}(\mathbb{Y}), \qquad \Lambda^{\hat{M}}(X) = \widehat{\Lambda}^{\hat{I}}(\mathbb{X}) U_{\hat{I}}{}^{\hat{M}}(\mathbb{Y}).$$

We use the characters $\hat{I}, \hat{J}, \hat{K}, \hat{L}$ and $\hat{H}$ for the indices of an intermediate theory with an $O(D,D)$ metric $\eta^{\hat{I}\hat{J}}$.

The matrix $U_{\hat{I}}{}^{\hat{M}}(\mathbb{Y})$ and its inverse $U^{\hat{I}}{}_{\hat{M}}(\mathbb{Y})$ are elements of $O(D,D)$, which give the GSS twist.

We obtain a GSS generalized fluxes:

$$\mathcal{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} = \widehat{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} + f_{\hat{I}\hat{J}\hat{K}} \widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}} \widehat{E}_{\hat{\bar{B}}}{}^{\hat{J}} \widehat{E}_{\hat{\bar{C}}}{}^{\hat{K}},$$

where $\widehat{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} = 3\widehat{\Omega}_{[\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}]} = 3\widehat{E}_{[\hat{\bar{A}}|}{}^{\hat{I}} \partial_{\hat{I}} \widehat{E}_{|\hat{\bar{B}}|}{}^{\hat{J}} \widehat{E}_{|\hat{\bar{C}}]\hat{J}}$ is a generalized flux obtained from $\widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}}$ in the external spacetime, and an internal flux is

$$f_{\hat{I}\hat{J}\hat{K}} := 3\widetilde{\Omega}_{[\hat{I}\hat{J}\hat{K}]} = 3U_{[\hat{I}|}{}^{\hat{M}} \partial_{\hat{M}} U_{|\hat{J}}{}^{\hat{N}} U_{\hat{K}]\hat{N}}.$$

In the GSS compactification, the internal flux $f_{\hat{I}\hat{J}\hat{K}}$ is assumed to be a constant.

Generalized Lie derivative and closure constraints

$$\widehat{\mathcal{L}}_{\widehat{\Lambda}(\mathbb{X})}\widehat{V}^{\hat{I}}(\mathbb{X}) = \mathcal{L}_{\widehat{\Lambda}(\mathbb{X})}\widehat{V}^{\hat{I}}(\mathbb{X}) + f^{\hat{I}}{}_{\hat{J}\hat{K}}\widehat{\Lambda}^{\hat{J}}(\mathbb{X})\widehat{V}^{\hat{K}}(\mathbb{X}).$$

The algebra of $\widehat{\mathcal{L}}$ closes if

$$\partial_{\hat{I}}\widehat{V}(\mathbb{X})\partial^{\hat{I}}\widehat{W}(\mathbb{X}) = 0, \quad f_{[\hat{I}\hat{J}}{}^{\hat{H}}f_{\hat{K}]\hat{L}\hat{H}} = 0,$$

the closure constraint for DFT fields and the Jacobi identity of the structure constant $f_{\hat{I}\hat{J}}{}^{\hat{K}}$. This theory is called a gauged DFT (GDFT).

Pre-QP manifold for GSS twist

We introduce a $2D$-dimensional intermediate coordinates of a graded tangent and cotangent space, denoted by $(\widehat{Q}^{\hat{I}}, \widehat{P}_{\hat{I}})$. The corresponding DFT basis is

$$\widehat{Q}'^{\hat{I}} := \frac{1}{\sqrt{2}}(\widehat{Q}^{\hat{I}} - \eta^{\hat{I}\hat{J}}\widehat{P}_{\hat{J}}) \quad , \quad \widehat{P}'_{\hat{I}} := \frac{1}{\sqrt{2}}(\widehat{P}_{\hat{I}} + \eta_{\hat{I}\hat{J}}\widehat{Q}^{\hat{J}}).$$

