L_∞-Algebras, the BV Formalism, and Classical Fields

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Outline

- L_∞-Algebras and the Batalin–Vilkovisky Formalism
- Higher Gauge Group(oid)s and Higher Principal Bundles
- Self-Dual Higher Gauge Theory
- Yang–Mills Theory
- Conclusions and Outlook

L_∞-Algebras

Q-Manifolds

- For a $\mathbb{Z}$-graded vector space $L = \bigoplus_k L_k$, we set $L[l] = \bigoplus_k (L[l])_k$ with $(L[l])_k := L_{k+l}$ for $l \in \mathbb{Z}$.

- A Q-manifold is a $\mathbb{Z}$-graded manifold quipped with a degree 1 vector field Q with $Q^2 = 0$ called homological vector field.

- Consider the de Rahm complex $(\Omega^\bullet(X), \mathrm{d})$ on a smooth manifold X. Using $\mathcal{C}^\infty(T[1]X) \cong \Omega^\bullet(X)$ it can be described equivalently by the Q-manifold $(T[1]X, Q)$ where $Q \leftrightarrow \mathrm{d}$.

L_∞-Algebroids

- Consider $L[1]$ for an ordinary vector space $L \equiv L_0$. Coordinates, denoted by ξ^α, on $L[1]$ are thus of degree 1 so that the most general degree 1 vector field Q is

$$Q := -\frac{1}{2}\xi^\alpha \xi^\beta f_{\alpha\beta}{}^\gamma \frac{\partial}{\partial \xi^\gamma}$$

  with $f_{\alpha\beta}{}^\gamma$ constant. Then, $Q^2 = 0$ is equivalent to requiring $f_{\alpha\beta}{}^\gamma$ to satisfy the Jacobi identity. Thus, the Q-manifold $(L[1], Q)$ describes a Lie algebra $(L, [-,-])$ with Q as its Chevalley–Eilenberg differential.

- Generally, an *n*-term L_∞-algebra is a Q-manifold concentrated in degrees $1, \ldots, n$ and $Q^2 = 0$ corresponds to higher or homotopy Jacobi identities

$$\sum_{j+k=i} \sum_{\sigma(j;i)} \pm \mu_{k+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(i)}) = 0$$

  for the higher brackets μ_i. If concentrated in degrees $0, \ldots, n$, we call it an *n*-term L_∞-algebroid.

Cyclic L_∞-Algebras

- A symplectic Q-manifold of degree k is a Q-manifold with a symplectic form ω of degree k such that Q is symplectic with respect to ω.
- For $k \neq -1$, Q is automatically Hamiltonian. Letting $\{-,-\}$ be the Poisson bracket induced by ω and S the Hamiltonian for Q, then for $k \neq -2$, the condition $Q^2 = 0$ is equivalent to $\{S, S\} = 0$, called the classical master equation.
- In the L_∞-language, a symplectic Q-manifold corresponds to a cyclic L_∞-algebra which is an L_∞-algebra L equipped with a graded symmetric non-degenerate bilinear pairing $\langle -,- \rangle : L \times L \to \mathbb{R}$ cyclic in the sense of

$$\langle \ell_1, \mu_i(\ell_2, \ldots, \ell_{i+1}) \rangle = \pm \langle \ell_{i+1}, \mu_i(\ell_1, \ldots, \ell_i) \rangle$$

Quasi-Isomorphisms

- A morphism of Q-manifolds is a map $\phi : (X, Q) \to (X', Q')$ such that $\phi \circ Q = Q' \circ \phi$.
- In the L_∞-language, a morphism of Q-manifolds corresponds to a map $\phi : (L, \mu_i) \to (L', \mu_i')$, called an L_∞-morphism, consisting of a collection of maps $\phi_i : L \times \cdots \times L \to L'$ such that

$$\sum_{j+k=i} \sum_{\sigma(j;i)} \pm\phi_{k+1}\big(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(i)}\big)$$

$$= \sum_{j=1}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma(k_1,\ldots,k_{j-1};i)} \pm\, \mu_j'\Big(\phi_{k_1}\big(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(k_1)}\big), \ldots, \phi_{k_j}\big(\ell_{\sigma(k_1+\cdots+k_{j-1}+1)}, \ldots, \ell_{\sigma(i)}\big)\Big)$$

- An L_∞-morphism is called an L_∞-quasi-isomorphism provided ϕ_1 induces an isomorphism $H^\bullet_{\mu_1}(L) \cong H^\bullet_{\mu_1'}(L')$.
- Every L_∞-algebra (L, μ_i) is quasi-isomorphic to an L_∞-algebra (L', μ_i') with $\mu_1' = 0$, known as the minimal model theorem.

