Rational higher structures and variations of topological actions

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Outline

The Whitehead tower

The **Whitehead tower** of a space X is a factorization of the point inclusion $* \to X$

$$* \simeq \lim_{n\to\infty} X_n \to \cdots \to X_2 \to X_1 \to X_0 \simeq X$$

such that

- the space X_k is $(k-1)$-connected
- the map $X_k \to X_{k-1}$ is a fibration which is an isomorphism on homotopy groups π_i for $i \geq k$

Examples:

- Tower for $K(G,n):\ \ * \to K(G,n)$
- Tower for $S^2_{\mathbb{Q}}:\ \ * \to K(\mathbb{Q},3) \to S^2_{\mathbb{Q}}$

Construction of the Whitehead tower

Constructing the Whitehead tower

- Inductively, X_n is $(n-1)$-connected
- There is an isomorphism $\pi_n(X) \cong H^n(X; \pi_n(X))$
- Choose a representative $u \in [X, K(\pi_n(X), n)]$
- X_{n+1} is then the homotopy pullback

$$
\begin{array}{ccc}
X_{n+1} & \longrightarrow & * \\
\downarrow & & \downarrow \\
X_n & \xrightarrow{u} & K(\pi_n(X), n)
\end{array}
$$

Whitehead tower for BSO

Using the fact that $\Omega BSO \simeq \mathrm{SO}$ and thus $\pi_i(B\mathrm{SO}) \cong \pi_{i-1}(\mathrm{SO})$

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\mathbb{Z},7) & \longrightarrow & B\mathrm{SO}\langle 9\rangle & & \\
 & & \downarrow & & \\
K(\mathbb{Z},3) & \longrightarrow & B\mathrm{SO}\langle 8\rangle & \xrightarrow{\frac{1}{6}p_2} & K(\mathbb{Z},8) \\
 & & \downarrow & & \\
K(\mathbb{Z}_2,1) & \longrightarrow & B\mathrm{SO}\langle 4\rangle & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},4) \\
 & & \downarrow & & \\
 & & B\mathrm{SO} & \xrightarrow{\omega_2} & K(\mathbb{Z}_2,2)
\end{array}
$$

$i \pmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Whitehead tower for BSO

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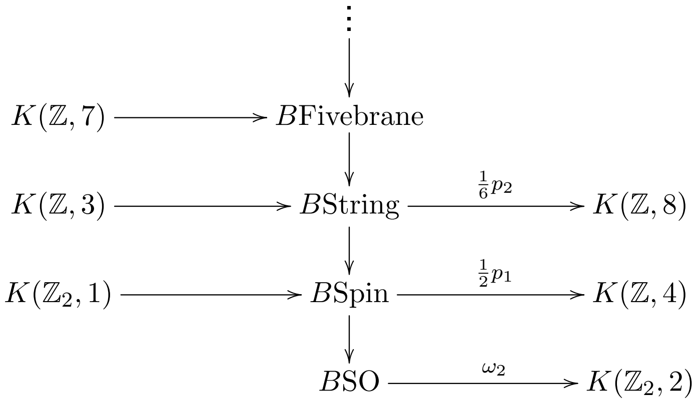


$i \pmod 8$	0	1	2	3	4	5	6	7
$\pi_i(\mathrm{O})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Connected Covers of SO(n)

SO(n):=$\left\{A \in M_n(\mathbb{R}) \middle| A^t A = I \text{ and } \det(A) = 1\right\}$

Spin(n): the universal cover of SO(n) and is a double covering.

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

String(n): introduced by Stolz, is the 6-connected cover of Spin(n) fitting in the short exact sequence

$$1 \to K(\mathbb{Z}, 2) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to 1$$

Following this pattern, Sati, Schreiber, and Stasheff define the **Fivebrane**(n) group as the 7-connected cover of String(n)

$$1 \to K(\mathbb{Z}, 6) \to \mathrm{Fivebrane}(n) \to \mathrm{String}(n) \to 1$$

Lifting the Structure Group

Given: Principal $\mathrm{O}(n)\langle k\rangle$-bundle $P \to M$.
A lifting of the structure group to $\mathrm{O}(n)\langle k+1\rangle$ is a lift of the classifying map

$$\begin{array}{ccc} & & B\mathrm{O}(n)\langle k+1\rangle \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ M & \xrightarrow{f_P} & B\mathrm{O}(n)\langle k\rangle \end{array}$$

- $k = 4$ corresponds to String structures
- $k = 8$ corresponds to Fivebrane structures
- $k = 12$ corresponds to Ninebrane structures