We can introduce three new types of canonical transformation functions using a new coordinate $\widehat{P}'_{\hat{I}}$,

$$\widehat{E} := \widehat{E}_{\hat{A}}{}^{\hat{I}}\eta^{\hat{A}\hat{B}}\widehat{P}'_{\hat{I}}\bar{P}'_{\hat{\bar{B}}}, \quad U := U_{\hat{I}}{}^{\hat{M}}\eta^{\hat{I}\hat{J}}\widehat{P}'_{\hat{J}}P'_{\hat{M}}, \quad \widehat{a} := \widehat{a}_{\hat{I}}{}^{\hat{J}}\eta^{\hat{I}\hat{K}}\widehat{P}'_{\hat{J}}\widehat{P}'_{\hat{K}}.$$

The GSS twist is produced by the canonical transformation U, where the parameter $U_{\hat{I}}{}^{\hat{M}}(\mathbb{Y})$ depends only on $\mathbb{Y}$, and the components of $U_{\hat{I}}{}^{\hat{M}}$ are non-trivial only when both indices lie in the internal directions.

Then, the canonical transformation $e^{-\frac{\pi}{2}\delta_U}$ provides the GSS twist of the generalized vielbein $\widehat{E}_{\hat{A}}{}^{\hat{I}}(\mathbb{X})$ and the gauge parameter $\Lambda^{\hat{I}}(\mathbb{X})$,

$$e^{-\frac{\pi}{2}\delta_U}(\widehat{E}_{\hat{A}}{}^{\hat{I}}(\mathbb{X})\widehat{P}'_{\hat{I}}) = \widehat{E}_{\hat{A}}{}^{\hat{I}}(\mathbb{X})U_{\hat{I}}{}^{\hat{M}}(\mathbb{Y})P'_{\hat{M}},$$
$$e^{-\frac{\pi}{2}\delta_U}(\widehat{\Lambda}^{\hat{I}}(\mathbb{X})\widehat{P}'_{\hat{I}}) = \widehat{\Lambda}^{\hat{I}}(\mathbb{X})U_{\hat{I}}{}^{\hat{M}}(\mathbb{Y})P'_{\hat{M}}.$$

Hamiltonian function and derived bracket

The twisted Hamiltonian function is given by

$$\begin{aligned}\Theta_{\mathrm{GSS}} =& e^{-\frac{\pi}{2}\delta_U}\Theta_0\\ =& U_{\hat{I}}{}^{\hat{M}}\Xi_{\hat{M}}\widehat{P}'^{\hat{I}} + \frac{1}{3!}f_{\hat{I}\hat{J}\hat{K}}\widehat{P}'^{\hat{I}}\widehat{P}'^{\hat{J}}\widehat{P}'^{\hat{K}} - \frac{1}{2}\widetilde{\Omega}_{\hat{I}\hat{J}\hat{K}}U^{\hat{J}}{}_{\hat{M}}U^{\hat{K}}{}_{\hat{N}}P'^{\hat{M}}P'^{\hat{N}}\widehat{P}'^{\hat{I}},\end{aligned}$$

where
$\widetilde{\Omega}_{\hat{I}\hat{J}\hat{K}} = U_{\hat{I}}{}^{\hat{M}} \partial_{\hat{M}} U_{\hat{J}}{}^{\hat{N}} U_{\hat{K}\hat{N}}$: internal Weitzenböck connection

$f_{\hat{I}\hat{J}\hat{K}} = 3\widetilde{\Omega}_{[\hat{I}\hat{J}\hat{K}]}$: internal flux

The generalized Lie derivative on the reduced theory is derived by the derived bracket,

$$\begin{aligned}
\mathcal{L}_\Lambda V =& - \{\{\Lambda, \Theta_0\}, V\} \\
=& - e^{-\frac{\pi}{2}\delta_U}\{\{\widehat{\Lambda}^{\hat{I}}(\mathbb{X})\widehat{P}'_{\hat{I}}, \Theta_{\mathrm{GSS}}\}, \widehat{V}^{\hat{J}}(\mathbb{X})\widehat{P}'_{\hat{J}}\} \\
=& U_{\hat{I}}{}^{\hat{M}}\left(\widehat{\mathcal{L}}_{\widehat{\Lambda}}\widehat{V}^{\hat{I}} + f_{\hat{J}\hat{K}}{}^{\hat{I}}\widehat{\Lambda}^{\hat{J}}\widehat{V}^{\hat{K}}\right)P'_{\hat{M}}.
\end{aligned}$$