Maurer–Cartan Theory

- For (L, μ_i) an L_∞-algebra, we call $a \in L_1$ a gauge potential and define its curvature as

$$f := \mu_1(a) + \tfrac{1}{2}\mu_2(a,a) + \cdots = \sum_{i\geq 1} \tfrac{1}{i!}\mu_i(a,\ldots,a)$$

- Due to the higher Jacobi identities, f satisfies a Bianchi identity

$$\mu_1(f) + \mu_2(a,f) + \cdots = \sum_{i\geq 0} \frac{(-1)^i}{i!}\mu_{i+1}(f,a,\ldots,a) = 0$$

- For $c_0 \in L_0$, gauge transformations act as

$$\delta_{c_0} a := \mu_1(a) + \mu_2(a,c_0) + \cdots = \sum_{i\geq 0} \tfrac{1}{i!}\mu_{i+1}(a,\ldots,a,c_0)$$

$$\delta_{c_0} f = -\mu_2(c_0,f) + \cdots = \sum_{i\geq 0} \frac{(-1)^i}{i!}\mu_{i+2}(f,a,\ldots,a,c_0)$$

  and there are higher gauge transformations with $c_{-k} \in L_{-k}$ and

$$\delta_{c_{-k-1}} c_{-k} := \sum_{i\geq 0} \tfrac{1}{i!}\mu_{i+1}(a,\ldots,a,c_{-k-1})$$

Maurer–Cartan Theory

- The equation $f = 0$ is called the Maurer–Cartan equation and solutions to this equation are called Maurer–Cartan elements.
- An L_∞-morphism $\phi_i : (L, \mu_i) \to (L', \mu_i')$ acts as

$$a \;\mapsto\; a' := \sum_{i \geq 1} \tfrac{1}{i!} \phi_i(a, \ldots, a)$$

$$f \;\mapsto\; f' = \sum_{i \geq 0} \frac{(-1)^i}{i!} \phi_{i+1}(f, a, \ldots, a)$$

  and provided a is a Maurer–Cartan element, gauge equivalence classes $[a]$ are mapped to gauge equivalence classes $[a']$. Thus, for a quasi-isomorphism between (L, μ_i) and (L', μ_i'), the corresponding moduli spaces of Maurer–Cartan elements are isomorphic.
- For $(L, \mu_i, \langle -, - \rangle)$ a cyclic L_∞-algebra with $\langle -, - \rangle$ of degree -3, the Maurer–Cartan equation follows from the gauge-invariant action functional

$$S = \tfrac{1}{2}\langle a, \mu_1(a)\rangle + \tfrac{1}{3!}\langle a, \mu_2(a, a)\rangle + \cdots = \sum_{i \geq 0} \tfrac{1}{(i+1)!} \langle a, \mu_i(a, \ldots, a)\rangle$$

Higher Chern–Simons Theory

- Let X be a d-dimensional compact orientable manifold without boundary and consider the de Rham complex $(\Omega^\bullet(X), \mathrm{d})$. Let $(L, \mu_i, \langle -,-\rangle)$ with $L = \bigoplus_{k=-n+1}^{0} L_k$ a finite-dimensional cyclic L_∞-algebra called the gauge algebra. The degree of $\langle -,-\rangle$ is necessarily $n-1$ i.e. $(L_k)^* \cong L_{-n+1-k}$. Next, we form the tensor product $(\Omega^\bullet(X,L), \mu_i', \langle -,-\rangle')$ by setting

$$\Omega^\bullet(X,L) := \bigoplus_{k=-n+1}^{d} \Omega^\bullet_k(X,L)~, \quad \Omega^\bullet_k(X,L) := \bigoplus_{i+j=k} \Omega^i(X) \otimes L_j$$

with

$$\mu_1'(\omega_1 \otimes \ell_1) := \mathrm{d}\omega_1 \otimes \ell_1 + (-1)^{|\omega_1|} \omega_1 \otimes \mu_1(\ell_1)~,$$

$$\begin{aligned} \mu_i'(\omega_1 \otimes \ell_1, \ldots, \omega_i \otimes \ell_i) \ &:= \ (-1)^{i \sum_{j=1}^{i} |\omega_j| + \sum_{j=0}^{i-2} |\omega_{i-j}| \sum_{k=1}^{i-j-1} |\ell_k|} \times \\ &\qquad \times (\omega_1 \wedge \ldots \wedge \omega_i) \otimes \mu_i(\ell_1, \ldots, \ell_i)~, \end{aligned}$$

$$\langle \omega_1 \otimes \ell_1, \omega_2 \otimes \ell_2 \rangle' := (-1)^{|\omega_2||\ell_1|} \int_X \omega_1 \wedge \omega_2 \ \langle \ell_1, \ell_2 \rangle$$

and $-3 = -d + n - 1$ so that $n = d - 2$.