Motivation

String Structures

- Killingback: existence of String structure required for anomaly cancellation in the worldvolume of a string
- Stolz-Teichner: String structures can be understood as a trivialization of a Chern-Simons theory
- Sati: C-field corresponds to a string structure
- Hopkins: String structure gives orientation for Witten genus
- Bunke: choice of string structure trivializes the pfaffian line bundle

Fivebrane Structures

- Setting: X a 10-dimensional spin manifold possibly equipped with gauge bundle.
- H_3 is (electric) NS field coupled to string and $H_7 = *H_3$ is (magnetic) dual field now coupled to fundamental 5-brane.
- Dual Green-Schwarz mechanism requires that the differential

$$dH_7 = \frac{1}{48} p_2(\omega) - ch_4(A)$$

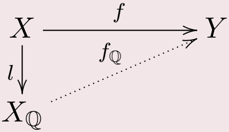
  vanish.
- Sati,Schreiber,Stasheff: This is equivalent to the choice of Fivebrane structure on a Spin manifold.

- We'll call a space **rational** if every homotopy group is a $\mathbb{Q}$ vector space.
- A key property of rational homotopy is the following universal property

Rationalization

The **rationalization** of a space X is a map $l : X \to X_{\mathbb{Q}}$ such that

- $l_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(X_{\mathbb{Q}})$ is an isomorphism,
- Every map $f : X \to Y$ where Y is a rational space, factors uniquely up to homotopy through l,



Fact: Every 1-connected space admits a rationalization

Examples:

- $K(\mathbb{Z}, n)$: Since $H^n(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}$, the generator gives a map

$$\iota : K(\mathbb{Z}, n) \to K(\mathbb{Q}, n)$$

- $B\mathrm{SO}$:
  - $\pi_i(B\mathrm{SO}_{\mathbb{Q}}) = \begin{cases} \mathbb{Q} & i = 4k, \\ 0 & i \neq 4k \end{cases}$
  - $H^*(B\mathrm{SO}_{\mathbb{Q}}; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, \ldots]$ where $|p_i| = 4i$ is the ith Pontrjagin class
- $S^4_{\mathbb{Q}} \simeq K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 7)$.

Rational Abelianization

By rationalizing, we lose information about torsion of the space. However the cohomology becomes more tractable and with regards to the Whitehead tower of $BSO(n)$

Theorem (Sati, W.)

Each element in the sequence of connected spaces $\{\mathrm{String}(n)_{\mathbb{Q}}, \mathrm{Fivebrane}(n)_{\mathbb{Q}}, \mathrm{Ninebrane}(n)_{\mathbb{Q}}, \ldots\}$ *is an abelian topological group, and this group structure is unique up to rational H-equivalence.*

The proof combines results of Lupton, Phillips, Schochet, and Smith and Wockel.

- For Lie group G, its Samelson product vanishes
- This is equivalent to G being rationally homotopy abelian.
- Spaces in Whitehead tower are rationally H-equivalent to a product of Eilenberg-MacLane spaces.
- Samelson product vanishes for each of these spaces.
- The standard loop multiplication on $\Pi K(G_i, n_i)$ is unique group-like, homotopy abelian H-structure (up to rational equivalence).

Gauge Group in Rational Homotopy Theory

For a G-principal bundle $P \to X$, the gauge group is

$$\mathcal{G}(P) := \{\eta \in \mathrm{Aut}(P) \mid \pi \circ \eta = \pi\},$$

i.e. the group of G-equivariant homeomorphisms of P.

- $\mathcal{G}(P) \cong \mathrm{Map}(P, G)^G$
- For $X \times G \to X$, $\mathcal{G}(P) \cong \mathrm{Map}(X, G)$.
- For G abelian, $\mathcal{G}(P) \cong \mathrm{Map}(X, G)$.

Pointed gauge group:

$$\mathcal{G}_1(P) := \{\eta \in \mathcal{G}(P) \mid \eta(*) = *\}$$

Gauge Group in Rational Homotopy Theory

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The Pointed gauge group is

$$\mathcal{G}_1(P) := \{\eta \in \mathcal{G}(P) \mid \eta(*) = *\}$$

- Félix, Oprea: For G a compact group, and X having the homotopy type of a finite CW complex, there are rational homotopy equivalences

$$\mathcal{G}(P) \simeq_{\mathbb{Q}} \mathrm{Map}(X, K), \quad \text{and} \quad \mathcal{G}_1(P) \simeq_{\mathbb{Q}} \mathrm{Map}_*(X, K)$$

- This works for $\mathrm{Spin}(n)\langle k\rangle$-bundles as well.
- Proof relies on structure of rational cohomology.
- Structure preserved when killing homotopy groups.