The closure condition for the derived bracket is provided by the weak master equation,

$$\{\{\{\{\Theta, \Theta\}, f\}, g\}, h\} = 0.$$

Then, the weak master equation for generalized vectors $\widehat{V}_1^{\hat{I}}(\mathbb{X})$ and $\widehat{V}_2^{\hat{I}}(\mathbb{X})$ leads closure conditions,

$$\eta^{\hat{I}\hat{J}} \partial_{\hat{I}} \widehat{V}_1^{\hat{K}}(\mathbb{X}) \partial_{\hat{J}} \widehat{V}_2^{\hat{L}}(\mathbb{X}) = 0, \quad f_{\hat{H}[\hat{I}\hat{J}} f^{\hat{H}}{}_{\hat{K}\hat{L}]} = 0.$$

Introduction of external generalized vielbein

By the canonical transformation function $\widehat{E}$, the twisted Hamiltonian function is

$$
\begin{aligned}
&e^{\frac{\pi}{2}\delta_{\widehat{E}}}\Theta_{\mathrm{GSS}} \\
=&E_{\hat{A}}{}^{\hat{I}}U_{\hat{I}}{}^{\hat{M}}\Xi_{\hat{M}}\bar{P}'^{\hat{A}}+\frac{1}{3!}(\widehat{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}+f_{\hat{I}\hat{J}\hat{K}}\widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}}\widehat{E}_{\hat{\bar{B}}}{}^{\hat{J}}\widehat{E}_{\hat{\bar{C}}}{}^{\hat{K}})\bar{P}'^{\hat{\bar{A}}}\bar{P}'^{\hat{\bar{B}}}\bar{P}'^{\hat{\bar{C}}} \\
&-\frac{1}{2}\widehat{\Omega}_{\hat{\bar{C}}\hat{\bar{A}}\hat{\bar{B}}}\widehat{E}^{\hat{\bar{A}}}{}_{\hat{I}}\widehat{E}^{\hat{\bar{B}}}{}_{\hat{J}}\widehat{P}'^{\hat{I}}\widehat{P}'^{\hat{J}}\bar{P}'^{\hat{\bar{C}}}-\frac{1}{2}\widetilde{\Omega}_{\hat{I}\hat{J}\hat{K}}U^{\hat{J}}{}_{\hat{M}}U^{\hat{K}}{}_{\hat{N}}E_{\hat{A}}{}^{\hat{I}}P'^{\hat{M}}P'^{\hat{N}}\bar{P}'^{\hat{A}}.
\end{aligned}
$$

We obtain correct $\widehat{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}$, $f_{\hat{M}\hat{N}\hat{R}}$ and $\mathcal{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}$,

$$
\begin{aligned}
\widehat{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} =&3\widehat{E}_{[\hat{\bar{A}}|}{}^{\hat{I}}\partial_{\hat{I}}\widehat{E}_{|\hat{\bar{B}}|}{}^{\hat{J}}\widehat{E}_{|\hat{\bar{C}}]\hat{J}}, \qquad f_{\hat{I}\hat{J}\hat{K}}=3U_{[\hat{I}|}{}^{\hat{M}}\partial_{\hat{M}}U_{|\hat{J}}{}^{\hat{N}}U_{\hat{K}]\hat{N}}, \\
\mathcal{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} =&\widehat{F}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}+f_{\hat{I}\hat{J}\hat{K}}\widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}}\widehat{E}_{\hat{\bar{B}}}{}^{\hat{J}}\widehat{E}_{\hat{\bar{C}}}{}^{\hat{K}}.
\end{aligned}
$$

§7. Covariantized pre-QP-manifold and DFT on group manifold

We generalize the formalism to a covariant pre-QP formulation.

$GL(2D)$ covariant formulation

Let $\widehat{M}$ be a $2D$-dimensional (curved) manifold with local coordinates $X^{\hat{M}} = (\tilde{x}_M, x^M)$ where $\hat{M}, \hat{N}, \cdots$ are $GL(2D)$ indices.