Higher Chern–Simons Theory

- For $d = 3$, we have $a = A \in \Omega^1(X, L_0)$ and $f = F \in \Omega^2(X, L_0)$ with $F := \mathrm{d}A + \frac{1}{2}[A, A]$ and
$$S = \int_X \left\{ \tfrac{1}{2}\langle A, \mathrm{d}A\rangle + \tfrac{1}{3!}\langle A, [A, A]\rangle \right\}$$
that is, ordinary Chern–Simons theory. The gauge transformations read as $\delta_c A := \mathrm{d}A + [A, c]$ and $\delta_c F = -[c, F]$.
- For $d = 4$, we have $a = A + B \in \Omega^1(X, L_0) \oplus \Omega^2(X, L_{-1})$ and $f = \mathcal{F} + H \in \Omega^2(X, L_0) \oplus \Omega^3(X, L_{-1})$ with
$$\mathcal{F} := \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \mu_1(B)\ , \quad H := \mathrm{d}B + \mu_2(A, B) - \tfrac{1}{3!}\mu_3(A, A, A)$$
and
$$S = \int_X \left\{ \langle B, \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \tfrac{1}{2}\mu_1(B)\rangle + \tfrac{1}{4!}\langle \mu_3(A, A, A), A\rangle \right\}$$
that is, higher Chern–Simons theory. The gauge transformations are
$$\begin{aligned}
\delta_{c,\Lambda} A &= \mathrm{d}c + \mu_2(A, c) + \mu_1(\Lambda)\ ,\\
\delta_{c,\Lambda} B &= -\mu_2(c, B) + \mathrm{d}\Lambda + \mu_2(A, \Lambda) + \tfrac{1}{2}\mu_3(c, A, A)\ ,\\
\delta_{c,\Lambda}\mathcal{F} &= -\mu_2(c, \mathcal{F})\ , \quad \delta_{c,\Lambda} H = -\mu_2(c, H) + \mu_2(\mathcal{F}, \Lambda) - \mu_3(\mathcal{F}, A, c)\ .
\end{aligned}$$

BRST-BV Operator

- For any cyclic L_∞-algebra $(L,\mu_i,\langle -,-\rangle)$ with $\langle -,-\rangle$ of degree k, we define a cyclic L_∞-algebra $(L',\mu_i',\langle -,-\rangle')$ by $L' := \mathscr{C}^\infty(L[1])\otimes L$ and

$$\mu_1'(\zeta\otimes\ell) := (-1)^{|\zeta|}\zeta\otimes\mu_1(\ell)\ ,$$

$$\mu_i'(\zeta_1\otimes\ell_1,\ldots,\zeta_i\otimes\ell_i) := (-1)^{i\sum_{j=1}^{i}|\zeta_i|+\sum_{j=2}^{i}|\zeta_j|\sum_{k=1}^{j-1}|\ell_k|}\times$$
$$\times(\zeta_1\cdots\zeta_i)\otimes\mu_i(\ell_1,\ldots,\ell_i)$$

$$\langle\zeta_1\otimes\ell_1,\zeta_2\otimes\ell_2\rangle' := (-1)^{k((|\zeta_1|+|\zeta_2|)+|\ell_1||\zeta_2|}(\zeta_1\zeta_2)\langle\ell_1,\ell_2\rangle$$

  which allows us to write the Q action on coordinate functions as

$$Q\xi\ =\ -\sum_{i\geq 1}\tfrac{1}{i!}\mu_i'(\xi,\ldots,\xi)$$

- To BRST quantise the Maurer–Cartan action, we need to introduce ghosts, ghosts-for-ghosts, etc so we get:

	a	c_0	c_{-1}	$\cdots$	c_{-k}	$\cdots$
L_∞-degree	1	0	-1	$\ldots$	$-k$	$\cdots$
ghost degree	0	1	2	$\cdots$	$k+1$	$\cdots$
field type	b	f	b	$\cdots$	f/b	$\cdots$

  Thus, the field space is $\mathfrak{F}_{\rm BRST}=L_{\rm trunc}[1]$ with $L_{\rm trunc}:=\bigoplus_{k\leq 1}L_k$.

BRST-BV Operator

- To write down the BRST operator, we consider $\mathscr{C}^\infty(L_{\text{trunc}}[1]) \otimes L_{\text{trunc}}$ and set

$$\mathsf{a} := a + \sum_{k\geq 0} c_{-k}\ ,\quad \mathsf{f} := \sum_{i\geq 1} \tfrac{1}{i!}\mu_i'(\mathsf{a},\ldots,\mathsf{a})$$

so that

$$Q_{\text{BRST}}\mathsf{a} = -\mathsf{f} \quad\Rightarrow\quad Q_{\text{BRST}}^2\mathsf{a} = 0 \text{ mod } f = 0$$

for $f = \sum_{i\geq 1}\frac{1}{i!}\mu_i(a,\ldots,a)$. Essentially, this is due to the fact that L_{trunc} is not an L_∞-algebra.