Gauge Group in Rational Homotopy Theory

Proposition (Sati, W.)

For a $\mathrm{Spin}(n)\langle k\rangle$*-principal bundle* $P \to X$ *over a finite CW complex, there are rational homotopy equivalences*

- $\mathcal{G}(P) \simeq_{\mathbb{Q}} \Pi_j \mathit{Map}(X, K(\mathbb{Q}, 4n_j)),$ *and*
- $\mathcal{G}_1(P) \simeq_{\mathbb{Q}} \Pi_j \mathit{Map}_*(X, K(\mathbb{Q}, 4n_j))$

- We also find that for X an n-dimensional manifold X and G k-connected, then $\mathcal{G}(P)$ is $(k-n)$-connected
- For $G = \mathrm{Spin}(n)\langle k\rangle_{\mathbb{Q}}$, we have

$$\pi_q(\mathcal{G}(P)) \cong \pi_{q+4}(\mathcal{G}(P))$$

Rational Whitehead Tower of BSO

$B\mathrm{SO}$ is 1-connected $\Rightarrow$ $B\mathrm{SO}_{\mathbb{Q}}$ is 1-connected,
so we can construct the Whitehead tower for $B\mathrm{SO}_{\mathbb{Q}}$.

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\mathbb{Q},11) & \longrightarrow & B\mathrm{SO}_{\mathbb{Q}}\langle 12\rangle & & \\
 & & \downarrow & & \\
K(\mathbb{Q},7) & \longrightarrow & B\mathrm{SO}_{\mathbb{Q}}\langle 8\rangle & \xrightarrow{(\frac{1}{6}p_2)_{\mathbb{Q}}} & K(\mathbb{Q},8) \\
 & & \downarrow & & \\
 & & B\mathrm{SO}_{\mathbb{Q}}\langle 4\rangle & \xrightarrow{(\frac{1}{2}p_1)_{\mathbb{Q}}} & K(\mathbb{Q},4)
\end{array}
$$

Fact: For every k, there is a map l_k such that $l_k : B\mathrm{SO}\langle k\rangle \rightarrow B\mathrm{SO}_{\mathbb{Q}}\langle k\rangle$ is the rationalization of $B\mathrm{SO}\langle k\rangle$.

Rational Whitehead Tower of BSO

BSO is 1-connected $\Rightarrow$ $BSO_{\mathbb{Q}}$ is 1-connected,
so we can construct the Whitehead tower for $BSO_{\mathbb{Q}}$.

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
K(\mathbb{Q},11) & \longrightarrow & B\mathrm{Fivebrane}_{\mathbb{Q}} & & \\
 & & \downarrow & & \\
K(\mathbb{Q},7) & \longrightarrow & B\mathrm{String}_{\mathbb{Q}} & \xrightarrow{(\frac{1}{6}p_2)_{\mathbb{Q}}} & K(\mathbb{Q},8) \\
 & & \downarrow & & \\
 & & B\mathrm{SO}_{\mathbb{Q}} & \xrightarrow{(\frac{1}{2}p_1)_{\mathbb{Q}}} & K(\mathbb{Q},4)
\end{array}
$$

Fact: For every k, there is a map l_k such that $l_k : B\mathrm{SO}\langle k\rangle \to B\mathrm{SO}_{\mathbb{Q}}\langle k\rangle$ is the rationalization of $B\mathrm{SO}\langle k\rangle$.

Rational Structures

- $\pi : P \to M$ is an $\mathrm{O}(n)\langle k\rangle$ bundle.
- Rationalizing the classifying map gives

$$f_{\mathbb{Q}} : M_{\mathbb{Q}} \to B\mathrm{O}(n)\langle k\rangle_{\mathbb{Q}}$$

- A **rational $\mathrm{O}(n)\langle k+1\rangle$-structure** is a lift

$$\begin{array}{ccccc} & & B\mathrm{O}(n)\langle k+1\rangle_{\mathbb{Q}} & & \\ & \overset{\tilde{f}_{\mathbb{Q}}}{\nearrow} & \downarrow & & \\ M & \xrightarrow{f_{\mathbb{Q}}} & B\mathrm{O}(n)\langle k\rangle_{\mathbb{Q}} & \xrightarrow{u_{\mathbb{Q}}} & K(\pi_{k+1} \otimes \mathbb{Q}, k+1) \end{array}$$