We define a basis $\Xi^{\nabla}_{\hat{M}}$ of degree 2, corresponding to the covariant derivative $\nabla_{\hat{M}}$, with affine connection Γ and spin connection W,

$$\Xi^{\nabla}_{\hat{M}} := \Xi_{\hat{M}} + \Gamma_{\hat{M}\hat{N}}{}^{\hat{P}} Q^{\hat{N}} P_{\hat{P}} + W_{\hat{M}\hat{I}}{}^{\hat{J}} \widehat{Q}^{\hat{I}} \widehat{P}_{\hat{J}}.$$

The Poisson bracket $\{-, \Xi^{\nabla}_{\hat{M}}\}$ with the vector fields $V^{\hat{M}} P_{\hat{M}}, \widehat{V}^{\hat{I}} \widehat{P}_{\hat{I}}$ and 1-forms

$\alpha_{\hat{M}} Q^{\hat{M}}, \widehat{\alpha}_{\hat{I}} \widehat{Q}^{\hat{I}}$ give their covariant derivative on $\widehat{M}$:

$$\{V^{\hat{M}}(X)P_{\hat{M}}, \Xi^{\nabla}_{\hat{N}}\} = \nabla_{\hat{N}} V^{\hat{M}}(X)P_{\hat{M}}, \quad \{\alpha_{\hat{M}}(X)Q^{\hat{M}}, \Xi^{\nabla}_{\hat{N}}\} = \nabla_{\hat{N}} \alpha_{\hat{M}}(X)Q^{\hat{M}},$$
$$\{\widehat{V}^{\hat{I}}(X)\widehat{P}_{\hat{I}}, \Xi^{\nabla}_{\hat{N}}\} = \nabla_{\hat{N}} \widehat{V}^{\hat{I}}(X)\widehat{P}_{\hat{I}}, \quad \{\widehat{\alpha}_{\hat{I}}(X)\widehat{Q}^{\hat{I}}, \Xi^{\nabla}_{\hat{N}}\} = \nabla_{\hat{N}} \widehat{\alpha}_{\hat{I}}(X)\widehat{Q}^{\hat{I}}.$$

If we require the vielbein postulate $\{E_{\hat{I}}{}^{\hat{N}} P_{\hat{N}} \widehat{Q}^{\hat{I}}, \Xi^{\nabla}_{\hat{M}}\} = 0$, i.e.

$$\nabla_{\hat{M}} E_{\hat{I}}{}^{\hat{N}} = 0,$$

we obtain a condition of generalized connections,

$$W_{\hat{M}\hat{I}}{}^{\hat{J}} E^{\hat{I}}{}_{\hat{N}} E_{\hat{J}}{}^{\hat{P}} - \Omega_{\hat{M}\hat{N}}{}^{\hat{P}} - \Gamma_{\hat{M}\hat{N}}{}^{\hat{P}} = 0,$$
$$W_{\hat{M}\hat{J}\hat{K}} + W_{\hat{M}\hat{K}\hat{J}} = 0.$$

Here

$$\nabla_{\hat{M}}\eta_{\hat{I}\hat{J}} = 0,$$

The covariant derivative of $\eta_{\hat{M}\hat{N}}$ automatically vanishes

$$\nabla_{\hat{M}}\eta_{\hat{N}\hat{P}} = \partial_{\hat{M}}\eta_{\hat{N}\hat{P}} - \Gamma_{\hat{M}\hat{N}}{}^{\hat{Q}}\eta_{\hat{Q}\hat{P}} - \Gamma_{\hat{M}\hat{P}}{}^{\hat{Q}}\eta_{\hat{N}\hat{Q}} = 0.$$

Hamiltonian function and generalized Lie derivative

A Hamiltonian function is covariantized as

$$\Theta_0^{\nabla} = \eta^{\hat{M}\hat{N}}\Xi^{\nabla}_{\hat{M}}P'_{\hat{N}}.$$

The generalized Lie derivative is defined by

$$-\{\{\Lambda, \Theta_0^{\nabla}\}, V\} = \mathcal{L}_{\Lambda}^{\nabla} V \ .$$

Closure condition

The closure condition of the generalized Lie derivative is the weak master equation:

$$\{\{\{\{\widehat{\Theta}_0^{\nabla}, \widehat{\Theta}_0^{\nabla}\}, \widehat{V}_1\}, \widehat{V}_2\}, \widehat{V}_3\} = 0.$$