- To fix this we simply transition to the Batalin–Vilkovisky formalism and define $\mathfrak{F}_{\text{BV}} := T^*[-1]\mathfrak{F}_{\text{BRST}}$. However, $\mathfrak{F}_{\text{BV}} \cong L[1]$ so that

$$Q_{\text{BV}}\mathsf{a} = -\mathsf{f} \quad\Rightarrow\quad Q_{\text{BV}}^2\mathsf{a} = 0$$

- Furthermore,

$$S = \sum_{i\geq 0} \tfrac{1}{(i+1)!}\langle \mathsf{a}, \mu_i'(\mathsf{a},\ldots,\mathsf{a})\rangle'$$

satisfies the classical master equation

$$\{S,S\} = -\langle \mathsf{f},\mathsf{f}\rangle' = 0$$

so that $Q_{\text{BV}} = \{S,-\}$. Note that S also satisfies formally the quantum master equation.

Yang–Mills Theory in the 2nd Order Formulation

- Let X be a d-dimensional compact oriented Riemannian manifold and let $\mathfrak{g}$ be a simple Lie algebra with inner product $\langle -,-\rangle$. Consider

$$\underbrace{\Omega^0(X,\mathfrak{g})}_{=:\,L'_0} \xrightarrow{\mu'_1:=\mathrm{d}} \underbrace{\Omega^1(X,\mathfrak{g})}_{=:\,L'_1} \xrightarrow{\mu'_1:=\mathrm{d}\star\mathrm{d}} \underbrace{\Omega^{d-1}(X,\mathfrak{g})}_{=:\,L'_2} \xrightarrow{\mu'_1:=\mathrm{d}} \underbrace{\Omega^{d}(X,\mathfrak{g})}_{=:\,L'_3}$$

with

$$\begin{gathered}
\mu'_1(c_1):=\mathrm{d}c_1\ ,\quad \mu'_1(A_1):=\mathrm{d}\star\mathrm{d}A_1\ ,\quad \mu'_1(A^+_1):=\mathrm{d}A^+_1\ ,\\
\mu'_2(c_1,c_2):=[c_1,c_2]\ ,\quad \mu'_2(c_1,A_1):=[c_1,A_1]\ ,\\
\mu'_2(c_1,A^+_2):=[c_1,A^+_2]\ ,\quad \mu'_2(c_1,c^+_2):=[c_1,c^+_2]\ ,\\
\mu'_2(A_1,A^+_2):=[A_1,A^+_2]\ ,\\
\mu'_2(A_1,A_2):=\mathrm{d}\star[A_1,A_2]+[A_1,\star\mathrm{d}A_2]+[A_2,\star\mathrm{d}A_1]\\
\mu'_3(A_1,A_2,A_3):=[A_1,\star[A_2,A_3]]+[A_2,\star[A_3,A_1]]+[A_3,\star[A_1,A_2]]
\end{gathered}$$

and

$$\langle\omega_1,\omega_2\rangle':=\pm\int_X\langle\omega_1,\omega_2\rangle$$

- Then, the Maurer–Cartan action becomes

$$S=\int_X\left\{\tfrac{1}{2}\langle F,\star F\rangle-\langle A^+,\nabla c\rangle+\tfrac{1}{2}\langle c^+,[c,c]\rangle\right\}$$

Yang–Mills Theory in the 1st Order Formulation

- Let X be a 4-dimensional compact oriented Riemannian manifold and let $\mathfrak{g}$ be a simple Lie algebra with inner product $\langle -, - \rangle$. Consider

$$\underbrace{\Omega^0(X,\mathfrak{g})}_{=:\,L'_0} \xrightarrow{\mu'_1 := \mathrm{d}} \underbrace{\Omega^2_+(X,\mathfrak{g}) \oplus \Omega^1(X,\mathfrak{g})}_{=:\,L'_1} \xrightarrow{\mu'_1 := (\varepsilon + \mathrm{d}) + P_+\mathrm{d}}$$

$$\underbrace{\Omega^2_+(X,\mathfrak{g}) \oplus \Omega^3(X,\mathfrak{g})}_{=:\,L'_2} \xrightarrow{\mu'_1 := 0 + \mathrm{d}} \underbrace{\Omega^4(X,\mathfrak{g})}_{=:\,L'_3}$$

with

$$\mu'_1(c_1) := \mathrm{d}c_1 \;, \quad \mu'_1(B_{+1} + A_1) := (\varepsilon B_{+1} + P_+ \mathrm{d}A_1) + \mathrm{d}B_{+1} \;, \quad \mu'_1(A^+_1) := \mathrm{d}A^+_1 \;,$$
$$\mu'_2(c_1, c_2) := [c_1, c_2] \;, \quad \mu'_2(c_1, B_{+1} + A_1) := [c_1, B_{+1}] + [c, A_1] \;,$$
$$\mu'_2(c_1, B^+_{+1} + A^+_1) := [c_1, B^+_{+1}] + [c, A^+_1] \;, \quad \mu'_2(c_1, c^+_2) := [c_1, c^+_2] \;,$$
$$\mu'_2(B_{+1} + A_1, B_{+2} + A_2) := P_+[A_1, A_2] + [A_1, B_{+2}] + [A_2, B_{+1}] \;,$$
$$\mu'_2(B_{+1} + A_1, B^+_{+2} + A^+_2) := [A_1, A^+_2] + [B_1, B^+_{+2}]$$

and

$$\langle \omega_1, \omega_2 \rangle' := \pm \int_X \langle \omega_1, \omega_2 \rangle$$