Rational Fivebrane Structures

- A **rational Fivebrane class** is a class $F_{\mathbb{Q}} \in H^7(P;\mathbb{Q})$ such that $\iota_x^* F_{\mathbb{Q}} = \tau(p_2)_{\mathbb{Q}}$ for every $x \in M$.

$$0 \longrightarrow H^7(M;\mathbb{Q}) \xrightarrow{\pi^*} H^7(P;\mathbb{Q}) \xrightarrow{\iota_*} H^7(\mathrm{O}(n)\langle k\rangle;\mathbb{Q}) \longrightarrow H^8(M;\mathbb{Q})$$

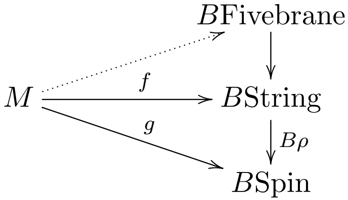
$$F_{\mathbb{Q}} \longmapsto \iota_* F_{\mathbb{Q}}$$

$$\tau(p_2)_{\mathbb{Q}} \longmapsto (p_2)_{\mathbb{Q}}$$

- Every Fivebrane class corresponds to a rational Fivebrane class
- Isomorphism classes of rational Fivebrane structures in 1-1 correspondence with rational Fivebrane classes
- The set of rational Fivebrane classes is a torsor for $H^7(M;\mathbb{Q})$

Underlying Spin Bundles

Let $\pi : P \to M$ be a String bundle and $f : M \to B\mathrm{String}$ its classifying map

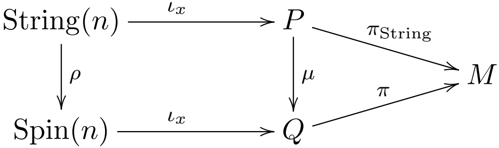


The map $g = B\rho \circ f$ classifies a Spin bundle over M.

We'll refer to the bundle $g^* E\mathrm{Spin} \to M$ of a String bundle $\pi : P \to M$ as **underlying Spin bundle** .

Rational Spin-Fivebrane Classes

Again $\pi : P \to M$ is a String(n) bundle, but now let $\tilde{\pi} : Q \to M$ be its underlying Spin(n) bundle



- There is a bundle map $\mu : P \to Q$
- $\rho^* : H^7(\mathrm{Spin}(n); \mathbb{Q}) \cong H^7(\mathrm{String}(n); \mathbb{Q})$
- Set $S := (\rho^*)^{-1}\tau(p_2)_{\mathbb{Q}}$ to be the image of transgression of the obstruction under the isomorphism ρ.
- There is an induced map $\mu^* : H^7(Q; \mathbb{Q}) \to H^7(P; \mathbb{Q})$

Definition

A **rational Spin-Fivebrane** class is a class $W_{\mathbb{Q}} \in H^7(Q; \mathbb{Q})$ such that $\iota_x^* W = S$

Rational Spin-Fivebrane Classes

- μ^* maps rational Spin-Fivebrane Classes to Fivebrane classes
- μ^* is also surjective
- Using spectral sequences, we calculate the kernel of this map.

Theorem (Sati, W)

- *Any rational Fivebrane class $F_{\mathbb{Q}}$ is the image of a rational Spin-Fivebrane class $W_{\mathbb{Q}}$*
- *Two rational Spin-Fivebrane classes correspond to the same rational Fivebrane structure if*

$$W_{\mathbb{Q}} - W'_{\mathbb{Q}} = \mathcal{S} \cdot \pi^* \phi_4$$

where S is the String class and $\phi_4 \in H^4(M; \mathbb{Q})$.

Applications

We will work in ”extended” spacetime which in our case is the Spin bundle.

- Fibered WZW models [Distler-Sharpe]
- T-duality in loop space [Bouwknegt, Han, Mathai]
- Double/Exceptional Field Theories [Berman, Cederwall, Ikeda]
- Supergerbes [Sati-Schreiber]
- Many others

NS5-brane action

Setting: NS5-brane extended on seven-dimensional Spin manifold X^7.

- Topological part of action functional is

$$S = \int_{X^7} C_3 \wedge G_4$$

- C_3 is the C-field and can be interpreted as a choice of String structure
- Cohomologically we consider the action as the pairing

$$S^{\text{coh}} = \langle [C_3] \cup [G_4], [X^7] \rangle$$

  where $[X^7]$ is the fundamental homology class.
- We interpret integrand as difference of two rational Spin-Fivebrane structures on X^7.