This condition leads to the following conditions for the spin connection ${W_{\hat{M}\hat{I}}}^{\hat{J}}$

and arbitrary generalized vectors $\widehat{V}_1, \widehat{V}_2$ and $\widehat{V}_3$,

$$
\begin{aligned}
&- 2(\partial^{\hat{M}}\widehat{V}_1^{\hat{J}}\widehat{V}_{2\hat{J}}\partial_{\hat{M}}\widehat{V}_3^{\hat{I}} - 2\partial^{\hat{M}}\widehat{V}_1^{[\hat{J}}\partial_{\hat{M}}\widehat{V}_2^{\hat{I}]}\widehat{V}_{3\hat{J}}) \\
&- 2\Big(2\Omega_{[\hat{I}\hat{J}]\hat{K}} - 3W_{[\hat{I}\hat{J}\hat{K}]}\Big)E^{\hat{K}\hat{M}} \\
&\times \Big[\partial_{\hat{M}}\widehat{V}_1^{\hat{L}}\widehat{V}_{2\hat{L}}\widehat{V}_3^{\hat{J}} - \partial_{\hat{M}}\widehat{V}_1^{\hat{L}}\widehat{V}_2^{\hat{J}}\widehat{V}_{3\hat{L}} + \widehat{V}_1^{\hat{J}}\partial_{\hat{M}}\widehat{V}_2^{\hat{L}}\widehat{V}_{3\hat{L}}\Big] \\
&+ 2\Big(2\Omega_{[\hat{L}\hat{J}]\hat{K}} - 3W_{[\hat{L}\hat{J}\hat{K}]}\Big)E^{\hat{K}\hat{M}} \\
&\times \Big[\partial_{\hat{M}}\widehat{V}_1^{\hat{I}}\widehat{V}_2^{\hat{L}}\widehat{V}_3^{\hat{J}} - \widehat{V}_1^{\hat{L}}\partial_{\hat{M}}\widehat{V}_2^{\hat{I}}\widehat{V}_3^{\hat{J}} + \widehat{V}_1^{\hat{L}}\widehat{V}_2^{\hat{J}}\partial_{\hat{M}}\widehat{V}_3^{\hat{I}}\Big] \\
&- 3!\widehat{V}_1^{\hat{I}}\widehat{V}_2^{\hat{J}}\widehat{V}_3^{\hat{K}}\Big[2R_{[\hat{I}\hat{J}\hat{K}\hat{L}]} - W_{\hat{H}[\hat{I}\hat{J}}W^{\hat{H}}{}_{\hat{K}\hat{L}]} \\
&\times -2(2W_{[\hat{I}\hat{J}}{}^{\hat{H}} - 2\Omega_{[\hat{I}\hat{J}}{}^{\hat{H}})W_{\hat{H}\hat{K}\hat{L}]}\Big] \\
&= 0.
\end{aligned}
$$

Twist

The possible twist functions made from $P', \widehat{P}'$ and $\bar{P}'$ are

$$A := A^{\hat{I}\hat{M}} P'_{\hat{M}} \widehat{P}'_{\hat{I}}, \quad \widehat{A} := \widehat{A}^{\hat{\bar{A}}\hat{J}} \widehat{P}'_{\hat{J}} \bar{P}'_{\hat{\bar{A}}}, \quad \mathcal{A} := \mathcal{A}^{\hat{\bar{A}}\hat{M}} P'_{\hat{M}} \bar{P}'_{\hat{\bar{A}}},$$

$$u := u_{\hat{P}}{}^{\hat{M}} \eta^{\hat{N}\hat{P}} P'_{\hat{M}} P'_{\hat{N}}, \quad \widehat{u} := \widehat{u}_{\hat{I}}{}^{\hat{J}} \eta^{\hat{K}\hat{I}} \widehat{P}'_{\hat{J}} \widehat{P}'_{\hat{K}}, \quad \bar{u} := \bar{u}_{\hat{\bar{A}}}{}^{\hat{\bar{B}}} \eta^{\hat{\bar{C}}\hat{\bar{A}}} \bar{P}'_{\hat{\bar{B}}} \bar{P}'_{\hat{\bar{C}}}.$$

Here $A^{\hat{I}\hat{M}}$, $\widehat{A}^{\hat{\bar{A}}\hat{J}}$ and $\mathcal{A}^{\hat{\bar{A}}\hat{M}}$ are $GL(2D)$ matrices and we can take them as vielbein $E_{\hat{I}}{}^{\hat{M}}, \widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}}$ and $\mathcal{E}_{\hat{\bar{A}}}{}^{\hat{M}}$.