Yang–Mills Theory in the 1st Order Formulation

- Then, the Maurer–Cartan action becomes

$$S = \int_X \Big\{ \langle F, B_+ \rangle + \tfrac{\varepsilon}{2} \langle B_+, B_+ \rangle - \\ - \langle A^+, \nabla c \rangle - \langle B_+^+, [B_+, c] \rangle + \tfrac{1}{2} \langle c^+, [c, c] \rangle \Big\}$$

- Both formulations, the first- and the second-order formulations, of Yang–Mills theory are, in fact, L_∞-quasi-isomorphic. Indeed, we have have ϕ with $Q_{\mathrm{YM_2BV}} \circ \phi = \phi \circ Q_{\mathrm{YM_1BV}}$ given by

$$\phi(c) := c\,, \quad \phi(B_+) := -\tfrac{1}{\varepsilon} F_+\,, \quad \phi(A) := A\,, \\ \phi(B_+^+) := 0\,, \quad \phi(A^+) = A^+\,, \quad \phi(c^+) := c^+$$

Higher Principal Bundles

Lie Quasi-Groupoids

- The finite counter part of L_∞-algebras (algebroids) are Lie quasi-groups (groupoids) which are special simplicial manifolds known as Kan manifolds.
- In particular, a simplicial manifold is a presheaf $\mathscr{X} : \Delta^{\mathrm{op}} \to \mathrm{Mfd}$ on the simplex category Δ. Morphisms between simplicial manifolds, known as simplicial maps, are the natural transformations between the defining functors.
- Letting Δ^p be the standard simplicial p-simplex, the simplicial p-simplices of a general simplicial manifold $\mathscr{X}$ are $\mathrm{hom}_{\mathrm{sSet}}(\Delta^p, \mathscr{X})$
- For each i, the (p, i)-horn Λ^p_i of Δ^p is the simplicial subset of Δ^p given by all faces of Δ^p except for the i-th one. The (p, i)-horns of a simplicial manifold $\mathscr{X}$ is the set $\mathrm{hom}_{\mathrm{sSet}}(\Lambda^p_i, \mathscr{X})$.
- The horns Λ^p_i of Δ^p can always be filled (i.e. completed) to Δ^p. For a simplicial manifold $\mathscr{X}$ this is, in general, not the case.
- A Kan manifold is a simplicial manifold for which the restrictions $\mathrm{hom}_{\mathrm{sSet}}(\Delta^p, \mathscr{X}) \to \mathrm{hom}_{\mathrm{sSet}}(\Lambda^p_i, \mathscr{X})$ surjective submersions.
- Importantly, for $\mathscr{Y}$ a Kan manifold, simplicial homotopy induces an equivalence relation on $\mathrm{hom}_{\mathrm{sSet}}(\mathscr{X}, \mathscr{Y})$.

Principal Bundles

- For G a Lie group, the delooping BG is the Lie groupoid $G \rightrightarrows *$, where the source and target maps are trivial, $\mathrm{id}_* = \mathbb{1}_G$, and the composition is group multiplication in G.
- Next, consider an ordinary cover $\bigcup_a \{(x,a)|x \in U_a\} \to X$ for a manifold X so that the morphisms of the corresponding Čech groupoid $\check{\mathcal{C}}$ are $\bigcup_{a,b}\{(x,a,b)|x \in U_a \cap U_b\}$ with the composition $(x,a,b)\circ(x,b,c) = (x,a,c)$.
- A principal G-bundle over X subordinate to the cover is a simplicial map $g : N(\check{\mathcal{C}}) \to N(BG)$. Explicitly,

$$g_a(x) := g^0(x,a) = * \,, \quad g_{ab}(x) := g^1(x,a,b) \in G \,,$$
$$g_{abc}(x) := g^2(x,a,b,c) = (g^1_{abc}(x), g^2_{abc}(x)) \in G \times G \,, \quad \text{etc.}$$

and being simplicial implies the constraints

$$g^1_{abc}(x) = g_{ab}(x) \,, \quad g^1_{abc}(x) g^2_{abc}(x) = g_{ac}(x) \,, \quad g^2_{abc}(x) = g_{bc}(x)$$

Higher Groupoid Bundles

- Since, in addition, homotopies yield equivalent bundles, we give the following definition ...
- For $\mathscr{G}$ a Lie quasi-groupoid, a Lie quasi-groupoid bundle or principal $\mathscr{G}$-bundle over X subordinate to a cover is a simplicial map $g : N(\check{\mathcal{C}}) \to \mathscr{G}$. Two such principal $\mathscr{G}$-bundles $g, \tilde{g} : N(\check{\mathcal{C}}) \to \mathscr{G}$ are called equivalent if and only if there is a simplicial homotopy between g and $\tilde{g}$.
- This can be generalised to higher bases spaces i.e. base spaces which are Kan simplicial manifolds.