Rational Spin-Ninebrane Structures

We have a similar story for Ninebrane structures. Let $P \to M$ be an $\mathrm{O}(n)\langle 11\rangle$-bundle.

- A rational ninebrane structure is a lift of the rationalized classifying map

$$\begin{array}{ccccc} & & BO(n)\langle 16\rangle_{\mathbb{Q}} & & \\ & \overset{\tilde{f}_{\mathbb{Q}}}{\nearrow} & \downarrow & & \\ M & \xrightarrow{f_{\mathbb{Q}}} & BO(n)\langle 12\rangle_{\mathbb{Q}} & \xrightarrow{(p_3)_{\mathbb{Q}}} & K(\pi_{12}\otimes\mathbb{Q}, 12) \end{array}$$

- A rational Ninebrane class is a class $N_{\mathbb{Q}} \in H^{11}(P;\mathbb{Q})$ such that $\iota_x^* N = \tau p_3$ for every $x \in M$.
- A rational Spin-Ninebrane class is a class $W \in H^{11}(Q;\mathbb{Q})$

Rational Spin-Ninebrane Classes

Let $\pi_{11} : P \to M$ be an $\mathrm{O}(n)\langle 11\rangle$-bundle and $\pi : Q \to M$ it's underlying Spin-bundle.

- There is an isomorphism $\rho^* : H^{11}(\mathrm{Spin}; \mathbb{Q}) \to H^{11}(\mathrm{O}(n)\langle 11\rangle; \mathbb{Q})$
- There is a bundle map $\rho : P \to Q$
- Under the pullback, rational Spin-Ninebrane classes are mapped to rational Ninebrane classes.

Theorem (Sati, W)

- *Any rational Ninebrane structure $\mathcal{N}_{\mathbb{Q}}$ is the image of a rational Spin-Ninebrane class $\mathcal{M}_{\mathbb{Q}} = \rho^* \mathcal{N}_{\mathbb{Q}}$.*
- *Two Spin-Ninebrane classes correspond to the same rational Ninebrane structure if*

$$\mathcal{N}_{\mathbb{Q}} - \mathcal{N}'_{\mathbb{Q}} = \mathcal{S} \cdot \pi^* \psi_8 + \mathcal{F} \cdot \pi^* \phi_4$$

where $\mathcal{S}$ is the String class and $\mathcal{F}$ is the Fivebrane structure class.

Dual Green-Schwarz anomaly cancellation

Setting: 10-dimensional manifold M^{10} with Spin bundle along with Yang-Mills gauge bundle E.

- Dual Green-Schwarz involves the dual $H_7 = \star H_3$.
- The action functional involves a term of the form

$$\mathcal{L} = H_7 \wedge J_4$$

  where $\phi_4 = [J_4] = p_1(TM) - ch_2(E) \in H^4(M^{10})$.
- $[H_7]$ corresponds to a rational Spin-Fivebrane class $\mathcal{F}$.
- Setting $\psi_8 = 0$ in this case, we then interpret the action functional as a difference

$$\mathcal{N}_{\mathbb{Q}} - \mathcal{N}'_{\mathbb{Q}} = \mathcal{F} \cdot \pi^* \phi_4$$

M-theory action

Setting: String manifold Y^{11}

- Topological action functional given by

$$\int_{Y^{11}} \left(\frac{1}{6} G_4 \wedge G_4 \wedge C_3 - I_8 \wedge C_3 \right)$$

- Interpret $[\frac{1}{6} G_4 \wedge G_4 - I_8]$ as a class $y_8 \in H^8(Y^{11}; \mathbb{Q})$.
- Again we interpret C_3 as a String structure $\mathcal{S}$ and set $\phi_4 = 0$
- So the integrand can be interpreted as a variation of the Spin-Ninebrane structure

$$\mathcal{N}_{\mathbb{Q}} - \mathcal{N}'_{\mathbb{Q}} = \mathcal{S} \cdot \pi^* y_8$$

Conclusion

Summary

- By studying the Whitehead tower rationally, Whitehead tower simplifies and structure groups become abelian
- Isomorphisms of specific degrees of cohomology allow us to classify structures using underlying Spin bundles
- Terms in the action functionals can be identified as variations of classes on the Spin bundle

Future Directions

- Lift from the rational to the integral setting
- Study corresponding partition functions

Thank You!