Applying the similar discussion to $\widehat{A}$, we can introduce the fluctuation vielbein $\widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}}$. When we take $\widehat{A}_{\hat{\bar{A}}}{}^{\hat{I}} = \frac{\pi}{2} \widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}}$, we obtain the canonical transformation rules as follows.

$$
\begin{aligned}
e^{\frac{\pi}{2}\delta_{\widehat{E}}}\widehat{P}'_{\hat{I}} =& \widehat{E}^{\hat{\bar{B}}}{}_{\hat{I}}\bar{P}'_{\hat{\bar{B}}},\\
e^{\frac{\pi}{2}\delta_{\widehat{E}}}\bar{P}'_{\hat{\bar{A}}} =& -\widehat{E}_{\hat{\bar{A}}}{}^{\hat{I}}\widehat{P}'_{\hat{I}},\\
e^{\frac{\pi}{2}\delta_{\widehat{E}}}\Xi^{\nabla}_{\hat{M}} =& \Xi_{\hat{M}} - \frac{1}{2}\mathcal{E}^{\hat{\bar{C}}}{}_{\hat{M}}\tilde{\Omega}^{\nabla}_{\hat{\bar{C}}\hat{\bar{A}}\hat{\bar{B}}}\widehat{E}^{\hat{\bar{A}}}{}_{\hat{I}}\widehat{E}^{\hat{\bar{B}}}{}_{\hat{J}}\widehat{P}'^{\hat{I}}\widehat{P}'^{\hat{J}} + \frac{1}{2}\mathcal{E}^{\hat{\bar{C}}}{}_{\hat{M}}\tilde{\Omega}^{\nabla}_{\hat{\bar{C}}\hat{\bar{A}}\hat{\bar{B}}}\bar{P}'^{\hat{\bar{A}}}\bar{P}'^{\hat{\bar{B}}}.
\end{aligned}
$$

Here we have defined $\widehat{\Omega}^{\nabla}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}} := \mathcal{E}_{\hat{\bar{A}}}{}^{\hat{M}}\nabla_{\hat{M}}\widehat{E}_{\hat{\bar{B}}}{}^{\hat{I}}\widehat{E}_{\hat{\bar{C}}\hat{I}}$. This is just the covariantized Weitzenböck connection $\widehat{\Omega}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}$. Twist of the Hamiltonian function $\bar{\Theta}^{\nabla}_{0}$ gives

$$
e^{\frac{\pi}{2}\delta_{\widehat{E}}}\widehat{\Theta}^{\nabla}_{0} = \mathcal{E}_{\hat{\bar{A}}}{}^{\hat{M}}\Xi^{\nabla}_{\hat{M}}\bar{P}'^{\hat{\bar{A}}} + \frac{1}{3!}\widehat{\mathcal{F}}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}\bar{P}'^{\hat{\bar{A}}}\bar{P}'^{\hat{\bar{B}}}\bar{P}'^{\hat{\bar{C}}} - \frac{1}{2}\widehat{\Omega}^{\nabla}_{\hat{\bar{A}}\hat{\bar{B}}\hat{\bar{C}}}\widehat{E}^{\hat{\bar{B}}}{}_{\hat{I}}\widehat{E}^{\hat{\bar{C}}}{}_{\hat{J}}\widehat{P}'^{\hat{I}}\widehat{P}'^{\hat{J}}\bar{P}'^{\hat{\bar{A}}}
$$

Pre-Bianchi identities

Now we can consider the pre-Bianchi identity for DFT on covariantized pre-QP-manifold. $\mathcal{B}$ as

$$\mathcal{B}(\Theta_F, \Theta_0, \alpha) := \{\Theta_F, \Theta_F\} - e^{\delta\alpha}\{\Theta_0, \Theta_0\} = 0.$$

gives the generalized Bianchi identities.