Higher Non-Abelian Deligne Cohomology

- Let $\mathscr{G}$ be a Lie 2-quasi group with the induced 2-term L_∞ algebra $L_{-1} \xrightarrow{\mu_1} L_0$. Let $\bigcup_a \{(x,a) | x \in U_a\} \to X$ be a cover. A Deligne cocycle describing a principal $\mathscr{G}$-bundle with connective structure consists of the transition functions $\{g_{ab}, g_{abc}, \Lambda_{ab}\}$ with $\Lambda_{ab} \in \Omega^1(U_a \cap U_b, L_{-1})$ and the connective structure $\{A_a, B_a\} \in \Omega^1(U_a, L_0) \oplus \Omega^2(U_a, L_{-1})$ with curvatures

$$\mathcal{F}_a := \mathrm{d}A_a + \tfrac{1}{2}\mu_2(A_a, A_a) + \mu_1(B_a) \ ,$$
$$H_a := \mathrm{d}B_a + \mu_2(A_a, B_a) - \tfrac{1}{3!}\mu_3(A_a, A_a, A_a)$$

6D Self-Dual Higher Gauge Theory

Twistor Space

- Consider $\mathcal{N} = (2,0)$ superspace $M := \mathbb{C}^{6|16}$ with coordinates (x^{AB}, η_I^A) with $A, B, \ldots = 1, \ldots, 4$ and $I, J, \ldots = 1, \ldots, 4$. Then,

$$P_{AB} := \partial_{AB} \ , \quad D_A^I := \partial_A^I - 2\Omega^{IJ}\eta_J^B\partial_{AB}$$

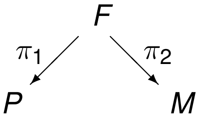
  have the non-vanishing (anti-)commutation relations

$$\{D_A^I, D_B^J\} = -4\Omega^{IJ}P_{AB}$$

- Define the correspondence space F to be $F := \mathbb{C}^{6|16} \times \mathbb{P}^3$ with coordinates $(x^{AB}, \eta_I^A, \lambda_A)$.
- Introduce a rank-3|12 distribution $\langle V^A, V^{IAB}\rangle \hookrightarrow TF$ by $V^A := \lambda_B\partial^{AB}$ and $V^{IAB} := \frac{1}{2}\varepsilon^{ABCD}\lambda_C D_D^I$ which is integrable. Hence, we have foliation $P := F/\langle V^A, V^{IAB}\rangle$.

Twistor Space

- On P, we may use coordinates (z^A, η_I, λ_A) with $z^A\lambda_A = \Omega^{IJ}\eta_I\eta_J$ and thus



with π_2 being the trivial projection and

$$\pi_1 : (x^{AB}, \eta_I^A, \lambda_A) \mapsto (z^A, \eta_I, \lambda_A) = \\ = ((x^{AB} + \Omega^{IJ}\eta_I^A\eta_J^B)\lambda_B, \eta_I^A\lambda_A, \lambda_A)$$

- A point $x \in M$ corresponds to a $\mathbb{P}^3$ in P, while a point $p \in P$ corresponds to a $3|12$-superplane

$$x^{AB} = x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D + 2\Omega^{IJ}\varepsilon^{CDE[A}\lambda_C\theta_{IDE}\eta_0{}_J^{B]} ,$$
$$\eta_I^A = \eta_0{}_I^A + \varepsilon^{ABCD}\lambda_B\theta_{ICD}$$

Penrose–Ward Transform: $P \overset{\pi_1}{\leftarrow} F \overset{\pi_2}{\rightarrow} M$

Let $\mathscr{G}$ be a Lie 2-quasi-group. There is a bijection between equivalence classes

(i) of holomorphic *M*-trivial principal $\mathscr{G}$-bundles on P and

(ii) of solutions to the constraint system on the chiral superspace M

$$
\begin{aligned}
F_A{}^B &= \mu_1(B_A{}^B)\ , \quad F_{AB}{}^I_C = \mu_1(B_{AB}{}^I_C)\ , \quad F^{IJ}_{AB} = \mu(B^{IJ}_{AB})\ ,\\
H^{AB} &= 0\ ,\\
H_A{}^{BI}{}_C &= \delta^B_C \psi^I_A - \tfrac{1}{4}\delta^B_A \psi^I_C\ ,\\
H_{AB}{}^{IJ}_{CD} &= \varepsilon_{ABCD}\phi^{IJ}\ , \quad \text{with} \quad \phi^{IJ}\Omega_{IJ} = 0\\
H^{IJK}_{ABC} &= 0
\end{aligned}
$$

This is a quasi-isomorphism of L_∞-algebras.

$4D$ Super Yang–Mills Theory

Ambitwistor Space

- Consider $M := \mathbb{C}^{4|12}$ with coordinates $(x^{\alpha\dot{\alpha}}, \theta^{i\alpha}, \eta_i^{\dot{\alpha}})$ where $\alpha, \dot{\alpha}, \ldots = 1, 2$ and $i, j, \ldots = 1, \ldots, 3$. Then,

$$P_{\alpha\dot{\alpha}} := \partial_{\alpha\dot{\alpha}} \ , \quad D_{i\alpha} := \partial_{i\alpha} + \eta_i^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \ , \quad D^i_{\dot{\alpha}} := \partial^i_{\dot{\alpha}} + \theta^{i\alpha} \partial_{\alpha\dot{\alpha}}$$

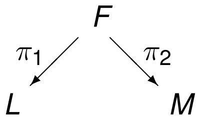
have the non-vanishing (anti-)commutation relations

$$\{D_{i\alpha}, D^j_{\dot{\alpha}}\} = 2\delta_i^j P_{\alpha\dot{\alpha}}$$

- Define $F := \mathbb{C}^{4|12} \times \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x^{\alpha\dot{\alpha}}, \theta^{i\alpha}, \eta_i^{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}})$.
- Introduce a rank-$1|6$ distribution $\langle V, V_i, V^i \rangle \hookrightarrow TF$ by $V := \mu^\alpha \lambda^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}$, $V_i := \mu^\alpha D_{i\alpha}$, and $V^i := \lambda^{\dot{\alpha}} D^i_{\dot{\alpha}}$ which is integrable. Hence, we have foliation $L := F / \langle V, V_i, V^i \rangle$.

Ambitwistor Space

- On L, we may use coordinates $(z^\alpha, w^{\dot\alpha}, \theta^i, \eta_i, \mu_\alpha, \lambda_{\dot\alpha})$ with $z^\alpha\mu_\alpha - w^{\dot\alpha}\lambda_{\dot\alpha} = 2\theta^i\eta_i$ and thus



with π_2 being the trivial projection and

$$\pi_1 : (x^{\alpha\dot\alpha}, \theta^{i\alpha}, \eta_i^{\dot\alpha}, \mu_\alpha, \lambda_{\dot\alpha}) \;\mapsto\; (z^\alpha, w^{\dot\alpha}, \theta^i, \eta_i, \mu_\alpha, \lambda_{\dot\alpha}) =$$
$$= ((x^{\alpha\dot\alpha} - \theta^{i\alpha}\eta_i^{\dot\alpha})\lambda_{\dot\alpha}, (x^{\alpha\dot\alpha} + \theta^{i\alpha}\eta_i^{\dot\alpha})\mu_\alpha, \theta^{i\alpha}\mu_\alpha, \eta_i^{\dot\alpha}\lambda_{\dot\alpha}, \mu_\alpha, \lambda_{\dot\alpha})$$

- A point $x \in M$ corresponds to a $\mathbb{P}^1 \times \mathbb{P}^1$ in L, while a point $p \in L$ corresponds to a 1|6-superline

$$x^{\alpha\dot\alpha} = x_0^{\alpha\dot\alpha} + t\mu^\alpha\lambda^{\dot\alpha} + t^i\mu^\alpha\eta_i^{\dot\alpha} - t_i\theta^{i\alpha}\lambda^{\dot\alpha}\ ,$$
$$\theta^{i\alpha} = \theta_0^{i\alpha} + t^i\mu^\alpha\ , \quad \eta_i^{\dot\alpha} = \eta_{0\,i}^{\dot\alpha} + t_i\lambda^{\dot\alpha}$$

Penrose–Ward Transform: $L \stackrel{\pi_1}{\leftarrow} F \stackrel{\pi_2}{\rightarrow} M$

Due to Witten and Isenberg–Yasskin–Green we have the following result. Let G be a Lie group. There is a bijection between equivalence classes

(i) of holomorphic M-trivial principal G-bundles on L and

(ii) of solutions to the constraint system of maximally supersymmetric Yang–Mills theory on M

$$F_{i\alpha j\beta} = \epsilon_{\alpha\beta}\epsilon_{ijk}\phi^k \ , \quad F^{ij}_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta}\epsilon^{ijk}\phi_k \ , \quad F_{i\alpha}{}^{j}_{\dot\beta} \ = \ 0$$

To prove this theorem, one makes use of the Čech description of holomorphic principal bundles. This is an intrinsically on-shell approach as the holomorphicity of the bundles encodes the equations of motion. How do we go off-shell?