Application to DFT$_{\rm WZW}$ Blumenhagen, Hassler, Luest, '14

We assume the background space as a group manifold G, so we can regard the coordinate $\widehat{P}'_{\hat{I}}$ of its tangent space TG as the generator of the Lie algebra of G by the injection map $j'^*(\widehat{P}'_{\hat{I}}) = T_{\hat{I}}$. Then, the derived bracket of $\widehat{P}'_{\hat{I}}$ should reproduce the Lie bracket:

$$-\{\{\widehat{P}'_{\hat{I}}, \widehat{\Theta}_0^{\nabla}\}, \widehat{P}'_{\hat{J}}\} = j'_*[T_I, T_J]_{\rm Lie}.$$

The left hand side is calculated as,

$$-\{\{\widehat{P}'_{\hat{I}}, \widehat{\Theta}_0^{\nabla}\}, \widehat{P}'_{\hat{J}}\} =(W^{\hat{K}}{}_{\hat{I}\hat{J}} + 2W_{[\hat{I}\hat{J}]}{}^{\hat{K}})\widehat{P}'_{\hat{K}},$$

and the right hand side is written by definition of a Lie algebra as

$$j'_*[T_I, T_J]_{\rm Lie} = F_{\hat{I}\hat{J}}{}^{\hat{K}}\widehat{P}'_{\hat{K}}.$$

Thus, the above equality leads the condition of the spin connection: $W^{\hat{K}}{}_{\hat{I}\hat{J}} + 2W_{[\hat{I}\hat{J}]}{}^{\hat{K}} = F_{\hat{K}\hat{I}}{}^{\hat{J}}$. This condition is solved by

$$W_{\hat{I}\hat{J}}{}^{\hat{K}} = \frac{1}{3}F_{\hat{I}\hat{J}}{}^{\hat{K}},$$

and this solution is just the one proposed in the DFT$_{\mathrm{WZW}}$ model.

With this spin connection, the derived bracket with $\widehat{\Theta}_0^{\nabla}$ reproduces the generalized Lie derivative of DFT$_{\mathrm{WZW}}$ as

$$-\{\{\widehat{\Lambda}, \widehat{\Theta}_0^{\nabla}\}, \widehat{V}\} = \Lambda^{\hat{J}} D_{\hat{J}} V^{\hat{I}} + (D^{\hat{I}}\Lambda_{\hat{J}} - D_{\hat{J}}\Lambda^{\hat{I}})V^{\hat{J}} + F^{\hat{I}}{}_{\hat{J}\hat{K}}\Lambda^{\hat{J}}V^{\hat{K}}.$$

Thus, the weak master equation yields the section condition and the Jacobi identity as the closure condition of generalized Lie derivative of DFT$_{\mathrm{WZW}}$.

§. Summary and outlook

• We have formulated DFT geometry by supergeometry in term of pre-QP-manifold.

A generalized Lie derivative is defined by a derived bracket,

$$\mathcal{L}_V V' = -\{\{V, \Theta\}, V'\},$$

and the closure condition (the weak master equation) is the weak master equation,

$$\{\{\{\{\Theta, \Theta\}, f\}, g\}, h\} = 0.$$

Generalized fluxes are introduced by twist on a pre-QP-manifold,

$$\Theta_F = e^{\delta_\alpha}\Theta_0,$$

taking a twisting function α properly. A generalized Bianchi identity is equivalently formulated by a pre-Bianchi identity,

$$\mathcal{B}(\Theta_F, \Theta_0, \alpha) = \{\Theta_F, \Theta_F\} - e^{\delta\alpha}\{\Theta_0, \Theta_0\} = 0.$$

We confirmed this formulation in the GSS compactification and DFT on group manifold.

Outlook

• Inclusion of a dilaton

• Characteristic classes of T^d bundles and nongeometric fluxes. (Q defines a complex and cohomology.)

• nonabelian/Poisson-Lie T-duality

• Geometry of exceptional field theory (T-duality + S-duality)

• Physics: action, quantization, etc.

Thank you for your attention!