- To go off-shell, we make use of the Dolbeault approach. In particular, a holomorphic principal G-bundle can be described by a smooth principal G-bundle equipped with a (0, 1)-connection locally given by a Lie(G)-valued (0, 1)-form $A^{0,1}$ subject to

$$F^{0,2} = \bar{\partial} A^{0,1} + \tfrac{1}{2}[A^{0,1}, A^{0,1}] = 0$$

- For a 3-dimensional Calabi–Yau manifold X, this equation is variational as it follows from the holomorphic Chern–Simons action functional

$$S = \int_X \Omega^{3,0} \wedge \left\{ \tfrac{1}{2}\langle A^{0,1}, \bar{\partial} A^{0,1}\rangle + \tfrac{1}{3!}\langle A^{0,1}, [A^{0,1}, A^{0,1}]\rangle \right\}$$

- Ambitwistor space is a Calabi–Yau supermanifold, however, its bosonic part is 5-dimensional, and so we cannot use this action functional.

Dolbeault Approach and Higher Gauge Theory

- We propose to consider higher holomorphic Chern–Simons theory.
- Let $\mathscr{G}$ be a Lie 3-quasi-group. Consider a smooth principal $\mathscr{G}$-bundle equipped with Lie($\mathscr{G}$)-valued $(0, p|0)$-forms $A^{0,1|0}$, $B^{0,2|0}$, and $C^{0,3|0}$ with

$$
\begin{aligned}
S := \int \Omega^{5|6,0} \wedge \Big\{ & \langle A^{0,1|0}, \bar\partial C^{0,3|0}\rangle + \langle B^{0,2|0}, \mu_1(C^{0,3|0})\rangle + \\
& + \tfrac{1}{2}\langle B^{0,2|0}, \bar\partial B^{0,2|0}\rangle + \tfrac{1}{2}\langle A^{0,1|0}, \mu_2(A^{0,1|0}, C^{0,3|0})\rangle + \\
& + \tfrac{1}{2}\langle A^{0,1|0}, \mu_2(B^{0,2|0}, B^{0,2|0})\rangle + \\
& + \tfrac{1}{3!}\langle A^{0,1|0}, \mu_3(A^{0,1|0}, A^{0,1|0}, B^{0,2|0})\rangle + \\
& + \tfrac{1}{5!}\langle A^{0,1|0}, \mu_4(A^{0,1|0}, A^{0,1|0}, A^{0,1|0}, A^{0,1|0})\rangle \Big\}
\end{aligned}
$$

where the fermionic integration is in the sense of Berezin.

- The corresponding equations of motion are

$$\bar{\partial}A^{0,1|0} + \tfrac{1}{2}\mu_2(A^{0,1|0}, A^{0,1|0}) + \mu_1(B^{0,2|0}) = 0\ ,$$

$$\begin{gathered}\bar{\partial}B^{0,2|0} + \mu_2(A^{0,1|0}, B^{0,2|0}) + \\ + \tfrac{1}{3!}\mu_3(A^{0,1|0}, A^{0,1|0}, A^{0,1|0}) + \mu_1(C^{0,3|0}) = 0\ ,\end{gathered}$$

$$\begin{gathered}\bar{\partial}C^{0,3|0} + \mu_2(A^{0,1|0}, C^{0,3|0}) + \tfrac{1}{2}\mu_2(B^{0,2|0}, B^{0,2|0}) + \\ + \tfrac{1}{2}\mu_3(A^{0,1|0}, A^{0,1|0}, B^{0,2|0}) + \tfrac{1}{4!}\mu_4(A^{0,1|0}, A^{0,1|0}, A^{0,1|0}, A^{0,1|0}) = 0\end{gathered}$$

- Recall the minimal model theorem that says that every L_∞-algebra is L_∞-quasi-isomorphic to an L_∞-algebra which has $\mu_1 = 0$. For this algebra, the first equation turns into

$$\bar{\partial}A^{0,1|0} + \tfrac{1}{2}\mu_2(A^{0,1|0}, A^{0,1|0}) = 0$$

and by means of the Penrose–Ward transform this will correspond to maximally supersymmetric Yang–Mills theory in four dimensions.

Conclusions and Outlook

Summary

Any field theory admits an L_∞-algebra and can be recast in Maurer–Cartan form, and the Batalin–Vilkovisky formalism provides the natural framework to discuss higher gauge theory.

The area of twistor geometry and categorified principal bundles can be fruitfully combined to formulate self-dual higher gauge theory in six dimensions. The advantage of twistor geometry is that the e.o.m. and the gauge transformations follow directly from complex algebraic data on twistor space. Higher gauge theory enables us to write down a twistor action principle for 4d maximally supersymmetric Yang–Mills theory.

Many open questions remain, quantisation of higher Chern–Simons theory, the choice of higher gauge group for the self-dual models, the explicit constructions of higher bundles, including the dimensional reductions.

Thank You!